THÈSE DE DOCTORAT DE L'ÉTABLISSEMENT UNIVERSITÉ BOURGOGNE FRANCHE-COMTÉ

PRÉPARÉE À L'UNIVERSITÉ DE BOURGOGNE

École doctorale numéro 553
Carnot-Pasteur

Doctorat de Mathématiques

Par

Clémentine Lemarié--Rieusset

Motivic knot theory

Thèse présentée et soutenue à Dijon, le 15 septembre 2023

Composition du Jury :

Aravind Asok	Professeur à l'University of Southern California (États-Unis)	Président et rapporteur
Oliver Röndigs	Professeur à l'Universität Osnabrück (Allemagne)	Rapporteur
Yonatan Harpaz	Chargé de recherche au CNRS (Université Sorbonne Paris Nord, France)	Examinateur
Lucy Moser-Jauslin	Professeure à l'Université de Bourgogne (France)	Examinatrice
Emmanuel Wagner	Professeur à l'Université Paris Cité (France)	Examinateur
Frédéric Déglise	Directeur de recherche au CNRS (ENS de Lyon, France)	Directeur de thèse
Adrien Dubouloz	Chargé de recherche au CNRS (Université de Bourgogne, France)	Codirecteur de thèse

ECOLE DOCTORALE
CARNOT - PASTEUR

Titre : Théorie motivique des nœuds

Mots clés : Théorie de l'homotopie motivique, Théorie des nœuds, Entrelacs, Groupes de Witt, K-théorie de Milnor-Witt, Complexe de Rost-Schmid.

Résumé : Dans ce manuscrit, nous créons une théorie en géométrie algébrique par analogie avec la théorie des nœuds. Étant donné que cette nouvelle théorie s'appuie sur la théorie de l'homotopie motivique (plus précisément, sur la théorie de l'intersection quadratique), nous la nommons *théorie motivique des nœuds*. Plus précisément, nous étudions l'*enlacement motivique*: comment deux F-sous-schémas fermés disjoints dans un F-schéma ambiant peuvent être enlacés (F étant un corps parfait). En théorie des nœuds, l'enlacement d'un entrelacs orienté à deux composantes (i.e. de deux nœuds orientés disjoints) est un entier qui compte combien de fois une des composantes tourne autour de l'autre composante. Nous définissons des analogues en géométrie algébrique des entrelacs orientés à deux composantes et de l'enlacement; nous appelons ces analogues de l'enlacement des *enlacements quadratiques*. Nos enlacements quadratiques ne sont pas nécessairement des entiers; ceux que nous étudions le plus sont des éléments du groupe de Witt du corps de base F, qui est un groupe de classes d'équivalence de formes bilinéaires symétriques sur F (ou de manière équivalente, de formes quadratiques sur F, quand la caractéristique de F est différente de 2). Dans un premier temps nous répondons aux questions qui émergent naturellement de ces enlacements quadratiques et dans un second temps nous créons des méthodes de calcul des enlacements quadratiques. Ces méthodes s'appuient sur des formules explicites pour les morphismes de résidus de la K-théorie de Milnor-Witt (qui permettent de calculer des morphismes de bord pour les complexes de Rost-Schmid) et pour le produit d'intersection de l'anneau de Rost-Schmid (et en particulier de l'anneau de Chow-Witt). Grâce à ces méthodes, nous calculons explicitement nos enlacements quadratiques sur des exemples. Certains de ces exemples sont inspirés de la théorie des nœuds, plus spécifiquement des entrelacs toriques (notamment les entrelacs de Hopf et de Salomon).

Title: Motivic knot theory

Keywords: Motivic homotopy theory, Knot theory, Links, Witt groups, Milnor-Witt K-theory, Rost-Schmid complex.

Abstract: In this thesis, we introduce a counterpart in algebraic geometry to knot theory. Since this new theory uses motivic homotopy theory (specifically, quadratic intersection theory), we name it *motivic knot theory*. We focus on *motivic linking*, which means that we study how two disjoint closed F-subschemes of an ambient F-scheme can be intertwined, i.e. *linked* together (where F is a perfect field). In knot theory, the linking number of an oriented link with two components (i.e. of two disjoint oriented knots) is an integer which counts how many times one of the components turns around the other component. We define counterparts in algebraic geometry to oriented links with two components and to the linking number; we call these latter counterparts *quadratic linking degrees*. Our quadratic linking degrees are not necessarily integers; the ones we study the most take values in the Witt group of the ground field F, which is a group of equivalence classes of symmetric bilinear forms over F (or equivalently, of quadratic forms over F, when the characteristic of F is different from 2). After answering questions which naturally arise from these quadratic linking degrees, we devise methods to compute them. These methods rely on explicit formulas for the residue morphisms of Milnor-Witt K-theory (from which boundary maps for the Rost-Schmid complexes are constructed) and for the intersection product of the Rost-Schmid ring (and in particular of the Chow-Witt ring). Using these methods, we explicitly compute our quadratic linking degrees on examples. Some of these examples are inspired by knot theory, specifically by torus links (including the Hopf and Solomon links).

UBFC
UNIVERSITÉ BOURGOGNE FRANCHE-COMTÉ

Université Bourgogne Franche-Comté
32 avenue de l'Observatoire
25000 Besançon

Introduction

Knot theory emerged in the end of the nineteenth century and is still widely studied today. Motivic knot theory is a new theory which begins with this thesis and is a counterpart in algebraic geometry to knot theory. We call it *motivic knot theory* because it relies heavily on motivic homotopy theory (specifically on quadratic intersection theory). Before we describe the contents of this thesis, let us recall some notions from these theories we have mentioned.

Knot theory

Knots in knot theory are similar to knots in everyday life, except that the two ends of the piece of string are glued together and the string has no thickness, so that a knot is an embedding of the circle $\mathbb{S}^1$ in $\mathbb{R}^3$, or rather in the 3-sphere $\mathbb{S}^3$ (which is $\mathbb{R}^3$ with a point at infinity). Knots have two possible orientations (see Figure 1.1 on page 23 for the orientations of the trivial knot (i.e. the circle), which is called "unknot") and oriented knots (i.e. knots with a fixed orientation) are important objects of study in knot theory.

In addition to (oriented) knots, knot theorists are also interested in (oriented) links, which are finite disjoint unions of (oriented) knots (which are called the components of the link). Of particular interest to us is the linking number of an oriented link with two components, which is an integer in $\mathbb{Z}$ which counts the number of times one of the components turns around the other component (the sign indicating the direction it turns in). The absolute value of the linking number does not depend on the orientations (but it is important to have orientations in order to compute it). The linking number has several applications outside of mathematics, one of which is in the study of DNA supercoiling (see for instance the article [BOS02]).

Motivic homotopy theory and quadratic intersection theory

Motivic homotopy theory began in 1999 with Morel and Voevodsky's article [MV99] and has already proved very useful (for instance, motivic homotopy theory was used to prove Milnor's conjecture and later on its generalisation the Bloch-Kato conjecture). This theory applies methods from algebraic topology to algebraic geometry, which is why it is particularly useful for our endeavor: creating a counterpart to knot theory in algebraic geometry. We are particularly interested in a theory which is central in motivic homotopy theory: quadratic intersection theory. In quadratic intersection theory, instead of considering $\mathbb{Z}$-linear combinations of subvarieties of a scheme, we consider subvarieties together with coefficients in Milnor-Witt K-theory graded rings (which are constructions with a deep relationship with motivic homotopy theory, see [Mor12, Corollary 1.25]) together with "twists" which are very useful for considerations pertaining to orientations. The term "quadratic" comes from the fact that for every perfect field F and for all $n < 0$, the n-th Milnor-Witt K-theory group $K_n^{\mathrm{MW}}(F)$ is canonically isomorphic to the Witt group $\mathrm{W}(F)$ and the ring $K_0^{\mathrm{MW}}(F)$ is canonically isomorphic to the Grothendieck-Witt ring $\mathrm{GW}(F)$; the Witt ring $\mathrm{W}(F)$ and Grothendieck-Witt ring $\mathrm{GW}(F)$ being constructed from symmetric bilinear forms on F, or equivalently from quadratic forms on F if the characteristic of F is different from 2. Milnor-Witt K-theory comes with residue morphisms from which "boundary maps" are constructed in quadratic intersection theory. These boundary maps, together with the intersection product in quadratic intersection theory, which is the product of the "Rost-Schmid ring"— which generalises the "Chow-Witt ring" and is the direct sum of the "Rost-Schmid groups"— are tools which are crucial for this thesis.

A bird's-eye view of the thesis

In this thesis, we define counterparts in algebraic geometry to oriented links with two components and to the linking number. In a sense, we answer the question "How many times does this closed F-subscheme turn around this other closed F-subscheme in this ambient F-scheme?" (where F is a perfect field and the two closed F-subschemes in question are disjoint). Our answer is an element of the Witt group $\mathrm{W}(F)$ (thus is an integer in the case $F = \mathbb{R}$, but not in general) or of the Grothendieck-Witt group $\mathrm{GW}(F)$ or of the first Milnor-Witt K-theory group $K_1^{\mathrm{MW}}(F)$, or is a couple of such elements (depending on the context).

The first counterpart to the linking number we present in this thesis is the ambient quadratic linking degree. It is thus named because it is obtained from an element (called the ambient quadratic linking class) of a Rost-Schmid group of the ambient F-scheme (similarly to the linking number which can be obtained from an element of a singular cohomology group of the ambient 3-sphere $\mathbb{S}^3$). In the cases which are studied in this thesis, the ambient quadratic linking degree is in the Witt group $\mathrm{W}(F)$ or in the Grothendieck-Witt group $\mathrm{GW}(F)$.

The other counterpart to the linking number (or rather to the linking couple, whose components are the linking number up to sign) we present in this thesis is the quadratic linking degree couple. This couple is obtained from an element (called the quadratic linking class) of a Rost-Schmid group of the link (similarly to the linking couple which is obtained from an element of a singular cohomology group of the link). In the cases which are studied in this thesis, each of the components of the quadratic linking degree couple is in $\mathrm{W}(F)$, in $\mathrm{GW}(F)$, or in $K_1^{\mathrm{MW}}(F)$.

Unlike the quadratic linking class and the ambient quadratic linking class which can be defined in a rather general context, the ambient quadratic linking degree requires knowledge of the Rost-Schmid group in which the ambient quadratic linking class lives (namely, an isomorphism between this group and a well-known group, such as $\mathrm{W}(F)$) and the quadratic linking degree couple requires knowledge of the Rost-Schmid group in which the quadratic linking class lives (namely, an isomorphism between this group and a well-known group, such as $\mathrm{W}(F) \oplus \mathrm{W}(F)$).

Thus, the ambient quadratic linking degree and the quadratic linking degree couple complete each other well, since the former only requires knowledge of Rost-Schmid groups of the ambient F-scheme (which is useful in situations in which the Rost-Schmid groups of the link are not well-known) while the latter only requires knowledge of a Rost-Schmid group of the link and the fact that some Rost-Schmid groups of the ambient F-scheme are zero (which is useful in situations in which the Rost-Schmid groups of the ambient F-scheme are not well-known).

Since the ambient quadratic linking degree and the quadratic linking degree couple depend on the orientation of the oriented link, we also define "invariants of the quadratic linking degree". These invariants are quantities computed from the ambient quadratic linking degree or the quadratic linking degree couple which do not depend on the orientation of the oriented link and thus anwer more accurately the question "How many times does this closed F-subscheme turn around this other closed F-subscheme in this ambient F-scheme?". In knot theory, the absolute value of the linking number is the answer to how many times one knot turns around another knot, and it answers completely this question since it is the only information we can get from the linking number which does not depend on the choices of orientations. In motivic knot theory, it is much harder to find interesting invariants of the quadratic linking degree. We find several such invariants by looking closely at the structure of the Witt group $\mathrm{W}(F)$ and at the structure of the Grothendieck-Witt group $\mathrm{GW}(F)$.

After we define these new mathematical objects and prove results which answer questions which naturally arise from these, we turn to computations. There were two main difficulties to overcome in order to be able to compute the ambient quadratic linking degree, the quadratic linking degree couple and their invariants:

- The quadratic linking class (which is an intermediate step both for the ambient quadratic linking degree and for the quadratic linking degree couple) is defined as the image by a boundary map of an intersection product in quadratic intersection theory, but the definitions of the intersection product and of the boundary map are not well-suited to computations. We gave an explicit definition (i.e. one well-suited to computations) of the residue morphisms of Milnor-Witt K-theory (and proved that it is equivalent to the classical definition) which enabled computations of the boundary maps in the situations in which we need them. A recent formula also enabled computations of the intersection product in some of the situations in which we need it, so that computing the quadratic linking class became possible in several situations.

- To get the ambient quadratic linking degree or the quadratic linking degree couple from the quadratic linking class, we need explicit (and computable) isomorphisms between some Rost-Schmid groups and well-known groups (such as $\mathrm{W}(F)$). This has taken some work (and will continue to take some work) since most results on the structure of Rost-Schmid groups are abstract results (in the sense that they show that a Rost-Schmid group is isomorphic to a well-known group in a way which does not provide an isomorphism between these groups).

In this thesis, we present methods to compute the quadratic linking class, the ambient quadratic linking class, the ambient quadratic linking degree and the quadratic linking degree couple in some cases (in which the ambient quadratic linking degree takes values in the Witt group $\mathrm{W}(F)$ and the quadratic linking degree couple takes values in $\mathrm{W}(F) \oplus \mathrm{W}(F)$). We then make explicit computations of these and of invariants of the quadratic linking degree on several examples. The first of these examples, the Hopf link, is a simple example over any perfect field. The second example is rather a family of examples, which we call binary links, over any perfect field of characteristic different from 2, which we have created in order to realise classes of binary quadratic forms in $\mathrm{W}(F)$ as ambient quadratic linking degrees (and also as components of quadratic linking degree couples) and to showcase the usefulness of an invariant of the quadratic linking degree we have defined. (Note that classes of unary quadratic forms in $\mathrm{W}(F)$ can be realised as ambient quadratic linking degrees (and also as components of quadratic linking degree couples) of variants of the Hopf link.) The third family of examples, which we consider over the field $\mathbb{R}$ of real numbers, is inspired by knot theory: it is a family of examples, indexed by $n \in \mathbb{N}$, which is a counterpart to a family of links in knot theory (which is also indexed by $n \in \mathbb{N}$), and verifies that the absolute value of the ambient quadratic linking degree (which is in $\mathrm{W}(\mathbb{R}) \simeq \mathbb{Z}$) of the n-th member of this family (which is equal to n) is equal to the absolue value of the linking number of its counterpart in knot theory. The same is true of the absolute value of each component of the quadratic linking degree couple of the n-th member of this family.

Outline of the thesis

We now discuss the contents of this thesis in some more detail.

Let us begin by highlighting the fact that there is a list of notations on page 17 which recalls usual notations. Notations which are specific to this thesis are introduced at the beginning of the section they are used in if they are local notations and in environments (especially Notation environments) if they are global notations. In any case, all important notations used in this thesis are referenced (with page numbers) in the Index of notations and all important words and phrases used in this thesis are referenced (with page numbers) in the General index (you may find these at the end of this thesis).

This thesis is divided into two parts.

Part I, *Mathematical background*, presents material which is important for the development of motivic knot theory. Most of this material is not new, with an important exception: Theorem 2.46 which enables the computation of the residue morphisms of Milnor-Witt K-theory (which we also included in our preprint [Lem23]). In Chapter 1, we present the aspects of knot theory and higher-dimensional knot theory which are important for the development of motivic knot theory. In Chapter 2, we present the Witt ring, the Grothendieck-Witt ring and the Milnor-Witt K-theory ring. In this chapter we also prove Theorem 2.46, which will be used to compute boundary maps in Chapters 6 and 7. In Chapter 3, we present the aspects of quadratic intersection theory which are useful for the development of motivic knot theory.

Part II, *Motivic linking*, is the beginning of motivic knot theory. In this part, we define oriented links with two components in algebraic geometry and we study their linking, i.e. how their components are intertwined. Everything in this part is new (note that a study of oriented links of type $(\mathbb{A}^2_F \setminus \{0\}, \mathbb{A}^2_F \setminus \{0\}, \mathbb{A}^4_F \setminus \{0\})$ was also included in our preprint [Lem23]).

In Chapter 4, we define counterparts in algebraic geometry to oriented links with two components (i.e. couples of disjoint oriented knots) and to the linking class (from which the linking number and the linking couple can be defined). We call this latter counterpart the *quadratic linking class*. The quadratic linking class is an interesting object of study because it contains the linking information of the oriented link (i.e. the information about how its components are intertwined) and does not depend on any convention, unlike the ambient quadratic linking degree and the quadratic linking degree couple. This allows the quadratic linking class to be defined in a coherent manner in a very wide variety of contexts. More precisely, we can associate a quadratic linking class to any couple (Z_1, Z_2) of disjoint irreducible smooth finite-type closed F-subschemes of same dimension in an irreducible smooth finite-type F-scheme X (with F a perfect field) which is equipped with orientation classes of the normal sheaves of Z_1 and Z_2 in X (in particular, the normal sheaves of Z_1 and Z_2 in X need to be orientable, which means that their determinants need to be isomorphic to squares). One of the reasons behind our statement that the quadratic linking class is defined in a coherent manner is Theorem 4.23: the pullback along a smooth morphism of the quadratic linking class of an oriented link with two components is the quadratic linking class of the pullback of this oriented link (under some minor additional assumptions). We end this chapter by studying some special settings in which the study of the quadratic linking class seems particularly interesting.

In Chapter 5, we define counterparts in algebraic geometry to the linking number and to the linking couple, which we call respectively the *ambient quadratic linking degree* and the *quadratic linking degree couple*. The inclusion of the first component of the oriented link in the ambient F-scheme induces a morphism of Rost-Schmid groups which takes the part of the quadratic linking class which lives over this first component to what we call the *ambient quadratic linking class*. Note that the inclusion of the second component of the oriented link in the ambient F-scheme induces a morphism of Rost-Schmid groups which takes the part of the quadratic linking class which lives over this second component to the opposite of the ambient quadratic linking class. The ambient quadratic linking class, like the quadratic linking class, does not depend on any convention. However, both of these are hard to understand (or rather their values are hard to understand), since it is difficult to compare elements of Rost-Schmid groups (especially in the case of the quadratic linking class, since it lives in a Rost-Schmid group of the link). This is why we introduce the ambient quadratic linking degree (respectively the quadratic linking degree couple), which is obtained from the ambient quadratic linking class (resp. the quadratic linking class) by an isomorphism between the Rost-Schmid group in which it lives and a well-known group. These are easier to understand (in the sense that comparisons of their values on different oriented links are easier to make), at the price of the introduction of a convention: the choice of the above-mentioned isomorphism (since there are several such isomorphisms in general). In the case of the ambient quadratic linking degree, this means that we fix a convention for the ambient F-scheme (an isomorphism between one of its Rost-Schmid groups and a well-known group) but in the case of the quadratic linking degree couple the situation is more complicated: we need to fix a convention for each link (an isomorphism between one of its Rost-Schmid groups and a well-known group) in a coherent manner (so that we can compare the quadratic linking degree couples of different links). This is why we introduce the notion of oriented links of type (Y_1, Y_2, X): oriented links in the ambient F-scheme X together with a parametrisation $\varphi_1 : Y_1 \to X$ of their first component (i.e. a closed immersion $\varphi_1 : Y_1 \to X$ whose image is their first component) and a parametrisation $\varphi_2 : Y_2 \to X$ of their second component (i.e. a closed immersion $\varphi_2 : Y_2 \to X$ whose image is their second component). By using the couple of orientation classes and the couple of parametrisations of an oriented link of type (Y_1, Y_2, X), we obtain an isomorphism between the twisted Rost-Schmid group in which its quadratic linking class lives and the direct sum of an untwisted Rost-Schmid group of Y_1 and of an untwisted Rost-Schmid group of Y_2, and it suffices to fix once and for all an isomorphism between this direct sum and a well-known group in order to have a quadratic linking degree couple which is defined in a coherent manner for all oriented links of type (Y_1, Y_2, X). We end this chapter with the creation of *invariants of the quadratic linking degree*, which are quantities computed from the ambient quadratic linking degree or from the quadratic linking degree couple which do not depend on the orientations (nor on the parametrisations in some cases) of the oriented link (of a certain type in the case of the quadratic linking degree couple). See Propositions 5.25 and 5.30 for simple invariants of the quadratic linking degree and Theorems 5.28 and 5.33 for more involved and potentially more interesting invariants of the quadratic linking degree. In Section 7.2, we give examples (over the field $\mathbb{Q}$ of rational numbers) which show the usefulness of Σ_2 (applied to the ambient quadratic linking degree or to a component of the quadratic linking degree couple (in $\mathrm{W}(\mathbb{Q})$)) which is the first of these more involved invariants. More precisely, we show that Σ_2 can distinguish between infinitely many oriented links. In Propositions 5.26 and 5.31, we create complete invariants of the quadratic linking degree over the field $\mathbb{R}$ of real numbers (for the ambient quadratic linking degree, and for the quadratic linking degree couple when none of its components is in $K_1^{\mathrm{MW}}(\mathbb{R})$). By "complete invariants", we mean invariants which capture all the information in the ambient quadratic linking degree or in the quadratic linking degree couple which does not depend on the orientations.

In Chapter 6, we give methods to compute the quadratic linking class (see Theorem 6.1), the ambient quadratic linking class (see Corollary 6.2), the ambient quadratic linking degree (see Theorem 6.3) and the quadratic linking degree couple (see Theorem 6.4) in the case $\mathbb{A}^2_F \setminus \{0\} \sqcup \mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$ under reasonable assumptions on the oriented link (and under assumptions on j_1 and j_2, which parametrise different (coherent) versions of the quadratic linking class etc.). We also list other cases (in the beginning of the chapter) in which similar theorems can be established (and have not been established yet due to lack of time).

In Chapter 7, we give examples of oriented links in the case $\mathbb{A}^2_F \setminus \{0\} \sqcup \mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$ and compute the quadratic linking class, the ambient quadratic linking class, the ambient quadratic linking degree, the quadratic linking degree couple and invariants of the quadratic linking degree on these examples. We begin by a simple example (the Hopf link, see Section 7.1) which is defined over any perfect field F, then we consider a family of examples (the binary links, see Section 7.2) which are defined over any perfect field of characteristic different from 2 and which show that the class of any binary quadratic form in $\mathrm{W}(F)$ can be realised as an ambient quadratic linking degree (and as a component of a quadratic linking degree couple); the Hopf link and its variants already show that the class of any unary quadratic form in $\mathrm{W}(F)$ can be realised as an ambient quadratic linking degree (and as a component of a quadratic linking degree couple). Finally, in Section 7.3, we consider a family of examples over $\mathbb{R}$ which is inspired by knot theory (specifically, by the torus links $T(2, 2n)$). More examples could be tackled but have not been tackled yet due to lack of time.

Finally, let us highlight the fact that there is a list of future works on page 16 which references (with page numbers) the Future work environments in this thesis.

Contents

List of Figures

List of Tables

List of Future works

List of Notations

We denote by:

$\mathbb{N}$	the positive integers (e.g. $1 \in \mathbb{N}$)
$\mathbb{N}_0$	the nonnegative integers (e.g. $0 \in \mathbb{N}_0$)
$\mathbb{Z}$	the integers (e.g. $-1 \in \mathbb{Z}$)
$\mathbb{Q}$	the rational numbers (e.g. $\frac{1}{2} \in \mathbb{Q}$)
$\mathbb{R}$	the real numbers (e.g. $\pi \in \mathbb{R}$)
$\mathbb{C}$	the complex numbers (e.g. $i \in \mathbb{C}$)
$\mathbb{R}^n$	the n-th (cartesian) power of $\mathbb{R}$ (where $n \in \mathbb{N}$)
$\mathbb{S}^{n-1}$	the set $\{(x_1, \ldots, x_n) \in \mathbb{R}^n, \sum_{i=1}^{n} x_i^2 = 1\}$ (a.k.a. unit $(n-1)$-sphere)
$K_1 \sqcup K_2$	the union of the disjoint subsets K_1 and K_2 of a set
$\{1, \ldots, n\}$	the set whose elements are the integers m such that $1 \leq m \leq n$
$\lfloor x \rfloor$	the floor of $x \in \mathbb{R}$ (i.e. the greatest integer n such that $n \leq x$)
$\lceil x \rceil$	the ceiling of $x \in \mathbb{R}$ (i.e. the least integer n such that $n \geq x$)
R^*	the group of units (a.k.a. invertible elements) of the ring R
$\mathrm{char}(R)$	the characteristic of the ring R (e.g. $\mathrm{char}(\mathbb{Q}) = 0$)
$\mathrm{Spec}(R)$	the spectrum of the ring R (as a scheme)
$\mathbb{A}^n_F$	the scheme $\mathrm{Spec}(F[x_1, \ldots, x_n])$ (a.k.a. affine n-space)
$\mathbb{A}^n_F \setminus \{0\}$	the affine n-space minus the origin (as a scheme)
$\mathrm{Proj}(R)$	the projective spectrum of the ring R (as a scheme)
$\mathbb{P}^n_F$	the scheme $\mathrm{Proj}(F[x_0, \ldots, x_n])$ (a.k.a. projective n-space)
$\mathcal{F}_p$	the stalk of the sheaf $\mathcal{F}$ at the point p (and $\mathcal{O}_{X,p} := (\mathcal{O}_X)_p$)
$\kappa(p)$	the residue field of the point p of a scheme
$\ker(f)$	the kernel of the group morphism f (i.e. $\ker(f) = \{x, f(x) = 0\}$)
$\mathrm{im}(f)$	the image of the group morphism f (i.e. $\mathrm{im}(f) = \{y, \exists\, x, f(x) = y\}$)

Part I

Mathematical background

Chapter 1

Knot theory

Before we develop motivic linking in Part II — a counterpart in algebraic geometry to classical linking — we introduce in this chapter knot theory (especially classical linking) to readers who are unfamiliar with it, in order to give the intuition behind motivic linking (and more generally motivic knot theory). In contrast to the following chapters, this chapter is rather informal, as its goal is to present the ideas in knot theory which are of particular interest for the development of motivic knot theory. If you wish to know more about knot theory, we recommend these five introductory books: [Ada94], [Cro04], [Lic97], [Mur96], [Rol90].

In Section 1.1 we paint the big picture of what knot theory consists of, while in Section 1.2 we give formal definitions of important notions in knot theory. In Section 1.3 we focus on the linking number, which is the link invariant to which we create (and study) counterparts in Part II. Section 1.4 focuses on torus links (from which all link classes which can be represented by complex algebraic varieties can be constructed), while Section 1.5 presents the fact that all link classes can be represented by real algebraic varieties, whose polynomial equations can be effectively determined. Finally, in Section 1.6, we present higher dimensional knot theory and a generalisation of the linking number.

1.1 What is knot theory?

You probably already encountered knots in your life (for instance, to tie your shoelaces). You also probably already encountered links (for instance, the links in a necklace or in a bracelet).

Knot theory is the study of knots and links. In knot theory, knots differ slightly from knots in real life: the two ends of the piece of string (or rope, etc.) are glued together and the string has no thickness. More formally, a knot is an embedding of the circle $\mathbb{S}^1$ in $\mathbb{R}^3$ (or in $\mathbb{S}^3$, see below for details) and a link is a finite disjoint union of knots.

The study of given knots or links goes back centuries, but the systematic study of knots and links began at the end of the nineteenth century. Indeed, that is when the classification of knots and links began. The goal of a classification of a collection of objects is to gather these objects together in classes which verify the following:

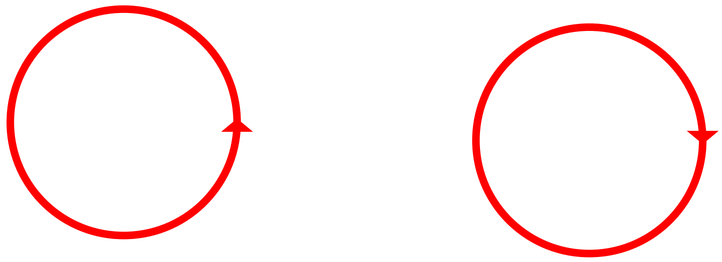
- When given an object of this collection, it is possible to determine to which class it belongs.
- The objects in a class verify the same properties (among the properties which interest you).

This means that once the classification is (at least partially) done, you can easily determine the properties of an object (which has been classified) by determining to which class it belongs then looking up the properties which are verified by this class of objects. This way of thinking (by classifying) is common in mathematics, but also in other sciences and in real life (with less precision).

In knot theory, the collection of objects is the collection of oriented links (which includes oriented knots). An oriented knot is a knot with a direction in which to follow the knot (such as the clockwise direction or the counterclockwise (a.k.a. trigonometric) direction for the circle) and an oriented link is a link whose components/knots are all oriented (thus a link with n components has 2^n possible orientations). The properties of oriented links which interest knot theorists are invariant under ambient isotopy (a relationship between oriented links, see below for details) hence the classes of oriented links are their classes for the equivalence relation of being ambient isotopic.

See Figure 1.1 for the two possible orientations of the unknot (the circle).

It is hard (perhaps impossible) to classify every link (or even every knot) in a meaningful way, so a link invariant (i.e. a characteristic of links which is invariant under ambient isotopy) which takes values in the nonnegative integers was chosen to order the classification (by ascending values), in order to set realistic classification goals (classifying all links with a value of this characteristic below a given value, then increasing this value to set a new goal when this goal is achieved). This characteristic (which in a sense is one way of measuring the complexity of a link) is the crossing number of a link: the minimum of the number of times a two-dimensional picture of the link crosses itself. It is important to take a minimum since two different



(a) The unknot with the trigonometric (a.k.a. counterclockwise) orientation.

(b) The unknot with the clockwise orientation.

Figure 1.1 – The unknot (a.k.a. circle) with the two orientations.

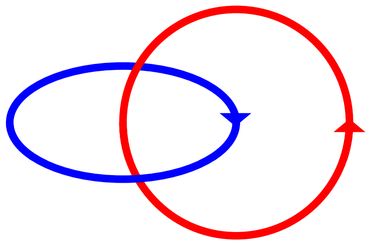


Figure 1.2 – The Hopf link is of crossing number 2 (you can see on this drawing that its crossing number is at most 2).

link diagrams (i.e. two-dimensional pictures of the link) may have different numbers of times the link diagram crosses itself. The links with crossing number 0 are called unlinks (one example of which is the unknot). There is no link with crossing number 1. The Hopf link has crossing number 2 (see Figure 1.2) and the trefoil knot has crossing number 3 (see Figure 1.3). When two strands of a knot cross in a picture, two lines are drawn around the strand which is on top (i.e. nearer to you) and when two different knots cross each other in a picture, the knot on top (i.e. nearer to you) is the one whose colour you see at the crossing.

As the author writes these lines, the links with crossing number at most 16 have been classified (see [HTW98] and [Hos05]). Tables with all the classes of links whose components are prime (i.e. are not connected sums of more than one knot), topologically linked (i.e. there is no homeomorphism H of $\mathbb{R}^3$ onto itself such that the image by H of one of the components of the link and the image by H of another of the components of the link

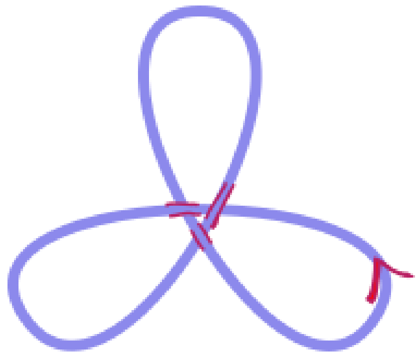


Figure 1.3 – The trefoil knot is of crossing number 3 (you can see on this drawing that its crossing number is at most 3).

can be separated by a plane) and whose crossing number is between 0 and 16 are available in Knotscape[1]. There are 2 518 665 such classes of links (including 1 701 936 classes of knots).

You can also find here[2] a table of the classes of prime knots with crossing number at most 12, except for the unknot, the mirror images of knots in the table, the knots with a reversed orientation from knots in the table, and the mirror images with a reversed orientation from knots in the table.

The knot which is denoted c_m in this table is the m-th knot of crossing number c (there are other names available by ticking squares in the Nomenclature section (and the meaning of each nomenclature is explained when clicking on it)). Several invariants of oriented knots are available in this table.

You can also find here[3] a table of the classes of links with prime components which are topologically linked and have crossing number at most 11, except for knots (which are in the previous table), the mirror images of links in the table, the links with a reversed orientation from links in the table, and the mirror images with a reversed orientation from links in the table.

The link which is denoted $Lckm\{\varepsilon_1, \dots, \varepsilon_p\}$ in this table is the m-th link of crossing number c which is alternating if $k = a$ (which means that there exists a diagram of this link such that each component goes over then under then over then under etc.), nonalternating if $k = n$; the ε_i (which are equal to 0 or 1) denote the changes in orientations from $Lckm\{0, \dots, 0\}$ (click on Name in the Nomenclature section for more information; there are

[1] https://web.math.utk.edu/~morwen/knotscape.html
[2] https://knotinfo.math.indiana.edu/
[3] https://linkinfo.sitehost.iu.edu/

other names available in the Nomenclature section). Several invariants of oriented links are available in this table.

Let us now go into details!

1.2 Knots and links

Knots are topological subspaces of $\mathbb{R}^3$ or of the 3-sphere $\mathbb{S}^3$ which are homeomorphic to $\mathbb{S}^1$ and verify an additional tameness property (for instance, smoothness). The 3-sphere $\mathbb{S}^3$ can be constructed by adding a point at infinity to $\mathbb{R}^3$, which is why it does not matter if we consider knots as being in $\mathbb{R}^3$ or in $\mathbb{S}^3$. In the following, we will consider knots as topological subspaces of the 3-sphere $\mathbb{S}^3$.

Definitions 1.1 (Knots and Links)**.**

- A knot is the image K of a smooth (i.e. indefinitely differentiable) map $\mathbb{S}^1 \to \mathbb{S}^3$ such that the induced map $\mathbb{S}^1 \to K$ is a homeomorphism.
- A link is a finite disjoint union of knots, which are called the components of the link.

Knot theorists are interested in equivalence classes of links for the following equivalence relation (which corresponds well to what happens when you move links around in real life).

Definition 1.2 (Ambient isotopy)**.** An ambient isotopy from a topological subspace N_1 of $\mathbb{S}^3$ to a topological subspace N_2 of $\mathbb{S}^3$ is a continuous map $H : \mathbb{S}^3 \times [0,1] \to \mathbb{S}^3$ such that, denoting for all $t \in [0,1]$ $H_t : \begin{cases} \mathbb{S}^3 & \to & \mathbb{S}^3 \\ x & \mapsto & H(x,t) \end{cases}$, H_0 is the identity, $H_1(N_1) = N_2$ and for all $t \in [0,1]$, H_t is a homeomorphism. If there is an ambient isotopy from N_1 to N_2 then N_1 and N_2 are said to be ambient isotopic.

This is indeed an equivalence relation (take $(x,t) \mapsto x$ for reflexivity, $(x,t) \mapsto H_t^{-1}(x)$ for symmetry and $(x,t) \mapsto H_2(H_1(x,t),t)$ for transitivity).

Let us now talk about orientation.

Similarly to the circle $\mathbb{S}^1$ which can be oriented in the clockwise direction or in the counterclockwise (a.k.a. trigonometric) direction, a knot has two possible orientations. The choice of an orientation of a knot K is the choice of a generator of the singular homology group $H_1(K) \simeq H_1(\mathbb{S}^1) \simeq \mathbb{Z}$.

Definition 1.3 ((Homological) oriented fundamental class)**.** An oriented knot is a knot K together with a generator of the singular homology group

$H_1(K)$ which is called the (homological) oriented fundamental class of K. An oriented link is a link whose components are oriented.

The homological oriented fundamental class (or oriented fundamental class for short) is sometimes simply called the fundamental class, but we will always call it the oriented fundamental class to stress out the fact that it depends on the orientation of the knot.

This definition may seem rather abstract compared to the informal talk on orientations which was made earlier, but it is equivalent to the more visual definition of an orientation, or rather of an orientation class, of a knot as the equivalence class of an orientation of its tangent bundle, i.e. of the datum for each point p of K of a basis (e_p) of the tangent space $T_pK \simeq \mathbb{R}$ of K at p such that the (e_p) vary continuously with p, for the following equivalence relation: $((e_p))_{p\in K}$ and $((e'_p))_{p\in K}$ are equivalent if for every point p in K there exists a positive real number $r_p > 0$ such that $e'_p = r_p.e_p$, which means visually that the arrow e_p and the arrow e'_p point in the same direction. We denote by $\overline{((e_p))_{p\in K}}$ the class of $((e_p))_{p\in K}$.

Indeed, a generator of $H_1(K)$ is the class of a continuous map $\sigma : [0,1] \to K$ which verifies that $\sigma(1) = \sigma(0)$ and that its restriction to $[0,1[$ is a bijection with K. The homological oriented fundamental class of K is the class in $H_1(K)$ of such a σ which goes in the direction pointed by the arrows of the orientation (class) of the tangent bundle of K, and conversely the orientation class of the tangent bundle of K is the one whose arrows point in the direction in which σ goes (as time moves from 0 to 1).

Note that since there is an orientation class $\overline{((a_p, b_p, c_p))_{p\in\mathbb{S}^3}}$ of the ambient space $\mathbb{S}^3$ (which verifies that at every point its tangent space is isomorphic to $\mathbb{R}^3$) which is fixed once and for all (by the "right-hand rule"), there is an equivalent definition of orientation which uses the normal bundle of the knot in the ambient space $\mathbb{S}^3$ instead of its tangent bundle. An orientation of the normal bundle of a knot K in $\mathbb{S}^3$ is the datum for each point p of K of a basis (f_p, g_p) of the normal space $(N_K\mathbb{S}^3)_p \simeq \mathbb{R}^2$ of K in $\mathbb{S}^3$ at p such that the (f_p, g_p) vary continuously with p. An orientation class of a knot K is an equivalence class of orientations of the normal bundle of K in $\mathbb{S}^3$ for the following equivalence relation: $((f_p, g_p))_{p\in K}$ and $((f'_p, g'_p))_{p\in K}$ are equivalent if for every point p in K there exists a 2×2 real matrix A_p with positive determinant such that $\begin{pmatrix} f'_p \\ g'_p \end{pmatrix} = A_p \begin{pmatrix} f_p \\ g_p \end{pmatrix}$. The relationship between $((e_p))_{p\in K}$ and $((f_p, g_p))_{p\in K}$ (when they give the same orientation class of K) is that for every point p in K, the basis (e_p, f_p, g_p) of the tangent space $T_p\mathbb{S}^3 = T_pK \oplus (N_K\mathbb{S}^3)_p$ of $\mathbb{S}^3$ at p verifies that there exists a 3×3

real matrix B_p with positive determinant such that $\begin{pmatrix} e_p \\ f_p \\ g_p \end{pmatrix} = B_p \begin{pmatrix} a_p \\ b_p \\ c_p \end{pmatrix}$.

Remark 1.4. Note that having an orientation class $\overline{((e_p))_{p\in K}}$ of a knot K is equivalent to having a cohomological oriented fundamental class of the knot K, i.e. a generator of the singular cohomology group $H^1(K) \simeq H^1(\mathbb{S}^1) \simeq \mathbb{Z}$. Indeed, the cohomological oriented fundamental class of the knot K is the class of the volume form ω such that for every point p in K: $\omega(p) = \det_{(e_p)}$ (the determinant in the basis (e_p)) and conversely the orientation class $\overline{(e_p)}$ of the tangent bundle of K is the one such that $\det_{(e_p)} = \omega(p)$.

Since ambient isotopy preserves orientation classes, we can consider equivalence classes of oriented links for ambient isotopy. This is what knot theorists strive to classify (see Section 1.1). Knot theorists also strive to compute link invariants: quantities which are computed from an oriented link and only depend on the equivalence class of the oriented link for ambient isotopy. In the next section, we consider such a link invariant for oriented links with two components: the linking number.

1.3 The linking number

The linking number is an invariant of oriented links with two components which counts the number of times one of the components turns around the other component. The sign of the linking number indicates in which direction this component turns around the other component. The linking number has several applications outside of mathematics, one of which is in the study of DNA supercoiling (in which the linking number is sometimes called the topological entanglement); see for instance the article [BOS02].

In Chapters 4 and 5 we will construct counterparts in algebraic geometry of the linking number and in Chapters 6 and 7 we will compute these counterparts. Before we do this, let us introduce the linking number.

See Figure 1.4 for an example of a link of linking number 1 (the Hopf link) and Figure 1.5 for an example of a link of linking number 2 (the Solomon link).

We say that a link with two components is topologically unlinked (or split) if there is a homeomorphism H of $\mathbb{R}^3 = \mathbb{S}^3 \setminus \{*\}$ onto itself such that the images by H of the two components of the link can be separated by a plane (where $*$ is a point which is not on the link). Note that links with two components which are topologically unlinked are of linking number 0 (see Subfigure 1.6a for an example) but the converse is false: the

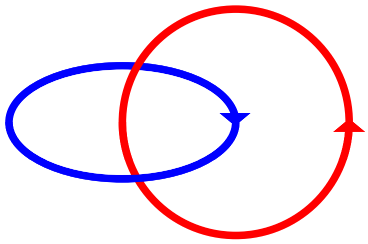


Figure 1.4 – The Hopf link is of linking number 1.

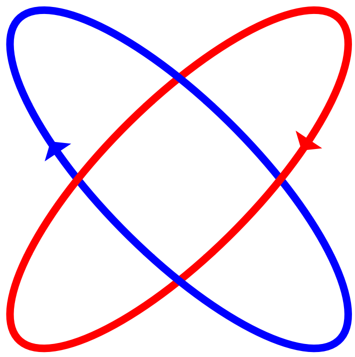
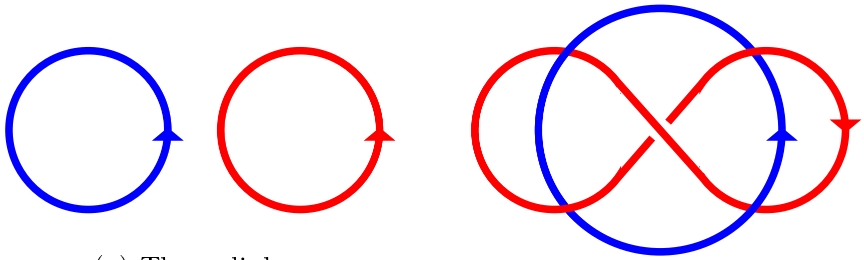


Figure 1.5 – The Solomon link is of linking number 2.

Whitehead link in Subfigure 1.6b is a counterexample. There is however a weaker notion which is equivalent to being of linking number 0: being homologically unlinked (or algebraically split). We say that a link with two components is homologically unlinked if one of the components of the link is the boundary of an orientable surface which is disjoint from the other component. See [BOS02] (in which the "linking number" is the opposite of the linking number (due to their choice of the "left-hand rule" instead of the more commonly used "right-hand rule") but this does not change the instances in which the linking number is equal to 0)).

The fact that being homologically unlinked implies being of linking number 0 will come directly from the following definition of the linking number, which uses the notion of Seifert surface of an oriented knot. A Seifert surface of an oriented knot K is a compact connected oriented surface whose oriented boundary is the oriented knot K. The following three steps give the linking number of two disjoint oriented knots K_1 and K_2 (i.e. of the



(a) The unlink

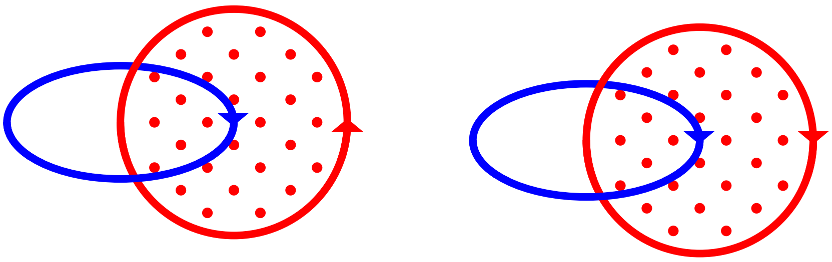
(b) The Whitehead link

Figure 1.6 – The unlink and the Whitehead link are both of linking number 0 but the Whitehead link is topologically linked.

oriented link $K_1 \sqcup K_2$).

- Pick a Seifert surface S_2 for K_2 such that the oriented intersection of K_1 with S_2 is a finite number of oriented points. This is always possible (and the oriented intersection of K_1 with S_2 is equal to the oriented intersection of S_2 with K_1).
- Let P be one of the oriented points mentioned above. We want to associate $\varepsilon_P \in \{-1, 1\}$ to P by taking into account the orientation of the oriented point P. To do this, place yourself so that near the point P, the oriented knot K_1 is coming towards you:
  - If the Seifert surface S_2 is oriented in a trigonometric (a.k.a. counterclockwise) manner, set $\varepsilon_P := 1$ (see for instance Subfigure 1.7a).
  - Otherwise (i.e. the Seifert surface S_2 is oriented in a clockwise manner), set $\varepsilon_P := -1$ (see for instance Subfigure 1.7b).
- The linking number is the sum (over the oriented points P of the oriented intersection of K_1 with S_2) of the ε_P.

See [Rol90, Chapter 5, Section D] for this definition of the linking number (more precisely, (2) and (5) at the beginning of the cited section, (2) being the visual definition (described in [Rol90] with a bicollar of a Seifert surface) and (5) being the more formal definition as the intersection number of S_2 with K_1 (see below for an even more formal variant of this definition)). The fact that ε_P is as described above comes from the fact that $\varepsilon_P = 1$ means that the orientation of the direct sum of tangent spaces



(a) The red (dotted) Seifert surface is oriented in a trigonometric (a.k.a. counterclockwise) manner.

(b) The red (dotted) Seifert surface is oriented in a clockwise manner.

Figure 1.7 – In the two examples above, near the intersection of the blue knot (on the left) with the red (dotted) Seifert surface (for the red knot), the blue knot is coming towards you.

$T_P(K_1) \oplus T_P(S_2)$, which is canonically isomorphic to the tangent space $T_P(\mathbb{S}^3)$, corresponds to the orientation given by the "right-hand rule".

Note that in [Rol90, Chapter 5, Section D], knots are considered to be polygonal rather than smooth, but this is inconsequential since every smooth knot is ambient isotopic to a polygonal knot (and vice versa) and the linking number is a link invariant.

The formal version of the definition above is as follows. We denote by L the oriented link whose components are K_1 and K_2, by N an open tubular neighbourhood of K_2 which is disjoint from K_1 and by E the complement of N in $\mathbb{S}^3$, i.e. $E := \mathbb{S}^3 \setminus N$. We can pick a Seifert surface S_2 of K_2 which induces a class $[S_2]$ in the singular cohomology group $H^1(E)$. The linking number of L is the cup-product of the class (denoted $[K_1]$) of K_1 in $H^2(E, \partial E)$ with $[S_2]$, or rather the image of this cup-product by the isomorphism $H^3(E, \partial E) \to \mathbb{Z}$ which is induced by the orientation of the ambient space $\mathbb{S}^3$ (more precisely, the isomorphism $H^3(E, \partial E) \to \mathbb{Z}$ in question is the Kronecker product with (or "evaluation on") the fundamental class $[E, \partial E]$, a.k.a. the cap product with the fundamental class $[E, \partial E]$; see [Bre97, Chapter VI]). Note that $[K_1] \cup [S_2] = (-1)^2 [S_2] \cup [K_1] = [S_2] \cup [K_1]$.

Note that even though this definition is non-symmetric, the linking number does not depend on the order of the components K_1 and K_2 (see [Rol90, Chapter 5, Section D, Theorem 6]). Further note that the linking number only depends on the oriented link (not on a choice of Seifert surface for one of the components); even better, it only depends on the class of the oriented link for ambient isotopy (see [Cro04, Theorem 3.8.2]), or even better, on its class for concordance (which is a weaker equivalence relation than being ambient isotopic; see [Rol90, Chapter 8, Section F], especially Exercise 13).

We will introduce a new definition of the linking number which will be more symmetric and which will only use classes in cohomology, not chains, so that it will be easier to see that the linking number only depends on the oriented link. To do this, we use Borel-Moore homology and singular cohomology (see [BM60] and [Mas78] for further information on these, as well as [Bre97] for further information on singular cohomology).

Notation 1.5. Let $A \subset M$ be Hausdorff topological spaces. We denote by $H_*^{\mathrm{BM}}(M, A)$ the Borel-Moore homology groups of the pair (M, A) and by $H^*(M, A)$ the singular cohomology groups of the pair (M, A). We denote $H_*^{\mathrm{BM}}(M) := H_*^{\mathrm{BM}}(M, \emptyset)$ and $H^*(M) := H^*(M, \emptyset)$.

We choose to work with these groups because they verify a Poincaré duality theorem which gives an isomorphism $H^k(M \setminus B, M \setminus A) \simeq H_{n-k}^{\mathrm{BM}}(A, B)$ whenever M is an oriented topological manifold of dimension n, $B \subset A$ are locally compact closed subspaces of M and $0 \leq k \leq n$. Note that this is different from the better-known Poincaré duality theorem for singular homology H_* and Čech cohomology $\check{H}^*$ which, under the same assumptions and the extra assumption that A and B are compact, gives an isomorphism $\check{H}^k(A, B) \simeq H_{n-k}(M \setminus B, M \setminus A)$. Indeed, in the former case the closed subspaces A and B of M are on the homology side of the isomorphism (and the open subspaces $M \setminus B$ and $M \setminus A$ are on the cohomology side) whereas in the latter case the closed subspaces A and B are on the cohomology side of the isomorphism (and the open subspaces $M \setminus B$ and $M \setminus A$ are on the homology side).

This Poincaré duality theorem between Borel-Moore homology and singular cohomology, together with the Borel-Moore homology long exact sequence, straightforwardly imply the following theorem, which we will use in our new definition of the linking number.

Theorem 1.6. *Let M be an oriented topological manifold and A be a locally compact closed submanifold of codimension c in M. We have the following long exact sequence, in which the maps are induced by the inclusions $A \to M$ and $M \setminus A \to M$ except for the maps ∂ which are the boundary maps (a.k.a. connecting morphisms):*

$$\dots \longrightarrow H^k(M) \longrightarrow H^k(M \setminus A) \xrightarrow{\partial} H^{k+1-c}(A) \longrightarrow H^{k+1}(M) \longrightarrow \dots$$

We directly get the following corollary.

Corollary 1.7. Let L be a link. We have the following long exact sequence, in which the maps are induced by the inclusions $L \to \mathbb{S}^3$ and $\mathbb{S}^3 \setminus L \to \mathbb{S}^3$ except for the maps ∂ which are the boundary maps:

$$\dots \longrightarrow H^k(\mathbb{S}^3) \longrightarrow H^k(\mathbb{S}^3 \setminus L) \xrightarrow{\partial} H^{k-1}(L) \longrightarrow H^{k+1}(\mathbb{S}^3) \longrightarrow \dots$$

In particular, the following sequence is exact:

$$H^1(\mathbb{S}^3) = 0 \longrightarrow H^1(\mathbb{S}^3 \setminus L) \xrightarrow{\partial} H^0(L) \longrightarrow H^2(\mathbb{S}^3) = 0$$

i.e. the boundary map $\partial : H^1(\mathbb{S}^3 \setminus L) \to H^0(L)$ is an isomorphism.

This corollary allows us to give the following definition.

Definition 1.8 (Couple of Seifert classes)**.** Let L be an oriented link with two components K_1 and K_2. Let $[o_{K_1}] \in H^0(K_1)$ (respectively $[o_{K_2}] \in H^0(K_2)$) be the element which corresponds to the oriented fundamental class of K_1 (resp. K_2), which was defined in Definition 1.3. The couple of Seifert classes of L is the (unique) couple (S_1, S_2) of elements of $H^1(\mathbb{S}^3 \setminus L)$ such that $\partial(S_1) = ([o_{K_1}], 0)$ and $\partial(S_2) = (0, [o_{K_2}])$ (via the isomorphism $H^0(L) \simeq H^0(K_1) \oplus H^0(K_2)$ induced by the inclusions of K_1 and K_2 in $L = K_1 \sqcup K_2$). We call S_1 the Seifert class of K_1 (relative to the link L) and S_2 the Seifert class of K_2 (relative to the link L).

By Poincaré duality, $S_1 \in H^1(\mathbb{S}^3 \setminus L) \simeq H_2^{\mathrm{BM}}(\mathbb{S}^3, L)$ is the class of some surfaces in $\mathbb{S}^3$ whose boundaries lie in the link L; in fact, it is precisely the class of the Seifert surfaces of K_1, and the same is true for S_2 and K_2. See Figure 1.8 for an example of a couple of Seifert surfaces (which in this simple example are disks).

Remark 1.9. If you reverse the orientation of K_1 (respectively of K_2) then $[o_{K_1}]$ (resp. $[o_{K_2}]$) is turned into its opposite hence S_1 (resp. S_2) is turned into its opposite since the boundary map is a group morphism.

Now we can define the linking class, from which we will define the linking number.

Definition 1.10 (Linking class)**.** Let L be an oriented link with two components K_1 and K_2 and let (S_1, S_2) be its couple of Seifert classes, as defined in Definition 1.8. The linking class of L is the image by the boundary map $\partial : H^2(\mathbb{S}^3 \setminus L) \to H^1(L)$ of the cup-product of S_1 with S_2, i.e. $\partial(S_1 \cup S_2)$.

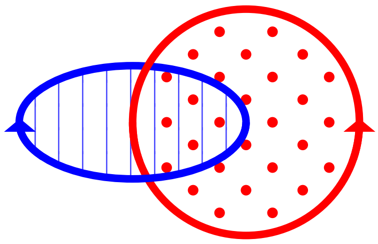


Figure 1.8 – The Hopf link with a Seifert surface hatched in blue for the blue component (on the left) and a Seifert surface dotted in red for the red component (on the right).

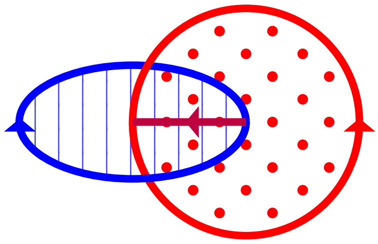


Figure 1.9 – The Hopf link and the oriented intersection of the blue (hatched) Seifert surface with the red (dotted) Seifert surface.

See Figure 1.9 for the oriented intersection of the blue Seifert surface (on the left) with the red Seifert surface (on the right), which is an oriented purple interval in this drawing. See Figure 1.10 for a portrayal of the linking class of the Hopf link (by two oriented green points, one of which lies on the blue component (which was chosen as first component) of the link and one of which lies on the red component of the link).

Note that the linking class contains as much information as the cup-product of S_1 with S_2, since the boundary map $\partial : H^2(\mathbb{S}^3 \setminus L) \to H^1(L)$ is injective (see Corollary 1.7 and note that $H^2(\mathbb{S}^3) = 0$).

Remark 1.11. The linking class is turned into its opposite if you reverse the order of the components, since $S_2 \cup S_1 = (-1)^1(S_1 \cup S_2) = -S_1 \cup S_2$ and the boundary map is a group morphism.

Remark 1.12. If you reverse the orientation of K_1 (respectively of K_2) then the linking class is turned into its opposite since S_1 (resp. S_2) is turned into its opposite (see Remark 1.9).

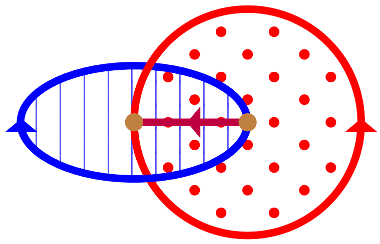


Figure 1.10 – The Hopf link and the oriented boundary of the oriented intersection of the blue (hatched) Seifert surfaces with the red (dotted) Seifert surface.

Let us now define the linking number.

Definition 1.13 (Linking number)**.** The linking number of the oriented link $L = K_1 \sqcup K_2$ is the image of the part of the linking class of L which is in $H^1(K_1)$ by the composite of the morphism $i_1 : H^1(K_1) \to H^3(\mathbb{S}^3)$ which is induced by the inclusion of K_1 in $\mathbb{S}^3$ and of the isomorphism $r : H^3(\mathbb{S}^3) \to \mathbb{Z}$ which corresponds to the "right-hand rule".

The fact that this definition of the linking number is equivalent to the definition which was made earlier follows from the stability property of the cup-product which is described in [Dol95, Chapter VII, 8.10]. Indeed, in our case this property tells us that the part of the quadratic linking class $\partial(S_1 \cup S_2)$ which is in $H^1(K_1)$ is sent to $[K_1] \cup [S_2]$ by the morphism $H^1(K_1) \to H^3(E, \partial E)$ which is induced by the inclusion of K_1 in E (since the oriented knot K_1 is the boundary of the Seifert surface S_1). Also note that the isomorphism $H^3(E, \partial E) \to \mathbb{Z}$ we mentioned earlier depends on the choice of the (oriented) fundamental class $[E, \partial E]$, i.e. on the orientation of E, and that we implicitly chose the orientation which is induced by the orientation of $\mathbb{S}^3$, i.e. by the isomorphism $r : H^3(\mathbb{S}^3) \to \mathbb{Z}$ which corresponds to the "right-hand rule".

Remark 1.14. If you reverse the orientation of K_1 (respectively of K_2) then the linking number is turned into its opposite since the linking class is turned into its opposite (see Remark 1.12).

Remark 1.15. Note that the image of the part of the linking class of the oriented link $L = K_1 \sqcup K_2$ which is in $H^1(K_2)$ by the composite of the morphism $i_2 : H^1(K_2) \to H^3(\mathbb{S}^3)$ which is induced by the inclusion of K_2 in $\mathbb{S}^3$ and of the isomorphism $r : H^3(\mathbb{S}^3) \to \mathbb{Z}$ which corresponds to the "right-hand rule" is the opposite of the linking number. Indeed, the linking class $\partial(S_1 \cup S_2)$ is in the kernel of the morphism $H^1(L) \to H^3(\mathbb{S}^3)$ which is induced by the inclusion of L in $\mathbb{S}^3$ (see Corollary 1.7) and this morphism is the composite of the isomorphism $H^1(L) \to H^1(K_1) \oplus H^1(K_2)$ (which is induced by the inclusions of K_1 and K_2 in $L = K_1 \sqcup K_2$) and of the direct sum of the morphisms $i_1 : H^1(K_1) \to H^3(\mathbb{S}^3)$ and $i_2 : H^1(K_2) \to H^3(\mathbb{S}^3)$. (Another way of proving this is to use the more general version of the stability property of the cup-product (see [Dol95, Chapter VII, 8.19(2)]) and to identify which part comes from $H^1(K_1)$ and which part comes from $H^1(K_2)$.) It follows from this and from the fact that the linking class is turned into its opposite if you reverse the order of the components (see Remark 1.11) that the linking number does not depend on the order of the components.

Note that a definition similar to our definition of the linking number is made between Exercise 8 and Exercise 9 in [Rol90, Chapter 5, Section D], with an important difference: in Rolfsen's definition, he considers Seifert surfaces in the four-dimensional disc $\mathbb{D}^4$ whose boundary is $\mathbb{S}^3$ and defines the linking number as the intersection number of these surfaces (which can be chosen so as to intersect in a finite number of points since they are surfaces in $\mathbb{D}^4$).

Remark 1.16. Note that the cohomological oriented fundamental classes $[\omega_{K_1}] \in H^1(K_1)$ of K_1 and $[\omega_{K_2}] \in H^1(K_2)$ of K_2 (see Remark 1.4) fix an isomorphism $h_1 : H^1(K_1) \to \mathbb{Z}$ (the isomorphism which sends $[\omega_{K_1}]$ to 1) and an isomorphism $h_2 : H^1(K_2) \to \mathbb{Z}$ (the isomorphism which sends $[\omega_{K_2}]$ to 1) respectively. Also note that the morphisms $i_1 : H^1(K_1) \to H^3(\mathbb{S}^3)$ and $i_2 : H^1(K_2) \to H^3(\mathbb{S}^3)$ are surjective since they are in the following exact sequences (see Theorem 1.6):

$$H^1(K_1) \xrightarrow{i_1} H^3(\mathbb{S}^3) \longrightarrow H^3(\mathbb{S}^3 \setminus K_1) = 0$$

$$H^1(K_2) \xrightarrow{i_2} H^3(\mathbb{S}^3) \longrightarrow H^3(\mathbb{S}^3 \setminus K_2) = 0$$

(where $H^3(\mathbb{S}^3 \setminus K_1) = 0$ and $H^3(\mathbb{S}^3 \setminus K_2) = 0$ since $\mathbb{S}^3 \setminus K_1$ and $\mathbb{S}^3 \setminus K_2$ are orientable connected noncompact manifolds). Therefore, the group morphisms $r \circ i_1 \circ (h_1)^{-1} : \mathbb{Z} \to \mathbb{Z}$ and $r \circ i_2 \circ (h_2)^{-1} : \mathbb{Z} \to \mathbb{Z}$ are surjective hence each is the identity of $\mathbb{Z}$ or the opposite (which sends $m \in \mathbb{Z}$ to $-m$). It follows from this, Definition 1.13 and Remark 1.15 that $h_1 \oplus h_2$ sends the linking class of $L = K_1 \sqcup K_2$ to (n, n), $(n, -n)$, $(-n, n)$ or $(-n, -n)$, where n is the linking number of L.

Definition 1.17 (Linking couple)**.** The linking couple of the oriented link $L = K_1 \sqcup K_2$ is the image of the linking class of L by the composite of the isomorphism $H^1(L) \to H^1(K_1) \oplus H^1(K_2)$ (which is induced by the inclusions of K_1 and K_2 in $L = K_1 \sqcup K_2$) and of the isomorphism $h_1 \oplus h_2 : H^1(K_1) \oplus H^1(K_2) \to \mathbb{Z} \oplus \mathbb{Z}$ (see Remark 1.16).

Remark 1.18. If you reverse the order of the components then the linking couple is either the same (if $r \circ i_1 \circ (h_1)^{-1}$ and $r \circ i_2 \circ (h_2)^{-1}$ are both the identity of $\mathbb{Z}$ or both the opposite) or is turned into its opposite (if $r \circ i_1 \circ (h_1)^{-1}$ is the identity of $\mathbb{Z}$ and $r \circ i_2 \circ (h_2)^{-1}$ is the opposite or vice versa).

Remark 1.19. If you reverse the orientation of the first component (respectively the second component) of the oriented link then the first component (resp. the second component) of the linking couple stays the same and the second component (resp. the first component) of the linking couple is turned into its opposite.

Finally, let us introduce link homotopy (which was defined by Milnor in [Mil54]).

Definition 1.20 (Link homotopy)**.** A link homotopy from an oriented link $L = K_1 \sqcup \cdots \sqcup K_n$ with $n \in \mathbb{N}$ components to an oriented link $L' = K'_1 \sqcup \cdots \sqcup K'_n$ with n components is the data of n continuous maps $H_1, \ldots, H_n : \mathbb{S}^1 \times [0,1] \to \mathbb{S}^3$ such that, denoting for all $i \in \{1, \ldots, n\}$ and $t \in [0,1]$, $H_{i,t} : \begin{cases} \mathbb{S}^1 & \to & \mathbb{S}^3 \\ x & \mapsto & H_i(x,t) \end{cases}$, for all $i \in \{1, \ldots, n\}$, $H_{i,0}(\mathbb{S}^1) = K_i$ and $H_{i,1}(\mathbb{S}^1) = K'_i$, and for all $t \in [0,1]$, the sets $H_{1,t}(\mathbb{S}^1), \ldots, H_{n,t}(\mathbb{S}^1)$ are pairwise disjoint (i.e. for all $i \neq j \in \{1, \ldots, n\}$, $H_{i,t}(\mathbb{S}^1) \cap H_{j,t}(\mathbb{S}^1) = \emptyset$). If there is a link homotopy from L to L' then L and L' are said to be link homotopic.

Note that link homotopy is an equivalence relation and that if $n = 1$ then it is merely homotopy (hence every two oriented knots are link homotopic). If $n \geq 2$ then two oriented links with n components are link homotopic if and only if you can deform one continuously into the other while keeping the n components pairwise disjoint. Note that any oriented link is link homotopic to an oriented link whose components are all unknotted circles (a.k.a. unknots). The class of an oriented link for link homotopy describes how the components of the link are "linked" together, how they turn around each other.

In the case of oriented links with two components, the linking number is a complete invariant for link homotopy, i.e. two oriented links with two

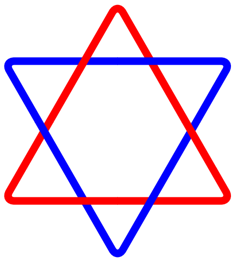


Figure 1.11 – The torus link $T(2,6)$ (without orientation on this drawing).

components are link homotopic if and only if they have the same linking number (see [Mil54, Section 5]).

Note that the unlink (see Subfigure 1.6a) has linking number 0, that the Hopf link (see Figure 1.4) has linking number 1 and that the Solomon link (see Figure 1.5) has linking number 2. The Hopf link (a.k.a. $T(2,2)$) and the Solomon link (a.k.a. $T(2,4)$) are part of a family of torus links $(T(2,2n))_{n\in\mathbb{N}}$ (see Figure 1.11 for $T(2,6)$; for $n \geq 3$, $T(2,2n)$ can be pictured as two intertwined n-gons) which verifies that for all $n \in \mathbb{N}$, $T(2,2n)$ is of linking number n (see the next section). Thus, the unlink, the family of torus links $(T(2,2n))_{n\in\mathbb{N}}$ and the family given by reversing the orientation of one of the components of $T(2,2n)$ (with $n \in \mathbb{N}$), make up a family of representatives for the link homotopy classes of oriented links with two components (see Remark 1.14). We present torus links in the following section.

1.4 Torus links

Torus links are links which can be drawn on the surface of a torus. They are indexed by couples of integers $(p,q) \in \mathbb{Z} \times \mathbb{Z}$. If $d \in \mathbb{N}$ is the greatest common divisor of p and q (by convention, $d := 1$ if $p = q = 0$) then $T(p,q)$ is an oriented link with d components, each of which wraps around the torus $\frac{p}{d}$ times meridionally and $\frac{q}{d}$ times longitudinally (the signs of p and q indicating the directions). For further details on the definition of torus links, see [Mur96, Chapter 7]. Note that if the greatest common divisor of p and q is 2, then the linking number of the oriented torus link $T(p,q)$ (which has two components) is equal to $\dfrac{pq}{4}$. See [BFS14, Theorem 4.2] but note that there is a typo there: their result should be divided by 2 (their proof consists in counting the number of crossings (which are all positive here) in the braid representation, but the linking number is the number of crossings divided by 2 (when all the crossings are positive), not the number of crossings).

For each $p, q \in \mathbb{N}$, the torus link $T(p, q)$ is a complex algebraic link, which means that there is a complex polynomial $R_{p,q} \in \mathbb{C}[u, v]$ and a positive real number $\varepsilon_{p,q} > 0$ such that:

- $R_{p,q}$ vanishes at the origin $(0, 0) \in \mathbb{C}^2$;
- the origin is an isolated singularity for $R_{p,q}$, i.e. the origin is a singularity for $R_{p,q}$ (which means that $\frac{\partial R_{p,q}}{\partial u}$ and $\frac{\partial R_{p,q}}{\partial v}$ both vanish at the origin) and there is an open neighbourhood $U_{p,q}$ of the origin in $\mathbb{C}^2$ such that $R_{p,q}$ has no singularity in $U_{p,q} \setminus \{(0, 0)\}$;
- for all $0 < \varepsilon \leq \varepsilon_{p,q}$, there is a diffeomorphism $h_{p,q,\varepsilon} : \mathbb{S}^3_\varepsilon \to \mathbb{S}^3$ such that $h_{p,q,\varepsilon}(V(R_{p,q}) \cap \mathbb{S}^3_\varepsilon) = T(p, q)$, where $\mathbb{S}^3_\varepsilon := \{(u, v) \in \mathbb{C}^2, |u|^2 + |v|^2 = \varepsilon^2\}$ and $V(R_{p,q}) := \{(u, v) \in \mathbb{C}^2, R_{p,q}(u, v) = 0\}$.

In fact, $R_{p,q}$ can be chosen to be the complex polynomial $u^p - v^q$. Thus, the torus link $T(p, q)$ is called the link of the singularity $(0, 0)$ of the complex curve defined by $u^p - v^q$. See the classical reference [Mil69] or the historical account [Dur99].

Note that not many links are complex algebraic links. Indeed, complex algebraic links are all unions of iterated torus links (see [Ore21] for the definition of iterated torus links). However, there is a reasonable way to define algebraic links in general so that every link is an algebraic link.

1.5 All links are algebraic

In their article [AK81], Akbulut and King defined algebraic links in a similar manner to complex algebraic links, with two important differences: the complex polynomial in two variables was replaced with two real polynomials in four variables and the isolated singularity was replaced with a weakly isolated singularity.

Definition 1.21 (Algebraic link)**.** A link L is an algebraic link if there are two real polynomials $P, Q \in \mathbb{R}[x, y, z, t]$ and a positive real number $\varepsilon_0 > 0$ such that:

- P and Q vanish at the origin $(0, 0, 0, 0) \in \mathbb{R}^4$;
- the origin is a weakly isolated singularity of (P, Q), i.e. the origin is a singularity of (P, Q) (which means that $\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial P}{\partial z}, \frac{\partial P}{\partial t}, \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial Q}{\partial z}, \frac{\partial Q}{\partial t}$

all vanish at the origin) and there is an open neighbourhood U of the origin in $\mathbb{R}^4$ such that (P,Q) has no singularity in $V(P,Q) \cap (U \setminus \{(0,0,0,0)\})$, where $V(P,Q) := \{(x,y,z,t) \in \mathbb{R}^4, P(x,y,z,t) = 0, Q(x,y,z,t) = 0\}$;

- for all $0 < \varepsilon \leq \varepsilon_0$, there is a diffeomorphism $h_\varepsilon : \mathbb{S}^3_\varepsilon \to \mathbb{S}^3$ such that $h_\varepsilon(V(P,Q) \cap \mathbb{S}^3_\varepsilon) = L$, where $\mathbb{S}^3_\varepsilon = \{(x,y,z,t) \in \mathbb{R}^4, x^2+y^2+z^2+t^2 = \varepsilon^2\}$.

Note that a complex algebraic link is an algebraic link (you can take the real part of the complex polynomial as P and the imaginary part of the complex polynomial as Q). In their article [AK81], Akbulut and King prove that every link is an algebraic link! However, their proof does not give explicit polynomials P and Q as in Definition 1.21. In his recent paper [Bod22], Bode provides an algorithm which gives explicit polynomials P and Q as in Definition 1.21.

1.6 Higher dimensional knot theory

In this section, we first consider the linking number of higher-dimensional (smooth) links with two components, then we mention the different contexts in which higher-dimensional knots and links are studied (to the best of our knowledge).

Definitions 1.22 ((Higher-dimensional) knots and links)**.**

- Smooth higher-dimensional knots are images of smooth maps from the m-sphere $\mathbb{S}^m$ to the n-sphere $\mathbb{S}^n$ for some integers $m, n \geq 1$.
- Smooth higher-dimensional links are finite disjoint unions of smooth higher-dimensional knots which go into the same sphere (but may come from spheres of different dimensions).

Definition 1.23 (Oriented fundamental class)**.** A higher-dimensional knot $K \simeq \mathbb{S}^m$ is oriented if a generator of $H^0(K) \simeq H^0(\mathbb{S}^m) \simeq \mathbb{Z}$ has been chosen; this generator is called the oriented fundamental class of K and is denoted $[o_K]$. A higher-dimensional link is oriented if all its components (i.e. the knots of which it is a union) are oriented.

Once an orientation of the ambient sphere (i.e. the sphere in which the considered higher-dimensional links live) has been fixed, a classical way to define the linking number of a higher-dimensional link $L = K_1 \sqcup K_2$ with two components is as the intersection number of K_1 with a "Seifert surface" of K_2 (which is not necessarily a surface anymore) or as the intersection number of a “Seifert surface” of K_1 with K_2 (which gives the same number up to a sign). For this intersection number to be well-defined (and not always zero), the sum of the dimensions of K_1 and K_2 needs to be one less than the dimension of the ambiant sphere: if $K_1 \simeq \mathbb{S}^m$ and $K_2 \simeq \mathbb{S}^n$, then they need to lie in $\mathbb{S}^{m+n+1}$. Indeed, we want the intersection of the dimension m chain K_1 and of a dimension $n+1$ “Seifert surface” of K_2 to be of dimension 0 in order to obtain an “intersection number” (by identifying the zeroth homology group of the ambient sphere with $\mathbb{Z}$). See [ST80, Section 77] for further details on this definition of the higher-dimensional linking number (and more generally [ST80, Chapter X] for a discussion of intersection numbers).

In the case where $m = n \geq 1$, we can give a definition of the higher-dimensional linking number which generalises Definition 1.13. Let us walk you through this generalisation.

Let $n \geq 1$. We fix an isomorphism $r : H^{2n+1}(\mathbb{S}^{2n+1}) \to \mathbb{Z}$ once and for all (in the case $n = 1$, we choose r to be isomorphism which is induced by the “right-hand rule”), which is the same as fixing an orientation of the ambient sphere $\mathbb{S}^{2n+1}$ once and for all. (If the other isomorphism $H^{2n+1}(\mathbb{S}^{2n+1}) \to \mathbb{Z}$ is chosen instead, then the linking number will be turned into its opposite.)

Let $K_1, K_2 \simeq \mathbb{S}^n$ be two disjoint oriented (higher-dimensional) knots in $\mathbb{S}^{2n+1}$ and $L = K_1 \sqcup K_2$ be the corresponding oriented (higher-dimensional) link with two components.

In order to define the couple of Seifert classes of L, we state the following corollary of Theorem 1.6, which is a direct application of this theorem.

Corollary 1.24. We have the following long exact sequence, in which the maps are induced by the inclusions $L \to \mathbb{S}^{2n+1}$ and $\mathbb{S}^{2n+1} \setminus L \to \mathbb{S}^{2n+1}$ except for the maps ∂ which are the boundary maps:

$$\dots \longrightarrow H^k(\mathbb{S}^{2n+1}) \longrightarrow H^k(\mathbb{S}^{2n+1} \setminus L) \xrightarrow{\partial} H^{k-n}(L) \longrightarrow H^{k+1}(\mathbb{S}^{2n+1}) \longrightarrow \dots$$

In particular, the following sequence is exact:

$$H^n(\mathbb{S}^{2n+1}) = 0 \longrightarrow H^n(\mathbb{S}^{2n+1} \setminus L) \xrightarrow{\partial} H^0(L) \longrightarrow H^{n+1}(\mathbb{S}^{2n+1}) = 0$$

i.e. the boundary map $\partial : H^n(\mathbb{S}^{2n+1} \setminus L) \to H^0(L)$ is an isomorphism.

The following definition generalises Definition 1.8.

Definition 1.25 (Couple of Seifert classes)**.** The couple of Seifert classes of L is the (unique) couple (S_1, S_2) of elements of $H^n(\mathbb{S}^{2n+1} \setminus L)$ such that

$\partial(S_1) = ([o_{K_1}], 0)$ and $\partial(S_2) = (0, [o_{K_2}])$ (via the isomorphism $H^0(L) \simeq H^0(K_1) \oplus H^0(K_2)$ induced by the inclusions of K_1 and K_2 in $L = K_1 \sqcup K_2$). We call S_1 the Seifert class of K_1 (relative to the link L) and S_2 the Seifert class of K_2 (relative to the link L).

Remark 1.26. If you reverse the orientation of K_1 (respectively of K_2) then $[o_{K_1}]$ (resp. $[o_{K_2}]$) is turned into its opposite hence S_1 (resp. S_2) is turned into its opposite since the boundary map is a group morphism.

Now we can define the linking class of L. The following definition generalises Definition 1.10.

Definition 1.27 (Linking class)**.** Let (S_1, S_2) be the couple of Seifert classes of L. The linking class of L is the image by the boundary map $\partial : H^{2n}(\mathbb{S}^{2n+1} \setminus L) \to H^n(L)$ of the cup-product of S_1 with S_2, i.e. $\partial(S_1 \cup S_2)$.

Note that the linking class contains as much information as the cup-product of S_1 with S_2, since the boundary map $\partial : H^{2n}(\mathbb{S}^{2n+1} \setminus L) \to H^n(L)$ is injective (see Corollary 1.24 and note that $H^{2n}(\mathbb{S}^{2n+1}) = 0$).

Remark 1.28. If you reverse the order of the components then the linking class is multiplied by $(-1)^{n^2}$ (i.e. it stays the same if n is even, it is turned into its opposite if n is odd). Indeed, $S_2 \cup S_1 = (-1)^{n^2}(S_1 \cup S_2)$ and the boundary map is a group morphism, hence $\partial(S_2 \cup S_1) = (-1)^{n^2}\partial(S_1 \cup S_2)$.

Remark 1.29. If you reverse the orientation of K_1 (respectively of K_2) then the linking class is turned into its opposite since S_1 (resp. S_2) is turned into its opposite (see Remark 1.26).

Let us now define the linking number of L. The following definition generalises Definition 1.13.

Definition 1.30 (Linking number)**.** The linking number of the oriented link L is the image of the part of the linking class of L which is in $H^n(K_1)$ by the composite of the morphism $i_1 : H^n(K_1) \to H^{2n+1}(\mathbb{S}^{2n+1})$ which is induced by the inclusion of K_1 in $\mathbb{S}^{2n+1}$ and of the isomorphism $r : H^{2n+1}(\mathbb{S}^{2n+1}) \to \mathbb{Z}$.

Remark 1.31. If you reverse the orientation of K_1 (respectively of K_2) then the linking number is turned into its opposite since the linking class is turned into its opposite (see Remark 1.29).

Remark 1.32. Note that the image of the part of the linking class of the oriented link L which is in $H^n(K_2)$ by the composite of the morphism i_2 :

$H^n(K_2) \to H^{2n+1}(\mathbb{S}^{2n+1})$ which is induced by the inclusion of K_2 in $\mathbb{S}^{2n+1}$ and of the isomorphism $r : H^{2n+1}(\mathbb{S}^{2n+1}) \to \mathbb{Z}$ is the opposite of the linking number. Indeed, the linking class $\partial(S_1 \cup S_2)$ is in the kernel of the morphism $H^n(L) \to H^{2n+1}(\mathbb{S}^{2n+1})$ which is induced by the inclusion of L in $\mathbb{S}^{2n+1}$ (see Corollary 1.24) and this morphism is the composite of the isomorphism $H^n(L) \to H^n(K_1) \oplus H^n(K_2)$ (which is induced by the inclusions of K_1 and K_2 in $L = K_1 \sqcup K_2$) and of the direct sum of the morphisms $i_1 : H^n(K_1) \to H^{2n+1}(\mathbb{S}^{2n+1})$ and $i_2 : H^n(K_2) \to H^{2n+1}(\mathbb{S}^{2n+1})$. It follows from this and from Remark 1.28 that the linking number is multiplied by $(-1)^{n^2+1}$ if you reverse the order of the components. In other words, if you reverse the order of the components, then the linking number stays the same if n is odd, and is turned into its opposite if n is even.

Remark 1.33. Note that the cohomological oriented fundamental classes $[\omega_{K_1}] \in H^n(K_1)$ of K_1 and $[\omega_{K_2}] \in H^n(K_2)$ of K_2 (which are defined similarly to what was done in Remark 1.4) fix an isomorphism $h_1 : H^n(K_1) \to \mathbb{Z}$ (the isomorphism which sends $[\omega_{K_1}]$ to 1) and an isomorphism $h_2 : H^n(K_2) \to \mathbb{Z}$ (the isomorphism which sends $[\omega_{K_2}]$ to 1) respectively. Also note that the morphisms $i_1 : H^n(K_1) \to H^{2n+1}(\mathbb{S}^{2n+1})$ and $i_2 : H^n(K_2) \to H^{2n+1}(\mathbb{S}^{2n+1})$ are surjective since they are in the following exact sequences (see Theorem 1.6):

$$H^n(K_1) \xrightarrow{i_1} H^{2n+1}(\mathbb{S}^{2n+1}) \longrightarrow H^{2n+1}(\mathbb{S}^{2n+1} \setminus K_1) = 0$$

$$H^n(K_2) \xrightarrow{i_2} H^{2n+1}(\mathbb{S}^{2n+1}) \longrightarrow H^{2n+1}(\mathbb{S}^{2n+1} \setminus K_2) = 0$$

(where $H^{2n+1}(\mathbb{S}^{2n+1} \setminus K_1) = 0$ and $H^{2n+1}(\mathbb{S}^{2n+1} \setminus K_2) = 0$ since $\mathbb{S}^{2n+1} \setminus K_1$ and $\mathbb{S}^{2n+1} \setminus K_2$ are orientable connected noncompact manifolds). Therefore, the group morphisms $r \circ i_1 \circ (h_1)^{-1} : \mathbb{Z} \to \mathbb{Z}$ and $r \circ i_2 \circ (h_2)^{-1} : \mathbb{Z} \to \mathbb{Z}$ are surjective hence each is the identity of $\mathbb{Z}$ or the opposite (which sends $m \in \mathbb{Z}$ to $-m$). It follows from this, Definition 1.30 and Remark 1.32 that $h_1 \oplus h_2$ sends the linking class of $L = K_1 \sqcup K_2$ to (l, l), $(l, -l)$, $(-l, l)$ or $(-l, -l)$, where l is the linking number of L.

Definition 1.34 (Linking couple)**.** The linking couple of the oriented link $L = K_1 \sqcup K_2$ is the image of the linking class of L by the composite of the isomorphism $H^1(L) \to H^1(K_1) \oplus H^1(K_2)$ (which is induced by the inclusions of K_1 and K_2 in $L = K_1 \sqcup K_2$) and of the isomorphism $h_1 \oplus h_2 : H^1(K_1) \oplus H^1(K_2) \to \mathbb{Z} \oplus \mathbb{Z}$ (see Remark 1.33).

Remark 1.35. If you reverse the order of the components then the linking couple is either the same (if n is odd and $r \circ i_1 \circ (h_1)^{-1}$ and $r \circ i_2 \circ (h_2)^{-1}$ are both the identity of $\mathbb{Z}$ or both the opposite or if n is even and $r \circ i_1 \circ (h_1)^{-1}$ is the identity of $\mathbb{Z}$ and $r \circ i_2 \circ (h_2)^{-1}$ is the opposite or vice versa) or is turned into its opposite (if n is odd and $r \circ i_1 \circ (h_1)^{-1}$ is the identity of $\mathbb{Z}$ and $r \circ i_2 \circ (h_2)^{-1}$ is the opposite or vice versa or if n is even and $r \circ i_1 \circ (h_1)^{-1}$ and $r \circ i_2 \circ (h_2)^{-1}$ are both the identity of $\mathbb{Z}$ or both the opposite).

Remark 1.36. If you reverse the orientation of the first component (respectively the second component) of the oriented link then the first component (resp. the second component) of the linking couple stays the same and the second component (resp. the first component) of the linking couple is turned into its opposite.

Let us now briefly mention the other higher-dimensional contexts in which knots and links are studied.

First, let us mention that in classical knot theory ($\mathbb{S}^1 \to \mathbb{S}^3$), there are three competing definitions of knots, which all give the same classes of knots for ambient isotopy (and the same is true for links). Knots can be defined as smooth knots (see Definition 1.1), as topological knots (topological subspaces of $\mathbb{S}^3$ which are homeomorphic to $\mathbb{S}^1$ and locally flat in all their points), or as piecewise-linear knots (a.k.a. combinatorial knots, a.k.a. polygonal knots in this case). In higher-dimensional cases ($\mathbb{S}^m \to \mathbb{S}^n$), these three competing definitions do not give the same classes of knots, hence there are three higher-dimensional knot theories: the theory of higher-dimensional smooth knots, the theory of higher-dimensional topological knots, and the theory of higher-dimensional piecewise-linear knots (a.k.a. combinatorial knots).

Although piecewise-linear knots $\mathbb{S}^m \to \mathbb{S}^n$ can be knotted (which means that there are at least two equivalence classes of piecewise-linear knots) only in codimension 2 and perhaps codimension 1 (the case of codimension 1 is an open problem if $n \geq 4$ as far as the author knows, whereas all knots $\mathbb{S}^1 \to \mathbb{S}^2$ or $\mathbb{S}^2 \to \mathbb{S}^3$ can be unknotted; see [Zee63] for these results), and a similar result exists for topological knots (see [Sta63]; this result existed before the corresponding result for piecewise-linear knots), this is not the case for smooth knots: for each integer $k \geq 2$, there are infinitely many equivalence classes of smooth knots $\mathbb{S}^{4k-1} \to \mathbb{S}^{6k}$ (see [Hae62]).

Note that in their article [AK81], Akbulut and King proved that every higher-dimensional smooth link is algebraic (similarly to what we discussed in Section 1.5). However, there is no constructive proof of this result (except for links $\mathbb{S}^1 \to \mathbb{S}^3$) as far as the author knows.

Finally, although we have only considered links $\mathbb{S}^m \to \mathbb{S}^n$, note that links $\mathbb{S}^m \to \mathbb{R}^n$ are also objects of interest. For an informal introduction to the case $\mathbb{S}^k \to \mathbb{R}^{k+2}$, see [Oga18].

Chapter 2

The Witt, Grothendieck-Witt and Milnor-Witt K-theory rings

In this chapter, we recall well-known facts about symmetric bilinear forms, quadratic forms and Milnor-Witt K-theory, which will play an important role in the following chapter on quadratic intersection theory and in all subsequent chapters. We also prove a new result: Theorem 2.46 which enables us to compute the residue morphisms of Milnor-Witt K-theory (we also included this theorem in our preprint [Lem23]). We use this theorem to compute the quadratic linking class and the quadratic linking degree (our counterparts of the linking class and of the linking number) in Chapters 6 and 7.

In Section 2.1 we consider symmetric bilinear forms and quadratic forms in order to construct the Witt ring $\mathrm{W}(F)$ and the Grothendieck-Witt ring $\mathrm{GW}(F)$ of a field F. In Section 2.2 we construct the Milnor-Witt K-theory (graded) ring $K_*^{\mathrm{MW}}(F)$ associated to a field F, which has a strong relationship to the Witt ring $\mathrm{W}(F)$ and the Grothendieck-Witt ring $\mathrm{GW}(F)$. Namely, the ring $K_0^{\mathrm{MW}}(F)$ in degree 0 is canonically isomorphic to the Grothendieck-Witt ring $\mathrm{GW}(F)$ and for each negative n, the group $K_n^{\mathrm{MW}}(F)$ in degree n is canonically isomorphic to the Witt group $\mathrm{W}(F)$. Furthermore, for all negative m, n, the product $K_m^{\mathrm{MW}}(F) \times K_n^{\mathrm{MW}}(F) \to K_{m+n}^{\mathrm{MW}}(F)$ corresponds via these isomorphisms to the product of the Witt ring of F.

2.1 The Witt ring and the Grothendieck-Witt ring

In this section, we define the Witt ring and the Grothendieck-Witt ring of a field, which arise from symmetric bilinear forms on finite-dimensional vector spaces over the field (which correspond to quadratic forms if the field is of characteristic different from 2). For further information on the Witt ring and the Grothendieck-Witt ring of a field, we recommend these five books (the first of which is in French): [dSP11], [EKM08], [Lam05], [MH73], [Sch85].

In the first subsection we construct the commutative semiring $\mathrm{Isom}(F)$ of isometry classes of (non-degenerate) symmetric bilinear forms on the field F. In the second subsection we use Grothendieck's construction to obtain a commutative ring $\mathrm{GW}(F)$ from $\mathrm{Isom}(F)$: the Grothendieck-Witt ring of F. In the third subsection we construct the Witt ring $\mathrm{W}(F)$ of F in two ways: from the Grothendieck-Witt ring of F (by taking out the hyperbolic plane) and directly (from Witt-equivalence). In the final subsection we give examples of Grothendieck-Witt rings and of Witt rings.

Throughout this section, F is a field and V, V' are F-vector spaces of finite dimension.

Symmetric bilinear forms and quadratic forms

Definitions 2.1 ((Symmetric) bilinear forms and quadratic forms)**.**

- A bilinear form on V is a bilinear map $b : V \times V \to F$.
- A bilinear form b on V is symmetric if for all $v, w \in V$:
$$b(v,w) = b(w,v)$$
- If $\mathrm{char}(F) \neq 2$, a quadratic form on V is a map $q : V \to F$ such that the map $b : \begin{cases} V \times V & \to & F \\ (x,y) & \mapsto & \frac{1}{2}(q(x+y) - q(x) - q(y)) \end{cases}$ is a symmetric bilinear form such that for all $x \in V$, $b(x,x) = q(x)$. We call b the polar form of q.

Remark 2.2. If $\mathrm{char}(F) \neq 2$ and b is a symmetric bilinear form on V then $q : \begin{cases} V & \to & F \\ x & \mapsto & b(x,x) \end{cases}$ is a quadratic form on V of polar form b.

Note that if $\mathrm{char}(F) \neq 2$ and $V = F^n$ for some $n \in \mathbb{N}$ then the quadratic forms on V are exactly the homogeneous polynomials of degree 2 in n variables on F.

Examples 2.3.
- If $V = \{0\}$ then the only symmetric bilinear form on V is $b_\bullet : (0,0) \mapsto 0$. If $\mathrm{char}(F) \neq 2$ then the only quadratic form on V is $q_\bullet : 0 \mapsto 0$ (and its polar form is $b_\bullet$).

- If $V = F$ then symmetric bilinear forms on V correspond to elements $a \in F$ in the following way: $b_a : (x, y) \mapsto axy$. If $\mathrm{char}(F) \neq 2$ then the quadratic form q_a of polar form b_a is simply $x \mapsto ax^2$.

- If $V = F^2$ then symmetric bilinear forms on V correspond to triples $(a, b, c) \in F^3$ in the following way: $b_{(a,b,c)} : ((x_1, y_1), (x_2, y_2)) \mapsto ax_1x_2 + b(x_1y_2 + x_2y_1) + cy_1y_2$. If $\mathrm{char}(F) \neq 2$ then the quadratic form $q_{(a,b,c)}$ of polar form $b_{(a,b,c)}$ is simply $(x, y) \mapsto ax^2 + 2bxy + cy^2$.

In what follows, we will only be interested in non-degenerate symmetric bilinear forms and quadratic forms.

Definitions 2.4 (Non-degenerate symmetric bilinear forms and rank)**.**

- The symmetric bilinear form b on V is non-degenerate if 0 is the only element x of V which verifies that for all $y \in V$, $b(x, y) = 0$. In this case, the rank of b is the dimension of V.

- If $\mathrm{char}(F) \neq 2$, the quadratic form q on V is non-degenerate if its polar form is non-degenerate. In this case, the rank of q is the rank of its polar form (i.e. the dimension of V).

In the examples above, $b_\bullet$ is non-degenerate (of rank 0), b_a is non-degenerate if and only if $a \neq 0$ (and is of rank 1 in this case) and $b_{(a,b,c)}$ is non-degenerate if and only if ($b \neq 0$ or ($a \neq 0$ and $c \neq 0$)) (and is of rank 2 in this case).

We want to say that two quadratic forms (or two symmetric bilinear forms) are "the same" if they are the same up to a change of coordinates, i.e. if they are isometric.

Definitions 2.5 (Isometry)**.**

- Two non-degenerate symmetric bilinear forms b on V and b' on V' are isometric if there exists a linear isomorphism $u : V \to V'$ such that for all $x, y \in V$, $b(x, y) = b'(u(x), u(y))$.

- If $\mathrm{char}(F) \neq 2$, two non-degenerate quadratic forms q on V and q' on V' are isometric if there exists a linear isomorphism $u : V \to V'$ such that for all $x \in V$, $q(x) = q'(u(x))$.

Note that two quadratic forms are isometric if and only if their polar forms are isometric.

Remark 2.6. Isometry is an equivalence relation on non-degenerate symmetric bilinear forms (set $u = \mathrm{Id}$ for reflexiveness, the inverse of the linear isomorphism for symmetry and the composite of the linear isomorphisms for transitivity).

Notation 2.7. We denote by $\mathrm{Isom}(F)$ the set of isometry classes of non-degenerate symmetric bilinear forms.

If $\mathrm{char}(F) \neq 2$, the set $\mathrm{Isom}(F)$ is in canonical bijection with the set of isometry classes of non-degenerate quadratic forms on F (by polarising (i.e. taking the polar form) / depolarising (i.e. associating the quadratic form $q : x \mapsto b(x, x)$ to the symmetric bilinear form b)).

The set $\mathrm{Isom}(F)$ can be endowed with a commutative semiring structure.

Definitions 2.8 (Orthogonal sum and tensor product)**.** Let $b : V \times V \to F$ and $b' : V' \times V' \to F$ be symmetric bilinear forms.

- The orthogonal sum of b and b' is the symmetric bilinear form $b \perp b' : (V \oplus V') \times (V \oplus V') \to F$ which verifies:

$$b \perp b' : ((x, x'), (y, y')) \mapsto b(x, y) + b'(x', y')$$

- The tensor product of b and b' is the symmetric bilinear form $b \otimes b' : (V \otimes V') \times (V \otimes V') \to F$ which verifies:

$$b \otimes b' : (\sum_{i \in I} x_i \otimes x'_i, \sum_{j \in J} y_j \otimes y'_j) \mapsto \sum_{(i,j) \in I \times J} b(x_i, y_j) \times b'(x'_i, y'_j)$$

Note that if $\mathrm{char}(F) \neq 2$ and $q : V \to F$ and $q' : V' \to F$ are quadratic forms of respective polar forms b and b' then we can define $q \perp q' : V \oplus V' \to F$ and $q \otimes q' : V \otimes V' \to F$ as the quadratic forms of respective polar forms $b \perp b'$ and $b \otimes b'$.

Remark 2.9. The orthogonal sum and the tensor product induce operations on the set $\mathrm{Isom}(F)$ which make it into a commutative semiring.

If $\mathrm{char}(F) \neq 2$ then we can construct a commutative semiring from quadratic forms on F which is canonically isomorphic (through polarising / depolarising) to $\mathrm{Isom}(F)$. This is also true of the commutative rings which are constructed from symmetric bilinear forms on F in the following subsections: the Grothendieck-Witt ring of F and the Witt ring of F. In these subsections we stop making comments about quadratic forms but the readers should keep in mind that if $\mathrm{char}(F) \neq 2$ then symmetric bilinear forms can be replaced with quadratic forms every step of the way (through polarising / depolarising).

The Grothendieck-Witt ring

The Grothendieck-Witt (commutative) ring of F is obtained from the commutative semiring $\mathrm{Isom}(F)$ by using Grothendieck's construction.

Definition 2.10 (Grothendieck-Witt ring)**.** The Grothendieck-Witt ring of F, denoted $\mathrm{GW}(F)$, is the Grothendieck ring associated to the commutative semiring $\mathrm{Isom}(F)$. More explicitly:

- As a set, $\mathrm{GW}(F)$ is the set of equivalence classes of elements of $\mathrm{Isom}(F) \times \mathrm{Isom}(F)$ for the following equivalence relation:
$$(b_1, b_2) \sim (b_1', b_2') \Leftrightarrow \exists d \in \mathrm{Isom}(F), b_1 \perp b_2' \perp d = b_1' \perp b_2 \perp d$$
The equivalence class of (b_1, b_2) is denoted by $b_1 - b_2$.
- The sum $+$ of $\mathrm{GW}(F)$ is given by:
$$(b_1 - b_2) + (b_1' - b_2') = (b_1 \perp b_1') - (b_2 \perp b_2')$$
- The product $\times$ of $\mathrm{GW}(F)$ is given by:
$$(b_1 - b_2) \times (b_1' - b_2') = (b_1 \otimes b_1' \perp b_2 \otimes b_2') - (b_1 \otimes b_2' \perp b_2 \otimes b_1')$$

Note that if $\mathrm{char}(F) \neq 2$ then Witt's cancellation theorem, which states that for all $b, b', d \in \mathrm{Isom}(F)$, $b \perp d = b' \perp d \Rightarrow b = b'$, gives as a corollary that the map $\begin{cases} \mathrm{Isom}(F) & \to & \mathrm{GW}(F) \\ b & \mapsto & b - 0 \end{cases}$ is injective. This is not the case if $\mathrm{char}(F) = 2$ (although the restriction of this map to classes of anisotropic symmetric bilinear forms b (i.e. symmetric bilinear forms b such that $b(x, x) = 0$ implies $x = 0$) is injective).

Remark 2.11. The rank is well-defined on $\mathrm{GW}(F)$ (by demanding that the rank of $b_1 - b_2$ be the rank of b_1 minus the rank of b_2).

Notation 2.12. Let $a \in F^*$. We denote by $\langle a \rangle \in \mathrm{GW}(F)$ the class of $b_a : \begin{cases} F \times F & \to & F \\ (x, y) & \mapsto & axy \end{cases}$.

Note that $\langle a \rangle \times \langle b \rangle = \langle ab \rangle$ for all $a, b \in F^*$.

Theorem 2.13 (Theorem 4.3 in Chapter II of [Lam05])**.** The elements of $\mathrm{GW}(F)$ are finite sums of elements of the form $\varepsilon\langle a \rangle$ with $\varepsilon \in \{-1, 1\}$ and $a \in F^*$. Furthermore, a presentation of the abelian group $\mathrm{GW}(F)$ is given by the generators $\langle a \rangle$ with $a \in F^*$ and the following relations:

- $\langle ab^2 \rangle = \langle a \rangle$ for all $a, b \in F^*$;
- $\langle a \rangle + \langle b \rangle = \langle a + b \rangle + \langle (a + b)ab \rangle$ for all $a, b \in F^*$ such that $a + b \in F^*$.

The Witt ring

There are two ways of defining the Witt ring. One way is to define the Witt ring as a quotient of the Grothendieck-Witt ring.

Definition 2.14 (Witt ring)**.** The Witt ring of F, denoted $\mathrm{W}(F)$, is the quotient of the Grothendieck-Witt ring of F by the ideal generated by the class of the hyperbolic plane $\mathbb{H} : \begin{cases} F^2 \times F^2 & \to & F \\ ((x_1, y_1), (x_2, y_2)) & \mapsto & x_1y_2 + x_2y_1 \end{cases}$.

Remark 2.15. The rank modulo 2 is well-defined on $\mathrm{W}(F)$.

Notation 2.16. Let $a \in F^*$. We denote by $< a > \in \mathrm{W}(F)$ the class of $b_a : \begin{cases} F \times F & \to & F \\ (x, y) & \mapsto & axy \end{cases}$.

Note that $< a > \times < b > = < ab >$ for all $a, b \in F^*$.

The following theorem follows immediately from Theorem 2.13.

Theorem 2.17. The elements of $\mathrm{W}(F)$ are finite sums of elements of the form $< a >$ with $a \in F^*$. Furthermore, a presentation of the abelian group $\mathrm{W}(F)$ is given by the generators $< a >$ with $a \in F^*$ and the following relations:

- $< ab^2 > = < a >$ for all $a, b \in F^*$;
- $< a > + < b > = < a + b > + < (a + b)ab >$ for all $a, b \in F^*$ such that $a + b \in F^*$;
- $< -1 > + < 1 > = 0$ (note that $< -1 > + < 1 >$ is the class of $\mathbb{H}$).

Note that for all $a \in F^*$, $< -a > + < a > = 0$ (which is why the elements of $\mathrm{W}(F)$ are finite sums of elements of the form $< a >$ rather than finite $\mathbb{Z}$-linear combinations).

Another way to define the Witt ring is to introduce Witt-equivalence of symmetric bilinear forms.

Definition 2.18 (Witt-equivalence)**.** Two non-degenerate symmetric bilinear forms b on V and b' on V' are Witt-equivalent if there exist integers $n, n' \in \mathbb{N}_0$ such that $b \perp n\mathbb{H}$ is isometric to $b' \perp n'\mathbb{H}$.

Remark 2.19. The orthogonal sum and the tensor product induce operations on the set of equivalence classes for Witt-equivalence which make it into a commutative ring. The morphism from $\mathrm{W}(F)$ to this commutative ring which for each $a \in F^*$ sends $< a >$ to the equivalence class of b_a is an isomorphism.

Examples of Witt rings and of Grothendieck-Witt rings

First note that if every element of F is a square (e.g. if F is a perfect field of characteristic 2 or if F is algebraically closed) then $\mathrm{GW}(F) \simeq \mathbb{Z}$ via the rank and $\mathrm{W}(F) \simeq \mathbb{Z}/2\mathbb{Z}$ via the rank modulo 2.

Examples 2.20.
- The rank $r : \mathrm{GW}(\mathbb{C}) \to \mathbb{Z}$ is a ring isomorphism.
- The morphism $r : \mathrm{W}(\mathbb{C}) \to \mathbb{Z}/2\mathbb{Z}$ induced by the rank is a ring isomorphism.
- If F is a finite field of characteristic 2 then the rank $r : \mathrm{GW}(F) \to \mathbb{Z}$ is a ring isomorphism.
- If F is a finite field of characteristic 2 then the morphism $r : \mathrm{W}(F) \to \mathbb{Z}/2\mathbb{Z}$ induced by the rank is a ring isomorphism.

For the real case, we need the following definitions.

Definition 2.21 (Group ring)**.** Let G be a group. The group ring $\mathbb{Z}[G]$ is the free abelian group $\bigoplus_{f\in G} \mathbb{Z}\lambda_f$ associated to G with the following product:

$$(\sum_{f\in G} n_f\lambda_f)(\sum_{g\in G} m_g\lambda_g) = \sum_{h\in G}(\sum_{\substack{f,g\in G\\ fg=h}} n_f m_g)\lambda_h$$

Definitions 2.22 (Signature couple and signature)**.** Let $a, b \in \mathbb{Z}$.

- The signature couple of $a\langle 1\rangle + b\langle -1\rangle \in \mathrm{GW}(\mathbb{R})$ is the couple $(a, b) \in \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$.
- The signature of $a < 1 > +b < -1 > \in \mathrm{W}(\mathbb{R})$ is $a - b \in \mathbb{Z}$.

Examples 2.23.
- The signature couple $\mathrm{GW}(\mathbb{R}) \to \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$ is a ring isomorphism.
- The signature $\mathrm{W}(\mathbb{R}) \to \mathbb{Z}$ is a ring isomorphism.

For the cases of finite fields, we need the following definitions.

Definition 2.24 (Binary group ring)**.** Let G be a group. The binary group ring $\mathbb{Z}/2\mathbb{Z}[G]$ is the $\mathbb{Z}/2\mathbb{Z}$-module $\bigoplus_{f\in G} \mathbb{Z}/2\mathbb{Z}\lambda_f$ associated to G with the following product:

$$(\sum_{f\in G} n_f\lambda_f)(\sum_{g\in G} m_g\lambda_g) = \sum_{h\in G}(\sum_{\substack{f,g\in G\\ fg=h}} n_f m_g)\lambda_h$$

Definitions 2.25 (Signature couple and signature for finite fields)**.** Let F be a finite field of characteristic different from 2 and $s \in F$ be a non-square (e.g. $s = -1$ if the cardinal of F is congruent to 3 modulo 4). Let $a, b \in \mathbb{Z}$.

- If the cardinal of F is congruent to 1 modulo 4 then the signature couple of $a < 1 > + b < s > \in \mathrm{W}(F)$ is the couple $(a, b) \in \mathbb{Z}/2\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$.
- If the cardinal of F is congruent to 3 modulo 4 then the signature of $a < 1 > + b < s > \in \mathrm{W}(F)$ is $a - b \in \mathbb{Z}/4\mathbb{Z}$.

Note that in the preceding definitions the choice of s is inconsequential since if s, s' are non-squares in a finite field F then $< s > = < s' > \in \mathrm{W}(F)$.

In the following example, $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is the abelian group endowed with the coordinatewise product $((n, a) \times (m, b) = (n \times m, a \times b))$, which makes it into a commutative ring.

Examples 2.26. Let F be a finite field of characteristic different from 2.

- Let $\varphi : \mathrm{GW}(F) \to \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ be the ring morphism which for each $a \in F^*$ sends $\langle a \rangle$ to $(1, 0)$ if a is a square in F, to $(1, 1)$ otherwise. The ring morphism $\varphi : \mathrm{GW}(F) \to \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is a ring isomorphism.
- If the cardinal of F is congruent to 1 modulo 4 then the signature couple $\mathrm{W}(F) \to \mathbb{Z}/2\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$ is a ring isomorphism.
- If the cardinal of F is congruent to 3 modulo 4 then the signature $\mathrm{W}(F) \to \mathbb{Z}/4\mathbb{Z}$ is a ring isomorphism.

Finally, let us consider the field $\mathbb{Q}$ of rational numbers. Note that for all $r \in \mathbb{Q}^*$, there exists an integer m with no square factor such that in $\mathrm{W}(\mathbb{Q})$ $< r > = < m >$. We denote by P the (ordered) set of prime numbers.

Example 2.27. The group morphism $\psi : \mathrm{W}(\mathbb{Q}) \to \mathrm{W}(\mathbb{R}) \oplus \bigoplus_{p\in P} \mathrm{W}(\mathbb{Z}/p\mathbb{Z})$ which for all $\varepsilon \in \{-1, 1\}$, $n \in \mathbb{N}_0$ and distinct prime numbers $p_1, \ldots, p_n$

sends $<\varepsilon \prod_{i=1}^{n} p_i>$ to $<\varepsilon \prod_{i=1}^{n} p_i> \in \mathrm{W}(\mathbb{R}) \oplus \bigoplus_{i=1}^{n} <\varepsilon \prod_{\substack{j=1 \\ j\neq i}}^{n} p_j> \in \mathrm{W}(\mathbb{Z}/p_i\mathbb{Z})$ is a group isomorphism. Since ψ is surjective, we can define a product $\cdot$ on $\mathrm{W}(\mathbb{R}) \oplus \bigoplus_{p\in P} \mathrm{W}(\mathbb{Z}/p\mathbb{Z})$ by $\psi(\alpha)\cdot\psi(\beta) = \psi(\alpha\beta)$; this makes the group $\mathrm{W}(\mathbb{R}) \oplus \bigoplus_{p\in P} \mathrm{W}(\mathbb{Z}/p\mathbb{Z})$ into a ring and ψ into a ring isomorphism.

We denote by $I(\mathbb{R}) \simeq 2\mathbb{Z}$ the kernel of the ring morphism $\mathrm{W}(\mathbb{R}) \to \mathbb{Z}/2\mathbb{Z}$ induced by the rank. Using the rank $r : \mathrm{GW}(\mathbb{Q}) \to \mathbb{Z}$ and the isomorphism ψ from the previous example, we determine the structure of $\mathrm{GW}(\mathbb{Q})$.

Example 2.28. We denote by $\pi : \mathrm{GW}(\mathbb{Q}) \to \mathrm{W}(\mathbb{Q})$ the ring morphism which sends $\langle a\rangle$ to $<a>$ for each $a \in \mathbb{Q}^*$. The group morphism $\psi' : \mathrm{GW}(\mathbb{Q}) \to \mathbb{Z} \oplus I(\mathbb{R}) \oplus \bigoplus_{p\in P} \mathrm{W}(\mathbb{Z}/p\mathbb{Z})$ which verifies for each $\alpha \in \mathrm{GW}(\mathbb{Q})$ $\psi'(\alpha) = (r(\alpha), \psi(\pi(\alpha - r(\alpha)\langle 1\rangle)))$ is a group isomorphism. Since ψ' is surjective, we can define a product $\cdot$ on $\mathbb{Z} \oplus I(\mathbb{R}) \oplus \bigoplus_{p\in P} \mathrm{W}(\mathbb{Z}/p\mathbb{Z})$ by $\psi'(\alpha)\cdot\psi'(\beta) = \psi'(\alpha\beta)$; this makes $\mathbb{Z} \oplus I(\mathbb{R}) \oplus \bigoplus_{p\in P} \mathrm{W}(\mathbb{Z}/p\mathbb{Z})$ into a ring and ψ' into a ring isomorphism.

2.2 The Milnor-Witt K-theory ring

In this section, we construct the Milnor-Witt K-theory ring associated to a field, which has a strong relationship to the Witt ring and the Grothendieck-Witt ring (see Theorem 2.33 and Corollary 2.34). In the first subsection, we introduce the Milnor-Witt K-theory of fields which was defined by Morel in [Mor12, Section 3.1] and recall some of its properties. In the second subsection, we consider the residue morphisms of Milnor-Witt K-theory and we prove Theorem 2.46 which gives an explicit definition (i.e. one which allows computations) of the noncanonical residue morphisms of Milnor-Witt K-theory. In other words, we give in Theorem 2.46 a formula to compute the noncanonical residue morphisms which were defined by Morel in [Mor12] (see Definition 2.36). Formulas to compute the canonical residue morphisms (see Definition 2.39) and the twisted canonical residue morphisms (see Definition 2.41) follow directly. This in turn enables us to compute the differentials of the Rost-Schmid complexes (see Definition 3.8) and their boundary maps (see Definition 3.18) in the cases which are useful to compute the quadratic linking class and the quadratic linking degree (see Chapters 6 and 7).

Throughout this section, F is a perfect field, $v : F^* \to \mathbb{Z}$ is a discrete valuation (of residue field $\kappa(v)$ and ring $\mathcal{O}_v$ (of maximal ideal $\mathfrak{m}_v$)) and π is a uniformizing parameter for v. For all $u \in \mathcal{O}_v^*$, we denote by $\overline{u}$ its class in $\kappa(v)$ (which is in $\kappa(v)^*$ since $u \in \mathcal{O}_v^*$).

Milnor-Witt K-theory

We start by defining the Milnor-Witt K-theory ring associated to F.

Definition 2.29 (Milnor-Witt K-theory)**.** The Milnor-Witt K-theory ring associated to F, denoted $K_*^{\mathrm{MW}}(F)$, is the $\mathbb{Z}$-graded ring with unit generated by elements $[a]$ of degree 1, for $a \in F^*$, and an element η of degree -1, subject to the relations:

- $[ab] = [a] + [b] + \eta[a][b]$ for all $a, b \in F^*$
- $[a][1-a] = 0$ for all $a \in F \setminus \{0,1\}$ (Steinberg relation)
- $\eta[a] = [a]\eta$ for all $a \in F^*$
- $\eta(\eta[-1]+2) = 0$, i.e. $\eta h = 0$ with $h := \eta[-1]+2$

This means that $K_*^{\mathrm{MW}}(F)$ is the quotient of the non-commutative polynomial ring with coefficients in $\mathbb{Z}$ and (non-commuting) indeterminates the $[a]$ for $a \in F^*$ and η (with $\eta^{k_0}[a_1]\eta^{k_1}\dots[a_n]\eta^{k_n}$ of degree $n-k$ with $n, k_0, \dots, k_n \in \mathbb{N}_0$ and $k = \sum_{i=0}^{n} k_i$) by the (homogeneous) ideal generated by the relations above.

Note that since $K_*^{\mathrm{MW}}(F)$ is a $\mathbb{Z}$-graded ring with unit, $K_0^{\mathrm{MW}}(F)$ inherits a ring (with unit) structure and all the $K_n^{\mathrm{MW}}(F)$ (with $n \in \mathbb{Z}$) inherit a $K_0^{\mathrm{MW}}(F)$-module structure (in particular, an abelian group structure).

Remark 2.30. This definition of the Milnor-Witt K-theory ring associated to F may seem abstract; the readers who are interested in motivic homotopy theory should see [Mor12, Corollary 1.25] for a more concrete definition.

We now introduce important notation.

Notation 2.31. Let $a \in F^*$. $\langle a \rangle := 1 + \eta[a] \in K_0^{\mathrm{MW}}(F)$. $\epsilon := -\langle -1 \rangle$.

For all $n \in \mathbb{N}_0$, $n_\epsilon := \sum_{i=1}^{n} \langle (-1)^{i-1} \rangle$ and $(-n)_\epsilon := \epsilon\, n_\epsilon$.

For all $a_1, \dots, a_n \in F^*$, $[a_1, \dots, a_n] := [a_1]\dots[a_n] \in K_n^{\mathrm{MW}}(F)$.

We recall the following facts which are very useful for computations.

Proposition 2.32.

1. For all $a, b \in F^*$, $\langle ab \rangle = \langle a \rangle \langle b \rangle$ (see [Mor12, Lemma 3.5]).
2. If $n \leq 0$ then any element of $K_n^{\mathrm{MW}}(F)$ can be written as a $\mathbb{Z}$-linear combination of elements of the form $\langle a \rangle \eta^{-n}$ with $a \in F^*$ (see [Mor12, Lemma 3.6]).
3. If $n \geq 1$ then any element of $K_n^{\mathrm{MW}}(F)$ can be written as a $\mathbb{Z}$-linear combination of elements of the form $[a_1, \ldots, a_n]$ with $a_1, \ldots, a_n \in F^*$ (see [Mor12, Lemma 3.6]).
4. For all $a \in F^*$, $[a, a] = [a, -1]$ (see [Mor12, Lemma 3.7]).
5. For all $\alpha \in K_m^{\mathrm{MW}}(F)$ and $\beta \in K_n^{\mathrm{MW}}(F)$, $\alpha\beta = \epsilon^{mn}\beta\alpha$ (see [Mor12, Corollary 3.8]).
6. For all $n \in \mathbb{Z}$ and $a \in F^*$, $[a^n] = n_\epsilon [a]$ (see [Mor12, Lemma 3.14]).

Note in particular that $K_0^{\mathrm{MW}}(F)$ is a commutative ring.

The following theorem and corollary give the relationship between the Milnor-Witt K-theory ring on the one hand and the Grothendieck-Witt ring and Witt ring on the other hand.

Theorem 2.33 (Lemma 3.10 in [Mor12])**.** For all $n \leq -1$, the morphism $\gamma_n : K_n^{\mathrm{MW}}(F) \to \mathrm{W}(F)$ which for all $a \in F^*$ sends $\langle a \rangle \eta^{-n}$ to $< a >$ is an isomorphism of abelian groups and the morphism $\gamma_0 : K_0^{\mathrm{MW}}(F) \to \mathrm{GW}(F)$ which for all $a \in F^*$ sends $\langle a \rangle$ to $\langle a \rangle$ is an isomorphism of commutative rings with unit.

Corollary 2.34. The product in negative degrees in Milnor-Witt K-theory corresponds to the product in the Witt ring via the isomorphisms described in Theorem 2.33. In other words, for all negative integers $m, n < 0$, the following diagram is commutative:

$$\begin{array}{ccc} K_m^{\mathrm{MW}}(F) \times K_n^{\mathrm{MW}}(F) & \xrightarrow{\times} & K_{m+n}^{\mathrm{MW}}(F) \\ \downarrow{\scriptstyle \gamma_m \times \gamma_n} & & \downarrow{\scriptstyle \gamma_{m+n}} \\ \mathrm{W}(F) \times \mathrm{W}(F) & \xrightarrow[\times]{} & \mathrm{W}(F) \end{array}$$

Proof. For all $a, b \in F^*$, $(\langle a \rangle \eta^{-m}) \times (\langle b \rangle \eta^{-n}) = (\langle a \rangle \times \langle b \rangle)\eta^{-(m+n)}$ and since $\gamma_0 : K_0^{\mathrm{MW}}(F) \to \mathrm{GW}(F)$ is an isomorphism of rings (see Theorem 2.33), $\gamma_0(\langle a \rangle \times \langle b \rangle) = \langle a \rangle \times \langle b \rangle \in \mathrm{GW}(F)$ which is sent to $< a > \times < b > \in \mathrm{W}(F)$ via the canonical ring morphism $\mathrm{GW}(F) \to \mathrm{W}(F)$ (see Definition

2.14). Furthermore, the composite of $\gamma_0 : K_0^{\mathrm{MW}}(F) \to \mathrm{GW}(F)$ and of the canonical morphism $\mathrm{GW}(F) \to \mathrm{W}(F)$ is equal to the composite of $\times\eta^{-(m+n)} : K_0^{\mathrm{MW}}(F) \to K_{m+n}^{\mathrm{MW}}(F)$ and of $\gamma_{m+n} : K_{m+n}^{\mathrm{MW}}(F) \to \mathrm{W}(F)$ since these two group morphisms send $\langle a\rangle \in K_0^{\mathrm{MW}}(F)$ to $< a > \in \mathrm{W}(F)$ for all $a \in F^*$. Therefore the diagram above is commutative. $\square$

Note that $\langle 1\rangle = 1$, thus $h = \langle 1\rangle + \langle -1\rangle$ (see Definition 2.29) corresponds via γ_0 to the hyperbolic plane.

Before we move on to residue morphisms, we introduce one last definition (which will be useful to turn noncanonical residue morphisms into canonical residue morphisms).

Definition 2.35 (Twisted Milnor-Witt K-theory)**.**

- The group ring $\mathbb{Z}[F^*]$ is the free abelian group $\bigoplus_{f\in F^*} \mathbb{Z}\lambda_f$ associated to F^* with the following product:

$$(\sum_{f\in F^*} n_f\lambda_f)(\sum_{g\in F^*} m_g\lambda_g) = \sum_{h\in F^*}(\sum_{\substack{f,g\in F^*\\ fg=h}} n_f m_g)\lambda_h$$

- Let L be an F-vector space of dimension 1. The $\mathbb{Z}[F^*]$-module $\mathbb{Z}[L\setminus\{0\}]$ is the free abelian group $\bigoplus_{e\in L\setminus\{0\}} \mathbb{Z}\xi_e$ associated to $L\setminus\{0\}$ with the following scalar product:

$$(\sum_{f\in F^*} n_f\lambda_f)\cdot(\sum_{g\in L\setminus\{0\}} m_g\xi_g) = \sum_{h\in L\setminus\{0\}}(\sum_{\substack{f\in F^*, g\in L\setminus\{0\}\\ f\cdot g=h}} n_f m_g)\xi_h$$

- Let $m \in \mathbb{Z}$ and L be an F-vector space of dimension 1. The L-twisted m-th Milnor-Witt K-theory abelian group of F, denoted $K_m^{\mathrm{MW}}(F,L)$, is the tensor product of the $\mathbb{Z}[F^*]$-modules $K_m^{\mathrm{MW}}(F)$ and $\mathbb{Z}[L\setminus\{0\}]$ (the scalar product of $K_m^{\mathrm{MW}}(F)$ being $(\sum_{f\in F^*} n_f\lambda_f)\cdot\alpha = \sum_{f\in F^*} n_f\langle f\rangle\alpha$):

$$K_m^{\mathrm{MW}}(F,L) = K_m^{\mathrm{MW}}(F) \otimes_{\mathbb{Z}[F^*]} \mathbb{Z}[L\setminus\{0\}]$$

Note that if we fix an isomorphism between L and F then we get an isomorphism of $\mathbb{Z}[F^*]$-modules between $K_m^{\mathrm{MW}}(F,L)$ and $K_m^{\mathrm{MW}}(F)$; nevertheless, $K_m^{\mathrm{MW}}(F,L)$ is a useful construction because there is no canonical isomorphism between $K_m^{\mathrm{MW}}(F,L)$ and $K_m^{\mathrm{MW}}(F)$ unless $L = F$ (since there is no canonical isomorphism between L and F unless $L = F$) and the introduction of $K_m^{\mathrm{MW}}(F,L)$ is what allows us to have canonical residue morphisms.

Residue morphisms of Milnor-Witt K-theory

We recall Morel's definition of the noncanonical residue morphism associated to the discrete valuation $v : F^* \to \mathbb{Z}$ and the uniformizing parameter π.

Definition 2.36 (The noncanonical residue morphism)**.** The residue morphism $\partial_v^\pi : K_*^{\mathrm{MW}}(F) \to K_{*-1}^{\mathrm{MW}}(\kappa(v))$ is the (only) morphism of graded groups which commutes to product by η and satisfies, for all $n \in \mathbb{N}_0, u_1, \ldots, u_n \in \mathcal{O}_v^*$:

$$\partial_v^\pi([\pi, u_1, \ldots, u_n]) = [\overline{u_1}, \ldots, \overline{u_n}] \text{ and } \partial_v^\pi([u_1, \ldots, u_n]) = 0.$$

(For $n = 0$, this means $\partial_v^\pi([\pi]) = 1$ and $\partial_v^\pi(1) = 0$.)

In [Mor12, Theorem 3.15], Morel proves that such a morphism exists and that it is unique. For an explicit definition, see Theorem 2.46. Before we define the canonical residue morphism associated to the discrete valuation $v : F^* \to \mathbb{Z}$ (which will not depend on a uniformizing parameter), we recall the following proposition and corollary.

Proposition 2.37 (Proposition 3.17 in [Mor12])**.** For all $u \in \mathcal{O}_v^*$ and $\alpha \in K_*^{\mathrm{MW}}(F)$, we have $\partial_v^\pi(\langle u\rangle\alpha) = \langle\overline{u}\rangle\partial_v^\pi(\alpha)$.

Corollary 2.38. Let $u' \in \mathcal{O}_v^*$ and $\pi' = u'\pi$. Then $\partial_v^\pi = \langle\overline{u'}\rangle\partial_v^{\pi'}$.

Proof. Note that $\langle\overline{u'}\rangle\partial_v^{\pi'} : K_*^{\mathrm{MW}}(F) \to K_{*-1}^{\mathrm{MW}}(\kappa(v))$ (which sends α to $\langle\overline{u'}\rangle\partial_v^{\pi'}(\alpha)$ for all $\alpha \in K_*^{\mathrm{MW}}(F)$) is a morphism of graded groups which commutes to product by η. Thus it suffices to prove that for all $n \in \mathbb{N}_0, u_1, \ldots, u_n \in \mathcal{O}_v^*$, $\langle\overline{u'}\rangle\partial_v^{\pi'}([u_1, \ldots, u_n]) = 0$ and $\langle\overline{u'}\rangle\partial_v^{\pi'}([\pi, u_1, \ldots, u_n]) = [\overline{u_1}, \ldots, \overline{u_n}]$. Note that $\partial_v^{\pi'}([u_1, \ldots, u_n]) = 0$ hence $\langle\overline{u'}\rangle\partial_v^{\pi'}([u_1, \ldots, u_n]) = 0$.

$$\begin{aligned}
\langle\overline{u'}\rangle\partial_v^{\pi'}([\pi, u_1, \ldots, u_n]) &= \partial_v^{\pi'}(\langle u'\rangle[\pi, u_1, \ldots, u_n]) \text{ by Proposition 2.37} \\
&= \partial_v^{\pi'}((1 + \eta[u'])[\pi][u_1, \ldots, u_n]) \text{ by definition of } \langle u'\rangle \\
&= \partial_v^{\pi'}((1 + \eta[u'])[\pi][u_1, \ldots, u_n]) + \partial_v^{\pi'}([u', u_1, \ldots, u_n]) \\
&= \partial_v^{\pi'}(([\pi] + \eta[u'][\pi] + [u'])[u_1, \ldots, u_n]) \\
&= \partial_v^{\pi'}([u'\pi][u_1, \ldots, u_n]) \text{ (see Definition 2.29)} \\
&= \partial_v^{\pi'}([\pi', u_1, \ldots, u_n]) \text{ by definition of } \pi' \\
&= [\overline{u_1}, \ldots, \overline{u_n}]
\end{aligned}$$

□

Recall Definition 2.35. In the following definition, we denote by $(\mathfrak{m}_v/\mathfrak{m}_v^2)^\vee$ the dual of the $\kappa(v)$-vector space $\mathfrak{m}_v/\mathfrak{m}_v^2$, by $\overline{\pi}$ the class of π in $\mathfrak{m}_v/\mathfrak{m}_v^2$ (which is nonzero since π is a uniformizing parameter for v) and by $(\overline{\pi}^*)$ the dual basis of $(\overline{\pi})$.

Definition 2.39 (The canonical residue morphism)**.** The canonical residue morphism $\partial_v : K_*^{\mathrm{MW}}(F) \to K_{*-1}^{\mathrm{MW}}(\kappa(v), (\mathfrak{m}_v/\mathfrak{m}_v^2)^\vee)$ is given by $\partial_v = \partial_v^\pi \otimes \overline{\pi}^*$.

Remark 2.40. Note that ∂_v does not depend on the choice of π, since if π' is another uniformizing parameter for v then there exists $u' \in \mathcal{O}_v^*$ such that $\pi' = u'\pi$ hence $\partial_v^\pi \otimes \overline{\pi}^* = \langle \overline{u'} \rangle \partial_v^{\pi'} \otimes \overline{\pi}^* = \partial_v^{\pi'} \otimes \overline{u'\pi}^* = \partial_v^{\pi'} \otimes \overline{\pi'}^*$ by Corollary 2.38.

We also introduce the twisted canonical residue morphism associated to the discrete valuation $v : F^* \to \mathbb{Z}$ and the rank one $\mathcal{O}_v$-module L.

Definition 2.41 (The twisted canonical residue morphism)**.** Let L be a rank one $\mathcal{O}_v$-module. The twisted canonical residue morphism

$$\partial_{v,L} : K_*^{\mathrm{MW}}(F, L \otimes_{\mathcal{O}_v} F) \to K_{*-1}^{\mathrm{MW}}(\kappa(v), (\mathfrak{m}_v/\mathfrak{m}_v^2)^\vee \otimes_{\kappa(v)} (L \otimes_{\mathcal{O}_v} \kappa(v)))$$

is the (only) morphism of graded groups which satisfies for all $\alpha \in K_*^{\mathrm{MW}}(F)$ and $l \in L$:

$$\partial_{v,L}(\alpha \otimes (l \otimes 1)) = \partial_v^\pi(\alpha) \otimes (\overline{\pi}^* \otimes (l \otimes 1))$$

These twisted canonical residue morphisms we have introduced will turn up in the definition of the differentials of the Rost-Schmid complexes (see Definition 3.8) and (through this definition) in the definition of the boundary maps of the Rost-Schmid complexes (see Definition 3.18). These twisted canonical residue morphisms will then be used (through the definition of boundary map) to define the quadratic linking class and the quadratic linking degree, which are central to this thesis. In order to compute the quadratic linking class and the quadratic linking degree, we thus need to be able to compute the twisted canonical residue morphisms. By definition, it suffices to be able to compute the noncanonical residue morphisms. The following lemmas and theorem allow us to do just that.

Lemma 2.42. *For all $n \in \mathbb{Z}$, $n_\epsilon = n + \lfloor \frac{n}{2} \rfloor \eta[-1]$.*

Proof. If $n \geq 0$ then $n_\epsilon = \sum_{i=1}^n \langle (-1)^{i-1} \rangle = \lceil \frac{n}{2} \rceil \langle 1 \rangle + \lfloor \frac{n}{2} \rfloor \langle -1 \rangle = \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor (1 + \eta[-1]) = n + \lfloor \frac{n}{2} \rfloor \eta[-1]$. If $n < 0$ then $n_\epsilon = \epsilon(-n)_\epsilon$ hence, from what we have previously shown, $n_\epsilon = \epsilon(-n + \lfloor \frac{-n}{2} \rfloor \eta[-1]) = (-1 - \eta[-1])(-n + \lfloor \frac{-n}{2} \rfloor \eta[-1]) = n + (-\lfloor \frac{-n}{2} \rfloor + n)\eta[-1] - \lfloor \frac{-n}{2} \rfloor \eta^2[-1,-1]$. By Definition 2.29, $\eta[-1,-1] = [1] - [-1] - [-1] = -2[-1]$ hence $n_\epsilon = n + (-\lfloor \frac{-n}{2} \rfloor + n + 2\lfloor \frac{-n}{2} \rfloor)\eta[-1] = n + (n + \lfloor \frac{-n}{2} \rfloor)\eta[-1] = n + \lfloor \frac{n}{2} \rfloor \eta[-1]$. □

Lemma 2.43. *For all $m, n \in \mathbb{Z}$, $(mn)_\epsilon = m_\epsilon n_\epsilon$.*

Proof. Let $m,n \in \mathbb{Z}$. By Lemma 2.42, $m_\epsilon n_\epsilon = (m + \lfloor \frac{m}{2} \rfloor \eta[-1])(n + \lfloor \frac{n}{2} \rfloor \eta[-1]) = mn + (n\lfloor \frac{m}{2} \rfloor + m\lfloor \frac{n}{2} \rfloor)\eta[-1] + \lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor \eta^2[-1,-1]$. By Definition 2.29, $\eta[-1,-1] = [1] - [-1] - [-1] = -2[-1]$ hence $m_\epsilon n_\epsilon = mn + (n\lfloor \frac{m}{2} \rfloor + m\lfloor \frac{n}{2} \rfloor - 2\lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor)\eta[-1] = mn + \lfloor \frac{mn}{2} \rfloor \eta[-1] = (mn)_\epsilon$ by Lemma 2.42. □

Notation 2.44. We denote by χ^{odd} the characteristic function of the set of odd numbers, i.e.

$$\chi^{\text{odd}} : \begin{cases} \mathbb{Z} & \to & \{0,1\} \\ m & \mapsto & \chi^{\text{odd}}(m) = \begin{cases} 1 & \text{if } m \text{ is odd} \\ 0 & \text{otherwise} \end{cases} \end{cases}.$$

Lemma 2.45. For all $m \in \mathbb{Z}$, $\eta m_\epsilon = \eta \chi^{\text{odd}}(m)$.

Proof. Note that $\eta\langle -1 \rangle = -\eta$ since $\eta(1 + \langle -1 \rangle) = 0$ (see Definition 2.29). It follows that if $m \geq 0$ then $\eta m_\epsilon = \lceil \frac{m}{2} \rceil \eta - \lfloor \frac{m}{2} \rfloor \eta = \eta \chi^{\text{odd}}(m)$. If $m < 0$ then $m_\epsilon = \epsilon(-m)_\epsilon$ hence, from what we have previously shown, $\eta m_\epsilon = \epsilon\eta\chi^{\text{odd}}(-m) = \epsilon\eta\chi^{\text{odd}}(m)$. Since $\epsilon\eta = -\langle -1 \rangle \eta = \eta$, $\eta m_\epsilon = \eta\chi^{\text{odd}}(m)$. □

We now give a formula to compute the noncanonical residue morphisms $\partial_v^\pi : K_n^{\text{MW}}(F) \to K_{n-1}^{\text{MW}}(\kappa(v))$ (see Definition 2.36). We restrict to generators of the group $K_n^{\text{MW}}(F)$ (see 2 and 3 in Proposition 2.32) since ∂_v^π is a group morphism (see Definition 2.36). Note that formulas to compute the canonical residue morphisms and the twisted canonical residue morphisms follow immediately from the definitions (see Definitions 2.39 and 2.41).

Theorem 2.46. For all $n \leq 0$, $m \in \mathbb{Z}$ and $u \in \mathcal{O}_v^*$:

$$\partial_v^\pi(\langle \pi^m u \rangle \eta^{-n}) = \langle \overline{u} \rangle \eta^{-n+1} \chi^{\text{odd}}(m)$$

For all $n \geq 1$, $m_1, \ldots, m_n \in \mathbb{Z}$ and $u_1, \ldots, u_n \in \mathcal{O}_v^*$:

$$\partial_v^\pi([\pi^{m_1}u_1, \ldots, \pi^{m_n}u_n]) =$$

$$\sum_{l=0}^{n-1} \sum_{\substack{J \subset \{1,\ldots,n\}, |J|=l \\ J=\{j_1<\cdots<j_l\}}} ((-1)^{\sum_{i=1}^{l} n-l+i-j_i} \prod_{k \in \{1,\ldots,n\} \setminus J} m_k)_\epsilon [\underbrace{-1, \ldots, -1}_{n-1-l \text{ terms}}, \overline{u_{j_1}}, \ldots, \overline{u_{j_l}}]$$

$$+ \sum_{p=1}^{n} \sum_{l=p}^{n} \sum_{\substack{J \subset \{1,\ldots,n\}, |J|=l \\ J=\{j_1<\cdots<j_l\}}} (\sum_{\substack{I \subset \{1,\ldots,l\} \\ |I|=p}} \eta^p \chi^{\text{odd}}(\prod_{i \in I} m_{j_i} \times \prod_{k \in \{1,\ldots,n\} \setminus J} m_k))[\underbrace{-1, \ldots, -1}_{n-1+p-l \text{ terms}}, \overline{u_{j_1}}, \ldots, \overline{u_{j_l}}]$$

Remark 2.47. This last formula may seem daunting, but for $n = 1$ it is merely

$$\partial_v^\pi([\pi^m u]) = m_\epsilon + \eta\chi^{\mathrm{odd}}(m)[\overline{u}] = \langle\overline{u}\rangle m_\epsilon$$

(similarly to the case $n \leq 0$ where $\partial_v^\pi(\langle\pi^m u\rangle\eta^{-n}) = \langle\overline{u}\rangle\eta^{-n+1}m_\epsilon$, see Lemma 2.45), for $n = 2$ it is merely

$$\begin{aligned}\partial_v^\pi([\pi^{m_1}u_1, \pi^{m_2}u_2]) = {} & (m_1m_2)_\epsilon[-1] + (-m_2)_\epsilon[\overline{u_1}] + (m_1)_\epsilon[\overline{u_2}] \\ & + \eta\chi^{\mathrm{odd}}(m_1m_2)[-1, \overline{u_1}] + \eta\chi^{\mathrm{odd}}(m_1)[-1, \overline{u_2}] \\ & + (\eta\chi^{\mathrm{odd}}(m_1) + \eta\chi^{\mathrm{odd}}(m_2))[\overline{u_1}, \overline{u_2}] \\ & + \eta^2\chi^{\mathrm{odd}}(m_1m_2)[-1, \overline{u_1}, \overline{u_2}]\end{aligned}$$

and so on (the number of terms growing (a priori) exponentially with n).

Remark 2.48. Note that for $n \geq 1$, the formula in Theorem 2.46 could be rewritten so that η does not appear (by using the fact that for all $a, b \in \kappa(v)^*$, $\eta[a, b] = [ab] - [a] - [b]$, see Definition 2.29). For $n = 1$ this gives:

$$\partial_v^\pi([\pi^m u]) = \left\lceil \frac{m}{2} \right\rceil \langle\overline{u}\rangle + \left\lfloor \frac{m}{2} \right\rfloor \langle -\overline{u}\rangle$$

for $n = 2$ this gives:

$$\begin{aligned}\partial_v^\pi([\pi^{m_1}u_1, \pi^{m_2}u_2]) = {} & (\chi^{\mathrm{odd}}(m_1m_2) - \chi^{\mathrm{odd}}(m_1))[-1] - \left\lceil \frac{m_1}{2} \right\rceil [\overline{u_1}] - \left\lfloor \frac{m_1}{2} \right\rfloor [-\overline{u_1}] \\ & + (\chi^{\mathrm{odd}}(m_1m_2) - \chi^{\mathrm{odd}}(m_1) + \left\lfloor \frac{m_2}{2} \right\rfloor)[\overline{u_2}] \\ & + (\chi^{\mathrm{odd}}(m_1) - \chi^{\mathrm{odd}}(m_1m_2) + \left\lceil \frac{m_2}{2} \right\rceil)[-\overline{u_2}] \\ & + \frac{m_1 - m_2 + \chi^{\mathrm{odd}}(m_1 - m_2)}{2}[\overline{u_1u_2}] \\ & + \frac{m_1 - m_2 - \chi^{\mathrm{odd}}(m_1 - m_2)}{2}[-\overline{u_1u_2}]\end{aligned}$$

and so on (the number of terms growing (a priori) exponentially with n).

Proof. Let $n \leq 0$, $m \in \mathbb{Z}$ and $u \in \mathcal{O}_v^*$.

$$\begin{aligned}\partial_v^\pi(\langle\pi^m u\rangle\eta^{-n}) &= \partial_v^\pi((1 + \eta[\pi^m u])\eta^{-n}) \\ &= \partial_v^\pi((1 + \eta([\pi^m] + [u] + \eta[\pi^m, u]))\eta^{-n}) && \text{by Definition 2.29} \\ &= \partial_v^\pi((1 + \eta m_\epsilon[\pi] + \eta[u] + \eta^2 m_\epsilon[\pi, u])\eta^{-n}) && \text{by 6 in Prop. 2.32} \\ &= \eta^{-n}\partial_v^\pi(1) + \eta^{-n+1}m_\epsilon\partial_v^\pi([\pi]) \\ &\quad + \eta^{-n+1}\partial_v^\pi([u]) + \eta^{-n+2}m_\epsilon\partial_v^\pi([\pi, u]) && \text{by Prop. 2.37 and Def. 2.36} \\ &= \eta^{-n+1}m_\epsilon + \eta^{-n+2}m_\epsilon[\overline{u}] && \text{by Def. 2.36} \\ &= (\eta^{-n+1} + \eta^{-n+2}[\overline{u}])\chi^{\mathrm{odd}}(m) && \text{by Lemma 2.45} \\ &= \langle\overline{u}\rangle\eta^{-n+1}\chi^{\mathrm{odd}}(m)\end{aligned}$$

Let $n \geq 1, m_1, \dots, m_n \in \mathbb{Z}$, $u_1, \dots, u_n \in \mathcal{O}_v^*$ and $N := \{1, \dots, n\}$.

$$\begin{aligned}[\pi^{m_1}u_1, \dots, \pi^{m_n}u_n] &= \prod_{i=1}^{n}([\pi^{m_i}] + [u_i] + \eta[\pi^{m_i}, u_i]) \text{ by Definition 2.29} \\ &= \prod_{i=1}^{n}((m_i)_\epsilon[\pi] + [u_i] + \eta(m_i)_\epsilon[\pi, u_i]) \text{ by 6 in Prop. 2.32}\end{aligned}$$

By developing this product and using 5 in Proposition 2.32 (ϵ-graded commutativity), as well as the fact that $\eta\epsilon = \eta$ (since $\eta(1 + \langle -1\rangle) = 0$ (see Definition 2.29)), we get that $[\pi^{m_1}u_1, \dots, \pi^{m_n}u_n]$ is equal to:

$$\begin{aligned}&\sum_{l=0}^{n} \sum_{\substack{J\subset\{1,\dots,n\},|J|=l \\ J=\{j_1<\dots<j_l\}}} \prod_{k\in N\setminus J} (m_k)_\epsilon \times \epsilon^{\sum_{i=1}^{l} n-l+i-j_i}[\pi, \dots, \pi, u_{j_1}, \dots, u_{j_l}] \\ &+\sum_{p=1}^{n}\sum_{l=p}^{n} \sum_{\substack{J\subset\{1,\dots,n\},|J|=l \\ J=\{j_1<\dots<j_l\}}} (\sum_{\substack{I\subset\{1,\dots,l\} \\ |I|=p}} \eta^p \times \prod_{i\in I}(m_{j_i})_\epsilon \times \prod_{k\in N\setminus J} (m_k)_\epsilon)[\pi, \dots, \pi, u_{j_1}, \dots, u_{j_l}]\end{aligned}$$

The index p corresponds to the number of terms coming from an $\eta(m_i)_\epsilon[\pi, u_i]$, the index l corresponds to the number of terms coming from a $[u_i]$ or an $\eta(m_i)_\epsilon[\pi, u_i]$ (which is why $l \geq p$), the set $J = \{j_1, \dots, j_l\}$ corresponds to the indices i of the terms coming from a $[u_i]$ or an $\eta(m_i)_\epsilon[\pi, u_i]$ (which is why the cardinality $|J|$ of J is equal to l) and the set I corresponds to the indices i of the j_i such that u_{j_i} comes from an $\eta(m_{j_i})_\epsilon[\pi, u_{j_i}]$ (rather than from a $[u_{j_i}]$), which is why the cardinality $|I|$ of I is equal to p.

Therefore, by 4 in Proposition 2.32 and Lemmas 2.43 and 2.45, we have that $[\pi^{m_1}u_1, \dots, \pi^{m_n}u_n]$ is equal to:

$$\begin{aligned}&\sum_{l=0}^{n} \sum_{\substack{J\subset\{1,\dots,n\},|J|=l \\ J=\{j_1<\dots<j_l\}}} ((-1)^{\sum_{i=1}^{l} n-l+i-j_i} \prod_{k\in N\setminus J} m_k)_\epsilon[\pi, -1, \dots, -1, u_{j_1}, \dots, u_{j_l}] \\ &+\sum_{p=1}^{n}\sum_{l=p}^{n} \sum_{\substack{J\subset\{1,\dots,n\},|J|=l \\ J=\{j_1<\dots<j_l\}}} (\sum_{\substack{I\subset\{1,\dots,l\} \\ |I|=p}} \eta^p\chi^{\text{odd}}(\prod_{i\in I} m_{j_i} \times \prod_{k\in N\setminus J} m_k))[\pi, -1, \dots, -1, u_{j_1}, \dots, u_{j_l}]\end{aligned}$$

Finally, by Definition 2.36 and Proposition 2.37, we get that $\partial_v^\pi([\pi^{m_1}u_1, \ldots, \pi^{m_n}u_n])$ is equal to:

$$\sum_{l=0}^{n-1} \sum_{\substack{J\subset\{1,\ldots,n\},|J|=l \\ J=\{j_1<\cdots<j_l\}}} ((-1)^{\sum_{i=1}^{l} n-l+i-j_i} \prod_{k\in N\setminus J} m_k)_\epsilon[-1, \ldots, -1, \overline{u_{j_1}}, \ldots, \overline{u_{j_l}}]$$

$$+\sum_{p=1}^{n}\sum_{l=p}^{n} \sum_{\substack{J\subset\{1,\ldots,n\},|J|=l \\ J=\{j_1<\cdots<j_l\}}} (\sum_{\substack{I\subset\{1,\ldots,l\} \\ |I|=p}} \eta^p\chi^{\mathrm{odd}}(\prod_{i\in I} m_{j_i} \times \prod_{k\in N\setminus J} m_k))[-1, \ldots, -1, \overline{u_{j_1}}, \ldots, \overline{u_{j_l}}]$$

Note that the term $l = n$ in the first double sum vanished because $\partial_v^\pi([u_1, \ldots, u_n]) = 0$ (see Definition 2.36). □

Chapter 3

Quadratic intersection theory

In this chapter, we present quadratic intersection theory, which is a quadratic refinement of classical intersection theory (in algebraic geometry) which is central in motivic homotopy theory (and will rely heavily on Chapter 2). This chapter will play an important role in all subsequent chapters.

In Section 3.1 we recall the reinvention and generalisation of Chow groups by Rost, which is the inspiration for quadratic intersection theory. In Section 3.2 we regroup important results on Rost-Schmid groups, in particular on Chow-Witt groups (the quadratic counterparts to Chow groups) and in Section 3.3 we focus on the intersection product in quadratic intersection theory and present a recent formula to compute it. Finally, in Section 3.4 we compute some useful Rost-Schmid groups.

3.1 Intersection theory à la Rost

In this section we present the reinvention and generalisation of Chow groups by Rost (see [Ros96]). For a more classical take on intersection theory, see [Ful98] and [EH16].

Throughout this section, F is a perfect field and X is a smooth finite-type F-scheme.

First, we define Milnor K-theory (which was introduced by Milnor in [Mil70]).

Definition 3.1 (Milnor K-theory)**.** The Milnor K-theory ring associated to F, denoted $K_*^{\mathrm{M}}(F)$, is the $\mathbb{Z}$-graded ring with unit defined as the quotient of the tensor algebra of F^* by the (homogeneous) ideal generated by the "Steinberg relations" $a \otimes (1-a)$ with $a \in F \setminus \{0,1\}$. The class in $K_*^{\mathrm{M}}(F)$ of $a_1 \otimes \cdots \otimes a_n$ (where $a_1, \ldots, a_n \in F^*$) is denoted $\{a_1, \ldots, a_n\}$.

Note that for all $n < 0$, $K_n^{\mathrm{M}}(F) = 0$, $K_0^{\mathrm{M}}(F) \simeq \mathbb{Z}$ and $K_1^{\mathrm{M}}(F) \simeq F^*$. The following definition is equivalent to Definition 3.1.

Equivalent Definition 3.2 (Milnor K-theory)**.** The Milnor K-theory ring associated to F, denoted $K_*^{\mathrm{M}}(F)$, is the $\mathbb{Z}$-graded ring with unit generated by the elements $\{a\}$ of degree 1, for $a \in F^*$ subject to the relations:

- $\{ab\} = \{a\} + \{b\}$ for all $a, b \in F^*$
- $\{a\}\{1-a\} = 0$ for all $a \in F \setminus \{0,1\}$ (Steinberg relation)

Note that $K_*^{\mathrm{M}}(F)$ is canonically isomorphic to the quotient of $K_*^{\mathrm{MW}}(F)$ (see Definition 2.29) by the (homogeneous) ideal generated by η (the element $\{a\} \in K_1^{\mathrm{M}}(F)$ corresponding to the class of $[a] \in K_1^{\mathrm{MW}}(F)$ for all $a \in F^*$).

Remark 3.3. Milnor K-theory has residue morphisms which are defined in a similar manner to the noncanonical residue morphisms in Milnor-Witt K-theory (see Definition 2.36) but which are canonical (i.e. they do not depend on a choice of uniformizing parameter).

Traditionally, the i-th Chow group $\mathrm{CH}^i(X)$ of X is defined as the group of rational equivalence classes of cycles of codimension i in X (which are $\mathbb{Z}$-linear combinations of subvarieties of codimension i in X). In his article [Ros96], Rost uses Milnor K-theory K_*^{M} to define for each $j \in \mathbb{Z}$ the following complex (where $X^{(i)}$ is the set of points of codimension i in X):

$$\dots \longrightarrow \bigoplus_{p \in X^{(i)}} K_{j-i}^{\mathrm{M}}(\kappa(p)) \xrightarrow{d_j^i} \bigoplus_{q \in X^{(i+1)}} K_{j-i-1}^{\mathrm{M}}(\kappa(q)) \longrightarrow \dots$$

whose cohomology groups are $A^i(X,j) := \ker(d_j^i)/\operatorname{im}(d_j^{i-1})$ and shows that the i-th Chow group $\mathrm{CH}^i(X)$ of X is equal to $A^i(X,i)$.

Remark 3.4. The morphisms d_j^i are constructed from the residue morphisms of Milnor K-theory.

This generalisation of Chow groups has several advantages, one of which is that Chow groups fit in the long exact sequences given by the exact triangle theorem in homological algebra (see [Rot88, Theorem 5.6]):

$$\dots \longrightarrow A^n(Z,m) \xrightarrow{i_*} A^{n+d_X-d_Z}(X, m+d_X-d_Z) \xrightarrow{j^*}$$

$$\xrightarrow{j^*} A^{n+d_X-d_Z}(U, m+d_X-d_Z) \xrightarrow{\partial} A^{n+1}(Z,m) \longrightarrow \dots$$

with $i : Z \to X$ a closed immersion and $j : U \to X$ an open immersion such that the image of U by j is the complement in X of the image of Z by i, where Z, X, U are smooth F-schemes of pure dimensions (denoted d_Z, d_X and $d_U = d_X$ respectively), the morphisms ∂ (which are called boundary maps) being the connecting morphisms of the exact triangle theorem.

In [Ros96, Remark 2.6], Rost suggested that his work could probably be developed similarly over the Witt ring rather than the Milnor K-theory ring (with additional difficulties), which was partially done by Schmid in [Sch98]. In [BM00], Barge and Morel developed similar work over the Milnor-Witt K-theory ring (which in this article was constructed from the Milnor K-theory ring and the Witt ring, see [Mor03, Theorem 6.4.5]) and created the Chow groups of oriented cycles, which were later called the Chow-Witt groups, and a complex which was later called the Rost-Schmid complex.

Remark 3.5. In [Ros96], Rost developed much more general machinery than the generalisation of Chow groups we have described above: Chow groups with coefficients in cycle modules. In [Fel21], Feld recently developed Chow-Witt groups with coefficients in Milnor-Witt cycle modules, finally completing the work announced in [Ros96, Remark 2.6] and in [Mor12, Remark 5.37].

3.2 The Rost-Schmid complex and Chow-Witt groups

In this section, we define the quadratic counterpart to the complex which was described in the preceding section and study its cohomology groups, some of which are the Chow-Witt groups which play the role of Chow groups in a quadratic setting. After we define the Rost-Schmid complex and its cohomology groups (including Chow-Witt groups), we study some of their properties (homotopy invariance, the existence of a localization long exact sequence, their equivalent definition as sheaf cohomology groups) then focus on their interactions with orientations and orientation classes.

Throughout this section, F is a perfect field and X is a smooth finite-type F-scheme.

Definitions

We need the following definition and notation to define the Rost-Schmid complex.

Definition 3.6 (Determinant of a locally free module)**.** The determinant of a locally free $\mathcal{O}_X$-module $\mathcal{V}$ of constant finite rank r, denoted $\det(\mathcal{V})$, is its r-th exterior power $\Lambda^r(\mathcal{V})$.

Notation 3.7. Let $i \in \mathbb{Z}$, $x \in X$ and $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module.

- We denote by $X^{(i)}$ the set of points of codimension i in X. Note that $X^{(i)}$ is empty if i is less than 0 or greater than the dimension of X.
- We denote by $\mathcal{N}_{x/X}$ the normal sheaf of x in X, i.e. the dual of $\mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}$, where $\mathfrak{m}_{X,x}$ is the maximal ideal of the local ring $\mathcal{O}_{X,x}$ of X at x. We denote by ν_x the determinant of $\mathcal{N}_{x/X}$.
- We denote by $\mathcal{L}_{|x}$ the tensor product of the $\mathcal{O}_{X,x}$-modules $\mathcal{L}_x$ and $\kappa(x)$, i.e. $\mathcal{L}_{|x} := \mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$.

We now give the definition of the Rost-Schmid complex that Morel gave in [Mor12, Chapter 5]. Note that an earlier (equivalent) definition of the Rost-Schmid complex was given in [BM00]. (The equivalence of these definitions follows from [Mor03, Theorem 6.4.5].) Recall Definition 2.35.

Definition 3.8 (Rost-Schmid complex)**.** Let $j \in \mathbb{Z}$ and $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The Rost-Schmid complex associated to X, j and $\mathcal{L}$, denoted $\mathcal{C}(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L}\})$, is the following:

$$\dots \longrightarrow \mathcal{C}^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L}\}) \xrightarrow{d^i_{X,j,\mathcal{L}}} \mathcal{C}^{i+1}(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L}\}) \longrightarrow \dots$$

where

$$\mathcal{C}^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L}\}) = \bigoplus_{x \in X^{(i)}} K^{\mathrm{MW}}_{j-i}(\kappa(x), \nu_x \otimes_{\kappa(x)} \mathcal{L}_{|x})$$

and $d^i_{X,j,\mathcal{L}}$ (which is called the differential of the Rost-Schmid complex) is the (only) morphism of groups $\mathcal{C}^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L}\}) \to \mathcal{C}^{i+1}(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L}\})$ which for each $x \in X^{(i)}$ and $k_x \in K^{\mathrm{MW}}_{j-i}(\kappa(x), \nu_x \otimes_{\kappa(x)} \mathcal{L}_{|x})$ maps k_x to $\sum_{y \in \overline{\{x\}}^{(1)}} \partial^x_y(k_x)$, with $\partial^x_y : K^{\mathrm{MW}}_{j-i}(\kappa(x), \nu_x \otimes_{\kappa(x)} \mathcal{L}_{|x}) \to K^{\mathrm{MW}}_{j-i-1}(\kappa(y), \nu_y \otimes_{\kappa(y)} \mathcal{L}_{|y})$ the twisted canonical residue morphism (see Definition 2.41) when $\overline{\{x\}}$ is smooth, and the adequate composition of twisted canonical residue morphisms and transfer morphisms (between y and points of the normalisation of $\overline{\{x\}}$) otherwise (see [Fel20, Section 7] (in which the (verified) axiom FD ensures that this sum is finite) or [Fas20, Subsection 2.1] or [Mor12, pp. 121-122] for further details). We denote $\mathcal{C}(X, \underline{K}_j^{\mathrm{MW}}) := \mathcal{C}(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{O}_X\})$ and for all $i \in \mathbb{Z}$, $\mathcal{C}^i(X, \underline{K}_j^{\mathrm{MW}}) := \mathcal{C}^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{O}_X\})$ and $d^i_{X,j} := d^i_{X,j,\mathcal{O}_X}$.

Theorem 3.9 (Theorem 5.31 in [Mor12])**.** The Rost-Schmid complex is a complex, i.e. for each $i, j \in \mathbb{Z}$ and each invertible $\mathcal{O}_X$-module $\mathcal{L}$:

$$d^{i+1}_{X,j,\mathcal{L}} \circ d^{i}_{X,j,\mathcal{L}} = 0$$

This theorem allows us to define the Rost-Schmid groups (and in particular the Chow-Witt groups) as follows.

Definition 3.10 (Rost-Schmid groups)**.** Let $i, j \in \mathbb{Z}$ and $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The i-th Rost-Schmid group associated to X, j and $\mathcal{L}$, denoted by $H^i(X, \underline{K}^{\mathrm{MW}}_j\{\mathcal{L}\})$, is the i-th cohomology group of the Rost-Schmid complex $\mathcal{C}(X, \underline{K}^{\mathrm{MW}}_j\{\mathcal{L}\})$, i.e.:

$$H^i(X, \underline{K}^{\mathrm{MW}}_j\{\mathcal{L}\}) := \ker(d^{i}_{X,j,\mathcal{L}}) / \operatorname{im}(d^{i-1}_{X,j,\mathcal{L}})$$

We denote $H^i(X, \underline{K}^{\mathrm{MW}}_j) := H^i(X, \underline{K}^{\mathrm{MW}}_j\{\mathcal{O}_X\})$.

Definition 3.11 (Chow-Witt groups)**.** Let $i \in \mathbb{Z}$ and $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. The i-th Chow-Witt group associated to X and $\mathcal{L}$, denoted by $\widetilde{\mathrm{CH}}^i(X, \mathcal{L})$, is the i-th cohomology group of the Rost-Schmid complex $\mathcal{C}(X, \underline{K}^{\mathrm{MW}}_i\{\mathcal{L}\})$, i.e. $\widetilde{\mathrm{CH}}^i(X, \mathcal{L}) := H^i(X, \underline{K}^{\mathrm{MW}}_i\{\mathcal{L}\})$. We denote $\widetilde{\mathrm{CH}}^i(X) := \widetilde{\mathrm{CH}}^i(X, \mathcal{O}_X)$.

Remark 3.12. As soon as i is less than 0 or greater than the dimension of X, we have $H^i(X, \underline{K}^{\mathrm{MW}}_j\{\mathcal{L}\}) = 0$ and $\widetilde{\mathrm{CH}}^i(X, \mathcal{L}) = 0$.

Note that if we quotient the $\mathcal{C}^i(X, \underline{K}^{\mathrm{MW}}_j\{\mathcal{L}\})$ by η then we obtain the complex which was described in the previous section, differentials included (i.e. the morphism d^i_j mentioned in the previous section is induced by the morphism $d^i_{X,j,\mathcal{L}}$ (which commutes to product by η)), hence we have morphisms $H^i(X, \underline{K}^{\mathrm{MW}}_j\{\mathcal{L}\}) \to A^i(X, j)$ and in particular morphisms $\widetilde{\mathrm{CH}}^i(X, \mathcal{L}) \to \mathrm{CH}^i(X)$.

Homotopy invariance

Let us now state the property of homotopy invariance of Rost-Schmid groups. We denote by $\mathbb{A}^1_X$ the product of F-schemes $\mathbb{A}^1_F \times_F X$.

Theorem 3.13 (Theorem 5.38 in [Mor12])**.** Let $\pi : \mathbb{A}^1_X \to X$ be the projection and $i, j \in \mathbb{Z}$. The induced morphism $\pi^* : H^i(X, \underline{K}^{\mathrm{MW}}_j) \to H^i(\mathbb{A}^1_X, \underline{K}^{\mathrm{MW}}_j)$ is an isomorphism.

Localization long exact sequence

In order to give the localization long exact sequence, we first define boundary triples and boundary maps (which were introduced by Feld in [Fel20] who was inspired by Rost's work in [Ros96]).

Definition 3.14 (Boundary triple)**.** A boundary triple is a triple (Z, X, U), or rather a 5-tuple (Z, i, X, j, U), with $i : Z \to X$ a closed immersion and $j : U \to X$ an open immersion such that the image of U by j is the complement in X of the image of Z by i, where Z, X, U are smooth finite-type F-schemes of pure dimensions. We denote by d_Z and d_X the dimensions of Z and X respectively and by ν_Z the determinant of the normal sheaf of Z in X (which is the dual of the $\mathcal{O}_Z$-module $\mathscr{I}_Z/\mathscr{I}_Z^2$, where $\mathscr{I}_Z$ is the ideal sheaf of Z in X).

Remark 3.15. Let (Z, i, X, j, U) be a boundary triple and $n, m \in \mathbb{Z}$. Since every point x of codimension n in X is either a point of codimension $n + d_Z - d_X$ in Z (in which case, $\det(\mathcal{N}_{x/X})$ is canonically isomorphic to $\det(\mathcal{N}_{x/Z}) \otimes_{\kappa(x)} (\nu_Z)_{|x}$) or a point of codimension n in U (in which case, $\det(\mathcal{N}_{x/X})$ is canonically isomorphic to $\det(\mathcal{N}_{x/U})$), we have a canonical isomorphism $\mathcal{C}^n(X, \underline{K}_m^{\mathrm{MW}}) \simeq \mathcal{C}^{n+d_Z-d_X}(Z, \underline{K}_{m+d_Z-d_X}^{\mathrm{MW}}\{\nu_Z\}) \oplus \mathcal{C}^n(U, \underline{K}_m^{\mathrm{MW}})$.

Notation 3.16. Let (Z, i, X, j, U) be a boundary triple and $n, m \in \mathbb{Z}$. We denote the projections by

$$\begin{aligned} i^* &: \mathcal{C}^n(X, \underline{K}_m^{\mathrm{MW}}) \to \mathcal{C}^{n+d_Z-d_X}(Z, \underline{K}_{m+d_Z-d_X}^{\mathrm{MW}}\{\nu_Z\}) \\ j^* &: \mathcal{C}^n(X, \underline{K}_m^{\mathrm{MW}}) \to \mathcal{C}^n(U, \underline{K}_m^{\mathrm{MW}}) \end{aligned}$$

and the inclusions by

$$\begin{aligned} i_* &: \mathcal{C}^{n+d_Z-d_X}(Z, \underline{K}_{m+d_Z-d_X}^{\mathrm{MW}}\{\nu_Z\}) \to \mathcal{C}^n(X, \underline{K}_m^{\mathrm{MW}}) \\ j_* &: \mathcal{C}^n(U, \underline{K}_m^{\mathrm{MW}}) \to \mathcal{C}^n(X, \underline{K}_m^{\mathrm{MW}}) \end{aligned}$$

Remark 3.17. Let (Z, i, X, j, U) be a boundary triple and $n, m \in \mathbb{Z}$. Note that the morphisms i_* and j^* commute with the differentials of the Rost-Schmid complexes and induce morphisms $i_* : H^n(Z, \underline{K}_m^{\mathrm{MW}}\{\nu_Z\}) \to H^{n+d_X-d_Z}(X, \underline{K}_{m+d_X-d_Z}^{\mathrm{MW}})$ (which is also induced by the pushforward along the closed immersion i, see [Fas20, Subsection 2.3]) and $j^* : H^n(X, \underline{K}_m^{\mathrm{MW}}) \to H^n(U, \underline{K}_m^{\mathrm{MW}})$ (which is also induced by the pullback along the open immersion j, see [Fas20, Subsection 2.4]).

Definition 3.18 (Boundary map)**.** Let (Z, i, X, j, U) be a boundary triple and $n, m \in \mathbb{Z}$. The boundary map associated to this boundary triple is the morphism

$$\partial : \mathcal{C}^{n+d_X-d_Z}(U, \underline{K}^{\mathrm{MW}}_{m+d_X-d_Z}) \to \mathcal{C}^{n+1}(Z, \underline{K}^{\mathrm{MW}}_m\{\nu_Z\})$$

induced by the differential $d^{n+d_X-d_Z}_{X,m+d_X-d_Z}$ of the Rost-Schmid complex $\mathcal{C}(X, \underline{K}^{\mathrm{MW}}_{m+d_X-d_Z})$:

$$\partial = i^* \circ d^{n+d_X-d_Z}_{X,m+d_X-d_Z} \circ j_*$$

The following theorem is a special case of the exact triangle theorem in homological algebra (see [Rot88, Theorem 5.6] and note that the boundary maps are the connecting morphisms by definition). Note its similarity to Theorem 1.6 (which is also a consequence of the exact triangle theorem (and of Poincaré duality)).

Theorem 3.19. Let (Z, i, X, j, U) be a boundary triple. The boundary maps induce morphisms $\partial : H^{n+d_X-d_Z}(U, \underline{K}^{\mathrm{MW}}_{m+d_X-d_Z}) \to H^{n+1}(Z, \underline{K}^{\mathrm{MW}}_m\{\nu_Z\})$ and we have the following long exact sequence, called the localization long exact sequence:

$$\cdots \longrightarrow H^n(Z, \underline{K}^{\mathrm{MW}}_m\{\nu_Z\}) \xrightarrow{i_*} H^{n+d_X-d_Z}(X, \underline{K}^{\mathrm{MW}}_{m+d_X-d_Z}) \xrightarrow{j^*}$$

$$\xrightarrow{j^*} H^{n+d_X-d_Z}(U, \underline{K}^{\mathrm{MW}}_{m+d_X-d_Z}) \xrightarrow{\partial} H^{n+1}(Z, \underline{K}^{\mathrm{MW}}_m\{\nu_Z\}) \longrightarrow \cdots$$

Rost-Schmid groups are sheaf cohomology groups

The reason Rost-Schmid groups are denoted as they are is the following: for each $j \in \mathbb{Z}$, there is a strongly $\mathbb{A}^1$-invariant sheaf $\underline{K}^{\mathrm{MW}}_j$ of abelian groups (this means that the morphisms $H^0_{Nis}(X, \underline{K}^{\mathrm{MW}}_j) \to H^0_{Nis}(\mathbb{A}^1_X, \underline{K}^{\mathrm{MW}}_j)$ and $H^1_{Nis}(X, \underline{K}^{\mathrm{MW}}_j) \to H^1_{Nis}(\mathbb{A}^1_X, \underline{K}^{\mathrm{MW}}_j)$ induced by the projection $\mathbb{A}^1_X := \mathbb{A}^1_F \times_F X \to X$ are isomorphisms, see [Mor12, Definition 1.7]) such that the Rost-Schmid cohomology groups $H^i(X, \underline{K}^{\mathrm{MW}}_j)$ are the Zariski sheaf cohomology groups $H^i_{Zar}(X, \underline{K}^{\mathrm{MW}}_j)$ with respect to the sheaf $\underline{K}^{\mathrm{MW}}_j$ as well as the Nisnevich sheaf cohomology groups $H^i_{Nis}(X, \underline{K}^{\mathrm{MW}}_j)$ with respect to the sheaf $\underline{K}^{\mathrm{MW}}_j$. See [Mor12, Chapter 3] for the construction of $\underline{K}^{\mathrm{MW}}_j$ (which is called the unramified Milnor-Witt K-theory in weight j) and the fact that it is a strongly $\mathbb{A}^1$-invariant sheaf of abelian groups.

Theorem 3.20 (Corollary 5.43 in [Mor12])**.** For all $i, j \in \mathbb{Z}$, there are canonical isomorphisms $H^i(X, \underline{K}^{\mathrm{MW}}_j) \simeq H^i_{Zar}(X, \underline{K}^{\mathrm{MW}}_j) \simeq H^i_{Nis}(X, \underline{K}^{\mathrm{MW}}_j)$.

Remark 3.21. It follows immediately from Theorems 3.20 and 3.13 that for all $j \in \mathbb{Z}$, the sheaf $\underline{K}_j^{\mathrm{MW}}$ is strictly $\mathbb{A}^1$-invariant (this means that for all $i \in \mathbb{N}_0$, the morphism $H^i_{Nis}(X, \underline{K}_j^{\mathrm{MW}}) \to H^i_{Nis}(\mathbb{A}^1_X, \underline{K}_j^{\mathrm{MW}})$ induced by the projection $\mathbb{A}^1_X \to X$ is an isomorphism, see [Mor12, Definition 1.7]).

Orientations and their induced isomorphisms

In all subsequent chapters (as well as Section 3.4), orientations will play a major role. Let us define orientations as Morel did in [Mor12, Definition 4.3].

Definition 3.22 (Orientation of a locally free module)**.** An *orientation* of a locally free $\mathcal{O}_X$-module $\mathcal{V}$ of constant finite rank r is an isomorphism $o : \det(\mathcal{V}) = \Lambda^r(\mathcal{V}) \to \mathcal{L} \otimes \mathcal{L}$ where $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module.

Two orientations $o : \det(\mathcal{V}) \to \mathcal{L} \otimes \mathcal{L}$ and $o' : \det(\mathcal{V}) \to \mathcal{L}' \otimes \mathcal{L}'$ are said to be equivalent if there exists an isomorphism $\psi : \mathcal{L} \to \mathcal{L}'$ such that $(\psi \otimes \psi) \circ o = o'$. The equivalence class of o, denoted $\bar{o}$, is called the *orientation class* of o.

Remark 3.23. Note that if $X = \mathrm{Spec}(F)$ then $\mathcal{V}$ is an F-vector space of dimension r and an orientation class of $\mathcal{V}$ corresponds to a basis of $\mathcal{V}$ up to multiplication by a matrix of determinant a square of a unit of F. In particular, if $X = \mathrm{Spec}(\mathbb{R})$ then an orientation class of $\mathcal{V}$ corresponds to a basis of $\mathcal{V}$ up to multiplication by a matrix of positive determinant, thus we recover the usual definition of orientation class.

Before we define the isomorphisms $\widetilde{o} : H^i(X, \underline{K}_j^{\mathrm{MW}}\{\det(\mathcal{V})\}) \to H^i(X, \underline{K}_j^{\mathrm{MW}})$ induced by the orientation $o : \det(\mathcal{V}) \to \mathcal{L}\otimes\mathcal{L}$, we need the following lemma.

Lemma 3.24. Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. For all $i, j \in \mathbb{Z}$, the morphism

$$\begin{cases} \mathcal{C}^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L} \otimes \mathcal{L}\}) & \to & \mathcal{C}^i(X, \underline{K}_j^{\mathrm{MW}}) \\ \displaystyle\sum_{x \in I} k_x \otimes (l_x \otimes l_x) & \mapsto & \displaystyle\sum_{x \in I} k_x \end{cases}$$

where I is a finite subset of $X^{(i)}$, $k_x \in K_{j-i}^{\mathrm{MW}}(\kappa(x), \nu_x)$ and $l_x \in \mathcal{L}_{|x} \setminus \{0\}$, is a well-defined isomorphism which commutes with differentials.

Proof. First note that elements of $\mathcal{C}^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L} \otimes \mathcal{L}\})$ are of the form $\sum_{x \in I} m_x \otimes t_x$ with I a finite subset of $X^{(i)}$, $m_x \in K_{j-i}^{\mathrm{MW}}(\kappa(x))$ and $t_x \in \mathbb{Z}[(\nu_x \otimes (\mathcal{L} \otimes \mathcal{L})_{|x}) \setminus \{0\}]$. Let $x \in I$. Since $\nu_x \otimes (\mathcal{L}\otimes\mathcal{L})_{|x}$ is a $\kappa(x)$-vector space of dimension 1, there exist $n_x \in K_{j-i}^{\mathrm{MW}}(\kappa(x))$ and $s_x \in (\nu_x \otimes (\mathcal{L} \otimes \mathcal{L})_{|x}) \setminus \{0\}$ such that $m_x \otimes t_x = n_x \otimes s_x$. By definition of $K_{j-i}^{\mathrm{MW}}(\kappa(x), \nu_x)$, there exist

$h_x \in K_{j-i}^{\mathrm{MW}}(\kappa(x), \nu_x)$ and $l_x, r_x \in \mathcal{L}_{|x} \setminus \{0\}$ such that $n_x \otimes s_x = h_x \otimes (l_x \otimes r_x)$. Since $\mathcal{L}_{|x}$ is a $\kappa(x)$-vector space of dimension 1, there exists $v_x \in \kappa(x)^*$ such that $r_x = v_x l_x$. It follows that $h_x \otimes (l_x \otimes r_x) = \langle v_x \rangle h_x \otimes (l_x \otimes l_x)$. Denoting $k_x := \langle v_x \rangle h_x$, we obtain $m_x \otimes t_x = k_x \otimes (l_x \otimes l_x)$. Thus, elements of $\mathcal{C}^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L} \otimes \mathcal{L}\})$ are of the form $\sum_{x \in I} k_x \otimes (l_x \otimes l_x)$ with I a finite subset of $X^{(i)}$, $k_x \in K_{j-i}^{\mathrm{MW}}(\kappa(x), \nu_x)$ and $l_x \in \mathcal{L}_{|x} \setminus \{0\}$.

To check that our map is well-defined, let us show that $\sum_{x \in I} k_x = \sum_{y \in J} k'_y$ in $\mathcal{C}^i(X, \underline{K}_j^{\mathrm{MW}})$ whenever $\sum_{x \in I} k_x \otimes (l_x \otimes l_x) = \sum_{y \in J} k'_y \otimes (l'_y \otimes l'_y)$ in $\mathcal{C}^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L} \otimes \mathcal{L}\})$, where I, J are finite subsets of $X^{(i)}$ and for all $x \in I$ and $y \in J$, $k_x \in K_{j-i}^{\mathrm{MW}}(\kappa(x), \nu_x)$, $k'_y \in K_{j-i}^{\mathrm{MW}}(\kappa(y), \nu_y)$, $l_x \in \mathcal{L}_{|x} \setminus \{0\}$ and $l'_y \in \mathcal{L}_{|y} \setminus \{0\}$. Since $\mathcal{C}^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L} \otimes \mathcal{L}\})$ is the direct sum over x in $X^{(i)}$ of $K_{j-i}^{\mathrm{MW}}(\kappa(x), \nu_x \otimes (\mathcal{L} \otimes \mathcal{L})_{|x})$, note that for all $x \in I \setminus (I \cap J)$, $k_x = 0$, for all $y \in J \setminus (I \cap J)$, $k'_y = 0$ and for all $z \in I \cap J$, $k_z \otimes (l_z \otimes l_z) = k'_z \otimes (l'_z \otimes l'_z)$. Let $x \in I \cap J$. Since $\mathcal{L}_{|x}$ is a one-dimensional $\kappa(x)$-vector space, there exists $u_x \in F^*$ such that $l'_x = u_x l_x$. Hence, $k'_x \otimes (l'_x \otimes l'_x) = \langle u_x^2 \rangle k'_x \otimes (l_x \otimes l_x) = k'_x \otimes (l_x \otimes l_x)$, thus $k'_x \otimes (l_x \otimes l_x) = k_x \otimes (l_x \otimes l_x)$ and finally $k'_x = k_x$ (since the tensor product is over $\mathbb{Z}[\kappa(x)^*]$). Similarly, the equality $k'_x \otimes (l'_x \otimes l'_x) = k'_x \otimes (l_x \otimes l_x)$ above gives straightforwardly that our map is a morphism and that the map

$$\begin{cases} \mathcal{C}^i(X, \underline{K}_j^{\mathrm{MW}}) & \to & \mathcal{C}^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L} \otimes \mathcal{L}\}) \\ \displaystyle\sum_{x \in I} k_x & \mapsto & \displaystyle\sum_{x \in I} k_x \otimes (l_x \otimes l_x) \end{cases}$$

is well-defined and is a morphism, which shows that our morphism is an isomorphism. The commutation with differentials is straightforward. □

Notation 3.25. Let $i, j \in \mathbb{Z}$ and $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. We denote by $\iota_{\mathcal{L},i,j} : H^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L} \otimes \mathcal{L}\}) \to H^i(X, \underline{K}_j^{\mathrm{MW}})$ the isomorphism induced by the isomorphism of Lemma 3.24. If $o : \det(\mathcal{V}) \to \mathcal{L} \otimes \mathcal{L}$ is an orientation then we denote by $\widetilde{o} : H^i(X, \underline{K}_j^{\mathrm{MW}}\{\det(\mathcal{V})\}) \to H^i(X, \underline{K}_j^{\mathrm{MW}})$ the isomorphism which is the composite of the isomorphism $H^i(X, \underline{K}_j^{\mathrm{MW}}\{\det(\mathcal{V})\}) \to H^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L} \otimes \mathcal{L}\})$ induced by o and of the isomorphism $\iota_{\mathcal{L},i,j}$.

In the following proposition, we show that the isomorphism $\widetilde{o}$ only depends on the orientation class $\overline{o}$ of o (see Definition 3.22).

Proposition 3.26. Let $i, j \in \mathbb{Z}$, $o : \det(\mathcal{V}) \to \mathcal{L} \otimes \mathcal{L}$ be an orientation and $\psi : \mathcal{L} \to \mathcal{L}'$ be an isomorphism. Then $\widetilde{(\psi \otimes \psi) \circ o} = \widetilde{o}$.

Proof. Note that the isomorphism $H^i(X, \underline{K}_j^{\mathrm{MW}}\{\det(\mathcal{V})\}) \to H^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L}' \otimes \mathcal{L}'\})$ induced by $(\psi \otimes \psi) \circ o$ is the composite of the isomorphism $H^i(X, \underline{K}_j^{\mathrm{MW}}\{\det(\mathcal{V})\}) \to H^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L} \otimes \mathcal{L}\})$ induced by o and of the isomorphism $H^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L} \otimes \mathcal{L}\}) \to$

$H^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L}' \otimes \mathcal{L}'\})$ induced by $\psi \otimes \psi$. Hence it suffices to show that the isomorphism $\iota_{\mathcal{L},i,j}$ is the composite of the isomorphism $H^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L} \otimes \mathcal{L}\}) \to H^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L}' \otimes \mathcal{L}'\})$ induced by $\psi \otimes \psi$ and of the isomorphism $\iota_{\mathcal{L}',i,j}$. This follows directly from the definitions of $\iota_{\mathcal{L},i,j}$ and $\iota_{\mathcal{L}',i,j}$ and the fact that the isomorphism $H^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L} \otimes \mathcal{L}\}) \to H^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L}' \otimes \mathcal{L}'\})$ induced by $\psi \otimes \psi$ sends $k_x \otimes (l_x \otimes l_x)$ to $k_x \otimes (\psi(l_x) \otimes \psi(l_x))$ for all $x \in X^{(i)}, k_x \in K_{j-i}^{\mathrm{MW}}(\kappa(x), \nu_x)$ and $l_x \in \mathcal{L}_{|x} \setminus \{0\}$. □

3.3 The intersection product

In this section we define the intersection product in quadratic intersection theory and recall some of its properties, then we present a formula to compute the intersection product.

Throughout this section, F is a perfect field and X is a smooth finite-type F-scheme.

Before we define the intersection product, we need to define the exterior product (a.k.a. cross product).

Definition 3.27 (The exterior product (or cross product))**.** Let X and X' be smooth finite-type F-schemes and $i, i', j, j' \in \mathbb{Z}$. The exterior product $\mu : \mathcal{C}^i(X, \underline{K}_j^{\mathrm{MW}}) \times \mathcal{C}^{i'}(X', \underline{K}_{j'}^{\mathrm{MW}}) \to \mathcal{C}^{i+i'}(X \times X', \underline{K}_{j+j'}^{\mathrm{MW}})$ (which is sometimes denoted $\times$ and called cross product) is the (only) morphism which for all $x \in X^{(i)}, k \in K_{j-i}^{\mathrm{MW}}(\kappa(x), \nu_x)$, $x' \in (X')^{(i')}, k' \in K_{j'-i'}^{\mathrm{MW}}(\kappa(x'), \nu_{x'})$, maps (k, k') to the sum over $l \in \{1, \ldots, n\}$ of $kk' \in K_{j+j'-(i+i')}^{\mathrm{MW}}(\kappa(z_l), \nu_{z_l})$, where $z_1, \ldots, z_l \in (X \times X')^{(i+i')}$ are such that $\kappa(x) \otimes_F \kappa(x') \simeq \prod_{l=1}^n \kappa(z_l)$.

The exterior product induces a well-defined product $\mu : H^i(X, \underline{K}_j^{\mathrm{MW}}) \times H^{i'}(X', \underline{K}_{j'}^{\mathrm{MW}}) \to H^{i+i'}(X \times X', \underline{K}_{j+j'}^{\mathrm{MW}})$ (see [Fel20, Section 11]). The intersection product is defined from the exterior product and the pull-back along the diagonal (see [Fas20, Subsection 3.3]), which is also known as the Gysin morphism induced by the diagonal (see [Fel20, Section 10]).

Definition 3.28 (The intersection product)**.** Let $\Delta : X \to X \times X$ be the diagonal. The intersection product $\cdot : H^i(X, \underline{K}_j^{\mathrm{MW}}) \times H^{i'}(X, \underline{K}_{j'}^{\mathrm{MW}}) \to H^{i+i'}(X, \underline{K}_{j+j'}^{\mathrm{MW}})$ is the composite of the exterior product $\mu : H^i(X, \underline{K}_j^{\mathrm{MW}}) \times H^{i'}(X, \underline{K}_{j'}^{\mathrm{MW}}) \to H^{i+i'}(X \times X, \underline{K}_{j+j'}^{\mathrm{MW}})$ with the pull-back (a.k.a. Gysin morphism) $\Delta^* : H^{i+i'}(X \times X, \underline{K}_{j+j'}^{\mathrm{MW}}) \to H^{i+i'}(X, \underline{K}_{j+j'}^{\mathrm{MW}})$.

We will give a more explicit definition of the intersection product below under some assumptions. Before we do this, let us state that the intersection product is a product, then expand on its graded commutativity.

Proposition 3.29 (Subsection 3.4 in [Fas20] or Theorem 11.6 in [Fel20]). The intersection product makes $\bigoplus_{i,j\in\mathbb{Z}} H^i(X, \underline{K}_j^{\mathrm{MW}})$ into a graded $K_0^{\mathrm{MW}}(F)$-algebra, which is called the Rost-Schmid ring. In particular, the intersection product makes $\bigoplus_{i\in\mathbb{Z}} \widetilde{\mathrm{CH}}^i(X)$ into a graded $K_0^{\mathrm{MW}}(F)$-algebra, which is called the Chow-Witt ring.

Recall that $\epsilon = -\langle -1\rangle \in K_0^{\mathrm{MW}}(F)$ and that Milnor-Witt K-theory is ϵ-commutative (see 5 in Proposition 2.32).

Proposition 3.30 (Subsection 3.4 in [Fas20]). Let $i, i', j, j' \in \mathbb{Z}$, $c_1 \in H^i(X, \underline{K}_j^{\mathrm{MW}})$ and $c_2 \in H^{i'}(X, \underline{K}_{j'}^{\mathrm{MW}})$. The intersection product of c_1 with c_2 is $\langle(-1)^{ii'}\rangle\epsilon^{(j-i)(j'-i')}$-commutative:

$$c_2 \cdot c_1 = \begin{cases} c_1 \cdot c_2 \text{ if } ii' \text{ is even and } (j-i)(j'-i') \text{ is even} \\ \epsilon(c_1 \cdot c_2) \text{ if } ii' \text{ is even and } (j-i)(j'-i') \text{ is odd} \\ -\epsilon(c_1 \cdot c_2) \text{ if } ii' \text{ is odd and } (j-i)(j'-i') \text{ is even} \\ -(c_1 \cdot c_2) \text{ if } ii' \text{ is odd and } (j-i)(j'-i') \text{ is odd} \end{cases}$$

We now present a formula to compute the intersection product under some assumptions (this will be very useful in Chapters 6 and 7). The following theorem has been proved by Déglise; the proof will be made available in the second part of his notes [Dég23]. In the meantime, we give a proof sketch of this theorem below.

Theorem 3.31. Let $n_1, n_2 \geq 0$ and D_1, D_2 be distinct smooth integral divisors in X. For all $i \in \{1, 2\}$, let g_i be a local parameter for D_i, i.e. g_i is a uniformizing parameter for $\mathcal{O}_{X,D_i}$. The intersection product of $\eta^{n_1} \otimes \overline{g_1}^* \in H^1(X, \underline{K}_{1-n_1}^{\mathrm{MW}})$ (over the generic point of D_1) with $\eta^{n_2} \otimes \overline{g_2}^* \in H^1(X, \underline{K}_{1-n_2}^{\mathrm{MW}})$ (over the generic point of D_2) is the class in $H^2(X, \underline{K}_{2-n_1-n_2}^{\mathrm{MW}})$ of the sum over the generic points x of the irreducible components of $D_1 \cap D_2$ of $(m_x)_\epsilon \langle u_x\rangle \eta^{n_1+n_2} \otimes (\overline{\pi_x}^* \otimes \overline{g_1}^*)$ (over the point x), where π_x is a uniformizing parameter for $\mathcal{O}_{X,x}/(g_1)$, u_x is a unit in $\mathcal{O}_{X,x}/(g_1)$ and $m_x \in \mathbb{Z}$, such that $g_2 = u_x \pi_x^{m_x} \in \mathcal{O}_{X,x}/(g_1)$.

The ideas of the proof are the following:

- Reduce the problem to the case where $D_1 = \mathrm{div}(g_1)$.
- Denoting by $i_1 : D_1 \to X$ the inclusion and by $\Theta_1 : H^0(D_1, \underline{K}_{-n_1}^{\mathrm{MW}}) \to H^0(D_1, \underline{K}_{-n_1}^{\mathrm{MW}}\{\nu_{D_1}\})$ (where ν_{D_1} is the determinant of the normal sheaf of D_1 in X) the isomorphism which sends η^{n_1} to $\eta^{n_1} \otimes \overline{g_1}^*$, check that $\eta^{n_1} \otimes \overline{g_1}^* \in H^1(X, \underline{K}_{1-n_1}^{\mathrm{MW}})$ is equal to $(i_1)_*(\Theta_1(\eta^{n_1}))$.

- Use the projection formula (Theorem 3.19 in [Fas20]) to show that $(i_1)_*(\Theta_1(\eta^{n_1}))\cdot(\eta^{n_2}\otimes\overline{g_2}^*)=(i_1)_*(\Theta_1(\eta^{n_1})\cdot(i_1)^*(\eta^{n_2}\otimes\overline{g_2}^*))$.

- Use Proposition 3.2.15 in [DFJ22], which states that if i is the closed immersion of a principal divisor $D=\operatorname{div}(\pi)$ and j is the complementary open immersion to i, then $i^!=\partial\circ\gamma_{[\pi]}\circ j^!$, to show that $(i_1)^*=\partial_1\circ\gamma_{[g_1]}\circ(j_1)^*$, with j_1 the complementary open immersion to i_1, ∂_1 the boundary map associated to the boundary triple $(D_1,i_1,X,j_1,X\setminus D_1)$ and $\gamma_{[g_1]}$ the multiplication by $[g_1]$.

- Deduce from the previous steps that $(\eta^{n_1}\otimes\overline{g_1}^*)\cdot(\eta^{n_2}\otimes\overline{g_2}^*)$ is equal to $(i_1)_*(\Theta_1(\eta^{n_1})\cdot(\partial_1\circ\gamma_{[g_1]}\circ(j_1)^*)(\eta^{n_2}\otimes\overline{g_2}^*))$ and conclude.

In Chapters 6 and 7 we use the following formula to compute the quadratic linking class and the quadratic linking degree.

Corollary 3.32. Let $n_1,n_2\geq 0$ and D_1,D_2 be distinct smooth integral divisors in X. For all $i\in\{1,2\}$, let g_i be a local parameter for D_i and f_i be a unit in $\kappa(D_i)=\mathcal{O}_{X,D_i}/\mathfrak{m}_{X,D_i}$ such that for all generic points x of irreducible components of $D_1\cap D_2$, $f_i\in\kappa(x)=\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ is a unit. The intersection product of $\langle f_1\rangle\eta^{n_1}\otimes\overline{g_1}^*\in H^1(X,\underline{K}^{\mathrm{MW}}_{1-n_1})$ (over the generic point of D_1) with $\langle f_2\rangle\eta^{n_2}\otimes\overline{g_2}^*\in H^1(X,\underline{K}^{\mathrm{MW}}_{1-n_2})$ (over the generic point of D_2) is the class in $H^2(X,\underline{K}^{\mathrm{MW}}_{2-n_1-n_2})$ of the sum over the generic points x of the irreducible components of $D_1\cap D_2$ of $(m_x)_\epsilon\langle f_1f_2u_x\rangle\eta^{n_1+n_2}\otimes(\overline{\pi_x}^*\otimes\overline{g_1}^*)$ (over the point x), where π_x is a uniformizing parameter for $\mathcal{O}_{X,x}/(g_1)$, u_x is a unit in $\mathcal{O}_{X,x}/(g_1)$ and $m_x\in\mathbb{Z}$ such that $g_2=u_x\pi_x^{m_x}\in\mathcal{O}_{X,x}/(g_1)$.

Proof. First note that, with the notations above, $f_i\in\kappa(x)$ is well-defined since if f_i and f_i' are two representatives in $\mathcal{O}_{X,D_i}$ of $f_i\in\kappa(D_i)$ (hence differ by an element of $\mathfrak{m}_{X,D_i}$) and if $f_i,f_i'\in\mathcal{O}_{X,x}$ are sent by the canonical morphism $\psi:\mathcal{O}_{X,x}\to\mathcal{O}_{X,D_i}$ to $f_i,f_i'\in\mathcal{O}_{X,D_i}$ respectively, then $f_i,f_i'\in\mathcal{O}_{X,x}$ differ by an element of $\mathfrak{m}_{X,x}$ (since $\psi^{-1}(\mathfrak{m}_{X,D_i})\subset\mathfrak{m}_{X,x}$).

Note that for all $i\in\{1,2\}$, $\langle f_i\rangle\eta^{n_i}\otimes\overline{g_i}^*=\eta^{n_i}\otimes\overline{f_ig_i}^*$ with f_ig_i a local parameter for D_i ($\overline{f_ig_i}\in\mathfrak{m}_{X,D_i}/\mathfrak{m}^2_{X,D_i}$ is well-defined since $f_i\in\mathcal{O}_{X,D_i}/\mathfrak{m}_{X,D_i}$ and $g_i\in\mathfrak{m}_{X,D_i}$ and (a representative of) $f_ig_i\in\mathfrak{m}_{X,D_i}$ is a generator of $\mathfrak{m}_{X,D_i}$ since (a representative of) f_i is a unit in $\mathcal{O}_{X,D_i}$ and g_i is a generator of $\mathfrak{m}_{X,D_i}$).

Therefore, by Theorem 3.31, the intersection product of $\langle f_1\rangle\eta^{n_1}\otimes\overline{g_1}^*$ with $\langle f_2\rangle\eta^{n_2}\otimes\overline{g_2}^*$ is the sum over the generic points x of the irreducible components of $D_1\cap D_2$ of $(m_x)_\epsilon\langle v_x\rangle\eta^{n_1+n_2}\otimes(\overline{\pi_x}^*\otimes\overline{f_1g_1}^*)$ (over the point x), where π_x is a uniformizing parameter for $\mathcal{O}_{X,x}/(f_1g_1)$, v_x is a unit in $\mathcal{O}_{X,x}/(f_1g_1)$ and $m_x\in\mathbb{Z}$ such that $f_2g_2=v_x\pi_x^{m_x}\in\mathcal{O}_{X,x}/(f_1g_1)$.

Note that since f_2 is a unit in $\kappa(x)$, $u_x := f_2^{-1} v_x$ is a unit in $\kappa(x)$ and $(m_x)_\epsilon \langle v_x \rangle \eta^{n_1+n_2} \otimes (\overline{\pi_x}^* \otimes \overline{f_1 g_1}^*) = (m_x)_\epsilon \langle f_2 u_x \rangle \eta^{n_1+n_2} \otimes (\overline{\pi_x}^* \otimes \overline{f_1 g_1}^*)$. Further note that since f_1 is a unit in $\kappa(x)$, the ideal $(f_1 g_1)$ is equal to the ideal (g_1) in $\mathcal{O}_{X,x}$ and $(m_x)_\epsilon \langle f_2 u_x \rangle \eta^{n_1+n_2} \otimes (\overline{\pi_x}^* \otimes \overline{f_1 g_1}^*) = (m_x)_\epsilon \langle f_1 f_2 u_x \rangle \eta^{n_1+n_2} \otimes (\overline{\pi_x}^* \otimes \overline{g_1}^*)$. Finally note that, by definition of u_x, $g_2 = u_x \pi_x^{m_x} \in \mathcal{O}_{X,x}/(g_1)$. □

It would be very useful to have more general intersection formulas in order to compute the quadratic linking class and the quadratic linking degree in more cases (especially in cases which give a quadratic linking degree in $\mathrm{GW}(F)$ rather than in $\mathrm{W}(F)$).

Future work 1 (More general formulas for the intersection product)**.** The formulas given in Theorem 3.31 and Corollary 3.32 can probably be generalised to the following settings (by increasing order of difficulty):

- The assumptions on the right-hand term of the intersection product could be greatly weakened (recall the asymmetry of the proof sketch of Theorem 3.31 (and of the resulting formula)).

- The smooth integral divisor D_1 could be replaced with a complete intersection of smooth integral divisors (by taking intersection products in a row, asking that each left-hand term be in $H^1(Y, \underline{K}_j^{\mathrm{MW}})$ with Y the intersection of the divisors which have already been considered and $j \leq 1$).

- The assumption that $n_1 \geq 0$ could probably be weakened, but the proof of the intersection formula would have to be different (since we would not have the special element η^{n_1} anymore). If we had a formula in the case where $n_1 = -1$ (and $n_2 \geq -1$) then we could compute the quadratic linking class and the quadratic linking degree in all the cases of codimension 2 links. Under the same assumptions as the ones of Corollary 3.32, a conjectural formula for the intersection product of $[f_1] \otimes \overline{g_1}^* \in H^1(X, \underline{K}_2^{\mathrm{MW}})$ (over the generic point of D_1) with $[f_2] \otimes \overline{g_2}^* \in H^1(X, \underline{K}_2^{\mathrm{MW}})$ (over the generic point of D_2) is the following: the class in $H^2(X, \underline{K}_4^{\mathrm{MW}})$ of the sum over the generic points x of the irreducible components of $D_1 \cap D_2$ of $(m_x)_\epsilon [f_2][f_1] \langle u_x \rangle \otimes (\overline{\pi_x}^* \otimes \overline{g_1}^*)$ (over the point x), with the same notations as in Corollary 3.32. Note that $[f_2][f_1] \neq [f_1][f_2]$ in general (in fact, $[f_2][f_1] = \epsilon[f_1][f_2]$; see 5 in Proposition 2.32). If $[f_1] \otimes \overline{g_1}^* \in H^1(X, \underline{K}_2^{\mathrm{MW}})$ is replaced with $\langle f_1 \rangle \eta^{n_1} \otimes \overline{g_1}^* \in H^1(X, \underline{K}_{1-n_1}^{\mathrm{MW}})$ (where $n_1 \geq 0$), conjecturally $(m_x)_\epsilon [f_2][f_1] \langle u_x \rangle \otimes (\overline{\pi_x}^* \otimes \overline{g_1}^*)$ (over the point x) should be replaced with $(m_x)_\epsilon [f_2] \langle f_1 u_x \rangle \eta^{n_1} \otimes (\overline{\pi_x}^* \otimes \overline{g_1}^*)$ (over the point x). If $[f_2] \otimes$

$\overline{g_2}^* \in H^1(X, \underline{K}_2^{\mathrm{MW}})$ is replaced with $\langle f_2\rangle\eta^{n_2} \otimes \overline{g_2}^* \in H^1(X, \underline{K}_{1-n_2}^{\mathrm{MW}})$, conjecturally $(m_x)_\epsilon[f_2][f_1]\langle u_x\rangle \otimes (\overline{\pi_x}^* \otimes \overline{g_1}^*)$ (over the point x) should be replaced with $(m_x)_\epsilon[f_1]\langle f_2 u_x\rangle\eta^{n_2} \otimes (\overline{\pi_x}^* \otimes \overline{g_1}^*)$ (over the point x).

3.4 Computations of Rost-Schmid groups

In the following chapters, we will need to know the Rost-Schmid groups of several smooth schemes. Furthermore, from Chapter 5 onwards (respectively Chapter 6 onwards), we will need *explicit* isomorphisms (resp. *computable* explicit isomorphisms) between these Rost-Schmid groups and well-known groups. In this section, we provide these explicit isomorphisms.

Throughout this section, F is a perfect field, X is a smooth finite-type F-scheme, and for each $n \in \mathbb{N}$, $\mathbb{A}_F^n = \mathrm{Spec}(F[x_1, \dots, x_n])$ and $\mathbb{P}_F^n = \mathrm{Proj}(F[x_0, \dots, x_n])$.

We begin with the following basic result.

Proposition 3.33. Let $i, j \in \mathbb{Z}$. The Rost-Schmid group $H^i(\mathrm{Spec}(F), \underline{K}_j^{\mathrm{MW}})$ is equal to $K_j^{\mathrm{MW}}(F)$ if $i = 0$, to 0 otherwise.

Proof. By definition, for all $i, j \in \mathbb{Z}$, $\mathcal{C}^i(\mathrm{Spec}(F), \underline{K}_j^{\mathrm{MW}})$ is equal to $K_j^{\mathrm{MW}}(F)$ if $i = 0$, to 0 otherwise. The result follows directly. □

Homotopy invariance (Theorem 3.13) gives us the following corollary.

Corollary 3.34. Let $n, i \in \mathbb{N}$, $j \in \mathbb{Z}$ and $\pi : \mathbb{A}_F^n \to \mathrm{Spec}(F)$ be the projection. The morphisms $\pi^* : 0 = H^i(\mathrm{Spec}(F), \underline{K}_j^{\mathrm{MW}}) \to H^i(\mathbb{A}_F^n, \underline{K}_j^{\mathrm{MW}})$ and $\pi^* : K_j^{\mathrm{MW}}(F) = H^0(\mathrm{Spec}(F), \underline{K}_j^{\mathrm{MW}}) \to H^0(\mathbb{A}_F^n, \underline{K}_j^{\mathrm{MW}})$ are isomorphisms.

The last two results together with the localization long exact sequence allow us to compute the Rost-Schmid groups of $\mathbb{A}_F^n \setminus \{0\}$ for $n \geq 2$. Recall Definitions 3.18 (boundary maps) and 3.22 (orientations) and Notation 3.25.

Proposition 3.35. Let $n \geq 2$ and $i, j \in \mathbb{Z}$ be integers. We denote by $\psi : \mathbb{A}_F^n \setminus \{0\} \to \mathbb{A}_F^n$ the inclusion, by $\pi : \mathbb{A}_F^n \to \mathrm{Spec}(F)$ the projection, by ∂ the boundary map associated to the boundary triple $(\{0\}, \mathbb{A}_F^n, \mathbb{A}_F^n \setminus \{0\})$ and by $o : \det(\mathcal{N}_{\{0\}/\mathbb{A}_F^n}) \to \mathcal{O}_{\{0\}} \otimes \mathcal{O}_{\{0\}}$ the orientation of the normal sheaf of $\{0\}$ in $\mathbb{A}_F^n$ which maps $\overline{x_1}^* \wedge \cdots \wedge \overline{x_n}^*$ to $1 \otimes 1$. The morphisms $\psi^* \circ \pi^* : K_j^{\mathrm{MW}}(F) = H^0(\mathrm{Spec}(F), \underline{K}_j^{\mathrm{MW}}) \to H^0(\mathbb{A}_F^n \setminus \{0\}, \underline{K}_j^{\mathrm{MW}})$ and $\widetilde{o} \circ \partial : H^{n-1}(\mathbb{A}_F^n \setminus \{0\}, \underline{K}_j^{\mathrm{MW}}) \to H^0(\{0\}, \underline{K}_{j-n}^{\mathrm{MW}}) = K_{j-n}^{\mathrm{MW}}(F)$ are isomorphisms and if $i \notin \{0, n-1\}$ then $H^i(\mathbb{A}_F^n \setminus \{0\}, \underline{K}_j^{\mathrm{MW}}) = 0$.

Proof. The localization long exact sequence (see Theorem 3.19) associated to the boundary triple $(\{0\}, \mathbb{A}^n_F, \mathbb{A}^n_F \setminus \{0\})$ gives the following exact sequences for all $j \in \mathbb{Z}$ and $i \notin \{0, n-1\}$:

$$0 \longrightarrow H^0(\mathbb{A}^n_F, \underline{K}^{\mathrm{MW}}_j) \xrightarrow{\psi^*} H^0(\mathbb{A}^n_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_j) \longrightarrow 0$$

$$0 \longrightarrow H^{n-1}(\mathbb{A}^n_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_j) \xrightarrow{\partial} H^0(\{0\}, \underline{K}^{\mathrm{MW}}_{j-n}\{\det(\mathcal{N}_{\{0\}/\mathbb{A}^n_F})\}) \longrightarrow 0$$

$$0 \longrightarrow H^i(\mathbb{A}^n_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_j) \longrightarrow 0$$

The result follows directly from this, Proposition 3.33 and Corollary 3.34. □

Notation 3.36. Let $n \geq 2$ and $j \leq n$ be integers. We denote by $\zeta_{n,j}$ the isomorphism which is the composite of the isomorphism $\widetilde{o} \circ \partial : H^{n-1}(\mathbb{A}^n_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_j) \to K^{\mathrm{MW}}_{j-n}(F)$ (see Proposition 3.35) and of the isomorphism γ_{j-n} (see Theorem 2.33).

Remark 3.37. Note that the localization long exact sequence associated to the boundary triple $(\{0\}, \mathbb{A}^1_F, \mathbb{A}^1_F \setminus \{0\})$ gives for all $j \in \mathbb{Z}$ and $i \neq 0$ the equality $H^i(\mathbb{A}^1_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_j) = 0$ and the following short exact sequence:

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^0(\mathbb{A}^1_F, \underline{K}^{\mathrm{MW}}_j) & \xrightarrow{\psi^*} & H^0(\mathbb{A}^1_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_j) & \xrightarrow{\partial} & H^0(\{0\}, \underline{K}^{\mathrm{MW}}_{j-1}\{\det(\mathcal{N}_{\{0\}/\mathbb{A}^1_F})\}) & \longrightarrow 0 \\ & \wr \uparrow \pi^* & & & & \wr \downarrow \widetilde{o} & \\ & K^{\mathrm{MW}}_j(F) & & & & K^{\mathrm{MW}}_{j-1}(F) & \end{array}$$

In particular, $H^0(\mathbb{A}^1_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_j) \neq 0$.

Notation 3.38. Let $n \in \mathbb{N}$.

$$Q_{2n} := \mathrm{Spec}(F[x_1, \ldots, x_n, y_1, \ldots, y_n, z]/(\sum_{i=1}^{n} x_i y_i - z(1+z)))$$

$$Q_{2n-1} := \mathrm{Spec}(F[x_1, \ldots, x_n, y_1, \ldots, y_n]/(\sum_{i=1}^{n} x_i y_i - 1))$$

In the following proposition and corollary, we explicitly compute the Rost-Schmid groups of Q_{2n-1}.

Proposition 3.39. Let $n \geq 2$ and $i, j \in \mathbb{Z}$ be integers. Let $p : Q_{2n-1} \to \mathbb{A}^n_F \setminus \{0\}$ be the projection on $x_1, \ldots, x_n$. The morphism $p^* : H^i(\mathbb{A}^n_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_j) \to H^i(Q_{2n-1}, \underline{K}^{\mathrm{MW}}_j)$ is an isomorphism.

Proof. This result is the direct application of [AF14, Lemma 4.5] and its proof to the strictly $\mathbb{A}^1$-invariant sheaf $\underline{K}_j^{\mathrm{MW}}$ (see Theorem 3.20 and Remark 3.21). □

We directly get the following corollary from Propositions 3.35 and 3.39.

Corollary 3.40. Let $n \geq 2$ and $i, j \in \mathbb{Z}$ be integers. With the same notations as in Propositions 3.35 and 3.39, the morphisms $p^* \circ \psi^* \circ \pi^* : K_j^{\mathrm{MW}}(F) = H^0(\mathrm{Spec}(F), \underline{K}_j^{\mathrm{MW}}) \to H^0(Q_{2n-1}, \underline{K}_j^{\mathrm{MW}})$ and $\widetilde{\sigma} \circ \partial \circ (p^*)^{-1} : H^{n-1}(Q_{2n-1}, \underline{K}_j^{\mathrm{MW}}) \to H^0(\{0\}, \underline{K}_{j-n}^{\mathrm{MW}}) = K_{j-n}^{\mathrm{MW}}(F)$ are isomorphisms, and if $i \notin \{0, n-1\}$ then $H^i(Q_{2n-1}, \underline{K}_j^{\mathrm{MW}}) = 0$.

Notation 3.41. Let $n \geq 2$ and $j \leq n$ be integers. We denote by $\varsigma_{2n-1,j}$ the composite of the isomorphism $\widetilde{\sigma} \circ \partial \circ (p^*)^{-1} : H^{n-1}(Q_{2n-1}, \underline{K}_j^{\mathrm{MW}}) \to K_{j-n}^{\mathrm{MW}}(F)$ (see Corollary 3.40) and of the isomorphism γ_{j-n} (see Theorem 2.33).

Remark 3.42. Since the F-scheme $Q_1 = \mathrm{Spec}(F[x,y]/(xy-1))$ is isomorphic to $\mathbb{A}_F^1 \setminus \{0\}$, it follows from Remark 3.37 that for all $j \in \mathbb{Z}$ and $i \neq 0$, $H^i(Q_1, \underline{K}_j^{\mathrm{MW}}) = 0$ and $H^0(Q_1, \underline{K}_j^{\mathrm{MW}}) \neq 0$.

In the following proposition, we compute (non-explicitly) the Rost-Schmid groups of Q_{2n}. For an explicit computation of the Rost-Schmid groups of Q_2, see Lemma 3.49 and Corollary 3.50.

Proposition 3.43. Let $n \geq 1$ and $i, j \in \mathbb{Z}$ be integers. The Rost-Schmid group $H^i(Q_{2n}, \underline{K}_j^{\mathrm{MW}})$ is isomorphic to $K_j^{\mathrm{MW}}(F)$ if $i = 0$, to $K_{j-n}^{\mathrm{MW}}(F)$ if $i = n$, to 0 otherwise.

Proof. This result is the direct application of [AF22, Proposition 1.1.5] to the strictly $\mathbb{A}^1$-invariant sheaf $\underline{K}_j^{\mathrm{MW}}$ (see Theorem 3.20 and Remark 3.21). □

By combining Corollary 3.40 and Proposition 3.43, we directly get the following corollary.

Corollary 3.44. Let $n \geq 2$ and $i, j \in \mathbb{Z}$ be integers. The Rost-Schmid group $H^i(Q_n, \underline{K}_j^{\mathrm{MW}})$ is isomorphic to $K_j^{\mathrm{MW}}(F)$ if $i = 0$, to $K_{j-\lceil \frac{n}{2} \rceil}^{\mathrm{MW}}(F)$ if $i = \lfloor \frac{n}{2} \rfloor$, to 0 otherwise.

Future work 2 (Explicit isomorphisms for the Rost-Schmid groups of Q_{2n})**.** In order to define the quadratic linking degree (in Chapter 5) in a setting with Q_{2n}, we need to *explicitly* compute the Rost-Schmid group $H^n(Q_{2n}, \underline{K}_j^{\mathrm{MW}})$, i.e. to exhibit for each $n \geq 2$ an isomorphism between $H^n(Q_{2n}, \underline{K}_j^{\mathrm{MW}})$ and $K_{j-n}^{\mathrm{MW}}(F)$. To do this, it suffices to show that the morphism $i_* : H^0(Y_n, \underline{K}^{\mathrm{MW}}_{j-n}\{\nu_{Y_n/Q_{2n}}\}) \to H^n(Q_{2n}, \underline{K}^{\mathrm{MW}}_j)$ is an isomorphism, where Y_n is the closed subscheme of the affine quadric $Q_{2n} = \mathrm{Spec}(F[x_1, \dots, x_n, y_1, \dots, y_n, z]/(\sum_{i=1}^n x_i y_i - z(1+z)))$ which is defined by the equations $x_1 = \dots = x_n = z = 0$ and $i : Y_n \to Q_{2n}$ is the inclusion. Indeed, the tuple $(x_1, \dots, x_n)$ induces an orientation of the normal sheaf of Y_n in Q_{2n}, hence an isomorphism $H^0(Y_n, \underline{K}^{\mathrm{MW}}_{j-n}) \to H^0(Y_n, \underline{K}^{\mathrm{MW}}_{j-n}\{\nu_{Y_n/Q_{2n}}\})$ (see Notation 3.25), the coordinates $y_1, \dots, y_n$ give an isomorphism between $\mathbb{A}^n_F$ and Y_n, hence an isomorphism $H^0(\mathbb{A}^n_F, \underline{K}^{\mathrm{MW}}_{j-n}) \to H^0(Y_n, \underline{K}^{\mathrm{MW}}_{j-n})$, and the projection $\pi : \mathbb{A}^n_F \to \mathrm{Spec}(F)$ gives an isomorphism $\pi^* : K^{\mathrm{MW}}_{j-n}(F) = H^0(\mathrm{Spec}(F), \underline{K}^{\mathrm{MW}}_{j-n}) \to H^0(\mathbb{A}^n_F, \underline{K}^{\mathrm{MW}}_{j-n})$ (see Corollary 3.34). The considerations in [ADF16, Section 2] should be useful to show that the morphism $i_* : H^0(Y_n, \underline{K}^{\mathrm{MW}}_{j-n}\{\nu_{Y_n/Q_{2n}}\}) \to H^n(Q_{2n}, \underline{K}^{\mathrm{MW}}_j)$ is an isomorphism.

We now give the Rost-Schmid groups of the projective space $\mathbb{P}^n_F$ when F is of characteristic different from 2. Recall Definition 3.1.

Theorem 3.45 (Theorem 11.7 in [Fas13])**.** Let $n \geq 1$ and $i, j \in \mathbb{Z}$ be integers. If F is of characteristic different from 2 then the Rost-Schmid group $H^i(\mathbb{P}^n_F, \underline{K}^{\mathrm{MW}}_j)$ is isomorphic to $K^{\mathrm{MW}}_j(F)$ if $i = 0$, to $K^{\mathrm{M}}_{j-i}(F)$ if $0 < i < n$, to $K^{\mathrm{M}}_{j-n}(F)$ if $i = n$ and n is even, to $K^{\mathrm{MW}}_{j-n}(F)$ if $i = n$ and n is odd, to 0 otherwise.

As before, we would like to *explicitly* compute the Rost-Schmid groups of $\mathbb{P}^n_F$ (and to drop the assumption on the characteristic of F). The following proposition and corollary do this for $n = 1$.

Proposition 3.46 (Subsection 3.4 in [Dég23])**.** Let $l \in \mathbb{Z}$, $i : \mathrm{Spec}(F) \to \mathbb{P}^1_F$ be the closed immersion of image the point $\infty := [1 : 0]$ and $j : \mathbb{A}^1_F \to \mathbb{P}^1_F$ be the open immersion of image $\mathbb{P}^1_F \setminus \{\infty\}$. The morphism $j^* : H^0(\mathbb{P}^1_F, \underline{K}^{\mathrm{MW}}_l) \to H^0(\mathbb{A}^1_F, \underline{K}^{\mathrm{MW}}_l)$ is an isomorphism and the morphism $i_* : H^0(\mathrm{Spec}(F), \underline{K}^{\mathrm{MW}}_{l-1}\{\nu_{\{\infty\}}\}) \to H^1(\mathbb{P}^1_F, \underline{K}^{\mathrm{MW}}_l)$ is an isomorphism (where $\nu_{\{\infty\}}$ is the normal sheaf of $\{\infty\}$ in $\mathbb{P}^1_F$).

Recall Definition 3.22 (orientations) and Notation 3.25. Propositions 3.46 and 3.33 and Corollary 3.34 directly give the following corollary.

Corollary 3.47. Let $l \in \mathbb{Z}$ and $\infty := [1 : 0]$ in $\mathbb{P}^1_F$.

1. The composite of the isomorphism $\widetilde{o}_\infty : K^{\mathrm{MW}}_{l-1}(F) = H^0(\mathrm{Spec}(F), \underline{K}^{\mathrm{MW}}_{l-1}) \to H^0(\mathrm{Spec}(F), \underline{K}^{\mathrm{MW}}_{l-1}\{\nu_{\{\infty\}}\})$ which is induced by the orientation $o_\infty : \nu_{\{\infty\}} \to \mathcal{O}_{\{\infty\}} \otimes \mathcal{O}_{\{\infty\}}$ of the normal sheaf $\nu_{\{\infty\}}$ of $\{\infty\}$ in $\mathbb{P}^1_F$ which maps $\frac{x_1}{x_0}$ to $1 \otimes 1$ and of the isomorphism $i_* : H^0(\mathrm{Spec}(F), \underline{K}^{\mathrm{MW}}_{l-1}\{\nu_{\{\infty\}}\}) \to H^1(\mathbb{P}^1_F, \underline{K}^{\mathrm{MW}}_l)$ which is induced by the inclusion of $\{\infty\}$ in $\mathbb{P}^1_F$ is an isomorphism.

2. The composite of the isomorphism $j^* : H^0(\mathbb{P}^1_F, \underline{K}^{\mathrm{MW}}_l) \to H^0(\mathbb{A}^1_F, \underline{K}^{\mathrm{MW}}_l)$ which is induced by the inclusion of $\mathbb{P}^1_F \setminus \{\infty\}$ in $\mathbb{P}^1_F$ and of the isomorphism $(\pi^*)^{-1} : H^0(\mathbb{A}^1_F, \underline{K}^{\mathrm{MW}}_l) \to H^0(\mathrm{Spec}(F), \underline{K}^{\mathrm{MW}}_l) = K^{\mathrm{MW}}_l(F)$ is an isomorphism.

3. If $k \notin \{0, 1\}$ then $H^k(\mathbb{P}^1_F, \underline{K}^{\mathrm{MW}}_l) = 0$.

Notation 3.48. Let $l \leq 0$ be an integer. We denote by ϱ_l the composite of the isomorphism $(i_* \circ \widetilde{o}_\infty)^{-1} : H^1(\mathbb{P}^1_F, \underline{K}^{\mathrm{MW}}_l) \to K^{\mathrm{MW}}_{l-1}(F)$ (see Corollary 3.47) and of the isomorphism $\gamma_{l-1} : K^{\mathrm{MW}}_{l-1}(F) \to \mathrm{W}(F)$ (see Theorem 2.33).

Future work 3 (Explicit isomorphisms for the Rost-Schmid groups of $\mathbb{P}^n_F$)**.** In order to define the quadratic linking degree (in Chapter 5) in a setting with $\mathbb{P}^n_F$ where n is odd, we need to *explicitly* compute the Rost-Schmid group $H^n(\mathbb{P}^n_F, \underline{K}^{\mathrm{MW}}_j)$, i.e. to exhibit for each odd integer $n \geq 3$ an isomorphism between $H^n(\mathbb{P}^n_F, \underline{K}^{\mathrm{MW}}_j)$ and $K^{\mathrm{MW}}_{j-n}(F)$. To do this, it suffices to show that the morphism $i_* : H^0(\mathrm{Spec}(F), \underline{K}^{\mathrm{MW}}_{j-n}\{\nu_{\{\infty\}}\}) \to H^n(\mathbb{P}^n_F, \underline{K}^{\mathrm{MW}}_j)$ is an isomorphism, where $i : \mathrm{Spec}(F) \to \mathbb{P}^n_F$ is the closed immersion of image the point $\infty := [1 : 0 : \ldots : 0]$. Indeed, the tuple $(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})$ induces an orientation of the normal sheaf of $\{\infty\}$ in $\mathbb{P}^n_F$, hence an isomorphism $K^{\mathrm{MW}}_{j-n}(F) = H^0(\mathrm{Spec}(F), \underline{K}^{\mathrm{MW}}_{j-n}) \to H^0(\mathrm{Spec}(F), \underline{K}^{\mathrm{MW}}_{j-n}\{\nu_{\{\infty\}}\})$. It may be possible to adapt the proof (in [Dég23, Subsection 3.4]) of Proposition 3.46 to prove that the morphism $i_* : H^0(\mathrm{Spec}(F), \underline{K}^{\mathrm{MW}}_{j-n}\{\nu_{\{\infty\}}\}) \to H^n(\mathbb{P}^n_F, \underline{K}^{\mathrm{MW}}_j)$ is an isomorphism. Note that [Yan21, Theorem 1.1] may be useful to show that the morphism $i_* : H^0(\mathrm{Spec}(F), \underline{K}^{\mathrm{MW}}_{j-n}\{\nu_{\{\infty\}}\}) \to H^n(\mathbb{P}^n_F, \underline{K}^{\mathrm{MW}}_j)$ is an isomorphism.

Finally, let us compute explicitly the Rost-Schmid groups of Q_2. Recall that $Q_2 = \mathrm{Spec}(F[x, y, z]/(xy - z(1 + z)))$. The following lemma can be proved in a similar way to Proposition 3.39 (since the morphism p which is defined in the following lemma is an $\mathbb{A}^1$-weak equivalence, hence an isomorphism in the $\mathbb{A}^1$-homotopy category).

Lemma 3.49. Let $i, j \in \mathbb{Z}$. Let $p : Q_2 \to \mathbb{P}^1_F$ be the morphism which sends (x, y, z) to $[x : z] = [1 + z : y]$ (note that $x, z, 1 + z, y$ cannot all be 0, so that for any (x, y, z), $[x : z]$ or $[1 + z : y]$ is well-defined). The morphism $p^* : H^i(\mathbb{P}^1_F, \underline{K}^{\mathrm{MW}}_j) \to H^i(Q_2, \underline{K}^{\mathrm{MW}}_j)$ is an isomorphism.

Lemma 3.49 and Corollary 3.47 directly give the following corollary.

Corollary 3.50. Let $l \in \mathbb{Z}$ and $\infty := [1 : 0]$ in $\mathbb{P}^1_F$. With the same notations as in Corollary 3.47 and Lemma 3.49, the morphisms $p^* \circ i_* \circ \widetilde{o}_\infty : K^{\mathrm{MW}}_{l-1}(F) \to H^1(Q_2, \underline{K}^{\mathrm{MW}}_l)$ and $(\pi^*)^{-1} \circ j^* \circ (p^*)^{-1} : H^0(Q_2, \underline{K}^{\mathrm{MW}}_l) \to K^{\mathrm{MW}}_l(F)$ are isomorphisms. If $k \notin \{0, 1\}$ then $H^k(Q_2, \underline{K}^{\mathrm{MW}}_l) = 0$.

Notation 3.51. Let $l \leq 1$ be an integer. We denote by ϕ_l the composite of the isomorphism $(p^* \circ i_* \circ \widetilde{o}_\infty)^{-1} : H^1(Q_2, \underline{K}_l^{\mathrm{MW}}) \to K_{l-1}^{\mathrm{MW}}(F)$ (see Corollary 3.50) and of the isomorphism γ_{l-1} (see Theorem 2.33). We denote $\phi_2 := (p^* \circ i_* \circ \widetilde{o}_\infty)^{-1} : H^1(Q_2, \underline{K}_2^{\mathrm{MW}}) \to K_1^{\mathrm{MW}}(F)$ (see Corollary 3.50).

Part II

Motivic linking

Chapter 4

The quadratic linking class

In this chapter and Chapter 5, we use quadratic intersection theory (see Chapter 3) — which is central in motivic homotopy theory — to study what we call motivic linking: a counterpart in algebraic geometry to classical linking (in knot theory and in higher-dimensional knot theory; see Chapter 1). More precisely, in this chapter we introduce and study counterparts in algebraic geometry to:

- oriented links with two components (see Definition 1.1 and its higher-dimensional generalisation Definition 1.22);
- the oriented fundamental class (see Definition 1.3 and its higher-dimensional generalisation Definition 1.23);
- the couple of Seifert classes (see Definition 1.8 and its higher-dimensional generalisation Definition 1.25);
- the linking class (see Definition 1.10 and its higher-dimensional generalisation Definition 1.27; we call its counterpart the quadratic linking class).

In Chapter 5 (which builds on this chapter), we will introduce and study counterparts in algebraic geometry to the linking number (see Definition 1.13 and its higher-dimensional generalisation Definition 1.30; we call its counterpart the ambient quadratic linking degree) and to the linking couple (see Definition 1.17 and its higher-dimensional generalisation Definition 1.34; we call its counterpart the quadratic linking degree (couple)).

In Section 4.1, we present the general context in which the above-mentioned counterparts, such as the quadratic linking class, can be defined, and study some general properties of these counterparts. In Section 4.2, we prove functoriality properties of the quadratic linking class in this general context. In Section 4.3, we explore which closed immersions between smooth models of motivic spheres are special cases of this general context and what properties are verified in these (quasi-affine) cases, and in Section 4.4 we present a (projective) special case of this general context which is not a closed immersion between smooth models of motivic spheres but which is also reminiscent of classical knot theory. Note that the case $(\mathbb{A}^2_F\setminus\{0\}, \mathbb{A}^2_F\setminus\{0\}, \mathbb{A}^4_F\setminus\{0\})$ was partially included in our preprint [Lem23].

4.1 The general case

In this section, we introduce oriented links with two components in algebraic geometry and define the quadratic linking class, before studying how the quadratic linking class changes when the order of the components of the oriented link or the orientations are changed.

Throughout this section, F is a perfect field, X is an irreducible smooth finite-type F-scheme of dimension d_X, Z_1 and Z_2 are disjoint irreducible smooth finite-type closed F-subschemes of X of same dimension d and $c := d_X - d$ is their codimension in X.

We denote by Z the (disjoint) union of Z_1 and Z_2 in X and by ν_Z (resp. ν_{Z_1},ν_{Z_2}) the determinant of the normal sheaf $\mathcal{N}_{Z/X}$ of Z (resp. Z_1,Z_2) in X, i.e. the dual of the $\mathcal{O}_Z$-module $\mathscr{I}_Z/\mathscr{I}_Z^2$ with $\mathscr{I}_Z$ the ideal sheaf of Z in X.

Defining the quadratic linking class

Similarly to oriented links with two components in knot theory which consist of a couple of closed subspaces of the topological 3-sphere $\mathbb{S}^3$ which are homeomorphic to the topological circle $\mathbb{S}^1$ (and verify a tameness property, such as smoothness), together with orientations of their normal bundles in $\mathbb{S}^3$ (see Definitions 1.1 and 1.3 as well as the discussion which follows this last definition), we define oriented links with two components as follows. See Definition 3.22 (orientations and orientation classes).

Definition 4.1 (Oriented link with two components)**.** The couple (Z_1, Z_2) of closed F-subschemes of X, together with a couple of orientation classes $(\overline{o_1}, \overline{o_2})$, where $o_1 : \nu_{Z_1} \to \mathcal{L}_1 \otimes \mathcal{L}_1$ is an orientation of the normal sheaf $\mathcal{N}_{Z_1/X}$ of Z_1 in X and $o_2 : \nu_{Z_2} \to \mathcal{L}_2 \otimes \mathcal{L}_2$ is an orientation of the normal sheaf $\mathcal{N}_{Z_2/X}$ of Z_2 in X, is called an *oriented link* $\mathscr{L}$ with two components. The closed F-subscheme Z_1 of X is called the *first component* of $\mathscr{L}$ and the closed F-subscheme Z_2 of X is called the *second component* of $\mathscr{L}$.

Remark 4.2. If $(Z_1, Z_2), (\overline{o_1}, \overline{o_2})$ is an oriented link with two components then in particular $\mathcal{N}_{Z_1/X}$ and $\mathcal{N}_{Z_2/X}$ are orientable (i.e. their determinants are isomorphic to squares, see Definition 3.22). If we were to define (nonoriented) links (Z_1, Z_2) with two components, we should first ask ourselves if it is better to require $\mathcal{N}_{Z_1/X}$ and $\mathcal{N}_{Z_2/X}$ to be orientable (so that a link can always give rise to an oriented link) or to have a more general definition of links. Also, note that even though we only defined oriented links with two components, similar definitions for (oriented) knots (i.e. (oriented) links with one component) and for (oriented) links with n components (with $n \in \mathbb{N}$) can be made.

For instance, oriented links with two components can be couples of disjoint closed F-subschemes of $\mathbb{A}^4_F \setminus \{0\}$ which are isomorphic to $\mathbb{A}^2_F \setminus \{0\}$ together with orientation classes (this is almost the definition we chose in our preprint [Lem23]; the only difference is that in our preprint we fixed isomorphisms $\varphi_1 : \mathbb{A}^2_F \setminus \{0\} \to Z_1$ and $\varphi_2 : \mathbb{A}^2_F \setminus \{0\} \to Z_2$ (which were used to define the quadratic linking degree (couple) but were not used to define the quadratic linking class)). See Section 4.3 for this special case (among others) and Chapter 7 for examples (especially Section 7.1 for a simple example: the Hopf link). Note that $\mathbb{A}^2_\mathbb{R} \setminus \{0\}(\mathbb{R})$, i.e. $\mathbb{R}^2 \setminus \{0\}$, and the topological circle $\mathbb{S}^1$ are of same homotopy type and that $\mathbb{A}^4_\mathbb{R} \setminus \{0\}(\mathbb{R})$, i.e. $\mathbb{R}^4 \setminus \{0\}$, and the topological 3-sphere $\mathbb{S}^3$ are of same homotopy type, so that the above-mentioned special case is quite close to the definition of oriented links with two components in classical knot theory. More generally, for $n \geq 1$, oriented links with two components can be couples of disjoint closed F-subschemes of $\mathbb{A}^{2n+2}_F \setminus \{0\}$ which are isomorphic to $\mathbb{A}^{n+1}_F \setminus \{0\}$ together with orientation classes (see Section 4.3 for this family of special cases). This family of special cases is quite close to the family of oriented links with two equidimensional components in higher-dimensional knot theory for which there is a linking class and a linking number: $\mathbb{S}^n \sqcup \mathbb{S}^n \to \mathbb{S}^{2n+1}$ (see Section 1.6). Indeed, on the one hand $\mathbb{A}^{n+1}_\mathbb{R} \setminus \{0\}(\mathbb{R})$, i.e. $\mathbb{R}^{n+1} \setminus \{0\}$, and the topological n-sphere $\mathbb{S}^n$ are of same homotopy type and on the other hand $\mathbb{A}^{2n+2}_\mathbb{R} \setminus \{0\}(\mathbb{R})$, i.e. $\mathbb{R}^{2n+2} \setminus \{0\}$, and the topological $(2n+1)$-sphere $\mathbb{S}^{2n+1}$ are of same homotopy type. Further note that on the one hand $\mathbb{A}^{n+1}_\mathbb{R} \setminus \{0\}(\mathbb{C})$, i.e. $\mathbb{R}^{2n+2} \setminus \{0\}$, and the topological $(2n+1)$-sphere $\mathbb{S}^{2n+1}$ are of same homotopy type and on the other hand $\mathbb{A}^{2n+2}_\mathbb{R} \setminus \{0\}(\mathbb{C})$, i.e. $\mathbb{R}^{4n+4} \setminus \{0\}$, and the topological $(4n+3)$-sphere $\mathbb{S}^{4n+3}$ are of same homotopy type (and $\mathbb{S}^{2n+1} \sqcup \mathbb{S}^{2n+1} \to \mathbb{S}^{4n+3}$ is simply a special case of $\mathbb{S}^m \sqcup \mathbb{S}^m \to \mathbb{S}^{2m+1}$). See Sections 4.3 and 4.4 for these and other families of special cases.

Now that we have defined oriented links with two components, we can define the oriented fundamental classes of their components. Recall that in Definitions 1.3 and 1.23, the oriented fundamental class of a knot K was the generator of the singular cohomology group $H^0(K)$ (or equivalently the generator of the singular homology group $H_n(K)$ if $K \simeq \mathbb{S}^n$) which corresponded to the orientation of K. We are going to define the oriented fundamental class of the component Z_i of the oriented link $\mathscr{L}$ as the element of the Rost-Schmid group $H^0(Z_i, \underline{K}_{j_i}^{\mathrm{MW}}\{\nu_{Z_i}\})$ (see Definition 3.10) which corresponds to the orientation of Z_i. Note that to do this, an integer j_i must be chosen (which was not the case in knot theory). Furthermore, we need j_i to be nonpositive to ensure that a special element can be isolated in $H^0(Z_i, \underline{K}_{j_i}^{\mathrm{MW}})$, so that we can define the oriented fundamental class as the element of $H^0(Z_i, \underline{K}_{j_i}^{\mathrm{MW}}\{\nu_{Z_i}\})$ which corresponds to this special element in $H^0(Z_i, \underline{K}_{j_i}^{\mathrm{MW}})$ via the isomorphism $\widetilde{o_i}$ (see Notation 3.25 and Proposition 3.26) induced by the orientation class $\overline{o_i}$ (similarly to the fact in knot theory that the oriented fundamental class is the element of $H^0(K)$ which corresponds to $1 \in \mathbb{Z}$ via the isomorphism induced by the orientation).

Definition 4.3 (Oriented fundamental class)**.** Let $i \in \{1,2\}$ and $j_i \leq 0$ be an integer. The *oriented fundamental class* of the ith component of $\mathscr{L}$ with respect to j_i is the (unique) element $[o_i]_{j_i}$ (denoted $[o_i]$ for short) of the Rost-Schmid group $H^0(Z_i, \underline{K}_{j_i}^{\mathrm{MW}}\{\nu_{Z_i}\})$ which is sent by the isomorphism $\widetilde{o_i}$ to the class in $H^0(Z_i, \underline{K}_{j_i}^{\mathrm{MW}})$ of the cycle whose coefficient over the generic point of Z_i is η^{-j_i}.

See Section 7.1 for simple examples of oriented fundamental classes.

Remark 4.4. Let $i \in \{1,2\}$ and $j'_i \leq j_i \leq 0$ be integers. Note that product by η (hence product by $\eta^{j_i - j'_i}$) commutes with the differentials of the Rost-Schmid complexes (since these are constructed from the residue morphisms of Milnor-Witt K-theory) so that the class $\eta^{j_i - j'_i}[o_i]_{j_i} \in H^0(Z_i, \underline{K}_{j'_i}^{\mathrm{MW}}\{\nu_{Z_i}\})$ is well-defined. Since, by definition of $\widetilde{o_i}$ (see Notation 3.25), the following diagram is commutative:

$$\begin{array}{ccc} H^0(Z_i, \underline{K}_{j_i}^{\mathrm{MW}}\{\nu_{Z_i}\}) & \xrightarrow{\widetilde{o_i}} & H^0(Z_i, \underline{K}_{j_i}^{\mathrm{MW}}) \\ \downarrow{\scriptstyle \times \eta^{j_i - j'_i}} & & \downarrow{\scriptstyle \times \eta^{j_i - j'_i}} \\ H^0(Z_i, \underline{K}_{j'_i}^{\mathrm{MW}}\{\nu_{Z_i}\}) & \xrightarrow[\widetilde{o_i}]{} & H^0(Z_i, \underline{K}_{j'_i}^{\mathrm{MW}}) \end{array}$$

and the morphisms $\widetilde{o_i}$ are isomorphisms (by definition, see Notation 3.25), the oriented fundamental class $[o_i]_{j'_i}$ (which is sent to $\eta^{-j'_i}$ by $\widetilde{o_i}$) is equal to $\eta^{j_i - j'_i}[o_i]_{j_i}$ (since $[o_i]_{j_i}$ is sent to η^{-j_i} by $\widetilde{o_i}$).

The following proposition will ensure that there exists a unique Seifert class for each component of the oriented link. Recall Definition 3.18 (boundary maps).

Proposition 4.5. Let $i \in \{1,2\}$ and $j_i \leq 0$ be an integer.

1. If $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_i+c}) = 0$ then the boundary map $\partial : H^{c-1}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_i+c}) \to H^0(Z, \underline{K}^{\mathrm{MW}}_{j_i}\{\nu_Z\})$ is injective.
2. If $H^{c}(X, \underline{K}^{\mathrm{MW}}_{j_i+c}) = 0$ then the boundary map $\partial : H^{c-1}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_i+c}) \to H^0(Z, \underline{K}^{\mathrm{MW}}_{j_i}\{\nu_Z\})$ is surjective.

Proof. The boundary map $\partial : H^{c-1}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_i+c}) \to H^0(Z, \underline{K}^{\mathrm{MW}}_{j_i}\{\nu_Z\})$ is part of the localization long exact sequence (see Theorem 3.19):

$$\cdots \longrightarrow H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_i+c}) \longrightarrow H^{c-1}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_i+c}) \xrightarrow{\partial}$$

$$\xrightarrow{\partial} H^0(Z, \underline{K}^{\mathrm{MW}}_{j_i}\{\nu_Z\}) \longrightarrow H^c(X, \underline{K}^{\mathrm{MW}}_{j_i+c}) \longrightarrow \cdots$$

□

We can now define the couple of Seifert classes (in a similar fashion to what was done in Definitions 1.8 and 1.25).

Definition 4.6 (Couple of Seifert classes)**.** Let $j_1, j_2 \leq 0$ be integers. We assume $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$, $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$, $H^{c}(X, \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$ and $H^{c}(X, \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$. The *couple of Seifert classes* of $\mathscr{L}$ with respect to (j_1, j_2) is the (unique) element $(\mathcal{S}_{o_1,j_1}, \mathcal{S}_{o_2,j_2})$ (denoted $(\mathcal{S}_1, \mathcal{S}_2)$ for short) of $H^{c-1}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1+c}) \oplus H^{c-1}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_2+c})$ such that $\partial(\mathcal{S}_1) = ([o_1], 0)$ and $\partial(\mathcal{S}_2) = (0, [o_2])$ (respectively via the isomorphism $H^0(Z, \underline{K}^{\mathrm{MW}}_{j_1}\{\nu_Z\}) \simeq H^0(Z_1, \underline{K}^{\mathrm{MW}}_{j_1}\{\nu_{Z_1}\}) \oplus H^0(Z_2, \underline{K}^{\mathrm{MW}}_{j_1}\{\nu_{Z_2}\})$ and via the isomorphism $H^0(Z, \underline{K}^{\mathrm{MW}}_{j_2}\{\nu_Z\}) \simeq H^0(Z_1, \underline{K}^{\mathrm{MW}}_{j_2}\{\nu_{Z_1}\}) \oplus H^0(Z_2, \underline{K}^{\mathrm{MW}}_{j_2}\{\nu_{Z_2}\})$ which are induced by the inclusions of Z_1 and Z_2 in $Z = Z_1 \sqcup Z_2$). We call $\mathcal{S}_{o_1,j_1}$ the *Seifert class* of the component K_1 (relative to the link $\mathscr{L}$ and j_1) and $\mathcal{S}_{o_2,j_2}$ the *Seifert class* of the component K_2 (relative to the link $\mathscr{L}$ and j_2).

See Section 7.1 for simple examples of couples of Seifert classes.

Remark 4.7. Note that the couple of Seifert classes of $\mathscr{L}$ with respect to (j_1, j_2) is well-defined as soon as there exist a unique preimage of $([o_1]_{j_1}, 0)$ by $\partial : H^{c-1}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1+c}) \to H^0(Z, \underline{K}^{\mathrm{MW}}_{j_1}\{\nu_Z\})$ and a unique preimage of $(0, [o_2]_{j_2})$ by $\partial : H^{c-1}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_2+c}) \to H^0(Z, \underline{K}^{\mathrm{MW}}_{j_2}\{\nu_Z\})$, even if the above-mentioned Rost-Schmid groups are nonzero.

Remark 4.8. Let $j_1, j_2 \leq 0$ be integers such that the oriented link $\mathscr{L}$ has a well-defined couple of Seifert classes with respect to (j_1, j_2) (see Remark 4.7). Let $j'_1 \leq j_1 \leq 0$ and $j'_2 \leq j_2 \leq 0$ be integers such that $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j'_1+c}) = 0$ and $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j'_2+c}) = 0$ (which ensures the unicity of the couple of Seifert classes with respect to (j'_1, j'_2) if it exists). By Remark 4.4, $[o_1]_{j'_1} = \eta^{j_1-j'_1}[o_1]_{j_1}$ and $[o_2]_{j'_2} = \eta^{j_2-j'_2}[o_2]_{j_2}$. Since the boundary map commutes to product by η (see Definition 3.18), it follows that $(\eta^{j_1-j'_1}\mathcal{S}_{o_1,j_1}, \eta^{j_2-j'_2}\mathcal{S}_{o_2,j_2})$ is the (well-defined) couple of Seifert classes of $\mathscr{L}$ with respect to (j'_1, j'_2).

We can now define the quadratic linking class as the boundary of the intersection of the Seifert classes, as was done in Definitions 1.10 and 1.27 for the linking class. See Definition 3.28 (intersection product).

Definition 4.9 (Quadratic linking class)**.** Let $j_1, j_2 \leq 0$ be integers. We assume $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$, $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$, $H^{c}(X, \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$ and $H^{c}(X, \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$. The *quadratic linking class* of $\mathscr{L}$ with respect to (j_1, j_2), denoted $\mathrm{Qlc}_{\mathscr{L},j_1,j_2}$ (or $\mathrm{Qlc}_{\mathscr{L}}$ for short), is the image of the intersection product of the Seifert class $\mathcal{S}_{o_1,j_1}$ with the Seifert class $\mathcal{S}_{o_2,j_2}$ by the boundary map $\partial : H^{2c-2}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2c}) \to H^{c-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\})$.

See Section 7.1 for simple examples of quadratic linking classes.

Remark 4.10. Note that the quadratic linking class of $\mathscr{L}$ with respect to (j_1, j_2) is well-defined as soon as the couple of Seifert classes of $\mathscr{L}$ with respect to (j_1, j_2) is well-defined (see Remark 4.7), even if the above-mentioned Rost-Schmid groups are nonzero.

Remark 4.11. Let $j_1, j_2 \leq 0$ be integers such that the oriented link $\mathscr{L}$ has a well-defined quadratic linking class with respect to (j_1, j_2) (see Remark 4.10). Let $j'_1 \leq j_1 \leq 0$ and $j'_2 \leq j_2 \leq 0$ be integers such that $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j'_1+c}) = 0$ and $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j'_2+c}) = 0$ (which ensures the unicity of the couple of Seifert classes with respect to (j'_1, j'_2) if it exists, hence the unicity of the quadratic linking class with respect to (j'_1, j'_2) if it exists). By Remark 4.8, $(\eta^{j_1-j'_1}\mathcal{S}_{o_1,j_1}, \eta^{j_2-j'_2}\mathcal{S}_{o_2,j_2})$ is the (well-defined) couple of Seifert classes $(\mathcal{S}_{o_1,j'_1}, \mathcal{S}_{o_2,j'_2})$ of $\mathscr{L}$ with respect to (j'_1, j'_2). It follows that if the following diagram is commutative (which is verified for instance under the assumptions of Corollary 3.32):

$$\begin{array}{ccc} H^{c-1}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1+c}) \times H^{c-1}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_2+c}) & \xrightarrow{\cdot} & H^{2c-2}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2c}) \\ \downarrow {\scriptstyle (\times\eta^{j_1-j'_1}, \times\eta^{j_2-j'_2})} & & \downarrow {\scriptstyle \times\eta^{j_1+j_2-(j'_1+j'_2)}} \\ H^{c-1}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j'_1+c}) \times H^{c-1}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j'_2+c}) & \xrightarrow{\cdot} & H^{2c-2}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j'_1+j'_2+2c}) \end{array}$$

then $\mathcal{S}_{o_1,j'_1}\cdot\mathcal{S}_{o_2,j'_2} = \eta^{j_1+j_2-(j'_1+j'_2)}(\mathcal{S}_{o_1,j_1}\cdot\mathcal{S}_{o_2,j_2})$, and since the boundary map commutes to product by η (see Definition 3.18), the oriented link $\mathscr{L}$ has a well-defined quadratic linking class $\mathrm{Qlc}_{\mathscr{L},j'_1,j'_2} = \eta^{j_1+j_2-(j'_1+j'_2)}\,\mathrm{Qlc}_{\mathscr{L},j_1,j_2}$.

In the following proposition, we show that the quadratic linking class is in a specific subgroup of $H^{c-1}(Z,\underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\})$.

Proposition 4.12. Let $i : Z \to X$ be the inclusion of the closed subscheme Z in X and $i_* : H^{c-1}(Z,\underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\}) \to H^{2c-1}(X,\underline{K}^{\mathrm{MW}}_{j_1+j_2+2c})$ be the morphism induced by its push-forward. Then $\mathrm{Qlc}_{\mathscr{L},j_1,j_2} \in \ker(i_*)$.

Proof. The boundary map which we used to define the quadratic linking class in Definition 4.9 is part of the localization long exact sequence (see Theorem 3.19):

$$\dots \longrightarrow H^{2c-2}(X,\underline{K}^{\mathrm{MW}}_{j_1+j_2+2c}) \longrightarrow H^{2c-2}(X\setminus Z,\underline{K}^{\mathrm{MW}}_{j_1+j_2+2c}) \xrightarrow{\partial}$$

$$\xrightarrow{\partial} H^{c-1}(Z,\underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\}) \xrightarrow{i_*} H^{2c-1}(X,\underline{K}^{\mathrm{MW}}_{j_1+j_2+2c}) \longrightarrow \dots$$

□

In the following proposition, we see that an additional assumption on a Rost-Schmid group of X ensures that no information is lost between the intersection product of the Seifert classes and the quadratic linking class.

Proposition 4.13. If $H^{2c-2}(X,\underline{K}^{\mathrm{MW}}_{j_1+j_2+2c}) = 0$ then the boundary map $\partial : H^{2c-2}(X\setminus Z,\underline{K}^{\mathrm{MW}}_{j_1+j_2+2c}) \to H^{c-1}(Z,\underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\})$ is injective.

Proof. This boundary map is part of the localization long exact sequence (see Theorem 3.19):

$$\dots \longrightarrow H^{2c-2}(X,\underline{K}^{\mathrm{MW}}_{j_1+j_2+2c}) \longrightarrow H^{2c-2}(X\setminus Z,\underline{K}^{\mathrm{MW}}_{j_1+j_2+2c}) \xrightarrow{\partial}$$

$$\xrightarrow{\partial} H^{c-1}(Z,\underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\}) \longrightarrow H^{2c-1}(X,\underline{K}^{\mathrm{MW}}_{j_1+j_2+2c}) \longrightarrow \dots$$

□

Remark 4.14. The additional assumption in Proposition 4.13 is verified in all the special cases of Sections 4.3 and 4.4 except $Z_1 \simeq Q_5 \simeq Z_2$ in $X \simeq Q_8$, $Z_1 \simeq Q_3 \simeq Z_2$ in $X \simeq Q_5$ and $Z_1 \simeq Q_2 \simeq Z_2$ in $X \simeq Q_4$ (recall Notation 3.38). Note that in these last two cases, $H^c(X,\underline{K}^{\mathrm{MW}}_{j_1+c}) \neq 0$ and

$H^c(X, \underline{K}^{\mathrm{MW}}_{j_2+c}) \neq 0$, so that the quadratic linking class does not necessarily exist (but it is well-defined if it exists since $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$ and $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$, see Proposition 4.5 and Definitions 4.6 and 4.9).

In the next two subsections, we determine how the quadratic linking class is affected by changes in the order of the components or in the orientation classes.

Changing the order of the components

Recall that the linking class of an oriented link $\mathbb{S}^n \sqcup \mathbb{S}^n \to \mathbb{S}^{2n+1}$ stayed the same if the order of the components of the oriented link was changed and n was even (i.e. the codimension $n+1$ was odd), and was turned into its opposite if the order of the components of the oriented link was changed and n was odd (i.e. the codimension $n+1$ was even); see Remark 1.28.

Proposition 4.15. Let $\mathscr{L}'$ be the link $(Z_2, Z_1), (\overline{o_2}, \overline{o_1})$. Then:

$$\mathrm{Qlc}_{\mathscr{L}',j_2,j_1} = \begin{cases} \mathrm{Qlc}_{\mathscr{L},j_1,j_2} & \text{if } c \text{ is odd and } (j_1 \text{ is odd or } j_2 \text{ is odd}) \\ \epsilon\, \mathrm{Qlc}_{\mathscr{L},j_1,j_2} & \text{if } c \text{ is odd and } j_1 \text{ is even and } j_2 \text{ is even} \\ -\epsilon\, \mathrm{Qlc}_{\mathscr{L},j_1,j_2} & \text{if } c \text{ is even and } (j_1 \text{ is odd or } j_2 \text{ is odd}) \\ -\mathrm{Qlc}_{\mathscr{L},j_1,j_2} & \text{if } c \text{ is even and } j_1 \text{ is even and } j_2 \text{ is even} \end{cases}$$

Proof. By Proposition 3.30 we have:

$$\mathcal{S}_2 \cdot \mathcal{S}_1 = \begin{cases} \mathcal{S}_1 \cdot \mathcal{S}_2 & \text{if } c \text{ is odd and } (j_1 \text{ is odd or } j_2 \text{ is odd}) \\ \epsilon\, \mathcal{S}_1 \cdot \mathcal{S}_2 & \text{if } c \text{ is odd and } j_1 \text{ is even and } j_2 \text{ is even} \\ -\epsilon\, \mathcal{S}_1 \cdot \mathcal{S}_2 & \text{if } c \text{ is even and } (j_1 \text{ is odd or } j_2 \text{ is odd}) \\ -\mathcal{S}_1 \cdot \mathcal{S}_2 & \text{if } c \text{ is even and } j_1 \text{ is even and } j_2 \text{ is even} \end{cases}$$

We deduce the result by using Proposition 2.37 and the fact that the boundary map is a group morphism (and is constructed from the residue morphisms for Milnor-Witt K-theory). □

Changing the orientation classes

In general, orientation classes can vary significantly. However, if the Picard group of the underlying scheme has no 2-torsion, then any two orientation classes differ from one another by the multiplication by a global invertible function (see [DDØ22, Theorem 6.1.6]). If this is the case for Z_1 and Z_2 and if their global invertible functions are exactly the units of the ground field (note that all these assumptions are verified in all the special cases of Sections 4.3 and 4.4), then we know how the quadratic linking class is changed by orientation changes.

Proposition 4.16. Let $a = (a_1, a_2)$ be a couple of elements of F^*. Let $\mathscr{L}_a$ be the link obtained from $\mathscr{L}$ by changing the orientation class $\overline{o_1}$ into $\overline{o_1 \circ (\times a_1)}$ and the orientation class $\overline{o_2}$ into $\overline{o_2 \circ (\times a_2)}$. Then

$$\mathrm{Qlc}_{\mathscr{L}_a} = \langle a_1 a_2 \rangle \, \mathrm{Qlc}_{\mathscr{L}}$$

Proof. Let $i \in \{1, 2\}$. Note that $\langle a_i^{-1} \rangle [o_i] = \langle a_i \rangle [o_i]$ is sent by $\widetilde{o_i \circ (\times a_i)}$ to the class in $H^0(Z_i, \underline{K}^{\mathrm{MW}}_{j_i})$ of the cycle whose coefficient over the generic point of Z_i is η^{-j_i} hence $[o_i \circ (\times a_i)] = \langle a_i \rangle [o_i]$. Therefore, by Proposition 2.37 and Definition 4.6, $\mathcal{S}_{o_i \circ (\times a_i)} = \langle a_i \rangle \mathcal{S}_{o_i}$ hence, by Proposition 3.29,

$$\mathcal{S}_{o_1 \circ (\times a_1)} \cdot \mathcal{S}_{o_2 \circ (\times a_2)} = \langle a_1 a_2 \rangle \mathcal{S}_{o_1} \cdot \mathcal{S}_{o_2}$$

It follows from this and Proposition 2.37 that $\partial(\mathcal{S}_{o_1 \circ (\times a_1)} \cdot \mathcal{S}_{o_2 \circ (\times a_2)}) = \langle a_1 a_2 \rangle \partial(\mathcal{S}_{o_1} \cdot \mathcal{S}_{o_2})$, i.e. $\mathrm{Qlc}_{\mathscr{L}_a} = \langle a_1 a_2 \rangle \, \mathrm{Qlc}_{\mathscr{L}}$. □

Note that in the case where the ground field is the field of real numbers (i.e. $F = \mathbb{R}$), this proposition is similar to what happens to the linking class (see Remark 1.29): the quadratic linking class is the same if a_1 and a_2 have the same sign (similarly to the linking class which is the same if both orientations are reversed (or if they are both left unchanged)) and is multiplied by $\langle -1 \rangle$ if a_1 and a_2 have different signs (similarly to the linking class which is multiplied by -1 if exactly one of the orientations is reversed).

Future work 4 (More general changes of orientation classes)**.** It would be interesting to know how the quadratic linking class can be affected by changes of orientation classes when the Picard groups of the components Z_1 and Z_2 of the oriented link have no 2-torsion but there are global invertible functions of Z_1 or of Z_2 which are not units of the ground field. It would also be interesting (but a priori even more difficult) to study this when one of these Picard groups has 2-torsion. This could be useful for the study of the ambient quadratic linking degree (see Definition 5.7).

In the following section, we consider some functoriality properties of the quadratic linking class.

4.2 Functoriality properties

In this section, we define the pullback of an oriented link with two components along a smooth (surjective) morphism and show that it is an oriented link with two components of quadratic linking class the pullback of the quadratic linking class of the original oriented link.

Assumptions and notations

Throughout this section, F is a perfect field, $\psi : X' \to X$ is a smooth surjective morphism between irreducible smooth finite-type F-schemes of respective dimensions $d_{X'}$ and d_X, Z_1 and Z_2 are disjoint irreducible smooth finite-type closed F-subschemes of X (of respective inclusions f_1, f_2 in X) of same dimension d and $c := d_X - d$ is their codimension in X. We set $Z_1' := \psi^*(Z_1)$ and $Z_2' := \psi^*(Z_2)$ and we assume Z_1' and Z_2' to be irreducible.

Note that Z_1' and Z_2' are disjoint closed F-subschemes of X' of codimension c in X'. We denote by f_1', f_2' their respective inclusions in X' and by d' their dimension (so that we also have $c = d_{X'} - d'$). We set $Z' := Z_1' \sqcup Z_2'$ and $Z := Z_1 \sqcup Z_2$. We denote by $\nu_{Z'}$ (resp. $\nu_{Z_1'}$,$\nu_{Z_2'}$) the determinant of the normal sheaf of Z' (resp. Z_1',Z_2') in X' and by ν_Z (resp. ν_{Z_1},ν_{Z_2}) the determinant of the normal sheaf of Z (resp. Z_1,Z_2) in X.

We denote by $\psi_1 : Z_1' \to Z_1$ and $\psi_2 : Z_2' \to Z_2$ the morphisms induced by ψ, thus we have the following commutative diagrams:

$$\begin{array}{ccc} Z_1' & \xrightarrow{f_1'} & X' \\ {\scriptstyle \psi_1}\downarrow & & \downarrow{\scriptstyle \psi} \\ Z_1 & \xrightarrow[f_1]{} & X \end{array} \qquad\qquad \begin{array}{ccc} Z_2' & \xrightarrow{f_2'} & X' \\ {\scriptstyle \psi_2}\downarrow & & \downarrow{\scriptstyle \psi} \\ Z_2 & \xrightarrow[f_2]{} & X \end{array}$$

We fix an orientation $o_1 : \nu_{Z_1} \to \mathcal{L}_1 \otimes \mathcal{L}_1$ of the normal sheaf of Z_1 in X and an orientation $o_2 : \nu_{Z_2} \to \mathcal{L}_2 \otimes \mathcal{L}_2$ of the normal sheaf of Z_2 in X.

The pullback of an oriented link is an oriented link

Lemma-Definition 4.17. Let $i \in \{1, 2\}$. There is a canonical isomorphism $\zeta : \nu_{Z_i'} \to (\psi_i)^*(\nu_{Z_i})$.

Proof. First note that there is a canonical isomorphism $\mathcal{N}_{Z_i'/X'} \to (\psi_i)^*(\mathcal{N}_{Z_i/X})$ since ψ_i is flat and $f_i : Z_i \to X$ is a regular closed imbedding (see [Ful98, B.7.4]; our normal sheaf $\mathcal{N}_{Z_i'/X'}$ is the sheaf of sections of Fulton's normal bundle $N_{Z_i'}X'$). Then note that flat pullback commutes with the determinant since it preserves the rank and commutes with the exterior product to conclude that there is a canonical isomorphism $\nu_{Z'_i} \to (\psi_i)^*(\nu_{Z_i})$. □

The fact that flat pullback commutes with the tensor product gives us the following lemma-definition.

Lemma-Definition 4.18. Let $i \in \{1,2\}$. There is a canonical isomorphism $\xi_{\mathcal{L}_i} : (\psi_i)^*(\mathcal{L}_i \otimes \mathcal{L}_i) \to (\psi_i)^*(\mathcal{L}_i) \otimes (\psi_i)^*(\mathcal{L}_i)$.

By using the two preceding lemma-definitions, we can construct orientations on Z'_1 and Z'_2 from o_1 and o_2. Better still, the classes of these orientations on Z'_1 and Z'_2 only depend on the orientation classes $\overline{o_1}$ and $\overline{o_2}$ rather than on the orientations o_1 and o_2.

Lemma-Definition 4.19. Let $i \in \{1,2\}$. The composite $o'_i := \xi_{\mathcal{L}_i} \circ (\psi_i)^*(o_i) \circ \zeta$ is an orientation of the normal sheaf of Z'_i in X' and its orientation class only depends on the orientation class of o_i.

Proof. By definition, $\xi_{\mathcal{L}_i} \circ (\psi_i)^*(o_i) \circ \zeta : \nu_{Z'_i} \to (\psi_i)^*(\mathcal{L}_i) \otimes (\psi_i)^*(\mathcal{L}_i)$ is an orientation if it is an isomorphism (since $(\psi_i)^*(\mathcal{L}_i)$ is an invertible $\mathcal{O}_{Z'_i}$-module, being the pullback along ψ_i of an invertible $\mathcal{O}_{Z_i}$-module). Since ζ and $\xi_{\mathcal{L}_i}$ are isomorphisms, it suffices to show that $(\psi_i)^*(o_i) : (\psi_i)^*(\nu_{Z_i}) \to (\psi_i)^*(\mathcal{L}_i \otimes \mathcal{L}_i)$ is an isomorphism. This follows from the fact that $o_i : \nu_{Z_i} \to \mathcal{L}_i \otimes \mathcal{L}_i$ is an isomorphism (since $(\psi_i)^*$ is a functor). Now let us show that the orientation class of o'_i only depends on the orientation class of o_i. Let $\varphi_i : \mathcal{L}_i \to \mathcal{L}''_i$ be an isomorphism of invertible $\mathcal{O}_{Z_i}$-modules. Note that $(\psi_i)^*((\varphi_i \otimes \varphi_i) \circ o_i) = (\psi_i)^*(\varphi_i \otimes \varphi_i) \circ (\psi_i)^*(o_i)$ since $(\psi_i)^*$ is a functor. In addition, $\xi_{\mathcal{L}''_i} \circ (\psi_i)^*(\varphi_i \otimes \varphi_i) = ((\psi_i)^*(\varphi_i) \otimes (\psi_i)^*(\varphi_i)) \circ \xi_{\mathcal{L}_i}$ by naturality of the commutation of flat pullback with the tensor product. It follows that $\xi_{\mathcal{L}''_i} \circ (\psi_i)^*((\varphi_i \otimes \varphi_i) \circ o_i) \circ \zeta = ((\psi_i)^*(\varphi_i) \otimes (\psi_i)^*(\varphi_i)) \circ \xi_{\mathcal{L}_i} \circ (\psi_i)^*(o_i) \circ \zeta = ((\psi_i)^*(\varphi_i) \otimes (\psi_i)^*(\varphi_i)) \circ o'_i$. Since $(\psi_i)^*(\varphi_i) : (\psi_i)^*\mathcal{L}_i \to (\psi_i)^*\mathcal{L}''_i$ is an isomorphism of invertible $\mathcal{O}_{Z'_i}$-modules (by pullback of φ_i), it follows that $\xi_{\mathcal{L}''_i} \circ (\psi_i)^*((\varphi_i \otimes \varphi_i) \circ o_i) \circ \zeta$ is in the same orientation class as o'_i. Thus the orientation class of o'_i only depends on the orientation class of o_i. □

Pullback

See [Fas20, Subsection 2.4] for more details on pullback.

Lemma 4.20. Let $i \in \{1,2\}$ and $j_i \leq 0$. We have $(\psi_i)^*([o_i]_{j_i}) = [o'_i]_{j_i}$.

Proof. By definition, $[o'_i]_{j_i}$ (respectively $[o_i]_{j_i}$) is the unique element of the Rost-Schmid group $H^0(Z'_i, \underline{K}^{\mathrm{MW}}_{j_i}\{\nu_{Z'_i}\})$ (resp. $H^0(Z_i, \underline{K}^{\mathrm{MW}}_{j_i}\{\nu_{Z_i}\})$) which is sent by $\widetilde{o'_i}$ (resp. $\widetilde{o_i}$) to the class in $H^0(Z'_i, \underline{K}^{\mathrm{MW}}_{j_i})$ (resp. $H^0(Z_i, \underline{K}^{\mathrm{MW}}_{j_i})$) of the cycle whose coefficient over the generic point of Z'_i (resp. Z_i) is η^{-j_i}. Thus, since $(\psi_i)^*(\eta^{-j_i}) = \eta^{-j_i}$ (see for instance [Fas20, Example 2.11] and note that ψ_i is smooth since ψ is), it suffices to show that the following diagram is commutative:

$$\begin{array}{ccc} H^0(Z'_i, \underline{K}^{\mathrm{MW}}_{j_i}\{\nu_{Z'_i}\}) & \xrightarrow{\widetilde{o'_i}} & H^0(Z'_i, \underline{K}^{\mathrm{MW}}_{j_i}) \\ \downarrow{\scriptstyle (\psi_i)^*} & & \downarrow{\scriptstyle (\psi_i)^*} \\ H^0(Z_i, \underline{K}^{\mathrm{MW}}_{j_i}\{\nu_{Z_i}\}) & \xrightarrow[\widetilde{o_i}]{} & H^0(Z_i, \underline{K}^{\mathrm{MW}}_{j_i}) \end{array}$$

This follows from the definitions of $\widetilde{o_i}$ and $\widetilde{o'_i}$ (see Notation 3.25 and Proposition 3.26) and the fact that $o'_i = \xi_{\mathcal{L}_i} \circ (\psi_i)^*(o_i) \circ \zeta$. □

We fix nonpositive integers $j_1, j_2 \leq 0$ and assume that $H^{c-1}(X', \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$, $H^{c-1}(X', \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$, $H^{c}(X', \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$, $H^{c}(X', \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$, $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$, $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$, $H^{c}(X, \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$ and $H^{c}(X, \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$.

Lemma 4.21. Let $i \in \{1, 2\}$. The pullback along ψ of the Seifert class $\mathcal{S}_{o_i}$ is equal to the Seifert class $\mathcal{S}_{o'_i}$.

Proof. Recall that the Seifert class $\mathcal{S}_{o'_i}$ is the only element of $H^{c-1}(X' \setminus Z', \underline{K}^{\mathrm{MW}}_{j_i+c})$ such that its image by the boundary map $\partial : H^{c-1}(X' \setminus Z', \underline{K}^{\mathrm{MW}}_{j_i+c}) \to H^0(Z', \underline{K}^{\mathrm{MW}}_{j_i}\{\nu_{Z'}\})$ is $([o'_1], 0)$ if $i = 1$, $(0, [o'_2])$ if $i = 2$. Thus it suffices to prove that $\partial(\psi^*(\mathcal{S}_{o_i}))$ is equal to $([o'_1], 0)$ if $i = 1$, $(0, [o'_2])$ if $i = 2$. Recall that by naturality of the boundary map (see [Rot88, Theorem 5.7] and note that the boundary map is the connecting morphism for Rost-Schmid cohomology by definition), it commutes with morphisms of Rost-Schmid complexes. It follows from the fact that the pullback ψ^* is a morphism of complexes (see Theorem 2.14 in [Fas20]) that $\partial(\psi^*(\mathcal{S}_{o_i})) = \psi^*(\partial(\mathcal{S}_{o_i})) = \psi^*([o_i]) = (\psi_i)^*([o_i])$ (by definition of the Seifert class $\mathcal{S}_{o_i}$ and of ψ_i). It follows from Lemma 4.20 that $\partial(\psi^*(\mathcal{S}_{o_i})) = [o'_i]$, hence $\psi^*(\mathcal{S}_{o_i}) = \mathcal{S}_{o'_i}$. □

Remark 4.22. We do not need the conditions $H^c(X', \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$ and $H^c(X', \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$ (which ensure the existence of $\mathcal{S}_{o'_1}$ and $\mathcal{S}_{o'_2}$) since we prove in Lemma 4.21 that the pullbacks along ψ of the Seifert classes $\mathcal{S}_{o_1}$ and $\mathcal{S}_{o_2}$ verify what is asked of the Seifert classes for o'_1 and o'_2 respectively.

Theorem 4.23. The pullback along ψ of the quadratic linking class of (Z_1, Z_2), $(\overline{o_1}, \overline{o_2})$ is the quadratic linking class of (Z'_1, Z'_2), $(\overline{o'_1}, \overline{o'_2})$.

Proof. By Proposition 3.13 in [Fas20], the pullback ψ^* is a ring morphism with respect to the intersection product, hence

$$\psi^*(\mathcal{S}_{o_1} \cdot \mathcal{S}_{o_2}) = \psi^*(\mathcal{S}_{o_1}) \cdot \psi^*(\mathcal{S}_{o_2})$$

By Lemma 4.21, it follows that:

$$\psi^*(\mathcal{S}_{o_1} \cdot \mathcal{S}_{o_2}) = \mathcal{S}_{o'_1} \cdot \mathcal{S}_{o'_2}$$

Recall that by naturality of the boundary map (see [Rot88, Theorem 5.7] and note that the boundary map is the connecting morphism for Rost-Schmid cohomology by definition), it commutes with morphisms of Rost-Schmid complexes. It follows from the fact that the pullback ψ^* is a morphism of complexes (see Theorem 2.14 in [Fas20]) that:

$$\psi^*(\partial(\mathcal{S}_{o_1} \cdot \mathcal{S}_{o_2})) = \partial(\psi^*(\mathcal{S}_{o_1} \cdot \mathcal{S}_{o_2})) = \partial(\mathcal{S}_{o'_1} \cdot \mathcal{S}_{o'_2})$$

In other words, the pullback along ψ of the quadratic linking class of (Z_1, Z_2), $(\overline{o_1}, \overline{o_2})$ is the quadratic linking class of (Z'_1, Z'_2), $(\overline{o'_1}, \overline{o'_2})$. □

Remark 4.24. The assumption that ψ is smooth, instead of flat, was only needed to invoke [Fas20, Theorem 2.14] (in Lemma 4.21 and Theorem 4.23) and [Fas20, Example 2.11] (in Lemma 4.20).

Pushforward

See [Fas20, Subsection 2.3] for more details on pushforward.

We denote by $\mathrm{Qlc}_{(Z_1,Z_2),(\overline{o_1},\overline{o_2})}$ (respectively $\mathrm{Qlc}_{(Z'_1,Z'_2),(\overline{o'_1},\overline{o'_2})}$) the quadratic linking class of the oriented link $(Z_1, Z_2), (\overline{o_1}, \overline{o_2})$ (resp. $(Z'_1, Z'_2), (\overline{o'_1}, \overline{o'_2})$) and by $1 = \langle 1\rangle \oplus \langle 1\rangle \in H^0(Z'_1, \underline{K}_0^{\mathrm{MW}}) \oplus H^0(Z'_2, \underline{K}_0^{\mathrm{MW}})$ the neutral element for the intersection product of the Rost-Schmid ring of $Z' = Z'_1 \sqcup Z'_2$.

Theorem 4.25. If we further assume that the morphism ψ is proper then the pushforward along ψ of the quadratic linking class $\mathrm{Qlc}_{(Z'_1,Z'_2),(\overline{o'_1},\overline{o'_2})}$ of (Z'_1, Z'_2), $(\overline{o'_1}, \overline{o'_2})$ is the intersection product $\psi_*(1) \cdot \mathrm{Qlc}_{(Z_1,Z_2),(\overline{o_1},\overline{o_2})}$.

Proof. It follows directly from Theorem 4.23 that $\psi_*(\mathrm{Qlc}_{(Z'_1,Z'_2),(\overline{o'_1},\overline{o'_2})}) = \psi_*(\psi^*(\mathrm{Qlc}_{(Z_1,Z_2),(\overline{o_1},\overline{o_2})})) = \psi_*(1 \cdot \psi^*(\mathrm{Qlc}_{(Z_1,Z_2),(\overline{o_1},\overline{o_2})}))$. Since ψ is proper, it follows from the projection formula (see Theorem 3.19 in [Fas20]) that $\psi_*(\mathrm{Qlc}_{(Z'_1,Z'_2),(\overline{o'_1},\overline{o'_2})}) = \psi_*(1) \cdot \mathrm{Qlc}_{(Z_1,Z_2),(\overline{o_1},\overline{o_2})}$. □

Applications

Recall that the Rost-Schmid ring is a graded $K_0^{\mathrm{MW}}(F)$-algebra (see Proposition 3.29) and that $K_0^{\mathrm{MW}}(F)$ is canonically isomorphic to the Grothendieck-Witt ring $\mathrm{GW}(F)$ of F (see Theorem 2.33).

Theorem 4.26. Let $F \subset K$ be a finite Galois extension, $\mathscr{L} = (Z_1 \subset X, Z_2 \subset X), (\overline{o_1}, \overline{o_2})$ be an oriented link with two components (over F), $\psi : X' := X \times_{\mathrm{Spec}(F)} \mathrm{Spec}(K) \to X$ be the canonical morphism, and $j_1, j_2 \leq 0$ be integers. We assume X', $Z_1' := \psi^*(Z_1)$ and $Z_2' := \psi^*(Z_2)$ to be irreducible and $H^{c-1}(X', \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$, $H^{c-1}(X', \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$, $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$, $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$, $H^{c}(X, \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$ and $H^{c}(X, \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$. The pullback along ψ of the quadratic linking class $\mathrm{Qlc}_{\mathscr{L}}$ of $\mathscr{L}$ is the quadratic linking class of $(Z_1', Z_2'), (\overline{o_1'}, \overline{o_2'})$ (see Lemma-Definition 4.19) and the pushforward along ψ of the quadratic linking class of (Z_1', Z_2'), $(\overline{o_1'}, \overline{o_2'})$ is equal to $\mathbb{T}_F^K . \mathrm{Qlc}_{\mathscr{L}}$, with $\mathbb{T}_F^K$ the class in $K_0^{\mathrm{MW}}(F) \simeq \mathrm{GW}(F)$ of the restriction to $F \times F$ of the trace form of K over F (which sends $(x, y) \in K \times K$ to the trace in F of (the multiplication by) xy).

Proof. The pullback along ψ of the quadratic linking class $\mathrm{Qlc}_{\mathscr{L}}$ of $\mathscr{L}$ is the quadratic linking class of $(Z_1', Z_2'), (\overline{o_1'}, \overline{o_2'})$ by Remark 4.22 and Theorem 4.23, since ψ is a smooth surjective morphism. Since ψ is also proper, by Theorem 4.25 we have that the pushforward along ψ of the quadratic linking class of (Z_1', Z_2'), $(\overline{o_1'}, \overline{o_2'})$ is the intersection product $\psi_*(1) \cdot \mathrm{Qlc}_{\mathscr{L}}$. The result follows from this and [Fas20, Example 1.23] (recall that F is perfect). □

Theorem 4.26 may be useful when computing the quadratic linking class of an oriented link (in some cases it can be used to have linear equations for the irreducible components which are considered when computing the intersection product of the Seifert classes).

Theorem 4.27. Let $h : \mathbb{A}_F^{n+1} \setminus \{0\} \to \mathbb{P}_F^n$ be the Hopf map (which sends $(x_0, \ldots, x_n)$ to $[x_0 : \ldots : x_n]$), $\mathscr{L} = (Z_1 \subset \mathbb{P}_F^n, Z_2 \subset \mathbb{P}_F^n), (\overline{o_1}, \overline{o_2})$ be an oriented link with two components, and $j_1, j_2 \leq 0$ be integers. We assume $Z_1' := \psi^*(Z_1)$ and $Z_2' := \psi^*(Z_2)$ to be irreducible and $H^{c-1}(X', \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$, $H^{c-1}(X', \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$, $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$, $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$, $H^{c}(X, \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$ and $H^{c}(X, \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$. The pullback along h of the quadratic linking class $\mathrm{Qlc}_{\mathscr{L}}$ of $\mathscr{L}$ is the quadratic linking class of $(Z_1', Z_2'), (\overline{o_1'}, \overline{o_2'})$ (see Lemma-Definition 4.19).

Proof. The result follows directly from Remark 4.22 and Theorem 4.23 since the Hopf map h is a smooth surjective morphism. □

Remark 4.28. Since the Hopf link (see Section 7.1) with the adequate orientation classes is the pullback along the Hopf map of the oriented link $\mathscr{L}$ whose components are defined respectively by the equations $x = 0, y = 0$ and by the equations $z = 0, t = 0$ in $\mathbb{P}^3_F$ (with some orientation classes), by Theorem 4.27 the quadratic linking class of this variant of the Hopf link, which is nonzero (see Section 7.1 and note that it is sent by a group isomorphism to the couple $(< a >, < b >) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$ for some $a, b \in F^*$), is the pullback along the Hopf map of the quadratic linking class of $\mathscr{L}$. In particular, the quadratic linking class of $\mathscr{L}$ is nonzero. Thus we have an example of an oriented link of the form $\mathbb{P}^1_F \sqcup \mathbb{P}^1_F \to \mathbb{P}^3_F$ whose quadratic linking class is nonzero without having had to make any computation in this projective setting.

4.3 Smooth models of motivic spheres

In this section, we explore which closed immersions of smooth models of motivic spheres give rise to a quadratic linking class.

Throughout this section, F is a perfect field, $\mathbb{G}_m$ is the multiplicative group scheme over F and S^1 is the simplicial circle over F.

Recall that a motivic sphere is a smash-product $S^i \wedge \mathbb{G}_m^{\wedge j}$ for some $i, j \in \mathbb{Z}$ (where $S^i := (S^1)^{\wedge i}$) and that a smooth model of $S^i \wedge \mathbb{G}_m^{\wedge j}$ is a smooth F-scheme which has the $\mathbb{A}^1$-homotopy type of $S^i \wedge \mathbb{G}_m^{\wedge j}$. See [ADF16] for further details.

Note that not all motivic spheres have smooth models. Indeed, in [ADF16, Proposition 2.3.1] it is shown that if $k > l$ then $S^k \wedge \mathbb{G}_m^{\wedge l}$ does not have a smooth model. However, it is shown in [ADF16, Theorem 2.2.5] that if $k = l$ then $S^k \wedge \mathbb{G}_m^{\wedge l}$ has a smooth model, and it is known since [MV99, Example 2.20 in Subsection 3.2] that if $k = l - 1$ then $S^k \wedge \mathbb{G}_m^{\wedge l}$ has a smooth model. More precisely, it is shown in [ADF16] that for every $l \in \mathbb{N}$, $\mathbb{A}^l_F \setminus \{0\}$ and Q_{2l-1} (see Notation 3.38) are smooth models of $S^{l-1} \wedge \mathbb{G}_m^{\wedge l}$ and Q_{2l} (see Notation 3.38) is a smooth model of $S^l \wedge \mathbb{G}_m^{\wedge l}$.

In what follows, we study closed immersions of Q_n or $\mathbb{A}^n_F \setminus \{0\}$ in Q_m or $\mathbb{A}^m_F \setminus \{0\}$.

Remark 4.29. There exist other smooth models of motivic spheres which could be studied, for instance the smooth affine scheme $Q_{f_1,\dots,f_n}$ which is $\mathrm{Spec}(F[x_1, \dots, x_n, y_1, \dots, y_n]/(\sum_{i=1}^n x_i f_i(y_1, \dots, y_n) - 1)$ where $n \in \mathbb{N}$ and $f_1, \dots, f_n \in F[y_1, \dots, y_n]$ are such that $\{f_1 = \dots = f_n = 0\}$ is a point in $\mathrm{Spec}(F[y_1, \dots, y_n])$ (see [AF14, Remark 4.13]). Note that $Q_{f_1,\dots,f_n}$ is not necessarily isomorphic to Q_{2n-1}; for instance if $n = 2$ and i, j are integers such that $i + j > 2$ then $Q_{y_1^i, y_2^j}$ is not isomorphic to Q_3. This follows from [DF14, Theorem 2.5] with the values $m_1 = 1, n_1 = 1, p_1 = 1, m_2 = i, n_2 = j, p_2 = 1$.

Let $m \geq 2$ be an integer and X be an F-scheme isomorphic to $\mathbb{A}^m_F \setminus \{0\}$ or Q_m. Let $n \in \mathbb{N}$ and Z_1, Z_2 be disjoint closed F-subschemes of X such that Z_1 is isomorphic to $\mathbb{A}^n_F \setminus \{0\}$ or Q_n and Z_2 is isomorphic to $\mathbb{A}^n_F \setminus \{0\}$ or Q_n. Thus, Z_1, Z_2 and X verify the assumptions in the beginning of Section 4.1: they are irreducible smooth finite-type F-schemes and Z_1 and Z_2 are disjoint closed F-subschemes of X of same dimension. We denote by $c := m - n$ the codimension of Z_1 in X (which is also the codimension of Z_2 in X), by Z the (disjoint) union of Z_1 and Z_2 in X and by ν_Z (resp. ν_{Z_1},ν_{Z_2}) the determinant of the normal sheaf of Z (resp. Z_1,Z_2) in X, i.e. the dual of the $\mathscr{O}_Z$-module $\mathscr{I}_Z/\mathscr{I}_Z^2$ with $\mathscr{I}_Z$ the ideal sheaf of Z in X.

From now on, we assume that ν_{Z_1} and ν_{Z_2} are orientable, and we fix an orientation class $\overline{o_1}$ of ν_{Z_1} and an orientation class $\overline{o_2}$ of ν_{Z_2}. We denote by $\mathscr{L}$ the oriented link $(Z_1, Z_2), (\overline{o_1}, \overline{o_2})$.

For each $i \in \{1, 2\}$ and integer $j_i \leq 0$, there exists a unique oriented fundamental class $[o_i]_{j_i}$ with respect to j_i (see Definition 4.3). The following lemma and theorem explore the existence and the unicity of the couple of Seifert classes and of the quadratic linking class. Recall Proposition 4.5 and Definitions 4.6 and 4.9.

Lemma 4.30. Let $j_1, j_2 \leq 0$ be integers.

1. The Rost-Schmid groups $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1+c})$ and $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_2+c})$ are equal to 0 if and only if $((X \simeq \mathbb{A}^m_F \setminus \{0\}$ and $c \notin \{1, m\})$ or $(X \simeq Q_m$ and $c \notin \{1, \lfloor \frac{m}{2} \rfloor + 1\}))$.
2. The Rost-Schmid groups $H^{c}(X, \underline{K}^{\mathrm{MW}}_{j_1+c})$ and $H^{c}(X, \underline{K}^{\mathrm{MW}}_{j_2+c})$ are equal to 0 if and only if $((X \simeq \mathbb{A}^m_F \setminus \{0\}$ and $c \notin \{0, m-1\})$ or $(X \simeq Q_m$ and $c \notin \{0, \lfloor \frac{m}{2} \rfloor\}))$.

Proof. By Proposition 3.35, for all $i, j \in \mathbb{Z}$: $H^i(\mathbb{A}^m_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_j) = 0$ if and only if $i \notin \{0, m-1\}$. By Corollary 3.44, for all $i, j \in \mathbb{Z}$: $H^i(Q_m, \underline{K}^{\mathrm{MW}}_j) = 0$ if and only if $i \notin \{0, \lfloor \frac{m}{2} \rfloor\}$. The results follow from applying this to $i = c-1$ and to $i = c$. □

The following theorem is a direct consequence of Proposition 4.5 and Lemma 4.30.

Theorem 4.31.

1. If $X \simeq \mathbb{A}^m_F \setminus \{0\}$, $m \geq n+2$ and $n \geq 2$ then for each couple of nonpositive integers (j_1, j_2) there exists a unique couple of Seifert classes of $\mathscr{L}$ with respect to j_1, j_2 and there exists a unique quadratic linking class of $\mathscr{L}$ with respect to j_1, j_2.

2. If $X \simeq Q_m$, $m \geq n+2$ and $n \notin \{m - \lfloor \frac{m}{2} \rfloor - 1, m - \lfloor \frac{m}{2} \rfloor\}$ then for each couple of nonpositive integers (j_1, j_2) there exists a unique couple of Seifert classes of $\mathscr{L}$ with respect to j_1, j_2 and there exists a unique quadratic linking class of $\mathscr{L}$ with respect to j_1, j_2.

3. If $X \simeq \mathbb{A}^m_F \setminus \{0\}$ and $m \geq n+2$ then for each couple of nonpositive integers (j_1, j_2) the couple of Seifert classes of $\mathscr{L}$ with respect to j_1, j_2 is unique if it exists and the quadratic linking class of $\mathscr{L}$ with respect to j_1, j_2 is unique if it exists.

4. If $X \simeq Q_m$, $m \geq n+2$ and $n \neq m - \lfloor \frac{m}{2} \rfloor - 1$ then for each couple of nonpositive integers (j_1, j_2) the couple of Seifert classes of $\mathscr{L}$ with respect to j_1, j_2 is unique if it exists and the quadratic linking class of $\mathscr{L}$ with respect to j_1, j_2 is unique if it exists.

There is another important property to check: when is the Rost-Schmid group in which the quadratic linking class of $\mathscr{L}$ lives (if it exists) nonzero? Recall that this Rost-Schmid group is the following:

$$H^{c-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\}) \simeq H^{c-1}(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_1}\}) \oplus H^{c-1}(Z_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_2}\})$$

Lemma 4.32. Let $j_1, j_2 \leq 0$ be integers.

1. The Rost-Schmid group $H^{c-1}(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_1}\})$ is different from 0 if and only if $((Z_1 \simeq \mathbb{A}^n_F \setminus \{0\}$ and $c \in \{1, n\})$ or $(Z_1 \simeq Q_n$ and $c \in \{1, \lfloor \frac{n}{2} \rfloor + 1\}))$.

2. The Rost-Schmid group $H^{c-1}(Z_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_2}\})$ is different from 0 if and only if $((Z_2 \simeq \mathbb{A}^n_F \setminus \{0\}$ and $c \in \{1, n\})$ or $(Z_2 \simeq Q_n$ and $c \in \{1, \lfloor \frac{n}{2} \rfloor + 1\}))$.

Proof. By Proposition 3.35 and Remark 3.37, for all $i, j \in \mathbb{Z}$: $H^i(\mathbb{A}^n_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_j) \neq 0$ if and only if $i \in \{0, n-1\}$. By Corollary 3.44 and Remark 3.42, for all for all $i, j \in \mathbb{Z}$: $H^i(Q_n, \underline{K}^{\mathrm{MW}}_j) \neq 0$ if and only if $i \in \{0, \lfloor \frac{n}{2} \rfloor\}$. The results follow from applying this to $i = c-1$ and from the fact that $H^{c-1}(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_1}\}) \simeq H^{c-1}(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c})$ (via $\widetilde{o_1}$) and $H^{c-1}(Z_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_2}\}) \simeq H^{c-1}(Z_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c})$ (via $\widetilde{o_2}$); see Notation 3.25. □

By combining Proposition 4.13, Theorem 4.31 and Lemma 4.32 (and recalling that if $n \geq 2$ then there is no closed immersion $\mathbb{A}^n_F \setminus \{0\} \to Q_m$ since $\mathbb{A}^n_F \setminus \{0\}$ is not affine and Q_m is affine), we get the following theorem.

Theorem 4.33.

1. If $n \geq 2$, $Z_1 \simeq \mathbb{A}^n_F \setminus \{0\}$ or $Z_2 \simeq \mathbb{A}^n_F \setminus \{0\}$, and $X \simeq \mathbb{A}^{2n}_F \setminus \{0\}$, then for each couple of nonpositive integers (j_1, j_2) there exists a unique couple of Seifert classes of $\mathscr{L}$ with respect to j_1, j_2 and there exists a unique quadratic linking class of $\mathscr{L}$ with respect to j_1, j_2, which is in the Rost-Schmid group $H^{n-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+n}\{\nu_Z\}) \neq 0$. Furthermore, the boundary map $\partial : H^{2n-2}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2n}) \to H^{n-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+n}\{\nu_Z\})$ is injective, which implies that the quadratic linking class of $\mathscr{L}$ contains as much information as the intersection product of the Seifert classes of $\mathscr{L}$.

2. If $n \geq 2$, $Z_1 \simeq Q_n$ or $Z_2 \simeq Q_n$, and $X \simeq \mathbb{A}^{n+\lfloor \frac{n}{2} \rfloor+1}_F \setminus \{0\}$, then for each couple of nonpositive integers (j_1, j_2) there exists a unique couple of Seifert classes of $\mathscr{L}$ with respect to j_1, j_2 and there exists a unique quadratic linking class of $\mathscr{L}$ with respect to j_1, j_2, which is in the Rost-Schmid group $H^{\lfloor \frac{n}{2} \rfloor}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+\lfloor \frac{n}{2} \rfloor+1}\{\nu_Z\}) \neq 0$. Furthermore, the boundary map $\partial : H^{2\lfloor \frac{n}{2} \rfloor}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2\lfloor \frac{n}{2} \rfloor+2}) \to H^{\lfloor \frac{n}{2} \rfloor}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+\lfloor \frac{n}{2} \rfloor+1}\{\nu_Z\})$ is injective, which implies that the quadratic linking class of $\mathscr{L}$ contains as much information as the intersection product of the Seifert classes of $\mathscr{L}$.

3. If $n \geq 5$, $Z_1 \simeq Q_n$, $Z_2 \simeq Q_n$ and $X \simeq Q_{n+\lfloor \frac{n}{2} \rfloor+1}$, then for each couple of nonpositive integers (j_1, j_2) there exists a unique couple of Seifert classes of $\mathscr{L}$ with respect to j_1, j_2 and there exists a unique quadratic linking class of $\mathscr{L}$ with respect to j_1, j_2, which is in the Rost-Schmid group $H^{\lfloor \frac{n}{2} \rfloor}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+\lfloor \frac{n}{2} \rfloor+1}\{\nu_Z\}) \neq 0$. Furthermore, if $n \geq 6$ then the boundary map $\partial : H^{2\lfloor \frac{n}{2} \rfloor}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2\lfloor \frac{n}{2} \rfloor+2}) \to H^{\lfloor \frac{n}{2} \rfloor}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+\lfloor \frac{n}{2} \rfloor+1}\{\nu_Z\})$ is injective, which implies that the quadratic linking class of $\mathscr{L}$ contains as much information as the intersection product of the Seifert classes of $\mathscr{L}$.

4. If $n \in \{2, 3, 4\}$, $Z_1 \simeq Q_n$, $Z_2 \simeq Q_n$ and $X \simeq Q_{n+\lfloor \frac{n}{2} \rfloor+1}$, then for each couple of nonpositive integers (j_1, j_2) the couple of Seifert classes of $\mathscr{L}$ with respect to j_1, j_2 is unique if it exists and the quadratic linking class of $\mathscr{L}$ with respect to j_1, j_2 is unique and is in the Rost-Schmid group $H^{\lfloor \frac{n}{2} \rfloor}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+\lfloor \frac{n}{2} \rfloor+1}\{\nu_Z\}) \neq 0$, if it exists. Furthermore, if

$n = 4$ then the boundary map $\partial : H^{2\lfloor \frac{n}{2} \rfloor}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2\lfloor \frac{n}{2} \rfloor+2}) \to H^{\lfloor \frac{n}{2} \rfloor}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+\lfloor \frac{n}{2} \rfloor+1}\{\nu_Z\})$ is injective, which implies that the quadratic linking class of $\mathscr{L}$ contains as much information as the intersection product of the Seifert classes of $\mathscr{L}$.

This last case is interesting if one can exhibit a couple of Seifert classes for $\mathscr{L}$ (which is the case under reasonable assumptions; see Chapter 6).

Let us now focus on orientation. The following proposition lets us see how restrictive the assumption that ν_{Z_1} and ν_{Z_2} are orientable is.

Proposition 4.34.

1. For all $n \geq 2$, the Picard group of $\mathbb{A}^n_F \setminus \{0\}$ is trivial (i.e. equal to 0) and every invertible $\mathcal{O}_{\mathbb{A}^n_F \setminus \{0\}}$-module is orientable.

2. For all $n \geq 3$, the Picard group of Q_n is trivial (i.e. equal to 0) and every invertible $\mathcal{O}_{Q_n}$-module is orientable.

3. For all $n \geq 1$, the Picard group of $\mathbb{P}^n_F$ is isomorphic to $\mathbb{Z}$. An invertible $\mathcal{O}_{\mathbb{P}^n_F}$-module is orientable if and only if it is even (as an integer).

4. The Picard group of Q_2 is isomorphic to $\mathbb{Z}$. An invertible $\mathcal{O}_{Q_2}$-module is orientable if and only if it is even (as an integer).

Proof. 1. The Picard group of $\mathbb{A}^n_F \setminus \{0\}$ is the divisor class group of $\mathbb{A}^n_F \setminus \{0\}$ (see [Har77, Corollary 6.16 in Chapter II]) and there is a surjective morphism from the divisor class group of $\mathbb{A}^n_F$, which is trivial (see [Har77, Example 6.3.1 in Chapter II]), to the divisor class group of $\mathbb{A}^n_F \setminus \{0\}$ (see [Har77, Proposition 6.5 in Chapter II]), hence the Picard group of $\mathbb{A}^n_F \setminus \{0\}$ is trivial. Since $\mathcal{O}_{\mathbb{A}^n_F \setminus \{0\}}$ is orientable (for instance, the multiplication $\mathcal{O}_{\mathbb{A}^n_F \setminus \{0\}} \otimes \mathcal{O}_{\mathbb{A}^n_F \setminus \{0\}} \to \mathcal{O}_{\mathbb{A}^n_F \setminus \{0\}}$ is an isomorphism), it follows that every invertible $\mathcal{O}_{\mathbb{A}^n_F \setminus \{0\}}$-module is orientable.

2. If $n \geq 3$ is odd then the projection $p : Q_n \to \mathbb{A}^{\frac{n+1}{2}}_F \setminus \{0\}$ on $x_1, \ldots, x_{\frac{n+1}{2}}$ is an $\mathbb{A}^1$-weak equivalence hence it induces an isomorphism between the Picard group $H^1(\mathbb{A}^{\frac{n+1}{2}}_F \setminus \{0\}, (\mathcal{O}_{\mathbb{A}^{\frac{n+1}{2}}_F \setminus \{0\}})^*)$ of $\mathbb{A}^{\frac{n+1}{2}}_F \setminus \{0\}$ and the Picard group $H^1(Q_n, (\mathcal{O}_{Q_n})^*)$ of Q_n (by a similar argument to the one used in the proof of Proposition 3.39). The result for n odd follows from the previous item.

   If $n \geq 4$ is even then the projection $p : Q_n \to \mathbb{A}^1_F$ on x_1 is a trivial fibre bundle outside of 0 (since if $x_1 \neq 0$ then $y_1 = \frac{z(1+z)-\sum_{i=2}^n x_i y_i}{x_1}$)

and its fibre over 0 is isomorphic to $\mathbb{A}^1_F \times_F Q_{n-2}$ (since $\sum_{i=2}^n x_i y_i = z(1+z)$ and there is no restriction on y_1) hence is integral, therefore the Picard group of Q_n is trivial and every invertible $\mathcal{O}_{Q_n}$-module is orientable (since $\mathcal{O}_{Q_n}$ is orientable; for instance the multiplication $\mathcal{O}_{Q_n} \otimes \mathcal{O}_{Q_n} \to \mathcal{O}_{Q_n}$ is an isomorphism).

3. See [Har77, Proposition 6.4 and Corollary 6.16 in Chapter II] for the fact that the Picard group of $\mathbb{P}^n_F$ is isomorphic to $\mathbb{Z}$. The fact that an invertible $\mathcal{O}_{\mathbb{P}^n_F}$-module is orientable if and only if it is even (as an integer) follows immediately from the definition of orientation (see Definition 3.22).

4. The morphism $p : Q_2 \to \mathbb{P}^1_F$ which sends (x, y, z) to $[x : z] = [1 + z : y]$ is an $\mathbb{A}^1$-weak equivalence hence it induces an isomorphism between the Picard group $H^1(\mathbb{P}^1_F, (\mathcal{O}_{\mathbb{P}^1_F})^*)$ of $\mathbb{P}^1_F$ and the Picard group $H^1(Q_2, (\mathcal{O}_{Q_2})^*)$ of Q_2 (by a similar argument to the one used for Lemma 3.49). The result follows from the previous item.

□

In the following proposition, we determine how changes of the orientation classes affect the quadratic linking class.

Proposition 4.35. Let $\mathscr{L}$ be as in one of the cases of Theorem 4.33 and such that $\mathscr{L}$ has a quadratic linking class. Let $a = (a_1, a_2)$ be a couple of elements of F^* and $\mathscr{L}_a$ be the link obtained from $\mathscr{L}$ by changing the orientation class $\overline{o_1}$ into $\overline{o_1 \circ (\times a_1)}$ and the orientation class $\overline{o_2}$ into $\overline{o_2 \circ (\times a_2)}$. Then $\mathscr{L}_a$ has a quadratic linking class and $\mathrm{Qlc}_{\mathscr{L}_a} = \langle a_1 a_2 \rangle \, \mathrm{Qlc}_{\mathscr{L}}$.

Proof. The result follows from Proposition 4.16 (note that in the proof of this proposition we show that $(\langle a_1 \rangle \mathcal{S}_{o_1}, \langle a_2 \rangle \mathcal{S}_{o_2})$ is a couple of Seifert classes for $\mathscr{L}_a$). □

Remark 4.36. Proposition 4.35 covers all possible changes of the orientation classes since the global invertible functions of Z_1 and of Z_2 are exactly the units of the ground field F (recall that for each $i \in \{1, 2\}$, $Z_i \simeq Q_m$ with $m \geq 2$ or $Z_i \simeq \mathbb{A}^m_F \setminus \{0\}$ with $m \geq 2$, and the global functions of $\mathbb{A}^m_F \setminus \{0\}$ extend uniquely to global functions of $\mathbb{A}^m_F$ by [GW10, Theorem 6.45 (Hartogs' theorem)]) and any two orientation classes on Z_1 or on Z_2 differ from one another by the multiplication by a global invertible function (see [DDØ22, Theorem 6.1.6]) since the Picard groups of Z_1 and of Z_2 have no 2-torsion (see Proposition 4.34).

In Tables 4.1 and 4.2, we recap the different cases we have discussed in this section. Specifically, Table 4.1 recaps the cases in which $X \simeq \mathbb{A}^m_F \setminus \{0\}$ and Table 4.2 recaps the cases in which $X \simeq Q_m$. The first column lists the different cases, the second column specifies whether the quadratic linking class always exists, the third column specifies whether all links are orientable, the fourth column specifies whether $H := H^{2c-2}(X, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2c}) = 0$ (recall that this equality ensures that the quadratic linking class contains as much information as the intersection product of the Seifert classes; see Proposition 4.13) and the fifth column gives a well-known group which is isomorphic to the group in which the quadratic linking class lives, namely $H^{c-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\})$ (see Proposition 3.35 and Corollary 3.44). In these two tables, χ^{even} denotes the characteristic function of the set of even numbers ($\chi^{\mathrm{even}}(n) = 1$ if n is even, $\chi^{\mathrm{even}}(n) = 0$ if n is odd).

We end this section with the following research lead. Note that the article [HWXZ21] may be useful for this investigation.

Future work 5 (Real realization and complex realization)**.** In the case $\mathbb{A}^2_{\mathbb{R}} \setminus \{0\} \sqcup \mathbb{A}^2_{\mathbb{R}} \setminus \{0\} \to \mathbb{A}^4_{\mathbb{R}} \setminus \{0\}$ (or more generally: $\mathbb{A}^n_{\mathbb{R}} \setminus \{0\} \sqcup \mathbb{A}^n_{\mathbb{R}} \setminus \{0\} \to \mathbb{A}^{2n}_{\mathbb{R}} \setminus \{0\}$ with $n \geq 2$), one may ask whether the real realization of the quadratic linking class is the linking class of the induced oriented link $\mathbb{S}^1 \sqcup \mathbb{S}^1 \to \mathbb{S}^3$ (respectively $\mathbb{S}^{n-1} \sqcup \mathbb{S}^{n-1} \to \mathbb{S}^{2n-1}$). One may also ask whether the complex realization of the quadratic linking class is the linking class of the induced oriented link $\mathbb{S}^3 \sqcup \mathbb{S}^3 \to \mathbb{S}^7$ (respectively $\mathbb{S}^{2n-1} \sqcup \mathbb{S}^{2n-1} \to \mathbb{S}^{4n-1}$).

Similar questions may be asked of the ambient quadratic linking degree (see Definition 5.7), compared with the linking number, and of the quadratic linking degree couple (see Definition 5.15), compared with the linking couple.

In the following section, we study another family of cases (which is summarised in Table 4.3) which give rise to a quadratic linking class. Unlike the cases in this section, the schemes in the next section are not smooth models of motivic spheres (except for the projective line $\mathbb{P}^1_F$ which is a smooth model of $S^1 \wedge \mathbb{G}_m$).

Case	∃QLC?	All links orient.?	$H = 0$?	Group isomorphic to $H^{c-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\})$
$\mathbb{A}^n_F \setminus \{0\} \sqcup \mathbb{A}^n_F \setminus \{0\} \to \mathbb{A}^{2n}_F \setminus \{0\}$ with $n \geq 2$	Yes	Yes	Yes	$K^{\mathrm{MW}}_{j_1+j_2}(F) \oplus K^{\mathrm{MW}}_{j_1+j_2}(F) \simeq \begin{cases} \mathrm{GW}(F) \oplus \mathrm{GW}(F) \text{ if } (j_1, j_2) = (0,0) \\ \mathrm{W}(F) \oplus \mathrm{W}(F) \text{ otherwise} \end{cases}$
$\mathbb{A}^n_F \setminus \{0\} \sqcup Q_n \to \mathbb{A}^{2n}_F \setminus \{0\}$ with $n \geq 3$	Yes	Yes	Yes	$K^{\mathrm{MW}}_{j_1+j_2}(F) \oplus 0 \simeq \begin{cases} \mathrm{GW}(F) \oplus 0 \text{ if } (j_1, j_2) = (0,0) \\ \mathrm{W}(F) \oplus 0 \text{ otherwise} \end{cases}$
$\mathbb{A}^2_F \setminus \{0\} \sqcup Q_2 \to \mathbb{A}^4_F \setminus \{0\}$	Yes	?	Yes	$K^{\mathrm{MW}}_{j_1+j_2}(F) \oplus K^{\mathrm{MW}}_{j_1+j_2+1}(F) \simeq \begin{cases} \mathrm{GW}(F) \oplus K^{\mathrm{MW}}_1(F) \text{ if } (j_1, j_2) = (0,0) \\ \mathrm{W}(F) \oplus \mathrm{GW}(F) \text{ if } (j_1, j_2) \in \{(-1,0),(0,-1)\} \\ \mathrm{W}(F) \oplus \mathrm{W}(F) \text{ otherwise} \end{cases}$
$\mathbb{A}^n_F \setminus \{0\} \sqcup Q_n \to \mathbb{A}^{n+\lfloor \frac{n}{2} \rfloor+1}_F \setminus \{0\}$ with $n \geq 3$	Yes	Yes	Yes	$0 \oplus K^{\mathrm{MW}}_{j_1+j_2+\chi^{\mathrm{even}}(n)}(F) \simeq \begin{cases} 0 \oplus K^{\mathrm{MW}}_1(F) \text{ if } (j_1, j_2) = (0,0) \text{ and } n \text{ is even} \\ 0 \oplus \mathrm{GW}(F) \text{ if } (j_1, j_2) = (0,0) \text{ and } n \text{ is odd or} \\ \qquad (j_1, j_2) \in \{(-1,0),(0,-1)\} \text{ and } n \text{ is even} \\ 0 \oplus \mathrm{W}(F) \text{ otherwise} \end{cases}$
$Q_n \sqcup Q_n \to \mathbb{A}^{n+\lfloor \frac{n}{2} \rfloor+1}_F \setminus \{0\}$ with $n \geq 2$	Yes	Yes if $n \neq 2$	Yes	$K^{\mathrm{MW}}_{j_1+j_2+\chi^{\mathrm{even}}(n)}(F) \oplus K^{\mathrm{MW}}_{j_1+j_2+\chi^{\mathrm{even}}(n)}(F) \simeq \begin{cases} K^{\mathrm{MW}}_1(F) \oplus K^{\mathrm{MW}}_1(F) \text{ if } (j_1, j_2) = (0,0) \text{ and } n \text{ is even} \\ \mathrm{GW}(F) \oplus \mathrm{GW}(F) \text{ if } (j_1, j_2) = (0,0) \text{ and } n \text{ is odd or} \\ \qquad (j_1, j_2) \in \{(-1,0),(0,-1)\} \text{ and } n \text{ is even} \\ \mathrm{W}(F) \oplus \mathrm{W}(F) \text{ otherwise} \end{cases}$

Table 4.1 – The quadratic linking class when the ambient space X is the affine space minus the origin.

Case	∃QLC?	All links orient.?	$H = 0$?	Group isomorphic to $H^{c-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\})$
$Q_n \sqcup Q_n \to Q_{n+\lfloor \frac{n}{2} \rfloor+1}$ with $n \geq 6$	Yes	Yes	Yes	$K^{\mathrm{MW}}_{j_1+j_2+\chi^{\mathrm{even}}(n)}(F) \oplus K^{\mathrm{MW}}_{j_1+j_2+\chi^{\mathrm{even}}(n)}(F) \simeq \begin{cases} K^{\mathrm{MW}}_1(F) \oplus K^{\mathrm{MW}}_1(F) \text{ if } (j_1,j_2)=(0,0) \text{ and } n \text{ is even} \\ \mathrm{GW}(F) \oplus \mathrm{GW}(F) \text{ if } (j_1,j_2)=(0,0) \text{ and } n \text{ is odd or } (j_1,j_2) \in \{(-1,0),(0,-1)\} \text{ and } n \text{ is even} \\ \mathrm{W}(F) \oplus \mathrm{W}(F) \text{ otherwise} \end{cases}$
$Q_5 \sqcup Q_5 \to Q_8$	Yes	Yes	No	$K^{\mathrm{MW}}_{j_1+j_2}(F) \oplus K^{\mathrm{MW}}_{j_1+j_2}(F) \simeq \begin{cases} \mathrm{GW}(F) \oplus \mathrm{GW}(F) \text{ if } (j_1,j_2)=(0,0) \\ \mathrm{W}(F) \oplus \mathrm{W}(F) \text{ otherwise} \end{cases}$
$Q_4 \sqcup Q_4 \to Q_7$	?	Yes	Yes	$K^{\mathrm{MW}}_{j_1+j_2+1}(F) \oplus K^{\mathrm{MW}}_{j_1+j_2+1}(F) \simeq \begin{cases} K^{\mathrm{MW}}_1(F) \oplus K^{\mathrm{MW}}_1(F) \text{ if } (j_1,j_2)=(0,0) \\ \mathrm{GW}(F) \oplus \mathrm{GW}(F) \text{ if } (j_1,j_2) \in \{(-1,0),(0,-1)\} \\ \mathrm{W}(F) \oplus \mathrm{W}(F) \text{ otherwise} \end{cases}$
$Q_3 \sqcup Q_3 \to Q_5$	?	Yes	No	$K^{\mathrm{MW}}_{j_1+j_2}(F) \oplus K^{\mathrm{MW}}_{j_1+j_2}(F) \simeq \begin{cases} \mathrm{GW}(F) \oplus \mathrm{GW}(F) \text{ if } (j_1,j_2)=(0,0) \\ \mathrm{W}(F) \oplus \mathrm{W}(F) \text{ otherwise} \end{cases}$
$Q_2 \sqcup Q_2 \to Q_4$	?	?	No	$K^{\mathrm{MW}}_{j_1+j_2+1}(F) \oplus K^{\mathrm{MW}}_{j_1+j_2+1}(F) \simeq \begin{cases} K^{\mathrm{MW}}_1(F) \oplus K^{\mathrm{MW}}_1(F) \text{ if } (j_1,j_2)=(0,0) \\ \mathrm{GW}(F) \oplus \mathrm{GW}(F) \text{ if } (j_1,j_2) \in \{(-1,0),(0,-1)\} \\ \mathrm{W}(F) \oplus \mathrm{W}(F) \text{ otherwise} \end{cases}$

Table 4.2 – The quadratic linking class when the ambient space X is the smooth affine quadric Q_m.

4.4 A projective case

In this section, we explore which closed immersions $\mathbb{P}^n \sqcup \mathbb{P}^n \to \mathbb{P}^m$ of projective spaces give rise to a quadratic linking class.

Throughout this section, F is a perfect field of characteristic different from 2 (this restriction is due to the same restriction in Theorem 3.45).

Let $m \geq 2$ be an integer and X be an F-scheme isomorphic to $\mathbb{P}^m_F$. Let $n \in \mathbb{N}$ and Z_1, Z_2 be disjoint closed F-subschemes of X isomorphic to $\mathbb{P}^n_F$. Thus, Z_1, Z_2 and X verify the assumptions in the beginning of Section 4.1: they are irreducible smooth finite-type F-schemes and Z_1 and Z_2 are disjoint closed F-subschemes of X of same dimension. We denote by $c := m - n$ the codimension of Z_1 in X (which is also the codimension of Z_2 in X), by Z the (disjoint) union of Z_1 and Z_2 in X and by ν_Z (resp. ν_{Z_1},ν_{Z_2}) the determinant of the normal sheaf of Z (resp. Z_1,Z_2) in X, i.e. the dual of the $\mathcal{O}_Z$-module $\mathscr{I}_Z/\mathscr{I}_Z^2$ with $\mathscr{I}_Z$ the ideal sheaf of Z in X.

From now on, we assume that ν_{Z_1} and ν_{Z_2} are orientable, and we fix an orientation class $\overline{o_1}$ of ν_{Z_1} and an orientation class $\overline{o_2}$ of ν_{Z_2}. We denote by $\mathscr{L}$ the oriented link $(Z_1, Z_2), (\overline{o_1}, \overline{o_2})$.

For each $i \in \{1, 2\}$ and integer $j_i \leq 0$, there exists a unique oriented fundamental class $[o_i]_{j_i}$ with respect to j_i (see Definition 4.3). The following lemma and theorem explore the existence and the unicity of the couple of Seifert classes and of the quadratic linking class. Recall Proposition 4.5 and Definitions 4.6 and 4.9.

Lemma 4.37. Let $j_1, j_2 \leq 0$ be integers.

1. The Rost-Schmid groups $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1+c})$ and $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_2+c})$ are equal to 0 if and only if $m \geq n + 2$ and $j_1 \leq -2$ and $j_2 \leq -2$.
2. The Rost-Schmid groups $H^{c}(X, \underline{K}^{\mathrm{MW}}_{j_1+c})$ and $H^{c}(X, \underline{K}^{\mathrm{MW}}_{j_2+c})$ are equal to 0 if and only if $m \geq n + 1$ and $j_1 \leq -1$ and $j_2 \leq -1$.

Proof. By Theorem 3.45, for all $i, j \in \mathbb{Z}$: $H^0(\mathbb{P}^m_F, \underline{K}^{\mathrm{MW}}_j) \simeq K^{\mathrm{MW}}_j(F)$ if $i = 0$, $H^i(\mathbb{P}^m_F, \underline{K}^{\mathrm{MW}}_j) \simeq K^{\mathrm{M}}_{j-i}(F)$ if $0 < i < m$, $H^m(\mathbb{P}^m_F, \underline{K}^{\mathrm{MW}}_j) \simeq K^{\mathrm{M}}_{j-m}(F)$ if m is even, $H^m(\mathbb{P}^m_F, \underline{K}^{\mathrm{MW}}_j) \simeq K^{\mathrm{MW}}_{j-m}(F)$ if m is odd, and $H^i(\mathbb{P}^m_F, \underline{K}^{\mathrm{MW}}_j) = 0$ otherwise. The results follow from applying this to $i = c-1$ and to $i = c$. □

The following theorem follows directly from Proposition 4.5 and Lemma 4.37.

Theorem 4.38. If $m \geq n + 2$ then for each couple of integers (j_1, j_2) such that $j_1 \leq -2$ and $j_2 \leq -2$, there exists a unique couple of Seifert classes of

$\mathscr{L}$ with respect to j_1, j_2 and there exists a unique quadratic linking class of $\mathscr{L}$ with respect to j_1, j_2.

There is another important property to check: when is the Rost-Schmid group in which the quadratic linking class of $\mathscr{L}$ lives (if it exists) nonzero? Recall that this Rost-Schmid group is the following:

$$H^{c-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\}) \simeq H^{c-1}(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_1}\}) \oplus H^{c-1}(Z_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_2}\})$$

Lemma 4.39. Let $j_1, j_2 \leq -2$ be integers.

1. The Rost-Schmid group $H^{c-1}(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_1}\})$ is different from 0 if and only if n is odd and $m = 2n+1$.
2. The Rost-Schmid group $H^{c-1}(Z_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_2}\})$ is different from 0 if and only if n is odd and $m = 2n+1$.

Proof. By Theorem 3.45, for all $i, j \in \mathbb{Z}$: $H^0(\mathbb{P}^n_F, \underline{K}^{\mathrm{MW}}_j) \simeq K^{\mathrm{MW}}_j(F)$ if $i = 0$, $H^i(\mathbb{P}^n_F, \underline{K}^{\mathrm{MW}}_j) \simeq K^{\mathrm{M}}_{j-i}(F)$ if $0 < i < n$, $H^n(\mathbb{P}^n_F, \underline{K}^{\mathrm{MW}}_j) \simeq K^{\mathrm{M}}_{j-n}(F)$ if n is even, $H^n(\mathbb{P}^n_F, \underline{K}^{\mathrm{MW}}_j) \simeq K^{\mathrm{MW}}_{j-n}(F)$ if n is odd, and $H^i(\mathbb{P}^n_F, \underline{K}^{\mathrm{MW}}_j) = 0$ otherwise. The results follow from applying this to $i = c-1$ and from the fact that $H^{c-1}(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_1}\}) \simeq H^{c-1}(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c})$ (via $\widetilde{o_1}$) and $H^{c-1}(Z_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_2}\}) \simeq H^{c-1}(Z_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c})$ (via $\widetilde{o_2}$); see Notation 3.25. □

By combining Proposition 4.13, Theorem 4.38 and Lemma 4.39, we get the following theorem.

Theorem 4.40. If $n \geq 1$ is odd, $Z_1 \simeq \mathbb{P}^n_F$, $Z_2 \simeq \mathbb{P}^n_F$ and $X \simeq \mathbb{P}^{2n+1}_F$, then for each couple of integers (j_1, j_2) such that $j_1 \leq -2$ and $j_2 \leq -2$, there exists a unique couple of Seifert classes of $\mathscr{L}$ with respect to j_1, j_2 and there exists a unique quadratic linking class of $\mathscr{L}$ with respect to j_1, j_2, which is in the Rost-Schmid group $H^n(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+n+1}\{\nu_Z\}) \neq 0$. Furthermore, the boundary map $\partial : H^{2n}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2n+2}) \to H^n(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+n+1}\{\nu_Z\})$ is injective, which implies that the quadratic linking class of $\mathscr{L}$ contains as much information as the intersection product of the Seifert classes of $\mathscr{L}$.

In the following proposition, we show that in the case of Theorem 4.40, the assumption we made earlier that ν_{Z_1} and ν_{Z_2} are orientable is not restrictive at all.

Proposition 4.41. Let $n \geq 1$ be an odd integer. Let $Z_1 \simeq \mathbb{P}^n_F$ and $Z_2 \simeq \mathbb{P}^n_F$ be disjoint closed F-subschemes of $X \simeq \mathbb{P}^{2n+1}_F$. Then ν_{Z_1} and ν_{Z_2} are orientable.

Proof. Let $i \in \{1, 2\}$. By [Ful98, Paragraph B.7.2] and the fact that Z_i and X are smooth F-schemes, we have the following short exact sequence:

$$0 \longrightarrow \mathcal{T}_{Z_i} \longrightarrow (\mathcal{T}_X)_{|Z_i} \longrightarrow \mathcal{N}_{Z_i/X} \longrightarrow 0$$

where $\mathcal{T}_{Z_i}$ is the tangent sheaf of Z_i, $\mathcal{T}_X$ is the tangent sheaf of X, $(\mathcal{T}_X)_{|Z_i}$ is the restriction to Z_i of the tangent sheaf of X, and $\mathcal{N}_{Z_i/X}$ is the normal sheaf of Z_i in X. Therefore, ν_{Z_i}, which by definition is the determinant of $\mathcal{N}_{Z_i/X}$, is isomorphic to the tensor product of the dual of the determinant of $\mathcal{T}_{Z_i}$ and of the restriction to Z_i of the determinant of $\mathcal{T}_X$. The result follows from the fact that these are squares of invertible $\mathcal{O}_{Z_i}$-modules, since $Z_i \simeq \mathbb{P}^n_F$ with n odd and $X \simeq \mathbb{P}^{2n+1}_F$ (note that $2n+1$ is odd). □

Finally, let us focus on what happens to the quadratic linking class when the orientation classes are changed.

Proposition 4.42. Let $\mathscr{L}$ be as in Theorem 4.40. Let $a = (a_1, a_2)$ be a couple of elements of F^* and $\mathscr{L}_a$ be the link obtained from $\mathscr{L}$ by changing the orientation class $\overline{o_1}$ into $\overline{o_1 \circ (\times a_1)}$ and the orientation class $\overline{o_2}$ into $\overline{o_2 \circ (\times a_2)}$. Then $\mathrm{Qlc}_{\mathscr{L}_a} = \langle a_1 a_2 \rangle \, \mathrm{Qlc}_{\mathscr{L}}$.

Proof. The result follows from Proposition 4.16. □

Remark 4.43. Proposition 4.42 covers all possible changes of the orientation classes since the global invertible functions of Z_1 and of Z_2 are exactly the units of the ground field F (recall that for each $i \in \{1, 2\}$, $Z_i \simeq \mathbb{P}^n_F$ with $n \geq 1$) and any two orientation classes on Z_1 or on Z_2 differ from one another by the multiplication by a global invertible function (see [DDØ22, Theorem 6.1.6]) since the Picard groups of Z_1 and of Z_2 have no 2-torsion (see Proposition 4.34).

In Table 4.3, we recap the case we have discussed in this section. The first column gives the case, the second column specifies whether the quadratic linking class always exists, the third column specifies whether all links are orientable, the fourth column specifies whether $H := H^{2c-2}(X, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2c}) = 0$ (recall that this equality ensures that the quadratic linking class contains as much information as the intersection product of the Seifert classes; see Proposition 4.13) and the fifth column gives a well-known group which is isomorphic to the group in which the quadratic linking class lives, namely $H^{c-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\})$ (see Theorem 3.45).

Case	$\exists$QLC?	All links orient.?	$H =$ 0 ?	Group isomorphic to $H^{c-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\})$
$\mathbb{P}^n_F \sqcup \mathbb{P}^n_F \to \mathbb{P}^{2n+1}_F$ with $n \geq 1$ odd, $j_1 \leq -2$ and $j_2 \leq -2$	Yes	Yes	Yes	$K^{\mathrm{MW}}_{j_1+j_2+1}(F) \oplus K^{\mathrm{MW}}_{j_1+j_2+1}(F) \simeq \mathrm{W}(F) \oplus \mathrm{W}(F)$

Table 4.3 – The quadratic linking class when the ambient space X is the projective space. Here the characteristic of F is different from 2.

We end this section with the following research lead. Note that the article [HWXZ21] may be useful for this investigation.

Future work 6 (Real realization in a projective setting)**.** In the case $\mathbb{P}^1_{\mathbb{R}} \sqcup \mathbb{P}^1_{\mathbb{R}} \to \mathbb{P}^3_{\mathbb{R}}$, one may ask whether the real realization of the quadratic linking class is equal to the linking classes of the oriented links $\mathbb{S}^1 \sqcup \mathbb{S}^1 \to \mathbb{S}^3$ such that their image via the projection $\mathbb{S}^3 \to \mathbb{P}^3_{\mathbb{R}}(\mathbb{R})$ is the induced map $\mathbb{P}^1_{\mathbb{R}}(\mathbb{R}) = \mathbb{S}^1 \sqcup \mathbb{P}^1_{\mathbb{R}}(\mathbb{R}) = \mathbb{S}^1 \to \mathbb{P}^3_{\mathbb{R}}(\mathbb{R})$. More generally, one may ask whether, in the case $\mathbb{P}^n_{\mathbb{R}} \sqcup \mathbb{P}^n_{\mathbb{R}} \to \mathbb{P}^{2n+1}_{\mathbb{R}}$ (with $n \geq 1$ odd), the real realization of the quadratic linking class is equal to the linking classes of the oriented links $\mathbb{S}^n \sqcup \mathbb{S}^n \to \mathbb{S}^{2n+1}$ which make the following diagram commute (where the map in the bottom of the diagram is the map induced by the oriented link $\mathbb{P}^n_{\mathbb{R}} \sqcup \mathbb{P}^n_{\mathbb{R}} \to \mathbb{P}^{2n+1}_{\mathbb{R}}$ and the vertical maps are the projections):

$$\begin{array}{ccc} \mathbb{S}^n \sqcup \mathbb{S}^n & \longrightarrow & \mathbb{S}^{2n+1} \\ \downarrow & & \downarrow \\ \mathbb{P}^n_{\mathbb{R}}(\mathbb{R}) \sqcup \mathbb{P}^n_{\mathbb{R}}(\mathbb{R}) & \longrightarrow & \mathbb{P}^{2n+1}_{\mathbb{R}}(\mathbb{R}) \end{array}$$

Similar questions may be asked of the ambient quadratic linking degree (see Future work 7), compared with the linking number, and of the quadratic linking degree couple (see Definition 5.21 and Future work 13), compared with the linking couple.

Chapter 5

The quadratic linking degree

In this chapter, we continue our study (which we started in Chapter 4) of what we call motivic linking: a counterpart in algebraic geometry to classical linking (in knot theory and in higher-dimensional knot theory; see Chapter 1).

More precisely, in this chapter we introduce and study counterparts in algebraic geometry to the linking number (see Definition 1.13 and its higher-dimensional generalisation Definition 1.30) and to the linking couple (see Definition 1.17 and its higher-dimensional generalisation Definition 1.34).

In Section 5.1, we define the ambient quadratic linking class and the ambient quadratic linking degree (our counterpart to the linking number) and study some of their properties. In particular, we study how changes of the orientation classes of the oriented link affect the ambient quadratic linking class and the ambient quadratic linking degree. In Section 5.2, we define oriented links of a certain type (for instance, oriented links of type $(\mathbb{A}^2_F \setminus \{0\}, \mathbb{A}^2_F \setminus \{0\}, \mathbb{A}^4_F \setminus \{0\})$, which are oriented links in $\mathbb{A}^4_F \setminus \{0\}$ whose components are isomorphic to $\mathbb{A}^2_F \setminus \{0\}$ and which are equipped with explicit isomorphisms (called parametrisations) between $\mathbb{A}^2_F \setminus \{0\}$ and each of their components) and the quadratic linking degree couple (our counterpart to the linking couple) of such links. We also study some of its properties, in particular how changes of the orientation classes and of the parametrisations affect the quadratic linking degree couple. In Section 5.3, we introduce invariants of the quadratic linking degree, which are quantities computed from the ambient quadratic linking degree or from the quadratic linking degree couple which do not depend on choices of orientation classes (nor, in some cases, on choices of parametrisations). This is similar to the absolute value of the linking number (or the absolute value of one of the components of the linking couple) which does not depend on the orientations of the components of the oriented link, but is more complicated in our case since the ambient quadratic linking degree takes values in the Witt ring $\mathrm{W}(F)$ of the ground field F or in the Grothendieck-Witt ring $\mathrm{GW}(F)$ of the ground field F, rather than in the ring of integers, and each component of the quadratic linking degree couple takes values in $\mathrm{W}(F)$, in $\mathrm{GW}(F)$ or in the first Milnor-Witt K-theory group $K_1^{\mathrm{MW}}(F)$ of the ground field F, rather than in the ring of integers, and since the effects of changes of the orientation classes are not merely changes of sign. Note that the case $(\mathbb{A}_F^2 \setminus \{0\}, \mathbb{A}_F^2 \setminus \{0\}, \mathbb{A}_F^4 \setminus \{0\})$ was partially included in our preprint [Lem23].

5.1 The ambient quadratic linking degree

In this section we define the ambient quadratic linking degree, which is a counterpart in algebraic geometry to the linking number (see Definition 1.13 and its higher-dimensional generalisation Definition 1.30).

Throughout this section, F is a perfect field.

We first define the ambient quadratic linking class (from which the ambient quadratic linking degree will be defined) and study some of its properties. Recall Definitions 4.1 (oriented links with two components) and 4.9 (the quadratic linking class).

Definition 5.1 (Ambient quadratic linking class)**.** Let $\mathscr{L} = ((Z_1 \subset X, Z_2 \subset X), (\overline{o_1}, \overline{o_2}))$ be an oriented link with two components and $j_1, j_2 \leq 0$ be integers such that $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$, $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$, $H^{c}(X, \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$ and $H^{c}(X, \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$ (where c is the codimension of Z_1 (or Z_2) in X). The *ambient quadratic linking class* of $\mathscr{L}$ with respect to (j_1, j_2), denoted $\mathrm{AQlc}_{\mathscr{L}, j_1, j_2}$ (or $\mathrm{AQlc}_{\mathscr{L}}$ for short), is the image of the part of the quadratic linking class of $\mathscr{L}$ with respect to (j_1, j_2) which is in the group $H^{c-1}(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_1}\})$ (where c is the codimension of Z_1 in X) by the morphism $(i_1)_* : H^{c-1}(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_1}\}) \to H^{2c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2c})$ induced by the push-forward of the inclusion $i_1 : Z_1 \to X$ of the closed subscheme Z_1 in X.

See Section 7.1 for simple examples of ambient quadratic linking classes.

Remark 5.2. Note that the ambient quadratic linking class of $\mathscr{L}$ with respect to (j_1, j_2) is well-defined as soon as the quadratic linking class of $\mathscr{L}$ with respect to (j_1, j_2) is well-defined (see Remark 4.10), even if the above-mentioned Rost-Schmid groups are nonzero.

Remark 5.3. Let $j_1, j_2 \leq 0$ be integers such that the oriented link $\mathscr{L}$ has a well-defined ambient quadratic linking class with respect to (j_1, j_2) (see

Remark 5.2). Let $j_1' \leq j_1 \leq 0$ and $j_2' \leq j_2 \leq 0$ be integers such that $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1'+c}) = 0$ and $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_2'+c}) = 0$ (which ensures the unicity of the quadratic linking class with respect to (j_1', j_2') if it exists, hence the unicity of the ambient quadratic linking class with respect to (j_1', j_2') if it exists). By Remark 4.11, if the following diagram is commutative (which is verified for instance under the assumptions of Corollary 3.32):

$$\begin{array}{ccc} H^{c-1}(X\setminus Z, \underline{K}^{\mathrm{MW}}_{j_1+c}) \times H^{c-1}(X\setminus Z, \underline{K}^{\mathrm{MW}}_{j_2+c}) & \xrightarrow{\cdot} & H^{2c-2}(X\setminus Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2c}) \\ \downarrow{\scriptstyle (\times\eta^{j_1-j_1'}, \times\eta^{j_2-j_2'})} & & \downarrow{\scriptstyle \times\eta^{j_1+j_2-(j_1'+j_2')}} \\ H^{c-1}(X\setminus Z, \underline{K}^{\mathrm{MW}}_{j_1'+c}) \times H^{c-1}(X\setminus Z, \underline{K}^{\mathrm{MW}}_{j_2'+c}) & \xrightarrow{\cdot} & H^{2c-2}(X\setminus Z, \underline{K}^{\mathrm{MW}}_{j_1'+j_2'+2c}) \end{array}$$

then the oriented link $\mathscr{L}$ has a well-defined quadratic linking class $\mathrm{Qlc}_{\mathscr{L},j_1',j_2'} = \eta^{j_1+j_2-(j_1'+j_2')}\,\mathrm{Qlc}_{\mathscr{L},j_1,j_2}$, hence it has a well-defined ambient quadratic linking class $\mathrm{AQlc}_{\mathscr{L},j_1',j_2'} = \eta^{j_1+j_2-(j_1'+j_2')}\,\mathrm{AQlc}_{\mathscr{L},j_1,j_2}$ (since $(i_1)_*$ commutes to product by η; see Remark 3.15, Notation 3.16 and Remark 3.17).

One may want to define the ambient quadratic linking class by considering the second component of the quadratic linking class rather than the first one: this gives the opposite of the ambient quadratic linking class (as is the case in classical knot theory, see Remark 1.15 and its higher-dimensional generalisation Remark 1.32).

Remark 5.4. Note that if $i : Z = Z_1 \sqcup Z_2 \to X$ (respectively $i_1 : Z_1 \to X$, $i_2 : Z_2 \to X$) is the inclusion of the closed subscheme Z (resp. Z_1,Z_2) in X then $i_* = (i_1)_* \oplus (i_2)_*$ via the isomorphism $H^{c-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\}) \simeq H^{c-1}(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_1}\}) \oplus H^{c-1}(Z_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_2}\})$ induced by the inclusions of Z_1 and Z_2 in Z. It follows from this and from Proposition 4.12, which states that $\mathrm{Qlc}_{\mathscr{L},j_1,j_2} \in \ker(i_*)$, that the image of the part of the quadratic linking class of $\mathscr{L}$ with respect to (j_1, j_2) which is in the group $H^{c-1}(Z_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_2}\})$ by the morphism $(i_2)_* : H^{c-1}(Z_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_2}\}) \to H^{2c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2c})$ is the opposite of the ambient quadratic linking class.

Let us now see what happens to the ambient quadratic linking class when we reverse the order of the components of the oriented link.

Proposition 5.5. Let $\mathscr{L} = ((Z_1 \subset X, Z_2 \subset X), (\overline{o_1}, \overline{o_2}))$ be an oriented link with two components and $j_1, j_2 \leq 0$ be integers such that $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$, $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$, $H^{c}(X, \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$ and $H^{c}(X, \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$. Let $\mathscr{L}'$

be the oriented link $(Z_2, Z_1), (\overline{o_2}, \overline{o_1})$. Then:

$$\mathrm{AQlc}_{\mathscr{L}',j_2,j_1} = \begin{cases} -\mathrm{AQlc}_{\mathscr{L},j_1,j_2} & \text{if } c \text{ is odd and } (j_1 \text{ is odd or } j_2 \text{ is odd}) \\ -\epsilon\,\mathrm{AQlc}_{\mathscr{L},j_1,j_2} & \text{if } c \text{ is odd and } j_1 \text{ is even and } j_2 \text{ is even} \\ \epsilon\,\mathrm{AQlc}_{\mathscr{L},j_1,j_2} & \text{if } c \text{ is even and } (j_1 \text{ is odd or } j_2 \text{ is odd}) \\ \mathrm{AQlc}_{\mathscr{L},j_1,j_2} & \text{if } c \text{ is even and } j_1 \text{ is even and } j_2 \text{ is even} \end{cases}$$

Proof. By Proposition 4.15:

$$\mathrm{Qlc}_{\mathscr{L}',j_2,j_1} = \begin{cases} \mathrm{Qlc}_{\mathscr{L},j_1,j_2} & \text{if } c \text{ is odd and } (j_1 \text{ is odd or } j_2 \text{ is odd}) \\ \epsilon\,\mathrm{Qlc}_{\mathscr{L},j_1,j_2} & \text{if } c \text{ is odd and } j_1 \text{ is even and } j_2 \text{ is even} \\ -\epsilon\,\mathrm{Qlc}_{\mathscr{L},j_1,j_2} & \text{if } c \text{ is even and } (j_1 \text{ is odd or } j_2 \text{ is odd}) \\ -\mathrm{Qlc}_{\mathscr{L},j_1,j_2} & \text{if } c \text{ is even and } j_1 \text{ is even and } j_2 \text{ is even} \end{cases}$$

The result follows from this, Remark 5.4 and the fact that $(i_1)_*$ and $(i_2)_*$ are group morphisms which commute to product by ϵ (see Remark 3.15, Notation 3.16 and Remark 3.17). □

Recall that if the Picard group of the underlying scheme has no 2-torsion, then any two orientation classes differ from one another by the multiplication by a global invertible function (see [DDØ22, Theorem 6.1.6]). If this is the case for Z_1 and Z_2 and if their global invertible functions are exactly the units of the ground field, then we know how the ambient quadratic linking class is changed by orientation changes.

Proposition 5.6. Let $\mathscr{L} = ((Z_1 \subset X, Z_2 \subset X), (\overline{o_1}, \overline{o_2}))$ be an oriented link with two components and $j_1, j_2 \leq 0$ be integers such that $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$, $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$, $H^{c}(X, \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$ and $H^{c}(X, \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$. Let $a = (a_1, a_2)$ be a couple of elements of F^*. Let $\mathscr{L}_a$ be the link obtained from $\mathscr{L}$ by changing the orientation class $\overline{o_1}$ into $\overline{o_1 \circ (\times a_1)}$ and the orientation class $\overline{o_2}$ into $\overline{o_2 \circ (\times a_2)}$. Then

$$\mathrm{AQlc}_{\mathscr{L}_a} = \langle a_1 a_2 \rangle\,\mathrm{AQlc}_{\mathscr{L}}$$

Proof. By Proposition 4.16, $\mathrm{Qlc}_{\mathscr{L}_a} = \langle a_1 a_2 \rangle\,\mathrm{Qlc}_{\mathscr{L}}$. The result follows from this and the fact that $(i_1)_*$ commutes to product by $\langle a_1 a_2 \rangle$ (see Remark 3.15, Notation 3.16 and Remark 3.17). □

Similarly to the linking number (see Definition 1.13 and its higher-dimensional generalisation Definition 1.30) which depends on an orientation of the ambient space (which is fixed once and for all), or equivalently on the choice of an isomorphism $H^3(\mathbb{S}^3) \to \mathbb{Z}$ (more generally, $H^{2n+1}(\mathbb{S}^{2n+1}) \to \mathbb{Z}$), the ambient quadratic linking degree will be defined as the image of the ambient quadratic linking class by an isomorphism (which will depend on the ambient space X but not on the oriented link) between $H^{2c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2c})$ and a well-known group. Before we fix these isomorphisms and define the ambient quadratic linking degree, let us see which cases are interesting.

Recall that in the cases which were studied in Section 4.3, the ambient space X was either $\mathbb{A}^{2n}_F \setminus \{0\}$, $\mathbb{A}^{n+\lfloor \frac{n}{2} \rfloor+1}_F \setminus \{0\}$ or $Q_{n+\lfloor \frac{n}{2} \rfloor+1}$, where $Z = Z_1 \sqcup Z_2$ was of dimension $n \geq 2$. Note that for all $n \geq 3$ and $j_1, j_2 \leq 0$, $H^{2n-1}(\mathbb{A}^{2n}_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2n}) \simeq K^{\mathrm{MW}}_{j_1+j_2}(F)$ and $H^{2\lfloor \frac{n}{2} \rfloor+1}(\mathbb{A}^{n+\lfloor \frac{n}{2} \rfloor+1}_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2\lfloor \frac{n}{2} \rfloor+2}) = 0$, and that for $n = 2$, $H^{2n-1}(\mathbb{A}^{2n}_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2n}) = H^{2\lfloor \frac{n}{2} \rfloor+1}(\mathbb{A}^{n+\lfloor \frac{n}{2} \rfloor+1}_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2\lfloor \frac{n}{2} \rfloor+2}) \simeq K^{\mathrm{MW}}_{j_1+j_2}(F)$. Also note that for all $n \geq 2$ and $j_1, j_2 \leq 0$, $H^{2\lfloor \frac{n}{2} \rfloor+1}(Q_{n+\lfloor \frac{n}{2} \rfloor+1}, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2\lfloor \frac{n}{2} \rfloor+2}) = 0$. Therefore, the only cases of Section 4.3 for which the ambient quadratic linking class is in a nonzero group are the ones which are in Table 5.1 (recall that the quadratic linking class is in $H^{c-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\})$ and that the ambient quadratic linking class is in $H^{2c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2c})$).

Recall that the case which was studied in Section 4.4 was $\mathbb{P}^n_F \sqcup \mathbb{P}^n_F \to \mathbb{P}^{2n+1}_F$ with $n \geq 1$ odd, $j_1 \leq -2$ and $j_2 \leq -2$ (with F of characteristic different from 2). Since $H^{2n+1}(\mathbb{P}^{2n+1}_F, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2n+2}) \simeq K^{\mathrm{MW}}_{j_1+j_2+1}(F)$, the ambient quadratic linking class is in a nonzero group in this case. See Table 5.2 for this case (recall that the quadratic linking class is in $H^{c-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\})$ and that the ambient quadratic linking class is in $H^{2c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2c})$).

In the cases of Table 5.1, we define the ambient quadratic linking degree as follows. Recall Definition 5.1 (the ambient quadratic linking class).

Definition 5.7 (Ambient quadratic linking degree)**.** Let $n \geq 2$ be an integer and $\mathscr{L} = ((Z_1 \subset \mathbb{A}^{2n}_F \setminus \{0\}, Z_2 \subset \mathbb{A}^{2n}_F \setminus \{0\}), (\overline{o_1}, \overline{o_2}))$ be an oriented link with two components of dimension n (i.e. Z_1 and Z_2 are of dimension n). The *ambient quadratic linking degree* of $\mathscr{L}$ with respect to a couple of nonpositive integers (j_1, j_2), denoted $\mathrm{AQld}_{\mathscr{L}, j_1, j_2}$ (or $\mathrm{AQld}_{\mathscr{L}}$ for short), is the image of the ambient quadratic linking class of $\mathscr{L}$ with respect to (j_1, j_2) by the isomorphism ζ_{2n, j_1+j_2+2n} (see Notation 3.36).

See Section 7.1 for simple examples of ambient quadratic linking degrees.

Case	Group isomorphic to $H^{c-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\})$	Group isomorphic to $H^{2c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2c})$
$\mathbb{A}^n_F \setminus \{0\} \sqcup \mathbb{A}^n_F \setminus \{0\} \to \mathbb{A}^{2n}_F \setminus \{0\}$ with $n \geq 2$	$K^{\mathrm{MW}}_{j_1+j_2}(F) \oplus K^{\mathrm{MW}}_{j_1+j_2}(F) \simeq \begin{cases} \mathrm{GW}(F) \oplus \mathrm{GW}(F) \text{ if } (j_1, j_2) = (0,0) \\ \mathrm{W}(F) \oplus \mathrm{W}(F) \text{ otherwise} \end{cases}$	$K^{\mathrm{MW}}_{j_1+j_2}(F) \simeq \begin{cases} \mathrm{GW}(F) \text{ if } (j_1, j_2) = (0,0) \\ \mathrm{W}(F) \text{ otherwise} \end{cases}$
$\mathbb{A}^2_F \setminus \{0\} \sqcup Q_2 \to \mathbb{A}^4_F \setminus \{0\}$	$K^{\mathrm{MW}}_{j_1+j_2}(F) \oplus K^{\mathrm{MW}}_{j_1+j_2+1}(F) \simeq \begin{cases} \mathrm{GW}(F) \oplus K^{\mathrm{MW}}_1(F) \text{ if } (j_1, j_2) = (0,0) \\ \mathrm{W}(F) \oplus \mathrm{GW}(F) \text{ if } (j_1, j_2) \in \{(-1,0),(0,-1)\} \\ \mathrm{W}(F) \oplus \mathrm{W}(F) \text{ otherwise} \end{cases}$	$K^{\mathrm{MW}}_{j_1+j_2}(F) \simeq \begin{cases} \mathrm{GW}(F) \text{ if } (j_1, j_2) = (0,0) \\ \mathrm{W}(F) \text{ otherwise} \end{cases}$
$Q_2 \sqcup Q_2 \to \mathbb{A}^4_F \setminus \{0\}$	$K^{\mathrm{MW}}_{j_1+j_2+1}(F) \oplus K^{\mathrm{MW}}_{j_1+j_2+1}(F) \simeq \begin{cases} K^{\mathrm{MW}}_1(F) \oplus K^{\mathrm{MW}}_1(F) \text{ if } (j_1, j_2) = (0,0) \\ \mathrm{GW}(F) \oplus \mathrm{GW}(F) \text{ if } (j_1, j_2) \in \{(-1,0),(0,-1)\} \\ \mathrm{W}(F) \oplus \mathrm{W}(F) \text{ otherwise} \end{cases}$	$K^{\mathrm{MW}}_{j_1+j_2}(F) \simeq \begin{cases} \mathrm{GW}(F) \text{ if } (j_1, j_2) = (0,0) \\ \mathrm{W}(F) \text{ otherwise} \end{cases}$

Table 5.1 – The ambient quadratic linking class for closed immersions of smooth models of motivic spheres.

Case	Group isomorphic to $H^{c-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\})$	Group isomorphic to $H^{2c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2c})$
$\mathbb{P}^n_F \sqcup \mathbb{P}^n_F \to \mathbb{P}^{2n+1}_F$ with $n \geq 1$ odd, $j_1 \leq -2$ and $j_2 \leq -2$	$K^{\mathrm{MW}}_{j_1+j_2+1}(F) \oplus K^{\mathrm{MW}}_{j_1+j_2+1}(F) \simeq \mathrm{W}(F) \oplus \mathrm{W}(F)$	$K^{\mathrm{MW}}_{j_1+j_2+1}(F) \simeq \mathrm{W}(F)$

Table 5.2 – The ambient quadratic linking class when the ambient space X is the projective space. Here the characteristic of F is different from 2.

Remark 5.8. Let $j_1' \leq j_1 \leq 0$ and $j_2' \leq j_2 \leq 0$ be integers. By Remark 5.3, if the following diagram is commutative (which is verified for instance under the assumptions of Corollary 3.32; here $X := \mathbb{A}_F^{2n} \setminus \{0\}$):

$$\begin{array}{ccc} H^{c-1}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1+c}) \times H^{c-1}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_2+c}) & \xrightarrow{\cdot} & H^{2c-2}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2c}) \\ \downarrow{\scriptstyle (\times\eta^{j_1-j_1'}, \times\eta^{j_2-j_2'})} & & \downarrow{\scriptstyle \times\eta^{j_1+j_2-(j_1'+j_2')}} \\ H^{c-1}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1'+c}) \times H^{c-1}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_2'+c}) & \xrightarrow[\cdot]{} & H^{2c-2}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1'+j_2'+2c}) \end{array}$$

then $\mathrm{AQlc}_{\mathscr{L},j_1',j_2'} = \eta^{j_1+j_2-(j_1'+j_2')}\,\mathrm{AQlc}_{\mathscr{L},j_1,j_2}$, hence:

$$\mathrm{AQld}_{\mathscr{L},j_1',j_2'} = \mathrm{AQld}_{\mathscr{L},j_1,j_2}$$

(with a slight abuse of notation when $(j_1, j_2) = (0, 0)$ and $(j_1', j_2') \neq (0, 0)$: in this case, the canonical morphism $\mathrm{GW}(F) \to \mathrm{W}(F)$ maps $\mathrm{AQld}_{\mathscr{L},j_1,j_2}$ to $\mathrm{AQld}_{\mathscr{L},j_1',j_2'}$). See Notation 3.36 and Theorem 2.33 and note that ∂ and $\widetilde{o}$ commute to product by η.

In classical knot theory, the alternative definition of the linking number (by considering the second component rather than the first one) gives the opposite of the linking number (see Remark 1.15 and its higher-dimensional generalisation Remark 1.32). This is also true of the ambient quadratic linking degree.

Remark 5.9. Let $i_2 : Z_2 \to X := \mathbb{A}_F^{2n} \setminus \{0\}$ be the inclusion of the closed subscheme Z_2 in X. It follows from Remark 5.4 that the image of the part of the quadratic linking class of $\mathscr{L}$ with respect to (j_1, j_2) which is in the Rost-Schmid group $H^{c-1}(Z_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_2}\})$ by the composite of the morphism $(i_2)_* : H^{c-1}(Z_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_2}\}) \to H^{2c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2c})$ and of the isomorphism ζ_{2n,j_1+j_2+2n} is the opposite of the ambient quadratic linking degree.

Similarly to the linking number which stays the same when the order of the components of the oriented link is reversed and the codimension is even, and is turned into its opposite when the order of the components of the oriented link is reversed and the codimension is odd (see Remark 1.32 and note that in this Remark, the codimension is $n + 1$), we have the following result for the ambient quadratic linking degree.

Proposition 5.10. Let $n \geq 2$ be an integer, $\mathscr{L} = ((Z_1 \subset \mathbb{A}_F^{2n} \setminus \{0\}, Z_2 \subset \mathbb{A}_F^{2n} \setminus \{0\}), (\overline{o_1}, \overline{o_2}))$ be an oriented link with two components of dimension n

(denote by $c := n$ the codimension) and $j_1, j_2 \leq 0$ be integers such that $(j_1, j_2) \neq (0,0)$. Let $\mathscr{L}'$ be the oriented link $(Z_2, Z_1), (\overline{o_2}, \overline{o_1})$. Then:

$$\mathrm{AQld}_{\mathscr{L}',j_2,j_1} = \begin{cases} -\mathrm{AQld}_{\mathscr{L},j_1,j_2} & \text{if } c \text{ is odd} \\ \mathrm{AQld}_{\mathscr{L},j_1,j_2} & \text{if } c \text{ is even} \end{cases}$$

$$\mathrm{AQld}_{\mathscr{L}',0,0} = \begin{cases} \langle -1 \rangle \mathrm{AQld}_{\mathscr{L},0,0} & \text{if } c \text{ is odd} \\ \mathrm{AQld}_{\mathscr{L},0,0} & \text{if } c \text{ is even} \end{cases}$$

Note that $\mathrm{AQld}_{\mathscr{L}',j_2,j_1} \in \mathrm{W}(F)$ whereas $\mathrm{AQld}_{\mathscr{L}',0,0} \in \mathrm{GW}(F)$.

Proof. By Proposition 5.5:

$$\mathrm{AQlc}_{\mathscr{L}',j_2,j_1} = \begin{cases} -\mathrm{AQlc}_{\mathscr{L},j_1,j_2} & \text{if } c \text{ is odd and } (j_1 \text{ is odd or } j_2 \text{ is odd}) \\ -\epsilon\,\mathrm{AQlc}_{\mathscr{L},j_1,j_2} & \text{if } c \text{ is odd and } j_1 \text{ is even and } j_2 \text{ is even} \\ \epsilon\,\mathrm{AQlc}_{\mathscr{L},j_1,j_2} & \text{if } c \text{ is even and } (j_1 \text{ is odd or } j_2 \text{ is odd}) \\ \mathrm{AQlc}_{\mathscr{L},j_1,j_2} & \text{if } c \text{ is even and } j_1 \text{ is even and } j_2 \text{ is even} \end{cases}$$

$$\mathrm{AQlc}_{\mathscr{L}',0,0} = \begin{cases} -\epsilon\,\mathrm{AQlc}_{\mathscr{L},0,0} & \text{if } c \text{ is odd} \\ \mathrm{AQlc}_{\mathscr{L},0,0} & \text{if } c \text{ is even} \end{cases}$$

The result follows from the fact that the ambient quadratic linking degree $\mathrm{AQld}_{\mathscr{L},j_1,j_2}$ (respectively $\mathrm{AQld}_{\mathscr{L},0,0}$) is the image of the ambient quadratic linking class $\mathrm{AQlc}_{\mathscr{L},j_1,j_2}$ (resp. $\mathrm{AQlc}_{\mathscr{L},0,0}$) by the composite of the morphism $\widetilde{o} \circ \partial$ (which commutes to product by ϵ) and of the morphism $\gamma_{j_1+j_2} : K^{\mathrm{MW}}_{j_1+j_2}(F) \to \mathrm{W}(F)$ which sends $\langle a \rangle \eta^{-(j_1+j_2)}$ to $< a >$ (resp. $\gamma_0 : K^{\mathrm{MW}}_0(F) \to \mathrm{GW}(F)$ which sends $\langle a \rangle$ to $\langle a \rangle$). □

Let us now see how the ambient quadratic linking degree is changed by some orientation changes. Note that when the oriented link corresponds to one of the cases in Table 5.1 then, by Remark 4.36, Proposition 5.11 covers all possible changes of the orientation classes.

Proposition 5.11. Let $n \geq 2$ be an integer, $\mathscr{L} = ((Z_1 \subset \mathbb{A}^{2n}_F \setminus \{0\}, Z_2 \subset \mathbb{A}^{2n}_F \setminus \{0\}), (\overline{o_1}, \overline{o_2}))$ be an oriented link with two components of dimension n and $j_1, j_2 \leq 0$ be integers such that $(j_1, j_2) \neq (0,0)$. Let $a = (a_1, a_2)$ be a couple of elements of F^* and $\mathscr{L}_a$ be the link obtained from $\mathscr{L}$ by changing the orientation class $\overline{o_1}$ into $\overline{o_1 \circ (\times a_1)}$ and the orientation class $\overline{o_2}$ into $\overline{o_2 \circ (\times a_2)}$. Then:

$$\mathrm{AQld}_{\mathscr{L}_a,j_1,j_2} = < a_1 a_2 > \mathrm{AQld}_{\mathscr{L},j_1,j_2}$$

$$\mathrm{AQld}_{\mathscr{L}_a,0,0} = \langle a_1 a_2 \rangle \mathrm{AQld}_{\mathscr{L},0,0}$$

Note that $\mathrm{AQld}_{\mathscr{L}_a,j_1,j_2} \in \mathrm{W}(F)$ whereas $\mathrm{AQld}_{\mathscr{L}_a,0,0} \in \mathrm{GW}(F)$.

Proof. By Proposition 5.6, $\mathrm{AQlc}_{\mathscr{L}_a,j_1,j_2} = \langle a_1a_2\rangle\,\mathrm{AQlc}_{\mathscr{L},j_1,j_2}$ and $\mathrm{AQlc}_{\mathscr{L}_a,0,0} = \langle a_1a_2\rangle\,\mathrm{AQlc}_{\mathscr{L},0,0}$. The result follows from the fact that the ambient quadratic linking degree $\mathrm{AQld}_{\mathscr{L},j_1,j_2}$ (respectively $\mathrm{AQld}_{\mathscr{L},0,0}$) is the image of the ambient quadratic linking class $\mathrm{AQlc}_{\mathscr{L},j_1,j_2}$ (resp. $\mathrm{AQlc}_{\mathscr{L},0,0}$) by the composite of the morphism $\widetilde{\sigma} \circ \partial$ (which commutes to product by $\langle a_1a_2\rangle$) and of the morphism $\gamma_{j_1+j_2} : K^{\mathrm{MW}}_{j_1+j_2}(F) \to \mathrm{W}(F)$ which sends $\langle a\rangle\eta^{-(j_1+j_2)}$ to $< a >$ (resp. $\gamma_0 : K^{\mathrm{MW}}_0(F) \to \mathrm{GW}(F)$ which sends $\langle a\rangle$ to $\langle a\rangle$). □

Remark 5.12. In the case where the ground field is the field of real numbers (i.e. $F = \mathbb{R}$), this proposition is similar to what happens to the linking number (see Remark 1.31): the ambient quadratic linking degree is the same if a_1 and a_2 have the same sign (similarly to the linking number which is the same if both orientations are reversed (or if they are both left unchanged)) and is multiplied by $< -1 >= -1 \in \mathrm{W}(F)$ or by $\langle -1\rangle \in \mathrm{GW}(F)$ if a_1 and a_2 have different signs (similarly to the linking number which is multiplied by -1 if exactly one of the orientations is reversed).

Future work 7 (Ambient quadratic linking degree in a projective setting)**.** Since we do not yet have an explicit isomorphism between the Rost-Schmid group $H^{2n+1}(\mathbb{P}^{2n+1}_F, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2n+2})$ and $K^{\mathrm{MW}}_{j_1+j_2+1}(F)$ (see Future work 3), we cannot define the ambient quadratic linking degree for the case in Table 5.2. When Future work 3 will be completed, we will be able to define the ambient quadratic linking degree for the case in Table 5.2 as the image of the ambient quadratic linking class by the composite of the explicit isomorphism between $H^{2n+1}(\mathbb{P}^{2n+1}_F, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2n+2})$ and $K^{\mathrm{MW}}_{j_1+j_2+1}(F)$ and of the isomorphism $\gamma_{j_1+j_2+1} : K^{\mathrm{MW}}_{j_1+j_2+1}(F) \to \mathrm{W}(F)$ (see Theorem 2.33).

5.2 The quadratic linking degree couple

In this section we define the quadratic linking degree (couple), which is a counterpart in algebraic geometry to the linking couple (see Definition 1.17 and its higher-dimensional generalisation Definition 1.34).

Throughout this section, F is a perfect field.

We begin by giving a definition of oriented links with two components which has more information than what was needed to define the quadratic linking class in Chapter 4.

Definition 5.13 (Oriented link with two components of a certain type)**.** An *oriented link* $\mathscr{L}$ with two components *of type* (Y_1, Y_2, X) is a couple of closed immersions $(\varphi_1 : Y_1 \to X, \varphi_2 : Y_2 \to X)$, where X, Y_1, Y_2 are irreducible smooth finite-type F-schemes and Y_1 and Y_2 are of same dimension, such that the image Z_1 of φ_1 and the image Z_2 of φ_2 are disjoint, together with a couple of orientation classes $(\overline{o_1}, \overline{o_2})$, where $o_1 : \nu_{Z_1} := \det(\mathcal{N}_{Z_1/X}) \to \mathcal{L}_1 \otimes \mathcal{L}_1$ is an orientation of the normal sheaf of Z_1 in X and $o_2 : \nu_{Z_2} := \det(\mathcal{N}_{Z_2/X}) \to \mathcal{L}_2 \otimes \mathcal{L}_2$ is an orientation of the normal sheaf of Z_2 in X. In other words, an oriented link $\mathscr{L}$ with two components of type (Y_1, Y_2, X) is an oriented link with two components (Z_1, Z_2), $(\overline{o_1}, \overline{o_2})$ in the sense of Definition 4.1 together with a *parametrisation* φ_1 of Z_1 and a *parametrisation* φ_2 of Z_2.

Remark 5.14. If $(Z_1, Z_2), (\overline{o_1}, \overline{o_2}), (\varphi_1, \varphi_2)$ is an oriented link with two components of a certain type then in particular $\mathcal{N}_{Z_1/X}$ and $\mathcal{N}_{Z_2/X}$ are orientable (i.e. their determinants are isomorphic to squares, see Definition 3.22). Similar considerations to the ones in Remark 4.2 apply to links of a certain type: (nonoriented) links $(Z_1, Z_2), (\varphi_1, \varphi_2)$ with two components of a certain type could be required to be orientable or not. Also, note that even though we only defined oriented links with two components of a certain type, similar definitions for (oriented) knots of a certain type (i.e. (oriented) links with one component of a certain type) and for (oriented) links with n components of a certain type (with $n \in \mathbb{N}$) can be made.

See Chapter 7 for examples (especially Section 7.1 for a simple example: the Hopf link).

In the following two subsections, we define quadratic linking degree couples by these three steps:

1. Apply the isomorphism $H^{c-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\}) \to H^{c-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c})$ induced by the orientation classes $\overline{o_1}$ and $\overline{o_2}$ to the quadratic linking class.

2. Apply the isomorphism $H^{c-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}) \to H^{c-1}(Y_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}) \oplus H^{c-1}(Y_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c})$ induced by the couple of isomorphisms of F-schemes $(\varphi_1 : Y_1 \to Z_1, \varphi_2 : Y_2 \to Z_2)$ to the result of the first step.

3. Apply an isomorphism (which only depends on Y_1 and Y_2, not on the specific oriented link) between $H^{c-1}(Y_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}) \oplus H^{c-1}(Y_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c})$ and a well-known group to the result of the second step to get the quadratic linking degree couple.

Note that step 3 will use an isomorphism which has been fixed once and for all but which is not canonical (similarly to what was done to define the ambient quadratic linking degree from the ambient quadratic linking class).

Smooth models of motivic spheres

In this subsection, we define the quadratic linking degree couple for oriented links with two components of type (Y_1, Y_2, X) with Y_1, Y_2, X smooth models of motivic spheres, as in Section 4.3. Recall Tables 4.1 and 4.2.

Definition 5.15 (Quadratic linking degree couple)**.** Let (j_1, j_2) be a couple of nonpositive integers and $\mathscr{L} = (Z_1, Z_2), (\overline{o_1}, \overline{o_2}), (\varphi_1, \varphi_2)$ be an oriented link with two components of type (Y_1, Y_2, X), with (Y_1, Y_2, X) equal to:

1. $(\mathbb{A}^n_F \setminus \{0\}, \mathbb{A}^n_F \setminus \{0\}, \mathbb{A}^{2n}_F \setminus \{0\})$ for some $n \geq 2$;
2. $(\mathbb{A}^n_F \setminus \{0\}, Q_n, \mathbb{A}^{2n}_F \setminus \{0\})$ for some $n \geq 3$;
3. $(Q_n, \mathbb{A}^n_F \setminus \{0\}, \mathbb{A}^{2n}_F \setminus \{0\})$ for some $n \geq 3$;
4. $(\mathbb{A}^2_F \setminus \{0\}, Q_2, \mathbb{A}^4_F \setminus \{0\})$;
5. $(Q_2, \mathbb{A}^2_F \setminus \{0\}, \mathbb{A}^4_F \setminus \{0\})$;
6. $(\mathbb{A}^n_F \setminus \{0\}, Q_n, \mathbb{A}^{n+\lfloor \frac{n}{2} \rfloor+1}_F \setminus \{0\})$ for some $n \geq 3$ odd;
7. $(Q_n, \mathbb{A}^n_F \setminus \{0\}, \mathbb{A}^{n+\lfloor \frac{n}{2} \rfloor+1}_F \setminus \{0\})$ for some $n \geq 3$ odd;
8. $(Q_2, Q_2, \mathbb{A}^4_F \setminus \{0\})$;
9. $(Q_n, Q_n, \mathbb{A}^{n+\lfloor \frac{n}{2} \rfloor+1}_F \setminus \{0\})$ for some $n \geq 3$ odd;
10. $(Q_n, Q_n, Q_{n+\lfloor \frac{n}{2} \rfloor+1})$ for some $n \geq 5$ odd;
11. (Q_3, Q_3, Q_5);
12. (Q_2, Q_2, Q_4).

In the cases 11 and 12, assume that $\mathscr{L}$ has a well-defined quadratic linking class with respect to (j_1, j_2) (see Remark 4.10; this is always verified in the other cases). The *quadratic linking degree couple* of $\mathscr{L}$ with respect to (j_1, j_2), denoted $\mathrm{Qld}_{\mathscr{L}, j_1, j_2}$ (or $\mathrm{Qld}_{\mathscr{L}}$ for short), is the image of the quadratic linking class of $\mathscr{L}$ with respect to (j_1, j_2) by the composite of four isomorphisms (in a nutshell, this quadratic linking degree couple is the image of $(\varphi_1^* \oplus \varphi_2^*)((\widetilde{o_1} \oplus \widetilde{o_2})(\varpi(\mathrm{Qlc}_{\mathscr{L}, j_1, j_2})))$ by an isomorphism which depends on the type of the oriented link $\mathscr{L}$). The first of these is the isomorphism $\varpi : H^{c-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\}) \to H^{c-1}(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_1}\}) \oplus H^{c-1}(Z_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_2}\})$ which is induced by the inclusions of Z_1, Z_2 in $Z := Z_1 \sqcup Z_2$ (where c is the codimension of Z_1 in X and $\nu_Z, \nu_{Z_1}, \nu_{Z_2}$

are the determinants of the normal sheaves of Z, Z_1, Z_2 in X respectively). The second of these isomorphisms is $\widetilde{o_1} \oplus \widetilde{o_2} : H^{c-1}(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_1}\}) \oplus H^{c-1}(Z_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_2}\}) \to H^{c-1}(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}) \oplus H^{c-1}(Z_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c})$ (see Notation 3.25). The third of these isomorphisms is $\varphi_1^* \oplus \varphi_2^* : H^{c-1}(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}) \oplus H^{c-1}(Z_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}) \to H^{c-1}(Y_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}) \oplus H^{c-1}(Y_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c})$. The last of these isomorphisms depends on the type of the oriented link $\mathscr{L}$ and is listed below in the same order as above (see Notations 3.36, 3.41 and 3.51):

1. $\zeta_{n,j_1+j_2+n} \oplus \zeta_{n,j_1+j_2+n}$
2. $\zeta_{n,j_1+j_2+n} \oplus 0$
3. $0 \oplus \zeta_{n,j_1+j_2+n}$
4. $\zeta_{2,j_1+j_2+2} \oplus \phi_{j_1+j_2+2}$
5. $\phi_{j_1+j_2+2} \oplus \zeta_{2,j_1+j_2+2}$
6. $0 \oplus \varsigma_{n,j_1+j_2+\lfloor \frac{n}{2} \rfloor+1}$
7. $\varsigma_{n,j_1+j_2+\lfloor \frac{n}{2} \rfloor+1} \oplus 0$
8. $\phi_{j_1+j_2+2} \oplus \phi_{j_1+j_2+2}$
9. $\varsigma_{n,j_1+j_2+\lfloor \frac{n}{2} \rfloor+1} \oplus \varsigma_{n,j_1+j_2+\lfloor \frac{n}{2} \rfloor+1}$
10. $\varsigma_{n,j_1+j_2+\lfloor \frac{n}{2} \rfloor+1} \oplus \varsigma_{n,j_1+j_2+\lfloor \frac{n}{2} \rfloor+1}$
11. $\varsigma_{3,j_1+j_2+2} \oplus \varsigma_{3,j_1+j_2+2}$
12. $\phi_{j_1+j_2+2} \oplus \phi_{j_1+j_2+2}$

See Section 7.1 for simple examples of quadratic linking degree couples.

Future work 8 (Additional quadratic linking degree couples)**.** Since we do not yet have an explicit isomorphism $\theta_{n,j_1+j_2+\lfloor \frac{n}{2} \rfloor+1} : H^{\lfloor \frac{n}{2} \rfloor}(Q_n, \underline{K}^{\mathrm{MW}}_{j_1+j_2+\lfloor \frac{n}{2} \rfloor+1}) \to K^{\mathrm{MW}}_{j_1+j_2+1}(F)$ when $n \geq 4$ is even (see Future work 2), we cannot define the quadratic linking degree couple for the following types (Y_1, Y_2, X):

13. $(\mathbb{A}^n_F \setminus \{0\}, Q_n, \mathbb{A}^{n+\lfloor \frac{n}{2} \rfloor+1}_F \setminus \{0\})$ for some $n \geq 4$ even;
14. $(Q_n, \mathbb{A}^n_F \setminus \{0\}, \mathbb{A}^{n+\lfloor \frac{n}{2} \rfloor+1}_F \setminus \{0\})$ for some $n \geq 4$ even;
15. $(Q_n, Q_n, \mathbb{A}^{n+\lfloor \frac{n}{2} \rfloor+1}_F \setminus \{0\})$ for some $n \geq 4$ even;
16. $(Q_n, Q_n, Q_{n+\lfloor \frac{n}{2} \rfloor+1})$ for some $n \geq 6$ even;

17. (Q_4, Q_4, Q_7) (note that in this case, there is a priori not necessarily a well-defined quadratic linking class).

When Future work 2 will be completed, we will be able to define the quadratic linking degree couple of an oriented link $\mathscr{L} = (Z_1, Z_2), (\overline{o_1}, \overline{o_2}), (\varphi_1, \varphi_2)$ of type (Y_1, Y_2, X) with respect to a couple of nonpositive integers (j_1, j_2) (for the five cases listed above) as the image of the quadratic linking class of $\mathscr{L}$ with respect to (j_1, j_2) by the composite of the first three isomorphisms which were described in Definition 5.15 and of the following isomorphism (which depends on the type of the oriented link $\mathscr{L}$ and is listed below in the same order as above; see Theorem 2.33 for $\gamma_{j_1+j_2+1}$ when $(j_1, j_2) \neq (0, 0)$, and conventionally γ_1 is the identity of $K_1^{\mathrm{MW}}(F)$):

13. $0 \oplus \gamma_{j_1+j_2+1} \circ \theta_{n, j_1+j_2+\lfloor \frac{n}{2} \rfloor+1}$

14. $\gamma_{j_1+j_2+1} \circ \theta_{n, j_1+j_2+\lfloor \frac{n}{2} \rfloor+1} \oplus 0$

15. $\gamma_{j_1+j_2+1} \circ \theta_{n, j_1+j_2+\lfloor \frac{n}{2} \rfloor+1} \oplus \gamma_{j_1+j_2+1} \circ \theta_{n, j_1+j_2+\lfloor \frac{n}{2} \rfloor+1}$

16. $\gamma_{j_1+j_2+1} \circ \theta_{n, j_1+j_2+\lfloor \frac{n}{2} \rfloor+1} \oplus \gamma_{j_1+j_2+1} \circ \theta_{n, j_1+j_2+\lfloor \frac{n}{2} \rfloor+1}$

17. $\gamma_{j_1+j_2+1} \circ \theta_{4, j_1+j_2+3} \oplus \gamma_{j_1+j_2+1} \circ \theta_{4, j_1+j_2+3}$

Note that in the cases 1, 4, 5 and 8 above, there is an ambient quadratic linking degree as well (see Definition 5.7), whereas in the other cases the ambient quadratic linking class is in the zero group (and we can conventionally say that the ambient quadratic linking degree is zero in these cases).

Future work 9 (Ambient quad. link. degree and quad. link. degree couple)**.** In classical knot theory, each component of the linking couple is the linking number up to a sign (see Remark 1.16 and its higher-dimensional generalisation Remark 1.33). Since the morphism $(i_1)_* : H^{c-1}(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_1}\}) \to H^{2c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2c})$ and the morphism $(i_2)_* : H^{c-1}(Z_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_2}\}) \to H^{2c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2c})$ induced by the inclusions $Z_1 \to X$ and $Z_2 \to X$ respectively are a priori neither the identity nor the opposite (a contrario from their counterparts in classical knot theory, which are surjective morphisms from a group isomorphic to $\mathbb{Z}$ to a group isomorphic to $\mathbb{Z}$), the components of the quadratic linking degree couple are a priori not the ambient quadratic linking degree up to a sign. It would be interesting to investigate the relationship between the ambient quadratic linking degree and the quadratic linking degree couple and especially to answer the following question: must the quadratic linking degree couple be zero when the ambient quadratic linking degree is zero? In particular, can the quadratic linking degree couple be nonzero in the cases 2, 3, 6, 7, 9, 10, 11, 12 and (when it will be defined) 13, 14, 15, 16, 17 ? (Note that the ambient quadratic linking degree is necessarily zero in all these cases.)

Remark 5.16. Let $\mathscr{L} = (Z_1, Z_2), (\overline{o_1}, \overline{o_2}), (\varphi_1, \varphi_2)$ be an oriented link with two components of type (Y_1, Y_2, X) in one of the twelve cases of Definition 5.15 and $j_1, j_2 \leq 0$ be integers such that $\mathscr{L}$ has a well-defined quadratic linking degree with respect to (j_1, j_2). Let $j_1' \leq j_1 \leq 0$ and $j_2' \leq j_2 \leq 0$ be integers such that $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1'+c}) = 0$ and $H^{c-1}(X, \underline{K}^{\mathrm{MW}}_{j_2'+c}) = 0$ (which ensures the unicity of the quadratic linking class with respect to (j_1', j_2') if it exists, hence the unicity of the quadratic linking degree with respect to (j_1', j_2') if it exists). By Remark 4.11, if the following diagram is commutative (which is verified for instance under the assumptions of Corollary 3.32):

$$\begin{array}{ccc} H^{c-1}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1+c}) \times H^{c-1}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_2+c}) & \xrightarrow{\cdot} & H^{2c-2}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2c}) \\ \downarrow{\scriptstyle (\times\eta^{j_1-j_1'}, \times\eta^{j_2-j_2'})} & & \downarrow{\scriptstyle \times\eta^{j_1+j_2-(j_1'+j_2')}} \\ H^{c-1}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1'+c}) \times H^{c-1}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_2'+c}) & \xrightarrow{\cdot} & H^{2c-2}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1'+j_2'+2c}) \end{array}$$

then the oriented link $\mathscr{L}$ has a well-defined quadratic linking class $\mathrm{Qlc}_{\mathscr{L},j_1',j_2'} = \eta^{j_1+j_2-(j_1'+j_2')}\,\mathrm{Qlc}_{\mathscr{L},j_1,j_2}$, hence it has a well-defined quadratic linking degree $\mathrm{Qld}_{\mathscr{L},j_1',j_2'}$ which is equal to $\mathrm{Qld}_{\mathscr{L},j_1,j_2}$, with the following conventions:

- an element α of $\mathrm{W}(F)$ and an element β of $\mathrm{GW}(F)$ are conventionally equal if the canonical morphism $\mathrm{GW}(F) \to \mathrm{W}(F)$ maps β to α;
- an element β of $\mathrm{GW}(F)$ and an element δ of $K_1^{\mathrm{MW}}(F)$ are conventionally equal if the morphism γ_0 maps $\eta\delta$ to β;
- an element α of $\mathrm{W}(F)$ and an element δ of $K_1^{\mathrm{MW}}(F)$ are conventionally equal if the morphism γ_{-1} maps $\eta^2\delta$ to α (note that for all $m \geq 2$, γ_{-1} maps $\eta^2\delta$ to α if and only if γ_{-m+1} maps $\eta^m\delta$ to α);
- as usual, a couple (a_1, a_2) is equal to a couple (b_1, b_2) if $a_1 = b_1, a_2 = b_2$.

Indeed, the isomorphisms ϖ, $\widetilde{o_1} \oplus \widetilde{o_2}$, $\varphi_1^* \oplus \varphi_2^*$, and ϕ_2 commute to product by η and the isomorphisms ζ, ς and ϕ_l with $l \leq 1$ are composites of isomorphisms which commute to product by η and of γ (recall Definition 5.15, Notations 3.36, 3.41, 3.51 and Theorem 2.33).

Let us now see what happens to the quadratic linking degree couple when the order of the components of the oriented link is reversed.

Proposition 5.17. Let $\mathscr{L} = (Z_1, Z_2), (\overline{o_1}, \overline{o_2}), (\varphi_1, \varphi_2)$ be an oriented link with two components of type (Y_1, Y_2, X) in one of the twelve cases of Definition 5.15 and $j_1, j_2 \leq 0$ be integers such that $\mathscr{L}$ has a well-defined quadratic linking degree couple $\mathrm{Qld}_{\mathscr{L},j_1,j_2} = (d_1, d_2)$ with respect to (j_1, j_2). Let $\mathscr{L}'$ be the oriented link $(Z_2, Z_1), (\overline{o_2}, \overline{o_1}), (\varphi_2, \varphi_1)$ of type (Y_2, Y_1, X). Then the oriented link $\mathscr{L}'$ has a well-defined quadratic linking degree couple $\mathrm{Qld}_{\mathscr{L}',j_2,j_1}$ which verifies (recall that $\epsilon = -\langle -1 \rangle \in K_0^{\mathrm{MW}}(F)$, and conventionally $\epsilon := -\langle -1 \rangle \in \mathrm{GW}(F)$ and $\epsilon := 1 \in \mathrm{W}(F)$):

$$\mathrm{Qld}_{\mathscr{L}',j_2,j_1} = \begin{cases} (d_2, d_1) & \text{if } c \text{ is odd and } (j_1 \text{ is odd or } j_2 \text{ is odd}) \\ (\epsilon d_2, \epsilon d_1) & \text{if } c \text{ is odd and } j_1 \text{ is even and } j_2 \text{ is even} \\ (-\epsilon d_2, -\epsilon d_1) & \text{if } c \text{ is even and } (j_1 \text{ is odd or } j_2 \text{ is odd}) \\ (-d_2, -d_1) & \text{if } c \text{ is even and } j_1 \text{ is even and } j_2 \text{ is even} \end{cases}$$

Proof. By Proposition 4.15:

$$\mathrm{Qlc}_{\mathscr{L}',j_2,j_1} = \begin{cases} \mathrm{Qlc}_{\mathscr{L},j_1,j_2} & \text{if } c \text{ is odd and } (j_1 \text{ is odd or } j_2 \text{ is odd}) \\ \epsilon \, \mathrm{Qlc}_{\mathscr{L},j_1,j_2} & \text{if } c \text{ is odd and } j_1 \text{ is even and } j_2 \text{ is even} \\ -\epsilon \, \mathrm{Qlc}_{\mathscr{L},j_1,j_2} & \text{if } c \text{ is even and } (j_1 \text{ is odd or } j_2 \text{ is odd}) \\ -\mathrm{Qlc}_{\mathscr{L},j_1,j_2} & \text{if } c \text{ is even and } j_1 \text{ is even and } j_2 \text{ is even} \end{cases}$$

The result follows from the fact that the isomorphisms ϖ, $\widetilde{o_1} \oplus \widetilde{o_2}$, $\varphi_1^* \oplus \varphi_2^*$, and ϕ_2 commute to product by ϵ and the isomorphisms ζ, ς and ϕ_l with $l \leq 1$ are composites of isomorphisms which commute to product by ϵ and of γ (recall Definition 5.15, Notations 3.36, 3.41, 3.51 and Theorem 2.33). □

Future work 10 (The two components of the quadratic linking degree couple)**.** It would be interesting to determine the relationship (if there is one) between the first component of the quadratic linking degree couple and the second component of the quadratic linking degree couple (it would allow us for instance to make Proposition 5.17 more precise). The fact that the quadratic linking class is in the kernel of i_* (see Proposition 4.12) may be useful for this (since the corresponding fact in knot theory is useful to show that the first component of the linking couple is the second component of the linking couple up to a sign, see Remark 1.16 and its higher-dimensional generalisation Remark 1.33).

Similarly to the linking couple whose first component (respectively second component) stays the same and whose second component (resp. first component) is turned into its opposite if the orientation of the first component (resp. second component) of the oriented link is reversed (see Remark

1.36), the quadratic linking degree couple is changed in the following way by orientation changes. Recall Remark 4.36.

Proposition 5.18. Let $\mathscr{L} = (Z_1, Z_2), (\overline{o_1}, \overline{o_2}), (\varphi_1, \varphi_2)$ be an oriented link with two components of type (Y_1, Y_2, X) in one of the twelve cases of Definition 5.15 and $j_1, j_2 \leq 0$ be integers such that $\mathscr{L}$ has a well-defined quadratic linking degree couple $\mathrm{Qld}_{\mathscr{L},j_1,j_2} = (d_1, d_2)$ with respect to (j_1, j_2). Let $a = (a_1, a_2)$ be a couple of elements of F^*. Let $\mathscr{L}_a$ be the link obtained from $\mathscr{L}$ by changing the orientation class $\overline{o_1}$ into $\overline{o_1 \circ (\times a_1)}$ and the orientation class $\overline{o_2}$ into $\overline{o_2 \circ (\times a_2)}$. Then $\mathscr{L}_a$ has a well-defined quadratic linking degree couple $\mathrm{Qld}_{\mathscr{L}_a,j_1,j_2}$ with respect to (j_1, j_2) which verifies:

$$\mathrm{Qld}_{\mathscr{L}_a,j_1,j_2} = (\langle a_2\rangle d_1, \langle a_1\rangle d_2)$$

(with slight abuses of notation: if $d_1 \in \mathrm{W}(F)$ then $\langle a_2\rangle$ should be replaced with $< a_2 >$, if $d_2 \in \mathrm{W}(F)$ then $\langle a_1\rangle$ should be replaced with $< a_1 >$, if d_1 is in the zero group (see cases 3 and 6) then conventionally $\langle a_2\rangle d_1 = 0$, if d_2 is in the zero group (see cases 2 and 7) then conventionally $\langle a_1\rangle d_2 = 0$).

Proof. By Proposition 4.16, $\mathrm{Qlc}_{\mathscr{L}_a,j_1,j_2} = \langle a_1 a_2\rangle \,\mathrm{Qlc}_{\mathscr{L},j_1,j_2}$ hence $\varpi(\mathrm{Qlc}_{\mathscr{L}_a,j_1,j_2}) = \langle a_1 a_2\rangle \varpi(\mathrm{Qlc}_{\mathscr{L},j_1,j_2})$ (see Definition 5.15). Thus, denoting $(\sigma_1, \sigma_2) := \varpi(\mathrm{Qlc}_{\mathscr{L},j_1,j_2})$, we have $\varpi(\mathrm{Qlc}_{\mathscr{L}_a,j_1,j_2}) = (\langle a_1 a_2\rangle\sigma_1, \langle a_1 a_2\rangle\sigma_2)$. Let $i \neq j \in \{1, 2\}$. Note that $\widetilde{o_i \circ (\times a_i)}(\langle a_1 a_2\rangle\sigma_i) = \langle a_i\rangle \widetilde{o_i}(\widetilde{\langle a_1 a_2\rangle\sigma_i}) = \langle a_i^2 a_j\rangle \widetilde{o_i}(\sigma_i) = \langle a_j\rangle \widetilde{o_i}(\sigma_i)$ (see Notation 3.25). Therefore, $\varphi_i^*(\widetilde{o_i \circ (\times a_i)}(\langle a_1 a_2\rangle\sigma_i)) = \varphi_i^*(\langle a_j\rangle \widetilde{o_i}(\sigma_i)) = \langle a_j\rangle \varphi_i^*(\widetilde{o_i}(\sigma_i))$ (since $a_j \in F^*$). Since the i-th component of the quadratic linking degree couple $\mathrm{Qld}_{\mathscr{L},j_1,j_2}$ (respectively $\mathrm{Qld}_{\mathscr{L}_a,j_1,j_2}$) is obtained from $\varphi_i^*(\widetilde{o_i}(\sigma_i))$ (respectively $\langle a_j\rangle \varphi_i^*(\widetilde{o_i}(\sigma_i))$) by applying the isomorphism from the relevant case in Definition 5.15 and since these isomorphisms commute to product by $\langle a_j\rangle$ (in the case of ϕ_2 (see Notation 3.51)) or are composites of isomorphisms which commute to product by $\langle a_j\rangle$ and of γ (in the case of ζ, ς and ϕ_l with $l \leq 1$ (see Notations 3.36, 3.41, 3.51 and Theorem 2.33)), the i-th component of $\mathrm{Qld}_{\mathscr{L}_a,j_1,j_2}$ is equal to $\langle a_j\rangle d_i$ (with the same slight abuses of notation as above). □

Let us now focus on changes of parametrisations.

Remark 5.19. Let $n \geq 2$ and ψ be an automorphism of $\mathbb{A}^n_F \setminus \{0\}$. By composing with the inclusion $\mathbb{A}^n_F \setminus \{0\} \to \mathbb{A}^n_F$, we get from ψ an n-tuple of elements of $\mathcal{O}_{\mathbb{A}^n_F}(\mathbb{A}^n_F \setminus \{0\})$. By [GW10, Theorem 6.45 (Hartogs' theorem)], the restriction $\mathcal{O}_{\mathbb{A}^n_F}(\mathbb{A}^n_F) \to \mathcal{O}_{\mathbb{A}^n_F}(\mathbb{A}^n_F \setminus \{0\})$ is an isomorphism, hence ψ is the restriction of an endomorphism of $\mathbb{A}^n_F$, which in fact is an automorphism of $\mathbb{A}^n_F$ (which preserves the origin) since the same arguments can be applied to the inverse of ψ. Since we have defined $\mathbb{A}^n_F$ as $\mathrm{Spec}(F[x_1,\dots,x_n])$ (thus fixing coordinates $x_1,\dots,x_n$), this automorphism of $\mathbb{A}^n_F$ has a Jacobian determinant, which we denote by J_ψ. Note that J_ψ is in F^* since $(F[x_1,\dots,x_n])^* = F^*$.

Proposition 5.20. Let $i \in \{1,2\}$. Let $\mathscr{L} = (Z_1,Z_2),(\overline{o_1},\overline{o_2}),(\varphi_1,\varphi_2)$ be an oriented link with two components of type (Y_1,Y_2,X) in one of the cases of Definition 5.15 such that $Z_i \simeq \mathbb{A}^n_F \setminus \{0\}$ and $X \simeq \mathbb{A}^{2n}_F \setminus \{0\}$. Let (j_1,j_2) be a couple of nonpositive integers, ψ_i be an automorphism of $\mathbb{A}^n_F \setminus \{0\}$ and $\mathscr{L}_{\psi_i}$ be the link obtained from $\mathscr{L}$ by changing $\varphi_i : \mathbb{A}^n_F \setminus \{0\} \to X$ into $\varphi_i \circ \psi_i : \mathbb{A}^n_F \setminus \{0\} \to X$. Then, denoting $(d_1,d_2) := \mathrm{Qld}_{\mathscr{L},j_1,j_2}$ and by k the only element of $\{1,2\} \setminus \{i\}$:

- the i-th component of $\mathrm{Qld}_{\mathscr{L}_{\psi_i},j_1,j_2}$ is equal to $\langle J_{\psi_i}\rangle d_i$;
- the k-th component of $\mathrm{Qld}_{\mathscr{L}_{\psi_i},j_1,j_2}$ is equal to d_k.

Proof. Recall that the quadratic linking class does not depend on the choice of parametrisations (see Definitions 4.1 and 4.9), thus $\mathrm{Qlc}_{\mathscr{L}_\psi,j_1,j_2} = \mathrm{Qlc}_{\mathscr{L},j_1,j_2}$. It follows immediately that $(\widetilde{o_1}\oplus\widetilde{o_2})(\varpi(\mathrm{Qlc}_{\mathscr{L}_\psi,j_1,j_2})) = (\widetilde{o_1}\oplus\widetilde{o_2})(\varpi(\mathrm{Qlc}_{\mathscr{L},j_1,j_2}))$ (see Definition 5.15). By [Fas20, Theorem 2.14], the following diagram is commutative (where the ψ_i on the right is the restriction to 0 of the (unique) extension of ψ_i to $\mathbb{A}^n_F$; see Remark 5.19):

$$\begin{array}{ccc} H^{n-1}(\mathbb{A}^n_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_{j_1+j_2+n}) & \xrightarrow{\partial} & H^0(\{0\}, \underline{K}^{\mathrm{MW}}_{j_1+j_2}\{\det(\mathcal{N}_{\{0\}/\mathbb{A}^n_F})\}) \\ \downarrow{\scriptstyle \psi_i^*} & & \downarrow{\scriptstyle \psi_i^*} \\ H^{n-1}(\mathbb{A}^n_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_{j_1+j_2+n}) & \xrightarrow[\partial]{} & H^0(\{0\}, \underline{K}^{\mathrm{MW}}_{j_1+j_2}\{\det(\mathcal{N}_{\{0\}/\mathbb{A}^n_F})\}) \end{array}$$

It follows from this and from the equality $(\varphi_i \circ \psi_i)^* = \psi_i^* \circ \varphi_i^*$ that, denoting $(\sigma_1,\sigma_2) := \varpi(\mathrm{Qlc}_{\mathscr{L},j_1,j_2})$, $\partial((\varphi_i \circ \psi_i)^*(\widetilde{o_i}(\sigma_i))) = \psi_i^*(\partial(\varphi_i^*(\widetilde{o_i}(\sigma_i))))$. The result follows from this and from the fact that for all $\alpha \in K^{\mathrm{MW}}_{j_1+j_2}(F)$, $\psi_i^*(\alpha \otimes (\overline{u_1}^* \wedge \cdots \wedge \overline{u_n}^*)) = \langle J_{\psi_i}\rangle \alpha \otimes (\overline{u_1}^* \wedge \cdots \wedge \overline{u_n}^*)$ (recall Definition 5.15 and Notation 3.36). □

Future work 11 (Changes of parametrisations for Q_n)**.** When $Z_i \simeq Q_n$ instead of $Z_i \simeq \mathbb{A}^n_F \setminus \{0\}$, it should be feasible to get a similar result as the one in Proposition 5.20 but with another definition of J_{ψ_i} (where ψ_i is an automorphism of Q_n). We should fix a volume form ω on Q_n (for instance, $\frac{1}{\frac{\partial f}{\partial x_1}} x_2 \wedge \cdots \wedge x_m \wedge y_1 \wedge \cdots \wedge y_m$ if $n = 2m-1$ and $f = \sum_{i=1}^m x_i y_i - 1$ (note that up to sign it is equal to $\frac{1}{\frac{\partial f}{\partial x_2}} x_1 \wedge x_3 \wedge \cdots \wedge x_m \wedge y_1 \wedge \cdots \wedge y_m$, etc.), or $\frac{1}{\frac{\partial g}{\partial x_1}} x_2 \wedge \cdots \wedge x_m \wedge y_1 \wedge \cdots \wedge y_m \wedge z$ if $n = 2m$ and $g = \sum_{i=1}^m x_i y_i - z(1+z)$ (note that up to sign it is equal to $\frac{1}{\frac{\partial f}{\partial x_2}} x_1 \wedge x_3 \wedge \cdots \wedge x_m \wedge y_1 \wedge \cdots \wedge y_m \wedge z$, etc.)) and define J_{ψ_i} (if possible) as the element of F^* such that for all $\alpha \in K_*^{\mathrm{MW}}(F)$, $\psi_i^*(\alpha \otimes \omega) = \langle J_{\psi_i} \rangle \alpha \otimes \omega$.

Before we move on to a projective case, let us mention the following future work.

Future work 12 ($\mathbb{A}^1$-ambient isotopy)**.** It would be interesting to define a notion of $\mathbb{A}^1$-ambient isotopy for which the ambient quadratic linking degree and the quadratic linking degree couple would be invariants or invariants up to multiplication (of each component) by some $\langle a \rangle$ with $a \in F^*$ (similarly to ambient isotopy (see Definition 1.2) for which the linking number is an invariant and for which each component of the linking couple is an invariant up to sign). A naïve version of $\mathbb{A}^1$-ambient isotopy could be constructed from natural transformations $h : \mathbb{A}_F^1 \to \mathrm{Aut}(X)$ such that $h_F(0) = \mathrm{Id}_X$ (the identity of X) and for all $i \in \{1, 2\}$, $(h_F(1))^*(Z_i) = Z_i'$ and $\overline{\xi_{\mathcal{L}_i} \circ (h_F(1))^*(o_i) \circ \zeta_i} = \overline{o_i'}$ (see Lemma-Definitions 4.17, 4.18 and 4.19)

A projective case

In this subsection, we define the quadratic linking degree couple for oriented links with two components of type $(\mathbb{P}_F^1, \mathbb{P}_F^1, \mathbb{P}_F^3)$.

Throughout this subsection, F is assumed of characteristic different from 2.

Recall Section 4.4 and Table 4.3.

Definition 5.21 (Quadratic linking degree couple)**.** Let $\mathscr{L} = (Z_1, Z_2), (\overline{o_1}, \overline{o_2}), (\varphi_1, \varphi_2)$ be an oriented link with two components of type $(\mathbb{P}_F^1, \mathbb{P}_F^1, \mathbb{P}_F^3)$ and $j_1, j_2 \leq -2$ be integers. The *quadratic linking degree couple* of $\mathscr{L}$ with respect to (j_1, j_2), denoted $\mathrm{Qld}_{\mathscr{L}, j_1, j_2}$ (or $\mathrm{Qld}_{\mathscr{L}}$ for short), is the image of the quadratic linking class of $\mathscr{L}$ with respect to (j_1, j_2) by the composite of four isomorphisms (in a nutshell, this quadratic linking degree couple is $(\varrho_{j_1+j_2+2} \oplus \varrho_{j_1+j_2+2})((\varphi_1^* \oplus \varphi_2^*)((\widetilde{o_1} \oplus \widetilde{o_2})(\varpi(\mathrm{Qlc}_{\mathscr{L}, j_1, j_2}))))$. The first of these is the isomorphism $\varpi : H^1(Z, \underline{K}_{j_1+j_2+2}^{\mathrm{MW}}\{\nu_Z\}) \to H^1(Z_1, \underline{K}_{j_1+j_2+2}^{\mathrm{MW}}\{\nu_{Z_1}\}) \oplus H^1(Z_2, \underline{K}_{j_1+j_2+2}^{\mathrm{MW}}\{\nu_{Z_2}\})$ which is induced by the inclusions of Z_1, Z_2 in $Z := Z_1 \sqcup Z_2$ (where $\nu_Z, \nu_{Z_1}, \nu_{Z_2}$ are the determinants of the normal sheaves of

Z, Z_1, Z_2 in $\mathbb{P}^3_F$ respectively). The second of these isomorphisms is $\widetilde{o_1} \oplus \widetilde{o_2} : H^1(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2}) \oplus H^1(Z_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2}) \to H^1(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2}) \oplus H^1(Z_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2})$ (see Notation 3.25). The third of these isomorphisms is $\varphi_1^* \oplus \varphi_2^* : H^1(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2}) \oplus H^1(Z_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2}) \to H^1(\mathbb{P}^1_F, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2}) \oplus H^1(\mathbb{P}^1_F, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2})$. The last of these isomorphisms is the isomorphism $\varrho_{j_1+j_2+2} \oplus \varrho_{j_1+j_2+2} : H^1(\mathbb{P}^1_F, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2}) \oplus H^1(\mathbb{P}^1_F, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2}) \to \mathrm{W}(F) \oplus \mathrm{W}(F)$ (see Notation 3.48).

Future work 13 (Additional projective quadratic linking degree couples)**.** Since we do not yet have an explicit isomorphism $\vartheta_{n,j_1+j_2+n+1} : H^n(\mathbb{P}^n_F, \underline{K}^{\mathrm{MW}}_{j_1+j_2+n+1}) \to K^{\mathrm{MW}}_{j_1+j_2+1}(F)$ when $n \geq 3$ is odd (see Future work 3), we cannot define the quadratic linking degree couple for oriented links of type $(\mathbb{P}^n_F, \mathbb{P}^n_F, \mathbb{P}^{2n+1}_F)$ with $n \geq 3$ odd. When Future work 3 will be completed, we will be able to define the quadratic linking degree couple of an oriented link $\mathscr{L} = (Z_1, Z_2), (\overline{o_1}, \overline{o_2}), (\varphi_1, \varphi_2)$ of type $(\mathbb{P}^n_F, \mathbb{P}^n_F, \mathbb{P}^{2n+1}_F)$ with respect to a couple of integers $j_1, j_2 \leq -2$ as the image of the quadratic linking class of $\mathscr{L}$ with respect to (j_1, j_2) by the composite of the first three isomorphisms which were described in Definition 5.21 and of the composite of the isomorphism $\vartheta_{n,j_1+j_2+n+1} \oplus \vartheta_{n,j_1+j_2+n+1}$ and of the isomorphism $\gamma_{j_1+j_2+1} \oplus \gamma_{j_1+j_2+1}$ (see Theorem 2.33). When this future work and Future work 7 will be completed, it would be interesting to investigate the relationship between the ambient quadratic linking degree and the quadratic linking degree couple and especially to answer the following question: must the quadratic linking degree couple be zero when the ambient quadratic linking degree is zero? This last research lead is similar to the one in Future work 9.

Remark 5.22. Let $\mathscr{L} = (Z_1, Z_2), (\overline{o_1}, \overline{o_2}), (\varphi_1, \varphi_2)$ be an oriented link with two components of type $(\mathbb{P}^1_F, \mathbb{P}^1_F, \mathbb{P}^3_F)$ and $j_1' \leq j_1 \leq -2$ and $j_2' \leq j_2 \leq -2$ be integers. By Remark 4.11, denoting $Z := Z_1 \sqcup Z_2$, if the following diagram is commutative (which is verified for instance under the assumptions of Corollary 3.32):

$$\begin{array}{ccc} H^1(\mathbb{P}^3_F \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1+2}) \times H^1(\mathbb{P}^3_F \setminus Z, \underline{K}^{\mathrm{MW}}_{j_2+2}) & \xrightarrow{\cdot} & H^2(\mathbb{P}^3_F \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+4}) \\ \downarrow{\scriptstyle (\times \eta^{j_1-j_1'}, \times \eta^{j_2-j_2'})} & & \downarrow{\scriptstyle \times \eta^{j_1+j_2-(j_1'+j_2')}} \\ H^1(\mathbb{P}^3_F \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1'+2}) \times H^1(\mathbb{P}^3_F \setminus Z, \underline{K}^{\mathrm{MW}}_{j_2'+2}) & \xrightarrow{\cdot} & H^2(\mathbb{P}^3_F \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1'+j_2'+4}) \end{array}$$

then the oriented link $\mathscr{L}$ has a well-defined quadratic linking class $\mathrm{Qlc}_{\mathscr{L},j_1',j_2'} = \eta^{j_1+j_2-(j_1'+j_2')} \mathrm{Qlc}_{\mathscr{L},j_1,j_2}$, hence it has a well-defined quadratic linking degree $\mathrm{Qld}_{\mathscr{L},j_1',j_2'}$ which is equal to $\mathrm{Qld}_{\mathscr{L},j_1,j_2}$. Indeed, the isomorphisms ϖ, $\widetilde{o_1} \oplus \widetilde{o_2}$ and $\varphi_1^* \oplus \varphi_2^*$ commute to product by η and the isomorphism $\varrho_{j_1+j_2+2} \oplus \varrho_{j_1+j_2+2}$ is the composite of an isomorphism which commutes to product by η and of the isomorphism $\gamma_{j_1+j_2+1} \oplus \gamma_{j_1+j_2+1} : K^{\mathrm{MW}}_{j_1+j_2+1}(F) \oplus K^{\mathrm{MW}}_{j_1+j_2+1}(F) \to \mathrm{W}(F) \oplus \mathrm{W}(F)$ (recall Definition 5.21, Notation 3.48 and Theorem 2.33).

Let us now see what happens to the quadratic linking degree couple when the order of the components of the oriented link is reversed.

Proposition 5.23. Let $\mathscr{L} = (Z_1, Z_2), (\overline{o_1}, \overline{o_2}), (\varphi_1, \varphi_2)$ be an oriented link with two components of type $(\mathbb{P}^1_F, \mathbb{P}^1_F, \mathbb{P}^3_F)$, $j_1, j_2 \leq -2$ be integers and $(d_1, d_2) := \mathrm{Qld}_{\mathscr{L}, j_1, j_2}$. Let $\mathscr{L}'$ be the oriented link $(Z_2, Z_1), (\overline{o_2}, \overline{o_1}), (\varphi_2, \varphi_1)$ of type $(\mathbb{P}^1_F, \mathbb{P}^1_F, \mathbb{P}^3_F)$. Then $\mathrm{Qld}_{\mathscr{L}', j_2, j_1} = (-d_2, -d_1)$.

Proof. By Proposition 4.15:

$$\mathrm{Qlc}_{\mathscr{L}', j_2, j_1} = \begin{cases} -\epsilon \, \mathrm{Qlc}_{\mathscr{L}, j_1, j_2} & \text{if } (j_1 \text{ is odd or } j_2 \text{ is odd}) \\ -\, \mathrm{Qlc}_{\mathscr{L}, j_1, j_2} & \text{if } j_1 \text{ is even and } j_2 \text{ is even} \end{cases}$$

The result follows from the fact that the isomorphisms ϖ, $\widetilde{o_1} \oplus \widetilde{o_2}$ and $\varphi_1^* \oplus \varphi_2^*$ commute to product by ϵ and that the isomorphism $\varrho_{j_1+j_2+2} \oplus \varrho_{j_1+j_2+2}$ is the composite of an isomorphism which commutes to product by ϵ and of the isomorphism $\gamma_{j_1+j_2+1} \oplus \gamma_{j_1+j_2+1} : K^{\mathrm{MW}}_{j_1+j_2+1}(F) \oplus K^{\mathrm{MW}}_{j_1+j_2+1}(F) \to \mathrm{W}(F) \oplus \mathrm{W}(F)$ (recall Definition 5.21, Notation 3.48 and Theorem 2.33). □

Future work 14 (The two comp. of the projective quad. link. degree couple)**.** It would be interesting to determine the relationship (if there is one) between the first component of the quadratic linking degree couple and the second component of the quadratic linking degree couple (it would allow us for instance to make Proposition 5.23 more precise). The fact that the quadratic linking class is in the kernel of i_* (see Proposition 4.12) may be useful for this (since the corresponding fact in knot theory is useful to show that the first component of the linking couple is the second component of the linking couple up to a sign, see Remark 1.16 and its higher-dimensional generalisation Remark 1.33).

Similarly to the linking couple whose first component (respectively second component) stays the same and whose second component (resp. first component) is turned into its opposite if the orientation of the first component (resp. second component) of the oriented link is reversed (see Remark 1.36), the quadratic linking degree couple is changed in the following way by orientation changes. Recall Remark 4.43.

Proposition 5.24. Let $\mathscr{L} = (Z_1, Z_2), (\overline{o_1}, \overline{o_2}), (\varphi_1, \varphi_2)$ be an oriented link with two components of type $(\mathbb{P}^1_F, \mathbb{P}^1_F, \mathbb{P}^3_F)$ and $j_1, j_2 \leq -2$ be integers. Let $a = (a_1, a_2)$ be a couple of elements of F^*. Let $\mathscr{L}_a$ be the link obtained from

$\mathscr{L}$ by changing the orientation class $\overline{o_1}$ into $\overline{o_1 \circ (\times a_1)}$ and the orientation class $\overline{o_2}$ into $\overline{o_2 \circ (\times a_2)}$. Then, denoting $(d_1, d_2) := \mathrm{Qld}_{\mathscr{L},j_1,j_2}$, we have $\mathrm{Qld}_{\mathscr{L}_a,j_1,j_2} = (\langle a_2\rangle d_1, \langle a_1\rangle d_2)$.

Proof. By Proposition 4.16, $\mathrm{Qlc}_{\mathscr{L}_a,j_1,j_2} = \langle a_1 a_2\rangle\, \mathrm{Qlc}_{\mathscr{L},j_1,j_2}$ hence $\varpi(\mathrm{Qlc}_{\mathscr{L}_a,j_1,j_2}) = \langle a_1 a_2\rangle \varpi(\mathrm{Qlc}_{\mathscr{L},j_1,j_2})$ (see Definition 5.15). Thus, denoting $(\sigma_1, \sigma_2) := \varpi(\mathrm{Qlc}_{\mathscr{L},j_1,j_2})$, we have $\varpi(\mathrm{Qlc}_{\mathscr{L}_a,j_1,j_2}) = (\langle a_1a_2\rangle\sigma_1, \langle a_1a_2\rangle\sigma_2)$. Let $i \neq j \in \{1,2\}$. Note that $\widetilde{o_i \circ (\times a_i)}(\langle a_1a_2\rangle\sigma_i) = \langle a_i\rangle\widetilde{o_i}(\langle a_1a_2\rangle\sigma_i) = \langle a_i^2 a_j\rangle\widetilde{o_i}(\sigma_i) = \langle a_j\rangle\widetilde{o_i}(\sigma_i)$ (see Notation 3.25). Therefore, $\varphi_i^*(\widetilde{o_i \circ (\times a_i)}(\langle a_1a_2\rangle\sigma_i)) = \varphi_i^*(\langle a_j\rangle\widetilde{o_i}(\sigma_i)) = \langle a_j\rangle\varphi_i^*(\widetilde{o_i}(\sigma_i))$ (since $a_j \in F^*$). Since the i-th component of the quadratic linking degree couple $\mathrm{Qld}_{\mathscr{L},j_1,j_2}$ (respectively $\mathrm{Qld}_{\mathscr{L}_a,j_1,j_2}$) is obtained from $\varphi_i^*(\widetilde{o_i}(\sigma_i))$ (respectively $\langle a_j\rangle\varphi_i^*(\widetilde{o_i}(\sigma_i))$) by applying the isomorphism $\varrho_{j_1+j_2+2}$ (see Definition 5.21) and since this isomorphism is the composite of an isomorphism which commutes to product by $\langle a_j\rangle$ and of $\gamma_{j_1+j_2+1} : K^{\mathrm{MW}}_{j_1+j_2+1}(F) \to \mathrm{W}(F)$ (see Notation 3.48 and Theorem 2.33), the i-th component of $\mathrm{Qld}_{\mathscr{L}_a,j_1,j_2}$ is equal to $\langle a_j\rangle d_i$. □

Let us finally mention the following future works.

Future work 15 (Changes of parametrisations for $\mathbb{P}^n_F$)**.** It should be feasible to get a similar result for the case $(\mathbb{P}^1_F, \mathbb{P}^1_F, \mathbb{P}^3_F)$ (or more generally the case $(\mathbb{P}^n_F, \mathbb{P}^n_F, \mathbb{P}^{2n+1}_F)$ with $n \geq 1$ odd, see Future work 13) as the one in Proposition 5.20. Indeed, the group of automorphisms of $\mathbb{P}^n_F$ is isomorphic to $\mathrm{PGL}_n(F) := \mathrm{GL}_{n+1}(F)/F^*$ (where $\mathrm{GL}_{n+1}(F)$ is the group of invertible $(n+1)\times(n+1)$ matrices with coefficients in F; see [Har77, Example 7.1.1]). Therefore, the Jacobian determinant J_ψ of an automorphism ψ of $\mathbb{P}^n_F$ is well-defined up to multiplication by the $(n+1)$-th power of an element of F^*, hence $\langle J_\psi\rangle$ is well-defined if n is odd.

Future work 16 ($\mathbb{A}^1$-ambient isotopy in a projective setting)**.** It would be interesting to define a notion of $\mathbb{A}^1$-ambient isotopy for the projective case studied in this subsection (see Future work 12 for more details).

5.3 Invariants of the quadratic linking degree

By construction, the ambient quadratic linking degree (see Definition 5.7) and the quadratic linking degree couple (see Definitions 5.15 and 5.21) depend on choices of orientation classes $(\overline{o_1}, \overline{o_2})$ and the quadratic linking degree couple depends on choices of parametrisations (φ_1, φ_2). Recall that the ambient quadratic linking degree is in the Witt ring $\mathrm{W}(F)$ of the ground field F or in the Grothendieck-Witt ring $\mathrm{GW}(F)$ of the ground field F and that the quadratic linking degree couple is a couple whose components are each in the zero group 0 or in $\mathrm{W}(F)$ or in $\mathrm{GW}(F)$ or in the first Milnor-Witt K-theory group $K_1^{\mathrm{MW}}(F)$ of the ground field F.

In the first (respectively second) subsection, we construct functions on the Witt ring $\mathrm{W}(F)$ (resp. the Grothendieck-Witt ring $\mathrm{GW}(F)$) which are invariant by multiplication by $< a >$ (resp. by $\langle a \rangle$) for all $a \in F^*$. When applied to the ambient quadratic linking degree or to a component of the quadratic linking degree couple (in the cases for which it is in $\mathrm{W}(F)$ (resp. in $\mathrm{GW}(F)$)), these functions provide quantities which are invariant by changes of the orientation classes $(\overline{o_1}, \overline{o_2})$ (see Propositions 5.11, 5.18 and 5.24; note that even when applied to a component of the quadratic linking degree couple, these quantities are probably also invariant by changes of parametrisations (φ_1, φ_2) (see Future works 11 and 15)) and which we call invariants of the quadratic linking degree.

Throughout this section, F is a perfect field.

Cases in the Witt ring $\mathrm{W}(F)$

Let us begin with the easiest (nontrivial) invariant.

Proposition 5.25.

1. Let $\mathscr{L} = (Z_1, Z_2), (\overline{o_1}, \overline{o_2})$ be an oriented link with two components satisfying the assumptions of Definition 5.7 and $(j_1, j_2) \neq (0, 0)$ be a couple of nonpositive integers. The rank modulo 2 of the ambient quadratic linking degree of $\mathscr{L}$ with respect to (j_1, j_2) (which is in $\mathrm{W}(F)$) is invariant under changes of the orientation classes $\overline{o_1}, \overline{o_2}$.

2. Let $\mathscr{L} = (Z_1, Z_2), (\overline{o_1}, \overline{o_2}), (\varphi_1, \varphi_2)$ be an oriented link with two components of type (Y_1, Y_2, X), (j_1, j_2) be a couple of nonpositive integers and $i \in \{1, 2\}$ satisfying the assumptions of Definition 5.15 or of Definition 5.21 and such that the i-th component of the quadratic linking degree couple of $\mathscr{L}$ with respect to (j_1, j_2) is in the Witt ring $\mathrm{W}(F)$ of F. The rank modulo 2 of the i-th component of the quadratic linking degree couple of $\mathscr{L}$ with respect to (j_1, j_2) is invariant under changes of the orientation classes $\overline{o_1}, \overline{o_2}$.

Proof. The results follow directly from Propositions 5.11, 5.18 and 5.24 since the rank modulo 2 of an element of the Witt ring $\mathrm{W}(F)$ is invariant under the multiplication by $< a >$ for all $a \in F^*$. □

In Chapter 7, there are examples of oriented links whose ambient quadratic linking degree is of rank modulo 2 equal to 0 as well as examples of oriented links whose ambient quadratic linking degree is of rank modulo 2 equal to 1.

Let us now give an invariant when the ground field is the field $\mathbb{R}$ of real numbers. Recall that $\mathrm{W}(\mathbb{R}) \simeq \mathbb{Z}$ (via the signature).

Proposition 5.26. Assume that $F = \mathbb{R}$.

1. Let $\mathscr{L} = (Z_1, Z_2), (\overline{o_1}, \overline{o_2})$ be an oriented link with two components satisfying the assumptions of Definition 5.7 and $(j_1, j_2) \neq (0, 0)$ be a couple of nonpositive integers. The absolute value of the ambient quadratic linking degree of $\mathscr{L}$ with respect to (j_1, j_2) (which is in $\mathrm{W}(\mathbb{R})$ which is isomorphic to $\mathbb{Z}$ via the signature) is invariant under changes of the orientation classes $\overline{o_1}, \overline{o_2}$.

2. Let $\mathscr{L} = (Z_1, Z_2), (\overline{o_1}, \overline{o_2}), (\varphi_1, \varphi_2)$ be an oriented link with two components of type (Y_1, Y_2, X), (j_1, j_2) be a couple of nonpositive integers and $i \in \{1, 2\}$ satisfying the assumptions of Definition 5.15 or of Definition 5.21 and such that the i-th component of the quadratic linking degree couple of $\mathscr{L}$ with respect to (j_1, j_2) is in the Witt ring $\mathrm{W}(\mathbb{R})$ of $\mathbb{R}$. The absolute value of the i-th component of the quadratic linking degree couple of $\mathscr{L}$ with respect to (j_1, j_2) (which is in $\mathrm{W}(\mathbb{R}) \simeq \mathbb{Z}$) is invariant under changes of the orientation classes $\overline{o_1}, \overline{o_2}$.

Proof. For all $a \in \mathbb{R}^*$, $< a >=< 1 >= 1$ or $< a >=< -1 >= -1$ since every real number is a square or the opposite of a square. The results follow directly from Propositions 5.11, 5.18 and 5.24 since the absolute value of an element of the Witt ring $\mathrm{W}(\mathbb{R}) \simeq \mathbb{Z}$ is invariant under the multiplication by 1 and under the multiplication by -1. □

Note that no better invariant of the quadratic linking degree can be given in the case where the ground field is $\mathbb{R}$ since the signs of the ambient quadratic linking degree and of each component of the quadratic linking degree couple can be changed by changing $\overline{o_1}$ into $\overline{o_1 \circ (\times(-1))}$ or $\overline{o_2}$ into $\overline{o_2 \circ (\times(-1))}$ (see Propositions 5.11, 5.18 and 5.24).

In Section 7.3, we give an example for each $n \in \mathbb{N}$ of a link of ambient quadratic linking degree whose absolute value is n (and in Section 7.2 we give (among others) examples of links of ambient quadratic linking degree 0).

We will now give a family of invariants in the general case. Before we do this, we need the following lemma-definition which is an inductive definition. For each $d \in \mathrm{W}(F)$, with k ranging over the nonnegative even integers, we define an abelian group $Q_{d,k}$ and an element $\Sigma_k(d) \in Q_{d,k}$.

Lemma-Definition 5.27. Let $d \in \mathrm{W}(F)$. There exists a unique sequence of abelian groups $Q_{d,k}$ and of elements $\Sigma_k(d) \in Q_{d,k}$, where k ranges over the nonnegative even integers, such that:

- $Q_{d,0} = \mathrm{W}(F)$ and $\Sigma_0(d) = 1 \in Q_{d,0}$;
- for each positive even integer k, $Q_{d,k}$ is the quotient group $Q_{d,k-2}/(\Sigma_{k-2}(d))$;
- for each positive even integer k, $\Sigma_k(d) = \sum_{1\leq i_1<\cdots<i_k\leq n} < \prod_{1\leq j\leq k} a_{i_j} > \in Q_{d,k}$ as soon as $n \in \mathbb{N}_0$ and $a_1, \dots, a_n \in F^*$ verify that $\sum_{i=1}^{n} < a_i >= d$.

Proof. Recall the following presentation of the abelian group $\mathrm{W}(F)$ (see Theorem 2.17): its generators are the $< a >$ for $a \in F^*$ and its relations are the following:

1. $< ab^2 >=< a >$ for all $a, b \in F^*$;
2. $< a > + < b >=< a+b > + < (a+b)ab >$ for all $a, b \in F^*$ such that $a + b \neq 0$;
3. $< -1 > + < 1 >= 0$.

We denote by G the free abelian group of generators the $< a >$ for $a \in F^*$, by G_1 the quotient of G by the first relation above and by G_2 the quotient of G_1 by the second relation above.

Let k be a nonnegative even integer such that for all nonnegative even integers $l < k$, $Q_{d,l}$ is an abelian group and $\Sigma_l(d) \in Q_{d,l}$ which verify the conditions of the statement. Note that the quotient of the abelian group $Q_{d,k-2}$ by its subgroup $(\Sigma_{k-2}(d))$ is well-defined, so we can fix $Q_{d,k} = Q_{d,k-2}/(\Sigma_{k-2}(d))$. Let $n \in \mathbb{N}_0$ and $a_1, \dots, a_n \in F^*$ be such that the class of $\sum_{i=1}^{n} < a_i > \in G$ in $\mathrm{W}(F)$ is d. Note that $\sum_{1\leq i_1<\cdots<i_k\leq n} < \prod_{1\leq j\leq k} a_{i_j} > \in Q_{d,k}$ is well-defined (since it is well-defined in G and $Q_{d,k}$ is obtained from G by quotienting several times). In fact, $\sum_{1\leq i_1<\cdots<i_k\leq n} < \prod_{1\leq j\leq k} a_{i_j} > \in Q_{d,k}$ only depends on the class of $\sum_{i=1}^{n} < a_i >$ in G_1 since for all $b \in F^*$, $\sum_{2\leq i_2<\cdots<i_k\leq n} < a_1 b^2 \prod_{2\leq j\leq k} a_{i_j} > + \sum_{2\leq i_1<\cdots<i_k\leq n} < \prod_{1\leq j\leq k} a_{i_j} > = \sum_{1\leq i_1<\cdots<i_k\leq n} < \prod_{1\leq j\leq k} a_{i_j} > \in$

$Q_{d,k-2}$ (since this equality is already true in $\mathrm{W}(F)$ and $Q_{d,k}$ is obtained from $\mathrm{W}(F)$ by quotienting several times) and similarly for other indices.

Furthermore, $\displaystyle\sum_{1\leq i_1<\cdots<i_k\leq n} <\prod_{1\leq j\leq k} a_{i_j} >\in Q_{d,k}$ only depends on the class of $\displaystyle\sum_{i=1}^{n} < a_i >$ in G_2 since if $a_1+a_2\neq 0$ then in $Q_{d,k}$:

$$\sum_{3\leq i_3<\cdots<i_k\leq n} < (a_1+a_2)^2 a_1 a_2 \prod_{3\leq j\leq k} a_{i_j} > + \sum_{3\leq i_1<\cdots<i_k\leq n} < \prod_{1\leq j\leq k} a_{i_j} >$$
$$+ \sum_{3\leq i_2<\cdots<i_k\leq n} < (a_1+a_2) \prod_{2\leq j\leq k} a_{i_j} > + \sum_{3\leq i_2<\cdots<i_k\leq n} < (a_1+a_2)a_1a_2 \prod_{2\leq j\leq k} a_{i_j} >$$
$$= \sum_{3\leq i_3<\cdots<i_k\leq n} < a_1a_2 \prod_{3\leq j\leq k} a_{i_j} > + \sum_{3\leq i_1<\cdots<i_k\leq n} < \prod_{1\leq j\leq k} a_{i_j} >$$
$$+ (< a_1+a_2 > + < (a_1+a_2)a_1a_2 >) \sum_{3\leq i_2<\cdots<i_k\leq n} < \prod_{2\leq j\leq k} a_{i_j} >$$
$$= \sum_{3\leq i_3<\cdots<i_k\leq n} < a_1a_2 \prod_{3\leq j\leq k} a_{i_j} > + \sum_{3\leq i_1<\cdots<i_k\leq n} < \prod_{1\leq j\leq k} a_{i_j} >$$
$$+ (< a_1 > + < a_2 >) \sum_{3\leq i_2<\cdots<i_k\leq n} < \prod_{2\leq j\leq k} a_{i_j} >$$
$$= \sum_{1\leq i_1<\cdots<i_k\leq n} < \prod_{1\leq j\leq k} a_{i_j} >$$

(since these equalities are already true in $\mathrm{W}(F)$) and similarly for other indices.

Finally, $\displaystyle\sum_{1\leq i_1<\cdots<i_k\leq n} < \prod_{1\leq j\leq k} a_{i_j} >\in Q_{d,k}$ only depends on the class of $\displaystyle\sum_{i=1}^{n} < a_i >$ in $\mathrm{W}(F)$, i.e. on d, since, with the convention that $\displaystyle\sum_{1\leq i_3<\cdots<i_2\leq n} < \prod_{3\leq j\leq 2} a_{i_j} >= 1$:

$$\sum_{1\leq i_1<\cdots<i_k\leq n} < \prod_{1\leq j\leq k} a_{i_j} > + (< 1 > + < -1 >) \sum_{1\leq i_2<\cdots<i_k\leq n} < \prod_{2\leq j\leq k} a_{i_j} >$$
$$+ < -1 > \sum_{1\leq i_3<\cdots<i_k\leq n} < \prod_{3\leq j\leq k} a_{i_j} >$$

is equal to $\displaystyle\sum_{1\leq i_1<\cdots<i_k\leq n} < \prod_{1\leq j\leq k} a_{i_j} > -\Sigma_{k-2}(d)$ in $Q_{d,k}$ (since this equality is already true in $\mathrm{W}(F)$) which is equal to $\displaystyle\sum_{1\leq i_1<\cdots<i_k\leq n} < \prod_{1\leq j\leq k} a_{i_j} >$ in $Q_{d,k} =$

$Q_{d,k-2}/(\Sigma_{k-2}(d))$. Thus we can fix $\Sigma_k(d) = \sum\limits_{1\leq i_1<\cdots<i_k\leq n} < \prod\limits_{1\leq j\leq k} a_{i_j} >\in Q_{d,k}$ (since there exist $a_1,\ldots,a_n \in F^*$ such that $\sum\limits_{i=1}^{n} < a_i >= d \in \mathrm{W}(F)$). □

It follows from Lemma-Definition 5.27 that we have a map

$$\Sigma_k : \mathrm{W}(F) \to \bigcup_{d\in \mathrm{W}(F)} Q_{d,k}$$

which verifies that for all $d \in \mathrm{W}(F)$, $\Sigma_k(d) \in Q_{d,k}$. This provides new invariants of the quadratic linking degree.

Theorem 5.28.

1. Let $\mathscr{L} = (Z_1, Z_2), (\overline{o_1}, \overline{o_2})$ be an oriented link with two components satisfying the assumptions of Definition 5.7, $(j_1, j_2) \neq (0, 0)$ be a couple of nonpositive integers and k be a positive even integer. The image by Σ_k of the ambient quadratic linking degree of $\mathscr{L}$ with respect to (j_1, j_2) is invariant under changes of the orientation classes $\overline{o_1}, \overline{o_2}$.

2. Let $\mathscr{L} = (Z_1, Z_2), (\overline{o_1}, \overline{o_2}), (\varphi_1, \varphi_2)$ be an oriented link with two components of type (Y_1, Y_2, X), (j_1, j_2) be a couple of nonpositive integers and $i \in \{1, 2\}$ satisfying the assumptions of Definition 5.15 or of Definition 5.21 and such that the i-th component of the quadratic linking degree couple of $\mathscr{L}$ with respect to (j_1, j_2) is in the Witt ring $\mathrm{W}(F)$ of F. Let k be a positive even integer. The image by Σ_k of the i-th component of the quadratic linking degree couple of $\mathscr{L}$ with respect to (j_1, j_2) is invariant under changes of the orientation classes $\overline{o_1}, \overline{o_2}$.

Proof. First, let us show that Σ_k is invariant under the multiplication by $< b >$ for all $b \in F^*$. Let $\sum\limits_{1\leq i\leq n} < a_i >\in \mathrm{W}(F)$. For all $b \in F^*$, $\Sigma_k(< b > \sum_{1\leq i\leq n} < a_i >) = \sum_{1\leq i_1<\cdots<i_k\leq n} < b^k \prod_{1\leq j\leq k} a_{i_j} >$ hence (b^k being a square since k is even) $\Sigma_k(< b > \sum_{1\leq i\leq n} < a_i >) = \sum_{1\leq i_1<\cdots<i_k\leq n} < \prod_{1\leq j\leq k} a_{i_j} >= \Sigma_k(\sum_{1\leq i\leq n} < a_i >)$. The results follow directly from this and from Propositions 5.11, 5.18 and 5.24. □

Note that even though these invariants are not interesting for some fields, e.g. the field $\mathbb{R}$ of real numbers (which verifies that for all $d \in \mathrm{W}(\mathbb{R})$, $Q_{d,2} = \mathrm{W}(\mathbb{R})/(1) = 0$ since $\mathrm{W}(\mathbb{R}) \simeq \mathbb{Z}$, hence all $Q_{d,k} = 0$ and all $\Sigma_k(d) = 0$ as soon as $k > 0$), they are interesting for other fields, e.g. the field $\mathbb{Q}$ of rational numbers. Indeed, in Section 7.2, we show that Σ_2 (applied to the ambient quadratic linking degree or to a component of the quadratic linking degree couple (over $\mathbb{Q}$)) distinguishes between infinitely many oriented links.

Before we move on to the cases in which the quadratic linking degrees are in $\mathrm{GW}(F)$ instead of being in $\mathrm{W}(F)$, let us make the following remark.

Remark 5.29. Let $p : \mathrm{GW}(F) \to \mathrm{W}(F)$ be the canonical morphism (which sends $\langle a \rangle$ to $< a >$ for all $a \in F^*$) and k be a positive even integer.

1. The composite of p and of the rank modulo 2 is invariant under the multiplication by $\langle a \rangle$ for all $a \in F^*$.
2. In the case $F = \mathbb{R}$, the composite of p (of the signature) and of the absolute value is invariant under the multiplication by $\langle a \rangle$ for all $a \in F^*$.
3. The composite of p and of Σ_k is invariant under the multiplication by $\langle a \rangle$ for all $a \in F^*$.

It follows from Propositions 5.11, 5.18 and 5.24 that these functions (applied to the ambient quadratic linking degree or to a component of the quadratic linking degree couple) provide invariants of the quadratic linking degree when it is in $\mathrm{GW}(F)$. However, we will provide better invariants than each of these in the following subsection.

Cases in the Grothendieck-Witt ring $\mathrm{GW}(F)$

Let us begin with the invariant which is a better version of the invariant which stems from the invariant of Proposition 5.25 (see Remark 5.29).

Proposition 5.30.

1. Let $\mathscr{L} = (Z_1, Z_2), (\overline{o_1}, \overline{o_2})$ be an oriented link with two components satisfying the assumptions of Definition 5.7. The rank of the ambient quadratic linking degree of $\mathscr{L}$ with respect to $(0, 0)$ (which is in $\mathrm{GW}(F)$) is invariant under changes of the orientation classes $\overline{o_1}, \overline{o_2}$.
2. Let $\mathscr{L} = (Z_1, Z_2), (\overline{o_1}, \overline{o_2}), (\varphi_1, \varphi_2)$ be an oriented link with two components of type (Y_1, Y_2, X), (j_1, j_2) be a couple of nonpositive integers and $i \in \{1, 2\}$ satisfying the assumptions of Definition 5.15 or of Definition 5.21 and such that the i-th component of the quadratic linking degree couple of $\mathscr{L}$ with respect to (j_1, j_2) is in the Grothendieck-Witt ring $\mathrm{GW}(F)$ of F. The rank of the i-th component of the quadratic linking degree couple of $\mathscr{L}$ with respect to (j_1, j_2) is invariant under changes of the orientation classes $\overline{o_1}, \overline{o_2}$.

Proof. These results follow directly from Propositions 5.11, 5.18 and 5.24 since the rank of an element of the Grothendieck-Witt ring $\mathrm{GW}(F)$ is invariant under the multiplication by $\langle a\rangle$ for all $a \in F^*$. □

Let us now give the invariant which is a better version of the invariant which stems from the invariant of Proposition 5.26 (when the ground field is the field $\mathbb{R}$ of real numbers; see Remark 5.29). Recall that $\mathrm{GW}(\mathbb{R}) \simeq \mathbb{Z}\oplus\mathbb{Z}$ (via the signature couple).

Proposition 5.31. Assume that $F = \mathbb{R}$.

1. Let $\mathscr{L} = (Z_1, Z_2), (\overline{o_1}, \overline{o_2})$ be an oriented link with two components satisfying the assumptions of Definition 5.7. The unordered pair (i.e. set of two elements) which underlies the ambient quadratic linking degree of $\mathscr{L}$ with respect to $(0,0)$ (which is in $\mathrm{GW}(\mathbb{R})$ which is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}$ via the signature couple) is invariant under changes of the orientation classes $\overline{o_1}, \overline{o_2}$.

2. Let $\mathscr{L} = (Z_1, Z_2), (\overline{o_1}, \overline{o_2}), (\varphi_1, \varphi_2)$ be an oriented link with two components of type (Y_1, Y_2, X), (j_1, j_2) be a couple of nonpositive integers and $i \in \{1, 2\}$ satisfying the assumptions of Definition 5.15 or of Definition 5.21 and such that the i-th component of the quadratic linking degree couple of $\mathscr{L}$ with respect to (j_1, j_2) is in the Grothendieck-Witt ring $\mathrm{GW}(\mathbb{R})$ of $\mathbb{R}$. The unordered pair (i.e. set of two elements) which underlies the i-th component of the quadratic linking degree couple of $\mathscr{L}$ with respect to (j_1, j_2) (which is in $\mathrm{GW}(\mathbb{R}) \simeq \mathbb{Z}\oplus\mathbb{Z}$) is invariant under changes of the orientation classes $\overline{o_1}, \overline{o_2}$.

Proof. For all $a \in \mathbb{R}^*$, $\langle a\rangle = \langle 1\rangle = 1$ or $\langle a\rangle = \langle -1\rangle$ since every real number is a square or the opposite of a square. The results follow directly from Propositions 5.11, 5.18 and 5.24 since in $\mathrm{GW}(\mathbb{R}) \simeq \mathbb{Z}\oplus\mathbb{Z}$, multiplication by 1 is the identity and multiplication by $\langle -1\rangle$ is the function which maps (a, b) to (b, a) for all $a, b \in \mathbb{Z}$ (and the sets $\{a, b\}$ and $\{b, a\}$ are equal). □

Note that no better invariant of the quadratic linking degree can be given in the case where the ground field is $\mathbb{R}$ since the components of the ambient quadratic linking degree and of each component of the quadratic linking degree couple can be switched by changing $\overline{o_1}$ into $\overline{o_1 \circ (\times(-1))}$ or $\overline{o_2}$ into $\overline{o_2 \circ (\times(-1))}$ (see Propositions 5.11, 5.18 and 5.24).

We will now give a family of invariants in the general case (which is a better version of the family of invariants which stems from the family of invariants of Theorem 5.28; see Remark 5.29). Before we do this, we need the following lemma-definition.

Lemma-Definition 5.32. Let k be a positive even integer. The map $\Sigma_k :$ $\mathrm{GW}(F) \to \mathrm{GW}(F)$ which maps d to $\Sigma_k(d) = \sum_{1\leq i_1<\cdots<i_k\leq n} (\prod_{1\leq l\leq k} \varepsilon_{i_l})\langle \prod_{1\leq j\leq k} a_{i_j}\rangle$, as soon as $n \in \mathbb{N}_0$, $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ and $a_1, \ldots, a_n \in F^*$ verify that $\sum_{i=1}^{n} \varepsilon_i \langle a_i\rangle = d$, is well-defined.

Proof. Recall the following presentation of the abelian group $\mathrm{GW}(F)$ (see Theorem 2.13): its generators are the $\langle a\rangle$ for $a \in F^*$ and its relations are the following:

1. $\langle ab^2\rangle = \langle a\rangle$ for all $a, b \in F^*$;

2. $\langle a\rangle + \langle b\rangle = \langle a + b\rangle + \langle (a + b)ab\rangle$ for all $a, b \in F^*$ such that $a + b \neq 0$.

We denote by G the free abelian group of generators the $\langle a\rangle$ for $a \in F^*$ and by G_1 the quotient of G by the first relation above. Let $d \in \mathrm{GW}(F)$ and $n \in \mathbb{N}_0$, $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$ and $a_1, \ldots, a_n \in F^*$ be such that the class of $\sum_{i=1}^{n} \varepsilon_i\langle a_i\rangle \in G$ in $\mathrm{GW}(F)$ is d. Let k be a positive even integer. Note that $\sum_{1\leq i_1<\cdots<i_k\leq n} (\prod_{1\leq l\leq k} \varepsilon_{i_l})\langle \prod_{1\leq j\leq k} a_{i_j}\rangle \in \mathrm{GW}(F)$ is well-defined (since it is well-defined in G and $\mathrm{GW}(F)$ is obtained from G by quotienting). In fact, $\sum_{1\leq i_1<\cdots<i_k\leq n} (\prod_{1\leq l\leq k} \varepsilon_{i_l})\langle \prod_{1\leq j\leq k} a_{i_j}\rangle \in \mathrm{GW}(F)$ only depends on the class of $\sum_{i=1}^{n} \varepsilon_i\langle a_i\rangle$ in G_1 since for all $b \in F^*$, $\sum_{2\leq i_2<\cdots<i_k\leq n} \varepsilon_1(\prod_{2\leq l\leq k} \varepsilon_{i_l})\langle a_1 b^2 \prod_{2\leq j\leq k} a_{i_j}\rangle +$ $\sum_{2\leq i_1<\cdots<i_k\leq n} (\prod_{1\leq l\leq k} \varepsilon_{i_l})\langle \prod_{1\leq j\leq k} a_{i_j}\rangle = \sum_{1\leq i_1<\cdots<i_k\leq n} (\prod_{1\leq l\leq k} \varepsilon_{i_l})\langle \prod_{1\leq j\leq k} a_{i_j}\rangle \in \mathrm{GW}(F)$ and similarly for other indices.

Finally, $\sum_{1\leq i_1<\cdots<i_k\leq n} (\prod_{1\leq l\leq k} \varepsilon_{i_l})\langle \prod_{1\leq j\leq k} a_{i_j}\rangle \in \mathrm{GW}(F)$ only depends on the class of $\sum_{i=1}^{n} \varepsilon_i\langle a_i\rangle$ in $\mathrm{GW}(F)$, i.e. on d, since if $a_1 + a_2 \neq 0$ and $\varepsilon_1 = \varepsilon_2 =: \varepsilon$

then in GW(F):

$$\begin{aligned}
&\sum_{3\leq i_3<\cdots<i_k\leq n}(\prod_{3\leq l\leq k}\varepsilon_{i_l})\langle(a_1+a_2)^2a_1a_2\prod_{3\leq j\leq k}a_{i_j}\rangle+\sum_{3\leq i_1<\cdots<i_k\leq n}(\prod_{1\leq l\leq k}\varepsilon_{i_l})\langle\prod_{1\leq j\leq k}a_{i_j}\rangle\\
&+\sum_{3\leq i_2<\cdots<i_k\leq n}\varepsilon(\prod_{2\leq l\leq k}\varepsilon_{i_l})(\langle(a_1+a_2)\prod_{2\leq j\leq k}a_{i_j}\rangle+\langle(a_1+a_2)a_1a_2\prod_{2\leq j\leq k}a_{i_j}\rangle)\\
&=\sum_{3\leq i_3<\cdots<i_k\leq n}(\prod_{3\leq l\leq k}\varepsilon_{i_l})\langle a_1a_2\prod_{3\leq j\leq k}a_{i_j}\rangle+\sum_{3\leq i_1<\cdots<i_k\leq n}(\prod_{1\leq l\leq k}\varepsilon_{i_l})\langle\prod_{1\leq j\leq k}a_{i_j}\rangle\\
&+(\langle a_1+a_2\rangle+\langle(a_1+a_2)a_1a_2\rangle)\sum_{3\leq i_2<\cdots<i_k\leq n}\varepsilon(\prod_{2\leq l\leq k}\varepsilon_{i_l})\langle\prod_{2\leq j\leq k}a_{i_j}\rangle\\
&=\sum_{3\leq i_3<\cdots<i_k\leq n}(\prod_{3\leq l\leq k}\varepsilon_{i_l})\langle a_1a_2\prod_{3\leq j\leq k}a_{i_j}\rangle+\sum_{3\leq i_1<\cdots<i_k\leq n}(\prod_{1\leq l\leq k}\varepsilon_{i_l})\langle\prod_{1\leq j\leq k}a_{i_j}\rangle\\
&+(\langle a_1\rangle+\langle a_2\rangle)\sum_{3\leq i_2<\cdots<i_k\leq n}\varepsilon(\prod_{2\leq l\leq k}\varepsilon_{i_l})\langle\prod_{2\leq j\leq k}a_{i_j}\rangle\\
&=\sum_{1\leq i_1<\cdots<i_k\leq n}(\prod_{1\leq l\leq k}\varepsilon_{i_l})\langle\prod_{1\leq j\leq k}a_{i_j}\rangle
\end{aligned}$$

and similarly for other indices. □

This provides new invariants of the quadratic linking degree.

Theorem 5.33.

1. Let $\mathscr{L}=(Z_1,Z_2),(\overline{o_1},\overline{o_2})$ be an oriented link with two components satisfying the assumptions of Definition 5.7 and k be a positive even integer. The image by Σ_k of the ambient quadratic linking degree of $\mathscr{L}$ with respect to $(0,0)$ is invariant under changes of the orientation classes $\overline{o_1},\overline{o_2}$.

2. Let $\mathscr{L}=(Z_1,Z_2),(\overline{o_1},\overline{o_2}),(\varphi_1,\varphi_2)$ be an oriented link with two components of type (Y_1,Y_2,X), (j_1,j_2) be a couple of nonpositive integers and $i\in\{1,2\}$ satisfying the assumptions of Definition 5.15 or of Definition 5.21 and such that the i-th component of the quadratic linking degree couple of $\mathscr{L}$ with respect to (j_1,j_2) is in the Grothendieck-Witt ring GW(F) of F. Let k be a positive even integer. The image by Σ_k of the i-th component of the quadratic linking degree couple of $\mathscr{L}$ with respect to (j_1,j_2) is invariant under changes of the orientation classes $\overline{o_1},\overline{o_2}$.

Proof. First, let us show that Σ_k is invariant under the multiplication by $\langle b\rangle$ for all $b \in F^*$. Let $\sum_{1\leq i\leq n} \varepsilon_i\langle a_i\rangle \in \mathrm{GW}(F)$. For all $b \in F^*$:

$$\begin{aligned}\Sigma_k(\langle b\rangle \sum_{1\leq i\leq n} \varepsilon_i\langle a_i\rangle) &= \sum_{1\leq i_1<\cdots<i_k\leq n} (\prod_{1\leq l\leq k} \varepsilon_{i_l})\langle b^k \prod_{1\leq j\leq k} a_{i_j}\rangle \\ &= \sum_{1\leq i_1<\cdots<i_k\leq n} (\prod_{1\leq l\leq k} \varepsilon_{i_l})\langle \prod_{1\leq j\leq k} a_{i_j}\rangle \\ &= \Sigma_k(\sum_{1\leq i\leq n} \varepsilon_i\langle a_i\rangle)\end{aligned}$$

since b^k is a square as k is even. The results follow directly from this and from Propositions 5.11, 5.18 and 5.24. □

Remark 5.34. Let $p : K_1^{\mathrm{MW}}(F) \to \mathrm{GW}(F)$ be the composite of the morphism $K_1^{\mathrm{MW}}(F) \to K_0^{\mathrm{MW}}(F)$ which is the multiplication by η and of the ring isomorphism $\gamma_0 : K_0^{\mathrm{MW}}(F) \to \mathrm{GW}(F)$ (see Theorem 2.33). Let k be a positive even integer.

1. The composite of p and of the rank is invariant under the multiplication by $\langle a\rangle$ for all $a \in F^*$.

2. In the case $F = \mathbb{R}$, the composite of p (of the signature couple) and of the map $(a, b) \mapsto \{a, b\}$ is invariant under the multiplication by $\langle a\rangle$ for all $a \in F^*$.

3. The composite of p and of Σ_k is invariant under the multiplication by $\langle a\rangle$ for all $a \in F^*$.

It follows from Propositions 5.18 and 5.24 that these functions (applied to a component of the quadratic linking degree couple) provide invariants of the quadratic linking degree when it is in $K_1^{\mathrm{MW}}(F)$ (however the first one is the trivial invariant since for all $a \in F^*$, $\gamma_0(\eta[a]) = \langle a\rangle - \langle 1\rangle$ is of rank 0).

Future work 17 (Invariants of the quad. link. degree in the case $K_1^{\mathrm{MW}}(F)$)**.** It would be interesting to devise better invariants of the quadratic linking degree for the cases in $K_1^{\mathrm{MW}}(F)$. Note that [Mor12, Definition 3.3 and Lemma 3.4] give a presentation of the abelian group $K_n^{\mathrm{MW}}(F)$ for $n \geq 1$ (in particular of $K_1^{\mathrm{MW}}(F)$) which may be useful for this. Also note that [Mor03, Theorem 6.4.5] gives, when the characteristic of F is different from 2, an isomorphism of graded rings between $K_*^{\mathrm{MW}}(F)$ and another graded ring which is constructed from Milnor K-theory (see Definition 3.1) and Witt K-theory. (Note that Witt K-theory $K_*^{\mathrm{W}}(F)$ verifies that for all $m \leq 0$,

$K_m^{\mathrm{W}}(F) = \mathrm{W}(F)$ and for all $n \geq 1$, $K_n^{\mathrm{W}}(F) = I(F)^n$, where $I(F)$ is the kernel of the ring morphism $\mathrm{W}(F) \to \mathbb{Z}/2\mathbb{Z}$ induced by the rank.) This isomorphism gives for instance the following isomorphism between the abelian group $K_1^{\mathrm{MW}}(\mathbb{R})$ and the fibre product of abelian groups $\mathbb{R}^* \times_{\mathbb{R}^*/(\mathbb{R}^*)^2 \simeq 2\mathbb{Z}/4\mathbb{Z}} 2\mathbb{Z}$ (which is the set of couples $(x, n) \in \mathbb{R}^* \times 2\mathbb{Z}$ such that either $x > 0$ and n is a multiple of 4 or $x < 0$ and n is not a multiple of 4 (hence is congruent to 2 modulo 4 since it is even) with the addition $(x, n) + (y, m) = (xy, n + m)$):

$$\sum_{i \in I} \varepsilon_i [a_i] \mapsto \sum_{i \in I} \varepsilon_i (a_i, -2\chi^{\mathrm{neg}}(a_i))$$

where I is a finite set, $\varepsilon_i \in \{-1, 1\}$, $a_i \in \mathbb{R}^*$ and χ^{neg} is the characteristic function of the negative numbers (i.e. χ^{neg} maps negative numbers to 1 and other numbers to 0; note that $-2\chi^{\mathrm{neg}}(a_i)$ is necessarily even and is a multiple of 4 precisely when $a_i > 0$). Note that $-(a_i, -2\chi^{\mathrm{neg}}(a_i)) = (a_i^{-1}, 2\chi^{\mathrm{neg}}(a_i))$. The inverse isomorphism (from $\mathbb{R}^* \times_{\mathbb{R}^*/(\mathbb{R}^*)^2 \simeq 2\mathbb{Z}/4\mathbb{Z}} 2\mathbb{Z}$ to $K_1^{\mathrm{MW}}(\mathbb{R})$) is the following (where $x \in \mathbb{R}^*$ (and $|x|$ is its absolute value) and $k \in \mathbb{Z}$ (even if $x > 0$, odd otherwise)):

$$(x, 2k) \mapsto \begin{cases} [x] & \text{if } k = 0 \\ -k[-|x|^{-\frac{1}{k}}] & \text{otherwise} \end{cases}$$

Chapter 6

Computing methods

In this chapter, we give methods to compute the quadratic linking class (see Definition 4.9), the ambient quadratic linking degree (see Definition 5.7) and the quadratic linking degree couple (see Definition 5.15) in the case $\mathbb{A}^2_F \setminus \{0\} \sqcup \mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$ with $j_1 \leq -1$ and $j_2 \leq -1$, under reasonable assumptions on the oriented link (which are verified in the examples of Chapter 7). Similar methods can be worked out for the following cases:

- the quadratic linking class, the ambient quadratic linking degree and the quadratic linking degree couple for oriented links with two components of type $(\mathbb{A}^2_F \setminus \{0\}, Q_2, \mathbb{A}^4_F \setminus \{0\})$, $(Q_2, \mathbb{A}^2_F \setminus \{0\}, \mathbb{A}^4_F \setminus \{0\})$, or $(Q_2, Q_2, \mathbb{A}^4_F \setminus \{0\})$ with $j_1 \leq -1$ and $j_2 \leq -1$ (recall Definitions 4.1 and 5.13);
- the quadratic linking class and the ambient quadratic linking degree for oriented links with two components of dimension 2 in $\mathbb{A}^4_F \setminus \{0\}$, with $j_1 \leq -1$ and $j_2 \leq -1$ (recall Definition 5.7);
- the quadratic linking class and the quadratic linking degree couple for oriented links with two components of type $(\mathbb{A}^3_F \setminus \{0\}, Q_3, \mathbb{A}^5_F \setminus \{0\})$, $(Q_3, \mathbb{A}^3_F \setminus \{0\}, \mathbb{A}^5_F \setminus \{0\})$, $(Q_3, Q_3, \mathbb{A}^5_F \setminus \{0\})$ with $j_1 \leq -1$ and $j_2 \leq -1$;
- the quadratic linking class and the quadratic linking degree couple for oriented links with two components of type $(\mathbb{P}^1_F, \mathbb{P}^1_F, \mathbb{P}^3_F)$ with $j_1 \leq -2$ and $j_2 \leq -2$ (recall Definition 5.21 and Remark 4.28));
- the quadratic linking class and the quadratic linking degree couple (when they are well-defined, which should be the case under reasonable assumptions on the oriented link) for oriented links with two components of type (Q_2, Q_2, Q_4) or (Q_3, Q_3, Q_5) with $j_1 \leq -1$ and $j_2 \leq -1$.

The codimension 2 assumption and the assumption that $j_1 \leq -1$ and $j_2 \leq -1$ which are in all these cases come from the fact that the method uses the formula to compute the intersection product which is in Corollary 3.32 on the couple of Seifert classes of the oriented link (see Definition 4.6). Note that in the case $(\mathbb{P}^1_F, \mathbb{P}^1_F, \mathbb{P}^3_F)$, it is assumed that $j_1 \leq -2$ and $j_2 \leq -2$ in order to have a well-defined quadratic linking class in a nonzero group. Note that under these assumptions, all of the ambient quadratic linking degrees and the quadratic linking degree couples take values in the Witt group $\mathrm{W}(F)$ of the perfect field F or in $\mathrm{W}(F) \oplus \mathrm{W}(F)$.

By Remarks 4.4 (recall Definition 4.3), 4.8, 4.11, 5.3 (recall Definition 5.1), 5.8, 5.16 and 5.22, in the cases above it suffices to give computing methods for $j_1 = -1$ and $j_2 = -1$ (except for the case $(\mathbb{P}^1_F, \mathbb{P}^1_F, \mathbb{P}^3_F)$, for which it suffices to give computing methods for $j_1 = -2$ and $j_2 = -2$).

As soon as Future work 1 is (at least partially) completed, more cases become effectively computable (in particular, cases with ambient quadratic linking degree in $\mathrm{GW}(F)$ or with a component of the quadratic linking degree couple in $\mathrm{GW}(F)$ or in $K_1^{\mathrm{MW}}(F)$).

Section 6.1 gives the assumptions and notations under which we can effectively compute the quadratic linking class and the ambient quadratic linking class of an oriented link $(Z_1 \subset \mathbb{A}^4_F \setminus \{0\}, Z_2 \subset \mathbb{A}^4_F \setminus \{0\}), (\overline{o_1}, \overline{o_2})$ such that $Z_1 \simeq \mathbb{A}^2_F \setminus \{0\}$ and $Z_2 \simeq \mathbb{A}^2_F \setminus \{0\}$. In Section 6.2, we compute the quadratic linking class with respect to $(-1, -1)$ and the ambient quadratic linking class with respect to $(-1, -1)$ of an oriented link which verifies the assumptions of Section 6.1. In Section 6.3, we compute the ambient quadratic linking degree with respect to $(-1, -1)$ of an oriented link which verifies the assumptions of Section 6.1. Finally, in Section 6.4, under the additional assumption that a choice of parametrisations (i.e. isomorphisms) $\varphi_1 : \mathbb{A}^2_F \setminus \{0\} \to Z_1$ and $\varphi_2 : \mathbb{A}^2_F \setminus \{0\} \to Z_2$ has been made, we compute the quadratic linking degree couple with respect to $(-1, -1)$ of an oriented link which verifies the assumptions of Section 6.1. Note that we also included these computing methods in our preprint [Lem23].

6.1 Assumptions and notations

In this section, we give the assumptions under which we will compute the quadratic linking class and the ambient quadratic linking class (in Section 6.2), the ambient quadratic linking degree (in Section 6.3) and the quadratic linking degree couple (in Section 6.4), as well as useful notations.

We assume that:

- F is a perfect field and $X = \mathbb{A}^4_F \setminus \{0\}$, where $\mathbb{A}^4_F = \mathrm{Spec}(F[x,y,z,t])$ (so that coordinates are fixed once and for all);
- Z_1 and Z_2 are disjoint closed F-subschemes of X and are isomorphic to $\mathbb{A}^2_F \setminus \{0\}$, where $\mathbb{A}^2_F = \mathrm{Spec}(F[u,v])$; we denote $Z := Z_1 \sqcup Z_2$;
- for each $i \in \{1,2\}$, the closure $\overline{Z_i}$ of Z_i in $\mathbb{A}^4_F$ is given by two equations

$$f_i(x,y,z,t) = 0, g_i(x,y,z,t) = 0$$

  where f_i and g_i are irreducible polynomials in $F[x,y,z,t]$;
- $\mathscr{L}$ is the oriented link $(Z_1, Z_2), (\overline{o_1} := \overline{o_{f_1,g_1}}, \overline{o_2} := \overline{o_{f_2,g_2}})$ (see below for the definition of the oriented classes $\overline{o_{f_1,g_1}}$ and $\overline{o_{f_2,g_2}}$);
- the subscheme of $X \setminus Z$ given by the equations $g_1 = 0, g_2 = 0$ is of codimension 2 in $X \setminus Z$;
- for each generic point p of an irreducible component of the subscheme of $X \setminus Z$ given by the equations $g_1 = 0, g_2 = 0$, the images of f_1 and of f_2 in the residue field $\kappa(p)$ (by the composite of the canonical morphism $F[x,y,z,t] = \mathcal{O}_{\mathbb{A}^4_F}(\mathbb{A}^4_F) \to \mathcal{O}_{\mathbb{A}^4_F,p}$ and of the canonical morphism $\mathcal{O}_{\mathbb{A}^4_F,p} \to \kappa(p)$) are units.

These last two assumptions are here to ensure that we can use the formula for the intersection product in Corollary 3.32 on the couple of Seifert classes of the oriented link $(Z_1, Z_2), (\overline{o_{f_1,g_1}}, \overline{o_{f_2,g_2}})$.

Let $i \in \{1,2\}$. Note that the conormal sheaf $\mathcal{C}_{Z_i/X} := \mathscr{I}_{Z_i}/\mathscr{I}^2_{Z_i}$ of Z_i in X (where $\mathscr{I}_{Z_i}$ is the ideal sheaf of Z_i in X) fits in the following short exact sequence (see [Ful98, Paragraph B.7.4] for a different formulation):

$$0 \longrightarrow (\mathcal{C}_{V(g_i)/\mathbb{A}^4_F})_{|Z_i} \longrightarrow \mathcal{C}_{Z_i/X} = (\mathcal{C}_{\overline{Z_i}/\mathbb{A}^4_F})_{|Z_i} \longrightarrow (\mathcal{C}_{V(f_i)/\mathbb{A}^4_F})_{|Z_i} \longrightarrow 0$$

so that the determinant of the dual of the conormal sheaf of Z_i in X, which we denote ν_{Z_i}, is canonically isomorphic to $\det(((\mathcal{C}_{V(f_i)/\mathbb{A}^4_F})_{|Z_i})^\vee) \otimes \det(((\mathcal{C}_{V(g_i)/\mathbb{A}^4_F})_{|Z_i})^\vee)$. We define o_{f_i,g_i} as the isomorphism $\nu_{Z_i} \to \mathcal{O}_{Z_i} \otimes \mathcal{O}_{Z_i}$ which maps $\overline{f_i}^* \wedge \overline{g_i}^*$ to $1 \otimes 1$ and $\overline{o_{f_i,g_i}}$ as the orientation class of o_{f_i,g_i} (see Definition 3.22).

Note that if we want to compute the quadratic linking class of $\mathscr{L} = (Z_1, Z_2), (\overline{o_1}, \overline{o_2})$ for some orientation classes $\overline{o_1}, \overline{o_2}$ rather than $\mathscr{L} = (Z_1, Z_2), (\overline{o_{f_1,g_1}}, \overline{o_{f_2,g_2}})$, we simply need to find $a_i \in F^*$ such that $\overline{o_i} = o_{f_i,g_i} \circ (\times a_i)$

(such an a_i exists by Remark 4.36) and use Proposition 4.16 (see also Propositions 5.6, 5.11 and 5.18). Also note that by definition $\overline{o_{f_i,g_i} \circ (\times a_i)} = \overline{o_{a_i^{-1} f_i, g_i}}$, so that $\overline{o_i} = \overline{o_{a_i^{-1} f_i, g_i}}$, and that if (p_i, q_i) is another couple of irreducible polynomials such that $\overline{Z_i}$ is given by the equations $p_i = 0, q_i = 0$ then $\overline{o_{p_i,q_i}} = \overline{o_{f_i,g_i} \circ (\times (J_i)^{-1})}$ with J_i the determinant of the 2×2 matrix A_i such that $A_i \begin{pmatrix} f_i \\ g_i \end{pmatrix} = \begin{pmatrix} p_i \\ q_i \end{pmatrix}$ (note that the coefficients of A_i are in $F[x, y, z, t]$ but $J_i \in (F[x, y, z, t])^* = F^*$ since A_i is invertible).

Now that we have presented our assumptions, let us turn to notations. We recall the following notations:

- we denote by $\chi^{\mathrm{odd}} : \mathbb{Z} \to \{0, 1\}$ the characteristic function of the set of odd numbers (i.e. $\chi^{\mathrm{odd}}(n) = 1$ if n is odd, $\chi^{\mathrm{odd}}(n) = 0$ if n is even);
- we denote $\epsilon := -\langle -1 \rangle \in K_0^{\mathrm{MW}}(F)$;
- for all $n \in \mathbb{N}_0$, we denote $n_\epsilon := \sum_{i=1}^{n} \langle (-1)^{i-1} \rangle \in K_0^{\mathrm{MW}}(F)$ and we denote $(-n)_\epsilon := \epsilon\, n_\epsilon \in K_0^{\mathrm{MW}}(F)$.

Let us now introduce notations which will be useful to explicitly compute the quadratic linking class, the ambient quadratic linking class, the ambient quadratic linking degree and the quadratic linking degree couple. Note that these will not depend on the choices of uniformizing parameters made below (see Definitions 4.9, 5.1, 5.7 and 5.15).

- We denote by I the **set** of generic points of irreducible components of the subscheme of $X \setminus Z$ given by the equations $g_1 = 0, g_2 = 0$.
- For every $p \in I$, we denote by π_p a **uniformizing parameter** of the discrete valuation ring $\mathcal{O}_{X \setminus Z, p}/(g_1)$, by u_p a **unit** in $\mathcal{O}_{X \setminus Z, p}/(g_1)$ and by $m_p \in \mathbb{Z}$ an **integer** such that $g_2 = u_p \pi_p^{m_p} \in \mathcal{O}_{X \setminus Z, p}/(g_1)$.
- For every $p \in I$ and $q \in \overline{\{p\}}^{(1)} \cap Z$, we denote by $\pi_{p,q}$ a **uniformizing parameter** of the discrete valuation ring $\mathcal{O}_{\overline{\{p\}}, q}$, by $u_{p,q}$ a **unit** in $\mathcal{O}_{\overline{\{p\}}, q}$ and by $m_{p,q} \in \mathbb{Z}$ an **integer** such that $f_1 f_2 u_p = u_{p,q} \pi_{p,q}^{m_{p,q}} \in \mathcal{O}_{\overline{\{p\}}, q}$.

6.2 The quadratic linking class

In this section, we compute the quadratic linking class with respect to $(-1, -1)$ and the ambient quadratic linking class with respect to $(-1, -1)$ of an oriented link which verifies the assumptions of Section 6.1.

Theorem 6.1. Under the assumptions and with the notations of Section 6.1, the cycle

$$\sum_{p\in I}\sum_{q\in\overline{\{p\}}^{(1)}\cap Z}\langle\overline{u_{p,q}}\rangle\eta\,\chi^{\mathrm{odd}}(m_pm_{p,q})\otimes(\overline{\pi_{p,q}}^*\otimes\overline{\pi_p}^*\otimes\overline{g_1}^*)$$

where $\langle\overline{u_{p,q}}\rangle\eta\,\chi^{\mathrm{odd}}(m_pm_{p,q})\otimes(\overline{\pi_{p,q}}^*\otimes\overline{\pi_p}^*\otimes\overline{g_1}^*)\in K_{-1}^{\mathrm{MW}}(\kappa(q),\nu_q\otimes(\nu_Z)_{|q})$ (with $\nu_q=\det(\mathcal{N}_{q/Z})$ (see Notation 3.7) and $\nu_Z=\det(\mathcal{N}_{Z/X})$), represents the quadratic linking class of $\mathscr{L}$ with respect to $(-1,-1)$ (which is in $H^1(Z,\underline{K}_0^{\mathrm{MW}}\{\nu_Z\})$, see Definition 4.9).

Proof. From Definition 4.3, the oriented fundamental class $[o_i]$ is the class in $H^0(Z_i,\underline{K}_{-1}^{\mathrm{MW}}\{\nu_{Z_i}\})$ of $\eta\otimes(\overline{f_i}^*\wedge\overline{g_i}^*)$ (over the generic point of Z_i). It follows from Definition 4.6 and Theorem 2.46 that the Seifert class $\mathcal{S}_i$ of Z_i is the class in $H^1(X\setminus Z,\underline{K}_1^{\mathrm{MW}})$ of $\langle f_i\rangle\otimes\overline{g_i}^*$ (over the generic point p_i of the hypersurface of $X\setminus Z$ of equation $g_i=0$). In the expression above, $\langle f_i\rangle\in K_0^{\mathrm{MW}}(\kappa(p_i))$ and $\overline{g_i}^*\in\mathbb{Z}[\det(\mathcal{N}_{\overline{\{p_i\}}/X\setminus Z})\setminus\{0\}]$; with a slight abuse of notation, we denoted by f_i the image in the fraction field of $F[x,y,z,t]/(g_i)$ of $f_i\in F[x,y,z,t]$. We will make similar slight abuses of notation below.

By Corollary 3.32, the intersection product of the Seifert class $\mathcal{S}_1$ of Z_1 with the Seifert class $\mathcal{S}_2$ of Z_2 is the class in $H^2(X\setminus Z,\underline{K}_2^{\mathrm{MW}})$ of the cycle:

$$\sum_{p\in I}(m_p)_\epsilon\langle f_1f_2u_p\rangle\otimes(\overline{\pi_p}^*\otimes\overline{g_1}^*)$$

The quadratic linking class is the image of this intersection product by the boundary map $\partial: H^2(X\setminus Z,\underline{K}_2^{\mathrm{MW}})\to H^1(Z,\underline{K}_0^{\mathrm{MW}}\{\nu_Z\})$ thus the cycle

$$\sum_{p\in I}\sum_{q\in\overline{\{p\}}^{(1)}\cap Z}(m_p)_\epsilon\partial_{v_q}^{\pi_{p,q}}(\langle f_1f_2u_p\rangle)\otimes(\overline{\pi_{p,q}}^*\otimes\overline{\pi_p}^*\otimes\overline{g_1}^*)$$

represents the quadratic linking class (note that we used Proposition 2.37 to extract $(m_p)_\epsilon$ from the morphism $\partial_{v_q}^{\pi_{p,q}}$). By Theorem 2.46 and Lemma 2.45, the cycle

$$\sum_{p\in I}\sum_{q\in\overline{\{p\}}^{(1)}\cap Z}\langle\overline{u_{p,q}}\rangle\eta\,\chi^{\mathrm{odd}}(m_pm_{p,q})\otimes(\overline{\pi_{p,q}}^*\otimes\overline{\pi_p}^*\otimes\overline{g_1}^*)$$

represents the quadratic linking class of $\mathscr{L}$. □

The following corollary is a direct consequence of Theorem 6.1.

Corollary 6.2. Under the assumptions and with the notations of Section 6.1, the cycle

$$\sum_{p\in I}\sum_{q\in\overline{\{p\}}^{(1)}\cap Z_1}\langle\overline{u_{p,q}}\rangle\eta\,\chi^{\text{odd}}(m_p m_{p,q})\otimes(\overline{\pi_{p,q}}^*\otimes\overline{\pi_p}^*\otimes\overline{g_1}^*)$$

where $\langle\overline{u_{p,q}}\rangle\eta\,\chi^{\text{odd}}(m_p m_{p,q})\otimes(\overline{\pi_{p,q}}^*\otimes\overline{\pi_p}^*\otimes\overline{g_1}^*)\in K_{-1}^{\text{MW}}(\kappa(q),\nu_q)$ (with $\nu_q=\det(\mathcal{N}_{q/X})$ (see Notation 3.7)), represents the ambient quadratic linking class of $\mathscr{L}$ with respect to $(-1,-1)$ (which is in $H^3(X,\underline{K}_2^{\text{MW}})$, see Definition 5.1).

Let us now compute the ambient quadratic linking degree from the ambient quadratic linking class.

6.3 The ambient quadratic linking degree

In this section, we compute the ambient quadratic linking degree with respect to $(-1,-1)$ of an oriented link which verifies the assumptions of Section 6.1.

Recall the notations in Section 6.1. We introduce the following additional notations:

- for every $p\in I$ and $q\in\overline{\{p\}}^{(1)}\cap Z_1$, we denote by $v'_{p,q,0}$ the **discrete valuation** of $\mathcal{O}_{\overline{\{q\}},0}$, by $\pi'_{p,q,0}$ a **uniformizing parameter** for $v'_{p,q,0}$, by $u'_{p,q,0}$ a **unit** in $\mathcal{O}_{\overline{\{q\}},0}$ and by $m'_{p,q,0}\in\mathbb{Z}$ an **integer** such that $\overline{u_{p,q}}=u'_{p,q,0}(\pi'_{p,q,0})^{m'_{p,q,0}}$;
- for every $p\in I$ and $q\in\overline{\{p\}}^{(1)}\cap Z_1$, we denote by $\lambda'_{p,q,0}$ an **element** of $K_0^{\text{MW}}(F)$ such that $\eta^2\otimes(\overline{\pi'_{p,q,0}}^*\otimes\overline{\pi_{p,q}}^*\otimes\overline{\pi_p}^*\otimes\overline{g_1}^*)=\lambda'_{p,q,0}\,\eta^2\otimes(\overline{x}^*\wedge\overline{y}^*\wedge\overline{z}^*\wedge\overline{t}^*)$. Note that such a $\lambda'_{p,q,0}$ exists since $\overline{\pi'_{p,q,0}}^*\otimes\overline{\pi_{p,q}}^*\otimes\overline{\pi_p}^*\otimes\overline{g_1}^*\in\mathbb{Z}[(\det(\mathcal{N}_{\{0\}/\mathbb{A}^4_F})_{|0})\setminus\{0\}]$.

Theorem 6.3. Under the assumptions and with the notations of Section 6.1 (and the notations above), the ambient quadratic linking degree of $\mathscr{L}$ with respect to $(-1,-1)$ is the following element of the Witt ring $\mathrm{W}(F)$:

$$\sum_{p\in I}\sum_{q\in\overline{\{p\}}^{(1)}\cap Z_1}\lambda'_{p,q,0}<\overline{u'_{p,q,0}}>\chi^{\text{odd}}(m_p m_{p,q} m'_{p,q,0})$$

(with the following abuse of notation: if $\lambda'_{p,q,0}=\sum_{i=1}^m\langle u_i\rangle\in K_0^{\text{MW}}(F)$ then the $\lambda'_{p,q,0}$ in the expression above is in fact $\sum_{i=1}^m<u_i>\in\mathrm{W}(F)$; in other words, the $\lambda'_{p,q,0}$ in the expression above should be replaced with the image of $\lambda'_{p,q,0}\eta^2$ by the isomorphism $\gamma_{-2} : K^{\mathrm{MW}}_{-2}(F) \to \mathrm{W}(F)$ (see Theorem 2.33)).

Proof. By Definition 5.7 and Notation 3.36, the first step consists in applying the boundary map $\partial : H^3(\mathbb{A}^4_F\backslash\{0\}, \underline{K}^{\mathrm{MW}}_2) \to H^0(\{0\}, \underline{K}^{\mathrm{MW}}_{-2}\{\det(\mathcal{N}_{\{0\}/\mathbb{A}^4_F})\})$ to the ambient quadratic linking class. By Corollary 6.2, the cycle

$$\sum_{p\in I} \sum_{q\in\overline{\{p\}}^{(1)}\cap Z_1} \partial^{\pi'_{p,q,0}}_{v'_{p,q,0}}(\langle\overline{u_{p,q}}\rangle\eta\,\chi^{\mathrm{odd}}(m_p m_{p,q})) \otimes (\overline{\pi'_{p,q,0}}^* \otimes \overline{\pi_{p,q}}^* \otimes \overline{\pi_p}^* \otimes \overline{g_1}^*)$$

represents the image of the ambient quadratic linking class by the boundary map. By Theorem 2.46, the cycle

$$\sum_{p\in I} \sum_{q\in\overline{\{p\}}^{(1)}\cap Z_1} \langle\overline{u'_{p,q,0}}\rangle\eta^2\,\chi^{\mathrm{odd}}(m_p m_{p,q} m'_{p,q,0}) \otimes (\overline{\pi'_{p,q,0}}^* \otimes \overline{\pi_{p,q}}^* \otimes \overline{\pi_p}^* \otimes \overline{g_1}^*)$$

represents the image of the ambient quadratic linking class by the boundary map. It follows that the cycle

$$\sum_{p\in I} \sum_{q\in\overline{\{p\}}^{(1)}\cap Z_1} \lambda'_{p,q,0}\langle\overline{u'_{p,q,0}}\rangle\eta^2\,\chi^{\mathrm{odd}}(m_p m_{p,q} m'_{p,q,0}) \otimes (\overline{x}^* \wedge \overline{y}^* \wedge \overline{z}^* \wedge \overline{t}^*)$$

represents the image of the ambient quadratic linking class by the boundary map. Therefore, by Definition 5.7 and Notation 3.36, the ambient quadratic linking degree of $\mathscr{L}$ is the following element of $\mathrm{W}(F)$:

$$\sum_{p\in I} \sum_{q\in\overline{\{p\}}^{(1)}\cap Z_1} \gamma_{-2}(\lambda'_{p,q,0}\eta^2) < \overline{u'_{p,q,0}} > \chi^{\mathrm{odd}}(m_p m_{p,q} m'_{p,q,0})$$

□

6.4 The quadratic linking degree couple

In this section, we compute the quadratic linking degree couple with respect to $(-1,-1)$ of an oriented link which verifies the assumptions of Section 6.1, together with a closed immersion $\varphi_1 : \mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$ whose image is Z_1 and a closed immersion $\varphi_2 : \mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$ whose image is Z_2. In other words, $(Z_1, Z_2), (\overline{o_1}, \overline{o_2}), (\varphi_1, \varphi_2)$ is an oriented link of type $(\mathbb{A}^2_F \setminus \{0\}, \mathbb{A}^2_F \setminus \{0\}, \mathbb{A}^4_F \setminus \{0\})$ (see Definition 5.13).

Recall the notations in Section 6.1. We introduce the following additional notations:

- for every $i \in \{1,2\}$, $p \in I$ and $q \in \overline{\{p\}}^{(1)} \cap Z_i$, we denote by $\tau_{p,q}$ an **element** of $\nu_q = \det(\mathcal{N}_{q/Z_i})$ (see Notation 3.7) such that $\overline{\pi_{p,q}}^* \otimes \overline{\pi_p}^* \otimes \overline{g_1}^* = \tau_{p,q} \otimes (\overline{f_i}^* \wedge \overline{g_i}^*)$. Note that such a $\tau_{p,q}$ exists since $\overline{\pi_{p,q}}^* \otimes \overline{\pi_p}^* \otimes \overline{g_1}^* \in \mathbb{Z}[(\nu_q \otimes_{\kappa(q)} (\nu_{Z_i})_{|q}) \setminus \{0\}]$;

- for every $i \in \{1,2\}$, $p \in I$ and $q \in \overline{\{p\}}^{(1)} \cap Z_i$, we denote by $v_{p,q,0}$ the **discrete valuation** of $\mathcal{O}_{\overline{\{\varphi_i^{-1}(q)\}},0}$, by $\pi_{p,q,0}$ a **uniformizing parameter** for $v_{p,q,0}$, by $u_{p,q,0}$ a **unit** in $\mathcal{O}_{\overline{\{\varphi_i^{-1}(q)\}},0}$ and by $m_{p,q,0} \in \mathbb{Z}$ an **integer** such that $\varphi_i^*(\overline{u_{p,q}}) = u_{p,q,0}\pi_{p,q,0}^{m_{p,q,0}}$;

- for every $i \in \{1,2\}$, $p \in I$ and $q \in \overline{\{p\}}^{(1)} \cap Z_i$, we denote by $\lambda_{p,q,0}$ an **element** of $K_0^{\mathrm{MW}}(F)$ such that $\eta^2 \otimes (\overline{\pi_{p,q,0}}^* \otimes \varphi_i^*(\tau_{p,q})) = \lambda_{p,q,0}\,\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$. Note that such a $\lambda_{p,q,0}$ exists since $\overline{\pi_{p,q,0}}^* \otimes \varphi_i^*(\tau_{p,q}) \in \mathbb{Z}[(\det(\mathcal{N}_{\{0\}/\mathbb{A}^2_F})_{|0}) \setminus \{0\}]$.

Theorem 6.4. Under the assumptions and with the notations of Section 6.1 (and the notations above), the quadratic linking degree couple of $(Z_1, Z_2), (\overline{o_1}, \overline{o_2}), (\varphi_1, \varphi_2)$ with respect to $(-1,-1)$ is the following couple of elements of the Witt ring $\mathrm{W}(F)$ (i.e. element of $\mathrm{W}(F) \oplus \mathrm{W}(F)$):

$$\sum_{p\in I} \sum_{q\in\overline{\{p\}}^{(1)}\cap Z_1} \lambda_{p,q,0} < \overline{u_{p,q,0}} > \chi^{\mathrm{odd}}(m_p m_{p,q} m_{p,q,0})$$
$$\oplus \sum_{p\in I} \sum_{q\in\overline{\{p\}}^{(1)}\cap Z_2} \lambda_{p,q,0} < \overline{u_{p,q,0}} > \chi^{\mathrm{odd}}(m_p m_{p,q} m_{p,q,0})$$

(with the following abuse of notation: if $\lambda_{p,q,0} = \sum_{i=1}^m \langle u_i \rangle \in K_0^{\mathrm{MW}}(F)$ then the $\lambda_{p,q,0}$ in the expression above is in fact $\sum_{i=1}^m < u_i > \in \mathrm{W}(F)$; in other words, the $\lambda_{p,q,0}$ in the expression above should be replaced with the image of $\lambda_{p,q,0}\eta^2$ by the isomorphism $\gamma_{-2} : K_{-2}^{\mathrm{MW}}(F) \to \mathrm{W}(F)$ (see Theorem 2.33)).

Proof. Recall from Definition 5.15 that the first step in computing the quadratic linking degree from the quadratic linking class consists in applying $(\widetilde{o_1} \oplus \widetilde{o_2}) \circ \varpi$. It follows from Theorem 6.1 and from the definitions of $\overline{o_1} = \overline{o_{f_1,g_1}}$ and $\overline{o_2} = \overline{o_{f_2,g_2}}$ that the couple of cycles

$$\sum_{p\in I} \sum_{q\in\overline{\{p\}}^{(1)}\cap Z_1} \langle\overline{u_{p,q}}\rangle\eta\,\chi^{\mathrm{odd}}(m_p m_{p,q}) \otimes \tau_{p,q}$$
$$\oplus \sum_{p\in I} \sum_{q\in\overline{\{p\}}^{(1)}\cap Z_2} \langle\overline{u_{p,q}}\rangle\eta\,\chi^{\mathrm{odd}}(m_p m_{p,q}) \otimes \tau_{p,q}$$

where $\langle\overline{u_{p,q}}\rangle\eta\,\chi^{\mathrm{odd}}(m_p m_{p,q})\otimes\tau_{p,q}\in K^{\mathrm{MW}}_{-1}(\kappa(q),\nu_q)$, represents $(\widetilde{o_1}\oplus\widetilde{o_2})(\varpi(\mathrm{Qlc}_{\mathscr{L}}))$.

It follows that the couple of cycles

$$\sum_{p\in I}\sum_{q\in\overline{\{p\}}^{(1)}\cap Z_1}\langle\varphi_1^*(\overline{u_{p,q}})\rangle\eta\,\chi^{\mathrm{odd}}(m_p m_{p,q})\otimes\varphi_1^*(\tau_{p,q})$$
$$\oplus\sum_{p\in I}\sum_{q\in\overline{\{p\}}^{(1)}\cap Z_2}\langle\varphi_2^*(\overline{u_{p,q}})\rangle\eta\,\chi^{\mathrm{odd}}(m_p m_{p,q})\otimes\varphi_2^*(\tau_{p,q})$$

where for all $i\in\{1,2\}$, $\langle\varphi_i^*(\overline{u_{p,q}})\rangle\eta\,\chi^{\mathrm{odd}}(m_p m_{p,q})\otimes\varphi_i^*(\tau_{p,q})\in K^{\mathrm{MW}}_{-1}(\kappa(\varphi_i^{-1}(q)),\nu_{\varphi_i^{-1}(q)})$, represents $(\varphi_1^*\oplus\varphi_2^*)(\widetilde{o_1}\oplus\widetilde{o_2})(\varpi(\mathrm{Qlc}_{\mathscr{L}}))$. This is the second step in computing the quadratic linking degree (see Definition 5.15).

Recall from Definition 5.15 and Notation 3.36 that the third step in computing the quadratic linking degree consists in applying the boundary map

$$\partial: H^1(\mathbb{A}^2_F\setminus\{0\},\underline{K}^{\mathrm{MW}}_0)\to H^0(\{0\},\underline{K}^{\mathrm{MW}}_{-2}\{\det(\mathcal{N}_{\{0\}/\mathbb{A}^2_F})\})$$

to each element of the couple above, which gives:

$$\sum_{p\in I}\sum_{q\in\overline{\{p\}}^{(1)}\cap Z_1}\partial^{\pi_{p,q,0}}_{v_{p,q,0}}(\langle\varphi_1^*(\overline{u_{p,q}})\rangle)\eta\,\chi^{\mathrm{odd}}(m_p m_{p,q})\otimes(\overline{\pi_{p,q,0}}^*\otimes\varphi_1^*(\tau_{p,q}))$$
$$\oplus\sum_{p\in I}\sum_{q\in\overline{\{p\}}^{(1)}\cap Z_2}\partial^{\pi_{p,q,0}}_{v_{p,q,0}}(\langle\varphi_2^*(\overline{u_{p,q}})\rangle)\eta\,\chi^{\mathrm{odd}}(m_p m_{p,q})\otimes(\overline{\pi_{p,q,0}}^*\otimes\varphi_2^*(\tau_{p,q}))$$

where for all $i\in\{1,2\}$, $\partial^{\pi_{p,q,0}}_{v_{p,q,0}}(\langle\varphi_i^*(\overline{u_{p,q}})\rangle)\eta\,\chi^{\mathrm{odd}}(m_p m_{p,q})\otimes(\overline{\pi_{p,q,0}}^*\otimes\varphi_i^*(\tau_{p,q}))\in K^{\mathrm{MW}}_{-2}(\kappa(0),\det(\mathcal{N}_{\{0\}/\mathbb{A}^2_F}))$.

By Theorem 2.46, for every $i\in\{1,2\}$ we have $\partial^{\pi_{p,q,0}}_{v_{p,q,0}}(\langle\varphi_i^*(\overline{u_{p,q}})\rangle)=\langle\overline{u_{p,q,0}}\rangle\eta\,\chi^{\mathrm{odd}}(m_{p,q,0})$ thus the third step gives:

$$\sum_{p\in I}\sum_{q\in\overline{\{p\}}^{(1)}\cap Z_1}\langle\overline{u_{p,q,0}}\rangle\,\eta^2\,\chi^{\mathrm{odd}}(m_p m_{p,q} m_{p,q,0})\otimes(\overline{\pi_{p,q,0}}^*\otimes\varphi_1^*(\tau_{p,q}))$$
$$\oplus\sum_{p\in I}\sum_{q\in\overline{\{p\}}^{(1)}\cap Z_2}\langle\overline{u_{p,q,0}}\rangle\,\eta^2\,\chi^{\mathrm{odd}}(m_p m_{p,q} m_{p,q,0})\otimes(\overline{\pi_{p,q,0}}^*\otimes\varphi_2^*(\tau_{p,q}))$$

It follows that the third step gives:

$$\sum_{p\in I}\sum_{q\in\overline{\{p\}}^{(1)}\cap Z_1}\lambda_{p,q,0}\langle\overline{u_{p,q,0}}\rangle\,\eta^2\,\chi^{\mathrm{odd}}(m_p m_{p,q} m_{p,q,0})\otimes(\overline{u}^*\wedge\overline{v}^*)$$
$$\oplus\sum_{p\in I}\sum_{q\in\overline{\{p\}}^{(1)}\cap Z_2}\lambda_{p,q,0}\langle\overline{u_{p,q,0}}\rangle\,\eta^2\,\chi^{\mathrm{odd}}(m_p m_{p,q} m_{p,q,0})\otimes(\overline{u}^*\wedge\overline{v}^*)$$

Therefore, by Definition 5.15 and Notation 3.36, the quadratic linking degree couple of $(Z_1, Z_2), (\overline{o_1}, \overline{o_2}), (\varphi_1, \varphi_2)$ with respect to $(-1, -1)$ is the following couple of elements of the Witt ring $\mathrm{W}(F)$:

$$\sum_{p \in I} \sum_{q \in \overline{\{p\}}^{(1)} \cap Z_1} \gamma_{-2}(\lambda_{p,q,0}\eta^2) < \overline{u_{p,q,0}} > \chi^{\mathrm{odd}}(m_p m_{p,q} m_{p,q,0})$$
$$\oplus \sum_{p \in I} \sum_{q \in \overline{\{p\}}^{(1)} \cap Z_2} \gamma_{-2}(\lambda_{p,q,0}\eta^2) < \overline{u_{p,q,0}} > \chi^{\mathrm{odd}}(m_p m_{p,q} m_{p,q,0})$$

□

Future work 18 (Additional computing methods)**.** In addition to the cases mentioned at the beginning of this chapter (and to the case tackled in this chapter), there are other cases in which the quadratic linking class etc. have been defined (and in which it would be interesting to compute the quadratic linking class etc.). Unlike previously mentioned cases, the following cases need Future work 1 to be (at least partially) completed before they can be tackled:

- the cases mentioned at the beginning of this chapter (except the case $(\mathbb{P}^1_F, \mathbb{P}^1_F, \mathbb{P}^3_F)$) and the case tackled in this chapter, with $j_1 = 0$ (and $j_2 \leq 0$) or $j_2 = 0$ (and $j_1 \leq 0$) instead of $j_1, j_2 \leq -1$;
- higher codimensional cases (see Definitions 5.7 and 5.15) with $j_1 \leq 0$ and $j_2 \leq 0$; note that Future works 2 and 3, once completed, would add other higher codimensional cases (as well as an ambient quadratic linking degree in the case $(\mathbb{P}^1_F, \mathbb{P}^1_F, \mathbb{P}^3_F)$, see Future work 7).

Chapter 7

Examples and computations

In this chapter, we compute the quadratic linking class, the ambient quadratic linking degree (and its invariants) and the quadratic linking degree couple (and its invariants) on examples, by using the methods given in Chapter 6.

In Section 7.1, we give a simple example over a perfect field F: the Hopf link. By changing its orientations, one can get any $< a > \in \mathrm{W}(F)$ as ambient quadratic linking degree (or as a component of the quadratic linking degree couple). In Section 7.2, we give a family of examples, which we call binary links, over a perfect field F of characteristic different from 2, such that, by changing their orientations, one can get any $< a > + < b > \in \mathrm{W}(F)$ (i.e. any class in $\mathrm{W}(F)$ of a binary quadratic form) as ambient quadratic linking degree (or as a component of the quadratic linking degree couple). Finally, in Section 7.3, we give for each $n \in \mathbb{N}$ a counterpart over $\mathbb{R}$ to the torus link $T(2, 2n)$ (whose linking number is n; see Section 1.4). The ambient quadratic linking degree of our counterpart of $T(2, 2n)$ is $-n \in \mathrm{W}(\mathbb{R}) \simeq \mathbb{Z}$ and its quadratic linking degree couple is $(n, -n) \in \mathrm{W}(\mathbb{R}) \oplus \mathrm{W}(\mathbb{R}) \simeq \mathbb{Z} \oplus \mathbb{Z}$. Note that we also included these computations in our preprint [Lem23].

7.1 The Hopf link

In this section, we present a simple example of oriented link with two components (see Definition 4.1) and compute oriented fundamental classes and Seifert classes for its components, as well as its quadratic linking class, its ambient quadratic linking class and its ambient quadratic linking degree. We then enrich this oriented link into an oriented link of type $(\mathbb{A}^2_F \setminus \{0\}, \mathbb{A}^2_F \setminus \{0\}, \mathbb{A}^4_F \setminus \{0\})$ (which means that we fix isomorphisms between $\mathbb{A}^2_F \setminus \{0\}$ and each of its components (which lie in $\mathbb{A}^4_F \setminus \{0\}$); see

Definition 5.13) and compute its quadratic linking degree couple.

Let F be a perfect field. Recall that $\mathbb{A}^4_F = \mathrm{Spec}(F[x,y,z,t])$ and $\mathbb{A}^2_F = \mathrm{Spec}(F[u,v])$ (so that coordinates are fixed once and for all). We define the Hopf link over F as follows:

- Z_1 is the intersection of the closed subscheme of $\mathbb{A}^4_F$ of ideal (x,y) and of $X := \mathbb{A}^4_F \setminus \{0\}$ (hence is a closed F-subscheme of X; in other words, Z_1 is the closed F-subscheme of $\mathbb{A}^4_F \setminus \{0\}$ given by the equations $x = 0, y = 0$);

- Z_2 is the intersection of the closed subscheme of $\mathbb{A}^4_F$ of ideal (z,t) and of X (hence is a closed F-subscheme of X; in other words, Z_2 is the closed F-subscheme of $\mathbb{A}^4_F \setminus \{0\}$ given by the equations $z = 0, t = 0$); note that Z_1 and Z_2 are disjoint (see [GW10, Proposition 3.35]);

- $\overline{o_1} = \overline{o_{x,y}}$ is the orientation class associated to the couple (x,y) (see Section 6.1; in other words, $\overline{o_1}$ is the class (see Definition 3.22) of the isomorphism $o_{x,y} : \nu_{Z_1} := \det(\mathcal{N}_{Z_1/X}) \to \mathcal{O}_{Z_1} \otimes \mathcal{O}_{Z_1}$ which maps $\overline{x}^* \wedge \overline{y}^*$ to $1 \otimes 1$);

- $\overline{o_2} = \overline{o_{z,t}}$ is the orientation class associated to the couple (z,t) (see Section 6.1; in other words, $\overline{o_2}$ is the class (see Definition 3.22) of the isomorphism $o_{z,t} : \nu_{Z_2} := \det(\mathcal{N}_{Z_2/X}) \to \mathcal{O}_{Z_2} \otimes \mathcal{O}_{Z_2}$ which maps $\overline{z}^* \wedge \overline{t}^*$ to $1 \otimes 1$).

The reason behind the name of the Hopf link will be made apparent in Section 7.3. See Table 7.1 for a recap in the case $j_1 = j_2 = -1$ of the computations made below (note that for the last three lines, closed immersions $\varphi_1, \varphi_2 : \mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$ need to be fixed; this is done below). The quadratic linking class is given in two different lines since it is used to compute the ambient quadratic linking degree on the one hand and the quadratic linking degree couple on the other hand. Note that the second column gives cycles which represent the classes in question (except for the ambient quadratic linking degree which is in the Witt ring $\mathrm{W}(F)$ and the quadratic linking degree couple which is in $\mathrm{W}(F) \oplus \mathrm{W}(F)$), without specifying the points over which these cycles live, but that in the case of this table these points are the obvious ones (for instance $\langle x \rangle \otimes \overline{y}^*$ (which represents a class in $H^1(X \setminus Z, \underline{K}_1^{\mathrm{MW}})$) lives over the generic point of the hypersurface of $X \setminus Z$ of equation $y = 0$).

Instead of applying Theorem 6.1 to get the quadratic linking class of the Hopf link, we go through the different steps which lead to the quadratic linking class in order to illustrate the proof of Theorem 6.1 and to give examples of the mathematical objects we have introduced in Chapter 4.

Oriented fund. cl.	$\eta \otimes (\overline{x}^* \wedge \overline{y}^*)$	|	$\eta \otimes (\overline{z}^* \wedge \overline{t}^*)$
Seifert classes	$\langle x\rangle \otimes \overline{y}^*$	|	$\langle z\rangle \otimes \overline{t}^*$
Int. prod. of Seif. cl.	$\langle xz\rangle \otimes (\overline{t}^* \wedge \overline{y}^*)$		
Quad. linking class	$-\langle z\rangle\eta \otimes (\overline{t}^* \wedge \overline{x}^* \wedge \overline{y}^*)$	$\oplus$	$\langle x\rangle\eta \otimes (\overline{y}^* \wedge \overline{z}^* \wedge \overline{t}^*)$
Amb. quad. link. cl.	$-\langle z\rangle\eta \otimes (\overline{t}^* \wedge \overline{x}^* \wedge \overline{y}^*)$		
Apply ∂	$-\eta^2 \otimes (\overline{x}^* \wedge \overline{y}^* \wedge \overline{z}^* \wedge \overline{t}^*)$		
Amb. quad. lk. deg.	-1		
Quad. linking class	$-\langle z\rangle\eta \otimes (\overline{t}^* \wedge \overline{x}^* \wedge \overline{y}^*)$	$\oplus$	$\langle x\rangle\eta \otimes (\overline{y}^* \wedge \overline{z}^* \wedge \overline{t}^*)$
Apply $(\widetilde{o_1} \oplus \widetilde{o_2}) \circ \varpi$	$-\langle z\rangle\eta \otimes \overline{t}^*$	$\oplus$	$\langle x\rangle\eta \otimes \overline{y}^*$
Apply $\varphi_1^* \oplus \varphi_2^*$	$-\langle u\rangle\eta \otimes \overline{v}^*$	$\oplus$	$\langle u\rangle\eta \otimes \overline{v}^*$
Apply $\partial \oplus \partial$	$-\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$	$\oplus$	$\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$
Quad. link. deg. cpl.	-1	$\oplus$	1

Table 7.1 – The ambient quadratic linking degree and the quadratic linking degree couple of the Hopf link.

First note that it follows immediately from Definition 4.3 (also recall Notation 3.25 and Proposition 3.26) that the oriented fundamental class $[o_1]_{j_1}$ of the first component of the Hopf link with respect to $j_1 \leq 0$ is the class in $H^0(Z_1, \underline{K}_{j_1}^{\mathrm{MW}}\{\nu_{Z_1}\})$ of the cycle $\eta^{-j_1} \otimes \overline{x}^* \wedge \overline{y}^*$ (over the generic point of Z_1), since by definition $\overline{o_1}$ is the orientation class of $o_{x,y}$. Similarly, the oriented fundamental class $[o_2]_{j_2}$ of the second component of the Hopf link with respect to $j_2 \leq 0$ is the class in $H^0(Z_2, \underline{K}_{j_2}^{\mathrm{MW}}\{\nu_{Z_2}\})$ of the cycle $\eta^{-j_2} \otimes \overline{z}^* \wedge \overline{t}^*$ (over the generic point of Z_2), since by definition $\overline{o_2}$ is the orientation class of $o_{z,t}$.

By Theorem 2.46 (see also Definitions 3.8 and 3.18), the image by the boundary map of the cycle $[x]\otimes\overline{y}^*$ (over the generic point of the hypersurface of $X \setminus Z$ of equation $y = 0$) is the cycle $1 \otimes \overline{x}^* \wedge \overline{y}^*$ (over the generic point of Z_1) and for each $j_1 \leq -1$ the image by the boundary map of the cycle $\langle x\rangle\eta^{-j_1-1}\otimes\overline{y}^*$ (over the generic point of the hypersurface of $X\setminus Z$ of equation $y = 0$) is the cycle $\langle x\rangle\eta^{-j_1}\otimes\overline{y}^*$ (over the generic point of Z_1). It follows from Definition 4.6 that the class in $H^1(X \setminus Z, \underline{K}_2^{\mathrm{MW}})$ of $[x] \otimes \overline{y}^*$ is the Seifert class $\mathcal{S}_{o_1,0}$ of the first component of the Hopf link with respect to 0 and that for each $j_1 \leq -1$, the class in $H^1(X\setminus Z, \underline{K}_{j_1+2}^{\mathrm{MW}})$ of $\langle x\rangle\eta^{-j_1-1} \otimes \overline{y}^*$ is the Seifert class $\mathcal{S}_{o_1,j_1}$ of the first component of the Hopf link with respect to j_1. Similarly, the Seifert class $\mathcal{S}_{o_2,0}$ of the second component of the Hopf link with respect to 0 is the class in $H^1(X \setminus Z, \underline{K}_2^{\mathrm{MW}})$ of the cycle $[z] \otimes \overline{t}^*$ (over the generic point of the hypersurface of $X \setminus Z$ of equation $t = 0$) and that for each $j_2 \leq -1$, the class in $H^1(X \setminus Z, \underline{K}_{j_2+2}^{\mathrm{MW}})$ of the cycle $\langle z\rangle\eta^{-j_2-1} \otimes \overline{t}^*$ (over the generic point of the hypersurface of $X\setminus Z$ of equation $t = 0$) is the

Seifert class $\mathcal{S}_{o_2,j_2}$ of the second component of the Hopf link with respect to j_2. Recalling Remark 4.8, it may seem surprising that on the one hand $\mathcal{S}_{o_2,0}$ is the class of $[z]\otimes\overline{t}^*$ and on the other hand $\mathcal{S}_{o_2,-1}$ is the class of $\langle z\rangle\otimes\overline{t}^*$ rather than the class of $\eta[z]\otimes\overline{t}^*$. This surprise evaporates when one remembers that we are talking of *classes* and not of cycles. Indeed, the cycle $\langle z\rangle\otimes\overline{t}^*$ and the cycle $\eta[z]\otimes\overline{t}^*$ have the same class in $H^1(X\setminus Z,\underline{K}_1^{\mathrm{MW}})$ since their difference is the cycle $1\otimes\overline{t}^*$ which is the image of the cycle $[t]$ (over the generic point of $X\setminus Z$) by the differential of the Rost-Schmid complex of $X\setminus Z$ (see Theorem 2.46 and Definition 3.8).

Recall from Definition 4.9 that the quadratic linking class with respect to a couple (j_1,j_2) of nonpositive integers is the image by the boundary map $\partial: H^2(X\setminus Z,\underline{K}^{\mathrm{MW}}_{j_1+j_2+4})\to H^1(Z,\underline{K}^{\mathrm{MW}}_{j_1+j_2+2}\{\nu_Z\})$ of the intersection product $\mathcal{S}_{o_1,j_1}\cdot\mathcal{S}_{o_2,j_2}$. With the formula for the intersection product which is in Corollary 3.32, we can compute $\mathcal{S}_{o_1,j_1}\cdot\mathcal{S}_{o_2,j_2}$ when $j_1\leq -1$ and $j_2\leq -1$. First note that the intersection of the hypersurfaces of $X\setminus Z$ of respective equations $y=0$ and $t=0$ is irreducible and is the closed subscheme of $X\setminus Z$ which is given by the equations $y=0,t=0$. It follows from Corollary 3.32 that the cycle $m_\epsilon\langle xzw\rangle\eta^{-(j_1+j_2+2)}\otimes(\overline{\pi}^*\otimes\overline{y}^*)$ (over the generic point p of the closed subscheme of $X\setminus Z$ which is given by the equations $y=0,t=0$) represents the intersection product $\mathcal{S}_{o_1,j_1}\cdot\mathcal{S}_{o_2,j_2}$, where π is a uniformizing parameter for $\mathcal{O}_{X\setminus Z,p}/(y)$, w is a unit in $\mathcal{O}_{X\setminus Z,p}/(y)$ and $m\in\mathbb{Z}$ is an integer such that $t=w\pi^m$ in $\mathcal{O}_{X\setminus Z,p}/(y)$. Note that the ring $\mathcal{O}_{X\setminus Z,p}/(y)$ is canonically isomorphic to $\mathcal{O}_{\mathbb{A}^4_F,p}/(y)$ hence to $F[x,y,z,t]_{(y,t)}/(y)$ hence to $F[x,z,t]_{(t)}$, i.e. to the localization of the ring $F[x,z,t]$ at the prime ideal (t). We can therefore take $\pi=t$, $m=1$ and $w=1$, so that the cycle $\langle xz\rangle\eta^{-(j_1+j_2+2)}\otimes(\overline{t}^*\wedge\overline{y}^*)$ (over the generic point p of the closed subscheme of $X\setminus Z$ which is given by the equations $y=0,t=0$) represents the intersection product $\mathcal{S}_{o_1,j_1}\cdot\mathcal{S}_{o_2,j_2}$. By Definition 3.18 (see also Defintion 3.8), it follows that for all $j_1,j_2\leq -1$, the quadratic linking class of the Hopf link with respect to (j_1,j_2) is the class in $H^1(Z,\underline{K}^{\mathrm{MW}}_{j_1+j_2+2}\{\nu_Z\})$ of the following cycle:

$$\partial^x_{v_x}(\langle xz\rangle\eta^{-(j_1+j_2+2)})\otimes(\overline{x}^*\wedge\overline{t}^*\wedge\overline{y}^*)\oplus\partial^z_{v_z}(\langle xz\rangle\eta^{-(j_1+j_2+2)})\otimes(\overline{z}^*\wedge\overline{t}^*\wedge\overline{y}^*)$$

where v_x (respectively v_z) is the discrete valuation of $\mathcal{O}_{\overline{\{p\}},q_x}$ (resp. $\mathcal{O}_{\overline{\{p\}},q_z}$) with q_x (resp. q_z) the generic point of the hypersurface of $\overline{\{p\}}$ of equation $x=0$ (resp. $z=0$). By Theorem 2.46, the quadratic linking class of the Hopf link with respect to (j_1,j_2) is the class in $H^1(Z,\underline{K}^{\mathrm{MW}}_{j_1+j_2+2}\{\nu_Z\})$ of the following cycle:

$$\langle z\rangle\eta^{-(j_1+j_2+1)}\otimes(\overline{x}^*\wedge\overline{t}^*\wedge\overline{y}^*)\oplus\langle x\rangle\eta^{-(j_1+j_2+1)}\otimes(\overline{z}^*\wedge\overline{t}^*\wedge\overline{y}^*)$$

hence of the following cycle:

$$-\langle z\rangle\eta^{-(j_1+j_2+1)}\otimes(\overline{t}^*\wedge\overline{x}^*\wedge\overline{y}^*)\oplus\langle x\rangle\eta^{-(j_1+j_2+1)}\otimes(\overline{y}^*\wedge\overline{z}^*\wedge\overline{t}^*)$$

Note that by Remark 4.11, we could have restricted ourselves to the case $(j_1,j_2)=(-1,-1)$ and deduced the other cases ($j_1\leq -1$ and $j_2\leq -1$) from it, but for expository purposes we chose to directly compute these cases.

Let us now turn to the ambient quadratic linking class and the ambient quadratic linking degree.

It follows from our computation of the quadratic linking class and Definition 5.1 (or Corollary 6.2) that the ambient quadratic linking class of the Hopf link with respect to (j_1,j_2) (where $j_1\leq -1$ and $j_2\leq -1$) is the class in $H^3(X,\underline{K}^{\mathrm{MW}}_{j_1+j_2+4})$ of the following cycle:

$$-\langle z\rangle\eta^{-(j_1+j_2+1)}\otimes(\overline{t}^*\wedge\overline{x}^*\wedge\overline{y}^*)$$

We could apply Theorem 6.3 to get the ambient quadratic linking degree but we will rather go through the different steps which lead to the ambient quadratic linking degree from the ambient quadratic linking class in order to illustrate the proof of this theorem. Recall Definition 5.7.

The first step consists in applying the boundary map $\partial : H^3(\mathbb{A}^4_F\setminus\{0\},\underline{K}^{\mathrm{MW}}_{j_1+j_2+4})\to H^0(\{0\},\underline{K}^{\mathrm{MW}}_{j_1+j_2}\{\det(\mathcal{N}_{\{0\}/\mathbb{A}^4_F})\})$ to the ambient quadratic linking class. This gives the class of the cycle $-\eta^{-(j_1+j_2)}\otimes(\overline{z}^*\wedge\overline{t}^*\wedge\overline{x}^*\wedge\overline{y}^*)$ which is the class of the cycle $-\eta^{-(j_1+j_2)}\otimes(\overline{x}^*\wedge\overline{y}^*\wedge\overline{z}^*\wedge\overline{t}^*)$. The second step consists in applying the isomorphism $H^0(\{0\},\underline{K}^{\mathrm{MW}}_{j_1+j_2}\{\det(\mathcal{N}_{\{0\}/\mathbb{A}^4_F})\})\to H^0(\{0\},\underline{K}^{\mathrm{MW}}_{j_1+j_2})=K^{\mathrm{MW}}_{j_1+j_2}(F)$ (denoted $\widetilde{o}$ in Notation 3.36) induced by the orientation of $\mathbb{A}^4_F\setminus\{0\}$, which gives $-\eta^{-(j_1+j_2)}$. The last step consists in applying the isomorphism $\gamma_{j_1+j_2}:K^{\mathrm{MW}}_{j_1+j_2}(F)\to \mathrm{W}(F)$, which gives -1 as ambient quadratic linking degree of the Hopf link.

Note that if we change one (or both) of the orientation classes of the Hopf link, then the ambient quadratic linking degree will be equal to $<a>\in \mathrm{W}(F)$ for some $a\in F^*$, and that all such values can be obtained by changing one of the orientation classes (see Proposition 5.11 and Remark 4.36). In any case, we have our invariants of the quadratic linking degree: the rank modulo 2 of the ambient quadratic linking degree of the Hopf link (and of all its variants for which one or both of the the orientation classes are changed) is equal to 1, all of the Σ_k map the ambient quadratic linking degree of the Hopf link to 0, and in the case $F=\mathbb{R}$, the absolute value of the ambient quadratic linking degree of the Hopf link (which is in $\mathrm{W}(\mathbb{R})\simeq\mathbb{Z}$ (via the signature)) is equal to 1.

Let us now turn to the quadratic linking degree couple. Recall that the quadratic linking degree couple (see Definition 5.15) is associated to oriented links of a certain type (see Definition 5.13), so that we need to introduce parametrisations φ_1 and φ_2. We set the following.

- $\varphi_1 : \mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$ is the morphism associated to the morphism of F-algebras $F[x,y,z,t] \to F[u,v]$ which maps x,y,z,t to $0,0,u,v$ respectively;
- $\varphi_2 : \mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$ is the morphism associated to the morphism of F-algebras $F[x,y,z,t] \to F[u,v]$ which maps x,y,z,t to $u,v,0,0$ respectively.

We could apply Theorem 6.4 to get the quadratic linking degree couple but we will rather go through the different steps which lead to the quadratic linking degree couple from the quadratic linking class in order to illustrate the proof of this theorem.

Recall Definition 5.15. By applying the isomorphism ϖ to the quadratic linking class, we consider that the cycle $-\langle z\rangle\eta^{-(j_1+j_2+1)}\otimes(\overline{t}^*\wedge\overline{x}^*\wedge\overline{y}^*)$ (over the generic point of the hypersurface of Z_1 given by the equation $t=0$) represents a class in $H^1(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2}\{\nu_{Z_1}\})$ rather than in $H^1(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2}\{\nu_Z\})$ and that the cycle $\langle x\rangle\eta^{-(j_1+j_2+1)}\otimes(\overline{y}^*\wedge\overline{z}^*\wedge\overline{t}^*)$ (over the generic point of the hypersurface of Z_2 given by the equation $y=0$) represents a class in $H^1(Z_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2}\{\nu_{Z_2}\})$ rather than in $H^1(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2}\{\nu_Z\})$. The isomorphism $\widetilde{o_1}$ maps the class of $-\langle z\rangle\eta^{-(j_1+j_2+1)}\otimes(\overline{t}^*\wedge\overline{x}^*\wedge\overline{y}^*)$ in $H^1(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2}\{\nu_{Z_1}\})$ to the class of $-\langle z\rangle\eta^{-(j_1+j_2+1)}\otimes\overline{t}^*$ in $H^1(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2})$ (since $\overline{o_1}$ is the orientation class of $o_{x,y}$) and the isomorphism $\widetilde{o_2}$ maps the class of $\langle x\rangle\eta^{-(j_1+j_2+1)}\otimes(\overline{y}^*\wedge\overline{z}^*\wedge\overline{t}^*)$ in $H^1(Z_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2}\{\nu_{Z_2}\})$ to the class of $\langle x\rangle\eta^{-(j_1+j_2+1)}\otimes\overline{y}^*$ in $H^1(Z_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2})$ (since $\overline{o_2}$ is the orientation class of $o_{z,t}$). By definition, φ_1^* maps the class of $-\langle z\rangle\eta^{-(j_1+j_2+1)}\otimes\overline{t}^*$ in $H^1(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2})$ to the class of $-\langle u\rangle\eta^{-(j_1+j_2+1)}\otimes\overline{v}^*$ in $H^1(\mathbb{A}^2_F\setminus\{0\}, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2})$ and φ_2^* maps the class of $\langle x\rangle\eta^{-(j_1+j_2+1)}\otimes\overline{y}^*$ in $H^1(Z_2, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2})$ to the class of $\langle u\rangle\eta^{-(j_1+j_2+1)}\otimes\overline{v}^*$ in $H^1(\mathbb{A}^2_F\setminus\{0\}, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2})$. We then apply the boundary map $\partial : H^1(\mathbb{A}^2_F\setminus\{0\}, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2}) \to H^0(\{0\}, \underline{K}^{\mathrm{MW}}_{j_1+j_2}\{\det(\mathcal{N}_{\{0\}/\mathbb{A}^2_F})\})$ to each of these, to get respectively the class of $-\eta^{-(j_1+j_2)}\otimes(\overline{u}^*\wedge\overline{v}^*)$ and the class of $\eta^{-(j_1+j_2)}\otimes(\overline{u}^*\wedge\overline{v}^*)$ in $H^0(\{0\}, \underline{K}^{\mathrm{MW}}_{j_1+j_2}\{\det(\mathcal{N}_{\{0\}/\mathbb{A}^2_F})\})$. By applying to each of these the isomorphism $H^0(\{0\}, \underline{K}^{\mathrm{MW}}_{j_1+j_2}\{\det(\mathcal{N}_{\{0\}/\mathbb{A}^2_F})\}) \to H^0(\{0\}, \underline{K}^{\mathrm{MW}}_{j_1+j_2}) = K^{\mathrm{MW}}_{j_1+j_2}(F)$ (denoted $\widetilde{o}$ in Notation 3.36) induced by the orientation of $\mathbb{A}^2_F\setminus\{0\}$, we get respectively $-\eta^{-(j_1+j_2)} \in K^{\mathrm{MW}}_{j_1+j_2}(F)$ and $\eta^{-(j_1+j_2)} \in K^{\mathrm{MW}}_{j_1+j_2}(F)$. Finally, by applying to each of these the isomorphism $\gamma_{j_1+j_2} : K^{\mathrm{MW}}_{j_1+j_2}(F) \to \mathrm{W}(F)$, we have that the quadratic linking degree couple of the Hopf link with respect to (j_1, j_2) is $(-1, 1) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$.

Note that if we change one (or both) of the orientation classes of the Hopf link, or one (or both) of the parametrisations φ_1, φ_2, then the quadratic linking degree couple will be equal to $(< a >, < b >) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$ for some $a, b \in F^*$, and that all such values can be obtained by changing one of the orientation classes (or by changing one of the parametrisations; see Proposition 5.18 and Remark 4.36 for orientation changes and Remark 5.19 and Proposition 5.20 for parametrisation changes). In any case, we have our invariants of the quadratic linking degree: the rank modulo 2 of each component of the quadratic linking degree couple of the Hopf link (and of all its variants for which one or both of the the orientation classes (or of the parametrisations) are changed) is equal to 1, all of the Σ_k map each component of the quadratic linking degree couple of the Hopf link to 0, and in the case $F = \mathbb{R}$, the absolute value of each component of the quadratic linking degree couple of the Hopf link (which is in $\mathrm{W}(\mathbb{R}) \simeq \mathbb{Z}$ (via the signature)) is equal to 1.

7.2 Binary links

In the previous section, we presented an oriented link (the Hopf link) of ambient quadratic linking degree $-1 \in \mathrm{W}(F)$ (with F a perfect field), whose variants (by changing one of its orientation classes) give examples of oriented links of ambient quadratic linking degree $< a > \in \mathrm{W}(F)$ for each $a \in F^*$. In this section, we present for each $a \in F^*$ (with F a perfect field of characteristic different from 2) an oriented link (which we call the binary link B_a) with ambient quadratic linking degree $1 + < a > \in \mathrm{W}(F)$. Its variants (by changing one of its orientation classes) give examples of oriented links of ambient quadratic linking degree $< b > + < ba > \in \mathrm{W}(F)$ for each $b \in F^*$ (see Proposition 5.11 and Remark 4.36). Thus, for each $b, c \in F^*$, we have an example of oriented link whose ambient quadratic linking degree is $< b > + < c >$ (take $B_{\frac{c}{b}}$ and change $\overline{o_1}$ into $\overline{o_1 \circ (\times b)}$).

Let F be a perfect field of characteristic different from 2 and $a \in F^*$. We define the binary link B_a over F as follows:

- Z_1 is the intersection of the closed subscheme of $\mathbb{A}^4_F$ of ideal $(f_1 := t - ((1+a)x - y)y, g_1 := z - x(x-y))$ and of $X := \mathbb{A}^4_F \setminus \{0\}$ (hence is a closed F-subscheme of X; in other words, Z_1 is the closed F-subscheme of $\mathbb{A}^4_F \setminus \{0\}$ given by the equations $f_1 = 0, g_1 = 0$);

- Z_2 is the intersection of the closed subscheme of $\mathbb{A}^4_F$ of ideal $(f_2 := t + ((1+a)x - y)y, g_2 := z + x(x-y))$ and of X (hence is a closed F-subscheme of X; in other words, Z_1 is the closed F-subscheme of $\mathbb{A}^4_F \setminus \{0\}$ given by the equations $f_2 = 0, g_2 = 0$); note that Z_1 and Z_2 are disjoint (see [GW10, Proposition 3.35]);

- $\overline{o_1} = \overline{o_{f_1,g_1}}$ is the orientation class associated to the couple (f_1, g_1) (see Section 6.1; in other words, $\overline{o_1}$ is the class (see Definition 3.22) of the isomorphism $o_{f_1,g_1} : \nu_{Z_1} := \det(\mathcal{N}_{Z_1/X}) \to \mathcal{O}_{Z_1} \otimes \mathcal{O}_{Z_1}$ which maps $\overline{f_1}^* \wedge \overline{g_1}^*$ to $1 \otimes 1$);

- $\overline{o_2} = \overline{o_{f_2,g_2}}$ is the orientation class associated to the couple (f_2, g_2) (see Section 6.1; in other words, $\overline{o_2}$ is the class (see Definition 3.22) of the isomorphism $o_{f_2,g_2} : \nu_{Z_2} := \det(\mathcal{N}_{Z_2/X}) \to \mathcal{O}_{Z_2} \otimes \mathcal{O}_{Z_2}$ which maps $\overline{f_2}^* \wedge \overline{g_2}^*$ to $1 \otimes 1$).

As in Section 7.1, we can compute the ambient quadratic linking degree and the quadratic linking degree couple of the binary links with respect to (j_1, j_2) for all $j_1 \leq -1$ and $j_2 \leq -1$. By using Remarks 4.4, 4.8, 4.11, 5.3, 5.8 and 5.16, we restrict ourselves to $(j_1, j_2) = (-1, -1)$.

See Table 7.2 for a recap of the computations made below (note that for the last three lines, closed immersions $\varphi_1, \varphi_2 : \mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$ need to be fixed; this is done below). The quadratic linking class is given in two different lines since it is used to compute the ambient quadratic linking degree on the one hand and the quadratic linking degree couple on the other hand. Note that the second column gives cycles which represent the classes in question (except for the ambient quadratic linking degree which is in the Witt ring $\mathrm{W}(F)$ and the quadratic linking degree couple which is in $\mathrm{W}(F) \oplus \mathrm{W}(F)$), without specifying the points over which these cycles live, but that in the case of this table these points are the obvious ones.

Instead of applying Theorem 6.1 to get the quadratic linking class of the binary link B_a, we go through the different steps which lead to the quadratic linking class in order to highlight a difficulty which arises from the fact that the equations which define the components of our link are no longer of degree 1 (compared with those for the Hopf link in Section 7.1).

There is no difficulty in computing the oriented fundamental classes ($\eta \otimes (\overline{f_1}^* \wedge \overline{g_1}^*)$ and $\eta \otimes (\overline{f_2}^* \wedge \overline{g_2}^*)$ respectively) and the Seifert classes ($\langle f_1 \rangle \otimes \overline{g_1}^*$ and $\langle f_2 \rangle \otimes \overline{g_2}^*$ respectively) of the binary link B_a. The difficulty (or rather difficulty lying in wait) appears when we want to compute the intersection product of the Seifert classes. By Corollary 3.32, the intersection product of $\langle f_1 \rangle \otimes \overline{g_1}^*$ (over the generic point of the hypersurface of $X \setminus Z$ of equation

$g_1 = 0$) with $\langle f_2\rangle \otimes \overline{g_2}^*$ (over the generic point of the hypersurface of $X \setminus Z$ of equation $g_2 = 0$) is the class in $H^2(X \setminus Z, \underline{K}_2^{\mathrm{MW}})$ of the cycle

$$(m_p)_\epsilon\langle f_1 f_2 u_p\rangle \otimes (\overline{\pi_p}^* \otimes \overline{g_1}^*) + (m_q)_\epsilon\langle f_1 f_2 u_q\rangle \otimes (\overline{\pi_q}^* \otimes \overline{g_1}^*)$$

where $(m_p)_\epsilon\langle f_1 f_2 u_p\rangle \otimes (\overline{\pi_p}^* \otimes \overline{g_1}^*)$ lives over the generic point p of the closed subscheme of $X\setminus Z$ given by the equations $z = 0, x = y$ (with π_p a uniformizing parameter for $\mathcal{O}_{X\setminus Z,p}/(g_1)$, u_p a unit in $\mathcal{O}_{X\setminus Z,p}/(g_1)$ and $m_p \in \mathbb{Z}$ an integer such that $g_2 = u_p\pi_p^{m_p} \in \mathcal{O}_{X\setminus Z,p}/(g_1)$) and $(m_q)_\epsilon\langle f_1 f_2 u_q\rangle \otimes (\overline{\pi_q}^* \otimes \overline{g_1}^*)$ lives over the generic point q of the closed subscheme of $X \setminus Z$ given by the equations $z = 0, x = 0$ (with π_q a uniformizing parameter for $\mathcal{O}_{X\setminus Z,q}/(g_1)$, u_q a unit in $\mathcal{O}_{X\setminus Z,q}/(g_1)$ and $m_q \in \mathbb{Z}$ an integer such that $g_2 = u_q\pi_q^{m_q} \in \mathcal{O}_{X\setminus Z,q}/(g_1)$).

The difficulty lying in wait for us lies in the choices of π_p and π_q. If one were to choose $\pi_p = g_2 \in \mathcal{O}_{X\setminus Z,p}/(g_1)$ (i.e. $\pi_p = 2x(x-y) \in F[x,y,z,t]_{(z,x-y)}/(z-x(x-y)) \simeq F[x,y,t]_{(x-y)}$) and $\pi_q = g_2 \in \mathcal{O}_{X\setminus Z,q}/(g_1)$ (i.e. $\pi_q = 2(x-y)x \in F[x,y,z,t]_{(z,x)}/(z-x(x-y)) \simeq F[x,y,t]_{(x)}$), computing the ambient quadratic linking degree or the quadratic linking degree couple would be horrendous. We choose $\pi_p = x-y \in F[x,y,z,t]_{(z,x-y)}/(z-x(x-y)) \simeq F[x,y,t]_{(x-y)}$ (hence $u_p = 2x$ and $m_p = 1$) and $\pi_q = x \in F[x,y,z,t]_{(z,x)}/(z-x(x-y)) \simeq F[x,y,t]_{(x)}$ (hence $u_q = 2(x-y)$ and $m_q = 1$) in order to have simple computations in what follows.

Thus, the intersection product of $\langle f_1\rangle \otimes \overline{g_1}^*$ with $\langle f_2\rangle \otimes \overline{g_2}^*$ is the class in $H^2(X \setminus Z, \underline{K}_2^{\mathrm{MW}})$ of the cycle

$$\langle f_1 f_2 2x\rangle \otimes (\overline{x-y}^* \otimes \overline{z-x(x-y)}^*) + \langle f_1 f_2 2(x-y)\rangle \otimes (\overline{x}^* \otimes \overline{z-x(x-y)}^*)$$

hence of the cycle

$$\langle f_1 f_2 2x\rangle \otimes (\overline{x-y}^* \otimes \overline{z}^*) + \langle f_1 f_2 2(x-y)\rangle \otimes (\overline{x}^* \otimes \overline{z}^*)$$

Therefore, the quadratic linking class is the class in $H^1(Z, \underline{K}_0^{\mathrm{MW}}\{\nu_Z\})$ of the cycle

$$\partial^{f_1}_{v_{1,f_1}}(\langle f_1 f_2 2x\rangle) \otimes (\overline{f_1}^* \otimes \overline{x-y}^* \otimes \overline{z}^*) + \partial^{f_1}_{v_{2,f_1}}(\langle f_1 f_2 2(x-y)\rangle) \otimes (\overline{f_1}^* \otimes \overline{x}^* \otimes \overline{z}^*)$$
$$+\partial^{f_2}_{v_{1,f_2}}(\langle f_1 f_2 2x\rangle) \otimes (\overline{f_2}^* \otimes \overline{x-y}^* \otimes \overline{z}^*) + \partial^{f_2}_{v_{2,f_2}}(\langle f_1 f_2 2(x-y)\rangle) \otimes (\overline{f_2}^* \otimes \overline{x}^* \otimes \overline{z}^*)$$

where v_{1,f_1} (respectively $v_{2,f_1}, v_{1,f_2}, v_{2,f_2}$) is the discrete valuation of $\mathcal{O}_{\overline{\{p\}},r_{1,1}}$ (resp. $\mathcal{O}_{\overline{\{q\}},r_{2,1}}$, $\mathcal{O}_{\overline{\{p\}},r_{1,2}}$, $\mathcal{O}_{\overline{\{q\}},r_{2,2}}$) with $r_{1,1}$ (resp. $r_{2,1}, r_{1,2}, r_{2,2}$) the generic point of the hypersurface of $\overline{\{p\}}$ (resp. $\overline{\{q\}}, \overline{\{p\}}, \overline{\{q\}}$) of equation $f_1 = 0$ (resp. $f_1 = 0$, $f_2 = 0$, $f_2 = 0$).

Oriented fund. classes	$\eta \otimes (\overline{f_1}^* \wedge \overline{g_1}^*)$	|	$\eta \otimes (\overline{f_2}^* \wedge \overline{g_2}^*)$
Seifert classes	$\langle f_1 \rangle \otimes \overline{g_1}^*$	|	$\langle f_2 \rangle \otimes \overline{g_2}^*$
Intersection product of Seifert classes	$\langle 2xf_1f_2 \rangle \otimes (\overline{x-y}^* \wedge \overline{z}^*)$ $+\langle 2(x-y)f_1f_2 \rangle \otimes (\overline{x}^* \wedge \overline{z}^*)$		
Quadratic linking class	$\langle ay \rangle \eta \otimes (\overline{t-ay^2}^* \wedge \overline{x-y}^* \wedge \overline{z}^*)$ $+\langle y \rangle \eta \otimes (\overline{t+y^2}^* \wedge \overline{x}^* \wedge \overline{z}^*)$	$\oplus$	$-\langle ay \rangle \eta \otimes (\overline{t+ay^2}^* \wedge \overline{x-y}^* \wedge \overline{z}^*)$ $-\langle y \rangle \eta \otimes (\overline{t-y^2}^* \wedge \overline{x}^* \wedge \overline{z}^*)$
Ambient quadratic linking class	$\langle ay \rangle \eta \otimes (\overline{t-ay^2}^* \wedge \overline{x-y}^* \wedge \overline{z}^*)$ $+\langle y \rangle \eta \otimes (\overline{t+y^2}^* \wedge \overline{x}^* \wedge \overline{z}^*)$		
Apply ∂	$-\langle a \rangle \eta^2 \otimes (\overline{x}^* \wedge \overline{y}^* \wedge \overline{z}^* \wedge \overline{t}^*)$ $-\eta^2 \otimes (\overline{x}^* \wedge \overline{y}^* \wedge \overline{z}^* \wedge \overline{t}^*)$		
Ambient quad. link. deg.	$-(1+ < a >)$		
Quadratic linking class	$\langle ay \rangle \eta \otimes (\overline{t-ay^2}^* \wedge \overline{x-y}^* \wedge \overline{z}^*)$ $+\langle y \rangle \eta \otimes (\overline{t+y^2}^* \wedge \overline{x}^* \wedge \overline{z}^*)$	$\oplus$	$-\langle ay \rangle \eta \otimes (\overline{t+ay^2}^* \wedge \overline{x-y}^* \wedge \overline{z}^*)$ $-\langle y \rangle \eta \otimes (\overline{t-y^2}^* \wedge \overline{x}^* \wedge \overline{z}^*)$
Apply $(\widetilde{o_1} \oplus \widetilde{o_2}) \circ \varpi$	$-\langle ay \rangle \eta \otimes \overline{x-y}^*$ $-\langle y \rangle \eta \otimes \overline{x}^*$	$\oplus$	$\langle ay \rangle \eta \otimes \overline{x-y}^*$ $+\langle y \rangle \eta \otimes \overline{x}^*$
Apply $\varphi_1^* \oplus \varphi_2^*$	$-\langle av \rangle \eta \otimes \overline{u-v}^*$ $-\langle v \rangle \eta \otimes \overline{u}^*$	$\oplus$	$\langle av \rangle \eta \otimes \overline{u-v}^*$ $+\langle v \rangle \eta \otimes \overline{u}^*$
Apply $\partial \oplus \partial$	$(1+\langle a \rangle)\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$	$\oplus$	$-(1+\langle a \rangle)\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$
Quad. link. deg. couple	$1+ < a >$	$\oplus$	$-(1+ < a >)$

Table 7.2 – The ambient quadratic linking degree and the quadratic linking degree couple of the binary link B_a with $a \in F^*$ (and F of characteristic different from 2).

By Theorem 2.46, the quadratic linking class of the binary link B_a with respect to $(-1,-1)$ is the class in $H^1(Z, \underline{K}_0^{\mathrm{MW}}\{\nu_Z\})$ of the following cycle:

$$\langle f_2 2x\rangle\eta\otimes(\overline{f_1}^*\otimes\overline{x-y}^*\otimes\overline{z}^*)+\langle f_2 2(x-y)\rangle\eta\otimes(\overline{f_1}^*\otimes\overline{x}^*\otimes\overline{z}^*)$$
$$+\langle f_1 2x\rangle\eta\otimes(\overline{f_2}^*\otimes\overline{x-y}^*\otimes\overline{z}^*)+\langle f_1 2(x-y)\rangle\eta\otimes(\overline{f_2}^*\otimes\overline{x}^*\otimes\overline{z}^*)$$

hence of the following cycle:

$$\langle ay\rangle\eta\otimes(\overline{t-ay^2}^*\wedge\overline{x-y}^*\wedge\overline{z}^*)+\langle y\rangle\eta\otimes(\overline{t+y^2}^*\wedge\overline{x}^*\wedge\overline{z}^*)$$
$$-\langle ay\rangle\eta\otimes(\overline{t+ay^2}^*\wedge\overline{x-y}^*\wedge\overline{z}^*)-\langle y\rangle\eta\otimes(\overline{t-y^2}^*\wedge\overline{x}^*\wedge\overline{z}^*)$$

From now on, there is no difficulty in the computations if one chooses y as uniformizing parameter (see Corollary 6.2 and Theorems 6.3 and 6.4). Note that in general, the choice of the uniformizing parameter $\pi'_{p,q,0}$ for the computation of the ambient quadratic linking degree (see Section 6.3) (or the choice of the uniformizing parameter $\pi_{p,q,0}$ for the computation of the quadratic linking degree couple (see Section 6.4)) affects the difficulty of the computations, so that it is not always a good idea to pick the first uniformizing parameter which comes to mind. See Table 7.2 for the ambient quadratic linking degree of the binary link B_a and the quadratic linking degree couple of the binary link B_a together with:

- $\varphi_1 : \mathbb{A}^2_F\setminus\{0\}\to\mathbb{A}^4_F\setminus\{0\}$ is the morphism associated to the morphism of F-algebras $F[x,y,z,t]\to F[u,v]$ which maps x,y,z,t to $u,v,u(u-v),((1+a)u-v)v$ respectively;
- $\varphi_2 : \mathbb{A}^2_F\setminus\{0\}\to\mathbb{A}^4_F\setminus\{0\}$ is the morphism associated to the morphism of F-algebras $F[x,y,z,t]\to F[u,v]$ which maps x,y,z,t to $u,v,-u(u-v),-((1+a)u-v)v$ respectively.

Let us now discuss the values of the invariants of the quadratic linking degree of the binary link B_a.

The rank modulo 2 of the ambient quadratic linking degree of B_a (which is $-(1+<a>)\in\mathrm{W}(F)$, see Table 7.2) is equal to 0 (thus the invariant presented in Proposition 5.25 distinguishes between the Hopf link and the binary links).

The image by Σ_2 of the ambient quadratic linking degree of the binary link B_a is $\langle a\rangle\in\mathrm{W}(F)/(1)$. For instance, if $F=\mathbb{Q}$, Σ_2 distinguishes between all the B_p with p prime numbers since if $p\neq q$ are prime numbers then $\langle p\rangle\in\mathrm{W}(\mathbb{Q})/(1)$ corresponds to $1\in\mathrm{W}(\mathbb{Z}/p\mathbb{Z})\subset\bigoplus_{r\text{ prime}}\mathrm{W}(\mathbb{Z}/r\mathbb{Z})$ and $\langle q\rangle\in\mathrm{W}(\mathbb{Q})/(1)$ corresponds to $1\in\mathrm{W}(\mathbb{Z}/q\mathbb{Z})\subset\bigoplus_{r\text{ prime}}\mathrm{W}(\mathbb{Z}/r\mathbb{Z})$ via the isomorphism $\mathrm{W}(\mathbb{Q})/(1)\to\bigoplus_{r\text{ prime}}\mathrm{W}(\mathbb{Z}/r\mathbb{Z})$ induced by the isomorphism

$W(\mathbb{Q}) \to W(\mathbb{R}) \oplus \bigoplus_{r \text{ prime}} W(\mathbb{Z}/r\mathbb{Z})$ described in Example 2.27. Thus the invariant induced by Σ_2 in Theorem 5.28 can distinguish between infinitely many oriented links.

In the case $F = \mathbb{R}$, the absolute value of the ambient quadratic linking degree of the binary link B_a (which is in $W(\mathbb{R}) \simeq \mathbb{Z}$ via the signature) is equal to 2 if $a > 0$ and is equal to 0 if $a < 0$ (hence the invariant presented in Proposition 5.26 distinguishes between the Hopf link and the binary links, as well as between the binary links with positive parameter and the binary links with negative parameter).

We get the same results for the invariants of each component of the quadratic linking degree couple of the binary link B_a (since the quadratic linking degree couple of B_a (together with φ_1, φ_2) is equal to $(1+ < a > , -(1+ < a >))$, see Table 7.2).

Future work 19 (The values of the quadratic linking degrees)**.** We created the binary links so that their ambient quadratic linking degree would be the class in $W(F)$ of a binary quadratic form, and so that by considering all our binary links with all their possible orientations, we could get the class in $W(F)$ of any binary quadratic form as ambient quadratic linking degree. Similarly, it seems feasible to construct ternary links (whose ambient quadratic linking degree would be the class in $W(F)$ of a ternary quadratic form), so that by considering all these ternary links with all their possible orientations, we would get most classes in $W(F)$ of ternary quadratic forms (if not all) as ambient quadratic linking degrees. More generally, the question arises as to which elements of $W(F)$ can be obtained as the ambient quadratic linking degree of an oriented link (or as a component of the quadratic linking degree couple of an oriented link). It would be interesting to exhibit for each positive even integer k examples of oriented links on which Σ_k (applied to the ambient quadratic linking degree or to a component of the quadratic linking degree couple) takes different values (similarly to what we have done above for Σ_2).

7.3 Torus links

In this section, we define counterparts over $\mathbb{R}$ to the torus links $T(2, 2n)$ (with $n \geq 1$ an integer) from knot theory (see Section 1.4) and compute their ambient quadratic linking degrees and their quadratic linking degree couples. Note that in knot theory $T(2, 2)$ is the Hopf link (see Figure 1.4), $T(2, 4)$ is the Solomon link (see Figure 1.5) and for each $n \geq 3$, $T(2, 2n)$ can be pictured as two intertwined n-gons (see Figure 1.11 for $T(2, 6)$). The similarity between the link we described in Section 7.1 (whose components are of respective equations $x = 0, y = 0$ and $z = 0, t = 0$ in $\mathbb{A}^4_F \setminus \{0\}$) and our counterpart of the Hopf link $T(2,2)$ (whose components are of respective equations $z = x, t = y$ and $z = -x, t = -y$ in $\mathbb{A}^4_{\mathbb{R}} \setminus \{0\}$) is the reason why we called the former the Hopf link.

Let $n \in \mathbb{N}$. Recall that in knot theory one of the components of $T(2,2n)$ is the intersection of $\{(a,b) \in \mathbb{C}^2, b = a^n\}$ with $\mathbb{S}^3_\varepsilon$, the 3-sphere of radius ε, and that the other component of $T(2,2n)$ is the intersection of $\{(a,b) \in \mathbb{C}^2, b = -a^n\}$ with $\mathbb{S}^3_\varepsilon$ (for $\varepsilon > 0$ small enough; see Section 1.4). By writing $a = x + iy$ and $b = z + it$ (with $x, y, z, t \in \mathbb{R}$), the equation $b = a^n$ becomes the system of equations

$$\begin{cases} t = \displaystyle\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} (-1)^k x^{n-2k-1} y^{2k+1} \\ z = \displaystyle\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (-1)^k x^{n-2k} y^{2k} \end{cases}$$

and the equation $b = -a^n$ becomes the system of equations

$$\begin{cases} t = -\displaystyle\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} (-1)^k x^{n-2k-1} y^{2k+1} \\ z = -\displaystyle\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (-1)^k x^{n-2k} y^{2k} \end{cases}$$

From now on, we denote

$$\Sigma_t(x,y) := \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} (-1)^k x^{n-2k-1} y^{2k+1}, f_1 := t - \Sigma_t(x,y), f_2 := t + \Sigma_t(x,y),$$

$$\Sigma_z(x,y) := \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (-1)^k x^{n-2k} y^{2k}, g_1 := z - \Sigma_z(x,y), g_2 := z + \Sigma_z(x,y).$$

Consequently, we define our counterpart over $\mathbb{R}$ to the torus link $T(2,2n)$ as follows:

- Z_1 is the intersection of the closed subscheme of $\mathbb{A}^4_{\mathbb{R}}$ of ideal (f_1, g_1) and of $X := \mathbb{A}^4_{\mathbb{R}} \setminus \{0\}$ (hence is a closed $\mathbb{R}$-subscheme of X; in other words, Z_1 is the closed $\mathbb{R}$-subscheme of $\mathbb{A}^4_{\mathbb{R}} \setminus \{0\}$ given by the equations $f_1 = 0, g_1 = 0$);

- Z_2 is the intersection of the closed subscheme of $\mathbb{A}^4_{\mathbb{R}}$ of ideal (f_2, g_2) and of X (hence is a closed $\mathbb{R}$-subscheme of X; in other words, Z_2 is the closed $\mathbb{R}$-subscheme of $\mathbb{A}^4_{\mathbb{R}} \setminus \{0\}$ given by the equations $f_2 = 0, g_2 = 0$); note that Z_1 and Z_2 are disjoint (see [GW10, Proposition 3.35]);

- $\overline{o_1} = \overline{o_{f_1,g_1}}$ is the orientation class associated to the couple (f_1, g_1) (see Section 6.1; in other words, $\overline{o_1}$ is the class (see Definition 3.22) of the isomorphism $o_{f_1,g_1} : \nu_{Z_1} := \det(\mathcal{N}_{Z_1/X}) \to \mathcal{O}_{Z_1} \otimes \mathcal{O}_{Z_1}$ which maps $\overline{f_1}^* \wedge \overline{g_1}^*$ to $1 \otimes 1$);

- $\overline{o_2} = \overline{o_{f_2,g_2}}$ is the orientation class associated to the couple (f_2, g_2) (see Section 6.1; in other words, $\overline{o_2}$ is the class (see Definition 3.22) of the isomorphism $o_{f_2,g_2} : \nu_{Z_2} := \det(\mathcal{N}_{Z_2/X}) \to \mathcal{O}_{Z_2} \otimes \mathcal{O}_{Z_2}$ which maps $\overline{f_2}^* \wedge \overline{g_2}^*$ to $1 \otimes 1$).

As in Section 7.1, we can compute the ambient quadratic linking degree and the quadratic linking degree couple of the torus links with respect to (j_1, j_2) for all $j_1 \leq -1$ and $j_2 \leq -1$. By using Remarks 4.4, 4.8, 4.11, 5.3, 5.8 and 5.16, we restrict ourselves to $(j_1, j_2) = (-1, -1)$.

For expository reasons, instead of applying Theorem 6.1 to get the quadratic linking class of the torus link $T(2, 2n)$, we go through the different steps which lead to the quadratic linking class.

There is no difficulty in computing the oriented fundamental classes ($\eta \otimes (\overline{f_1}^* \wedge \overline{g_1}^*)$ and $\eta \otimes (\overline{f_2}^* \wedge \overline{g_2}^*)$ respectively) and the Seifert classes ($\langle f_1 \rangle \otimes \overline{g_1}^*$ and $\langle f_2 \rangle \otimes \overline{g_2}^*$ respectively) of the torus link $T(2, 2n)$.

The first difficulty lies in determining the irreducible components of the intersection of the hypersurfaces of $X \setminus Z$ of respective equations $g_1 = 0$ and $g_2 = 0$, i.e. of the closed subscheme of $X \setminus Z$ given by the equations $z - \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (-1)^k x^{n-2k} y^{2k}, z + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (-1)^k x^{n-2k} y^{2k}$. This boils down to finding the irreducible factors of $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (-1)^k x^{n-2k} y^{2k}$. This may seem difficult, but if we remember that this is the real part of $(x + iy)^n$, then the following line of reasoning leads us to the irreducible factors we are seeking.

Denoting $x + iy = \rho e^{i\theta}$ with $x, y, \theta \in \mathbb{R}$ and $\rho > 0$, we have:

$$\begin{aligned} \Re((x+iy)^n) = 0 &\Leftrightarrow \cos(n\theta) = 0 \\ &\Leftrightarrow \theta = \frac{(2j+1)\pi}{2n} \quad \text{mod } 2\pi \text{ for some } j \in \{0, \dots, 2n-1\} \\ &\Leftrightarrow x = \tan\left(\frac{(n-1-2j)\pi}{2n}\right) y \text{ for some } j \in \{0, \dots, n-1\} \end{aligned}$$

For this last equivalence, visualize the usual orthogonal triangle (of sides of lengths $|x|$, $|y|$ and ρ) and take the tangent of the angle between the side of length $|y|$ and the hypotenuse (also note that the tangent is π-periodic).

From now on, for every $j \in \{0, \dots, n-1\}$, we denote $\theta_j := \dfrac{(n-1-2j)\pi}{2n}$.

Since $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k}(-1)^k x^{n-2k} y^{2k}$ is equal to $\prod_{j=0}^{n-1}(x - \tan(\theta_j)y)$ and since the $\tan(\theta_j)$, with $j \in \{0, \dots, n-1\}$, are distinct (as they are the roots of the polynomial $(x+i)^n + (x-i)^n$ which is coprime with its derivative), the closed subscheme of $X \setminus Z$ given by the equations $z - \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k}(-1)^k x^{n-2k} y^{2k}$, $z + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k}(-1)^k x^{n-2k} y^{2k}$ has n irreducible components, whose generic points we denote by $P_0, \dots, P_{n-1}$, where for all $j \in \{0, \dots, n-1\}$, the component of generic point P_j is given in $X \setminus Z$ by the equations $z = 0, x = \tan(\theta_j)y$.

It follows from Corollary 3.32 that the intersection product of $\langle f_1 \rangle \otimes \overline{g_1}^*$ (over the generic point of the hypersurface of $X \setminus Z$ of equation $g_1 = 0$) with $\langle f_2 \rangle \otimes \overline{g_2}^*$ (over the generic point of the hypersurface of $X \setminus Z$ of equation $g_2 = 0$) is the class in $H^2(X \setminus Z, \underline{K}_2^{\mathrm{MW}})$ of the cycle

$$\sum_{j=0}^{n-1} (m_j)_\epsilon \langle f_1 f_2 u_j \rangle \otimes (\overline{\pi_j}^* \otimes \overline{g_1}^*)$$

where $(m_j)_\epsilon \langle f_1 f_2 u_j \rangle \otimes (\overline{\pi_j}^* \wedge \overline{g_1}^*)$ lives over P_j (which corresponds to the equations $z = 0, x = \tan(\theta_j)y$), π_j is a uniformizing parameter for $\mathcal{O}_{X \setminus Z, P_j}/(g_1)$, u_j is a unit in $\mathcal{O}_{X \setminus Z, P_j}/(g_1)$ and $m_j \in \mathbb{Z}$ is an integer such that $g_2 = u_j \pi_j^{m_j} \in \mathcal{O}_{X \setminus Z, P_j}/(g_1)$. The second difficulty (or rather difficulty lying in wait) in our computations lies in the choice of π_j (similarly to the difficulty highlighted in Section 7.2). We choose $\pi_j = x - \tan(\theta_j)y \in \mathbb{R}[x,y,z,t]_{(z, x - \tan(\theta_j)y)}/(z - \prod_{i=0}^{n-1}(x - \tan(\theta_i)y)) \simeq \mathbb{R}[x,y,t]_{(x - \tan(\theta_j)y)}$ (hence $u_q = 2\prod_{i \neq j, i=0}^{n-1}(x - \tan(\theta_i)y)$ and $m_q = 1$) in order to have simple computations in what follows.

Thus, the intersection product of $\langle f_1 \rangle \otimes \overline{g_1}^*$ with $\langle f_2 \rangle \otimes \overline{g_2}^*$ is the class in $H^2(X \setminus Z, \underline{K}_2^{\mathrm{MW}})$ of the cycle

$$\sum_{j=0}^{n-1} \langle f_1 f_2\, 2 \prod_{i \neq j, i=0}^{n-1} (x - \tan(\theta_i)y) \rangle \otimes (\overline{x - \tan(\theta_j)y}^* \otimes \overline{z - \prod_{i=0}^{n-1}(x - \tan(\theta_i)y)}^*)$$

hence of the cycle

$$\sum_{j=0}^{n-1} \langle f_1 f_2\, 2 \prod_{i \neq j, i=0}^{n-1} (x - \tan(\theta_i)y) \rangle \otimes (\overline{x - \tan(\theta_j)y}^* \otimes \overline{z}^*)$$

It follows from Theorem 2.46 that the quadratic linking class of the torus link $T(2,2n)$ with respect to $(-1,-1)$ is the class in $H^1(Z, \underline{K}_0^{\mathrm{MW}}\{\nu_Z\})$ of the cycle

$$\sum_{j=0}^{n-1} \langle f_2 2 \prod_{i\neq j, i=0}^{n-1} (x-\tan(\theta_i)y)\rangle \eta \otimes (\overline{f_1}^* \otimes \overline{x-\tan(\theta_j)y}^* \otimes \overline{z}^*)$$
$$+\sum_{k=0}^{n-1} \langle f_1 2 \prod_{l\neq k, l=0}^{n-1} (x-\tan(\theta_l)y)\rangle \eta \otimes (\overline{f_2}^* \otimes \overline{x-\tan(\theta_k)y}^* \otimes \overline{z}^*)$$

Let us now turn to the ambient quadratic linking class and the ambient quadratic linking degree.

It follows from our computation of the quadratic linking class and Definition 5.1 (or Corollary 6.2) that the ambient quadratic linking class of the torus link $T(2,2n)$ with respect to $(-1,-1)$ is the class in $H^3(X, \underline{K}_2^{\mathrm{MW}})$ of the cycle

$$\sum_{j=0}^{n-1} \langle f_2 2 \prod_{i\neq j, i=0}^{n-1} (x-\tan(\theta_i)y)\rangle \eta \otimes (\overline{f_1}^* \otimes \overline{x-\tan(\theta_j)y}^* \otimes \overline{z}^*)$$

hence of the cycle

$$\sum_{j=0}^{n-1} \langle (\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} (-1)^k (\tan(\theta_j))^{n-2k-1}) (\prod_{i\neq j, i=0}^{n-1} (\tan(\theta_j)-\tan(\theta_i)))y\rangle \eta$$
$$\otimes (\overline{f_1}^* \otimes \overline{x-\tan(\theta_j)y}^* \otimes \overline{z}^*)$$

We could apply Theorem 6.3 to get the ambient quadratic linking degree but, for expository purposes, we will rather go through the different steps which lead to the ambient quadratic linking degree from the ambient quadratic linking class. Recall Definition 5.7.

The first step consists in applying the boundary map $\partial : H^3(\mathbb{A}_F^4 \setminus \{0\}, \underline{K}_2^{\mathrm{MW}}) \to H^0(\{0\}, \underline{K}_{-2}^{\mathrm{MW}}\{\det(\mathcal{N}_{\{0\}/\mathbb{A}_F^4})\})$ to the ambient quadratic linking class. This gives the class in $H^0(\{0\}, \underline{K}_{-2}^{\mathrm{MW}}\{\det(\mathcal{N}_{\{0\}/\mathbb{A}_F^4})\})$ of the following cycle:

$$\sum_{j=0}^{n-1} \langle (\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} (-1)^k (\tan(\theta_j))^{n-2k-1}) \prod_{i\neq j, i=0}^{n-1} (\tan(\theta_j)-\tan(\theta_i))\rangle \eta^2$$
$$\otimes (\overline{y}^* \otimes \overline{f_1}^* \otimes \overline{x-\tan(\theta_j)y}^* \otimes \overline{z}^*)$$

hence of the following cycle:

$$\sum_{j=0}^{n-1}\langle -(\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2k+1}(-1)^k(\tan(\theta_j))^{n-2k-1})\prod_{i\neq j,i=0}^{n-1}(\tan(\theta_j)-\tan(\theta_i))\rangle\eta^2 \otimes(\overline{x}^*\wedge\overline{y}^*\wedge\overline{z}^*\wedge\overline{t}^*)$$

The second step consists in applying the isomorphism $H^0(\{0\},\underline{K}^{\mathrm{MW}}_{-2}\{\det(\mathcal{N}_{\{0\}/\mathbb{A}^4_F})\}) \to H^0(\{0\},\underline{K}^{\mathrm{MW}}_{-2}) = K^{\mathrm{MW}}_{-2}(\mathbb{R})$ (denoted $\widetilde{o}$ in Notation 3.36) induced by the orientation of $\mathbb{A}^4_F\setminus\{0\}$, which gives:

$$\sum_{j=0}^{n-1}\langle -(\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2k+1}(-1)^k(\tan(\theta_j))^{n-2k-1})\prod_{i\neq j,i=0}^{n-1}(\tan(\theta_j)-\tan(\theta_i))\rangle\eta^2$$

The last step consists in applying the isomorphism $\gamma_{-2}: K^{\mathrm{MW}}_{-2}(\mathbb{R})\to \mathrm{W}(\mathbb{R})$, which gives

$$\sum_{j=0}^{n-1} < -(\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2k+1}(-1)^k(\tan(\theta_j))^{n-2k-1})\prod_{i\neq j,i=0}^{n-1}(\tan(\theta_j)-\tan(\theta_i)) >$$

Recall that in $\mathrm{W}(\mathbb{R})$, $< a >= 1$ if a is positive and $< a >= -1$ if a is negative. Since $\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2k+1}(-1)^k(\tan(\theta_j))^{n-2k-1}$ is the imaginary part of $(\tan(\theta_j)+i)^n$, it has the same sign as $\sin(\frac{(2j+1)\pi}{2})$ hence is positive if j is even and negative if j is odd. Also note that for all $k\in\{0,\dots,n-1\}$, $-\frac{\pi}{2}<\theta_k<\frac{\pi}{2}$ so that for all $i<j$, $\tan(\theta_j)-\tan(\theta_i)$ is negative and for all $i>j$, $\tan(\theta_j)-\tan(\theta_i)$ is positive, hence $\prod_{i\neq j,i=0}^{n-1}(\tan(\theta_j)-\tan(\theta_i))$ is positive if j is even and negative if j is odd. Therefore, for all $j\in\{0,\dots,n-1\}$:

$$< -(\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2k+1}(-1)^k(\tan(\theta_j))^{n-2k-1})\prod_{i\neq j,i=0}^{n-1}(\tan(\theta_j)-\tan(\theta_i)) >= -1$$

and it follows that $-n\in\mathrm{W}(\mathbb{R})\simeq\mathbb{Z}$ (via the signature) is the ambient quadratic linking degree of the torus link $T(2,2n)$.

Note that if we change one of the orientation classes of the torus link $T(2,2n)$ then we get n as ambient quadratic linking degree (similarly to the linking number of the topological torus link $T(2,2n)$ which is equal to n) and that if we change both orientation classes then we get $-n$ as ambient quadratic linking degree (see Proposition 5.11 and Remark 4.36). In any case, the absolute value of the ambient quadratic linking degree of the torus link $T(2,2n)$ is equal to n.

Let us now turn to the quadratic linking degree couple. Recall that the quadratic linking degree couple (see Definition 5.15) is associated to oriented links of a certain type (see Definition 5.13), so that we need to introduce parametrisations φ_1 and φ_2. We set the following.

- $\varphi_1 : \mathbb{A}^2_{\mathbb{R}} \setminus \{0\} \to \mathbb{A}^4_{\mathbb{R}} \setminus \{0\}$ is the morphism associated to the morphism of $\mathbb{R}$-algebras $\mathbb{R}[x, y, z, t] \to \mathbb{R}[u, v]$ which maps x, y, z, t to $u, v, \Sigma_z(u, v), \Sigma_t(u, v)$ respectively;
- $\varphi_2 : \mathbb{A}^2_{\mathbb{R}} \setminus \{0\} \to \mathbb{A}^4_{\mathbb{R}} \setminus \{0\}$ is the morphism associated to the morphism of $\mathbb{R}$-algebras $\mathbb{R}[x, y, z, t] \to \mathbb{R}[u, v]$ which maps x, y, z, t to $u, v, -\Sigma_z(u, v), -\Sigma_t(u, v)$ respectively.

The computations are similar to the ones for the ambient quadratic linking degree and give $(n, -n) \in \mathrm{W}(\mathbb{R}) \oplus \mathrm{W}(\mathbb{R}) \simeq \mathbb{Z} \oplus \mathbb{Z}$ as quadratic linking degree couple. Thus, each component of the quadratic linking degree couple of the torus link $T(2, 2n)$ has the same absolute value as the ambient quadratic linking degree of the torus link $T(2, 2n)$.

Future work 20 (More general torus links)**.** It would not be much more difficult to compute the ambient quadratic linking degree of a counterpart over $\mathbb{R}$ to $T(2p, 2q)$ with p and q coprime (see Section 1.4). Note that when $p, q \geq 2$ the components of $T(2p, 2q)$ would no longer be isomorphic to $\mathbb{A}^2_{\mathbb{R}} \setminus \{0\}$ but we do not need this to compute the ambient quadratic linking degree (see Definition 5.7). It would also be interesting to study counterparts to other links than torus links (see Section 1.5).

We end this chapter with the two following future works.

Future work 21 (Examples over other specific fields than $\mathbb{R}$)**.** We plan to study examples over specific fields other than the field $\mathbb{R}$ of real numbers and see whether we can get any element of the Witt ring of these fields as the ambient quadratic linking degree of an oriented link. (Note that this is the case for $\mathbb{R}$ since the ambient quadratic linking degree of the binary link B_a is equal to 0 if a is negative (see Section 7.2) and for each $n \in \mathbb{N}$, the ambient quadratic linking degree of $T(2, 2n)$ is equal to $-n$ and the ambient quadratic linking degree of $T(2, 2n)$ with its first orientation class reversed is equal to n.)

Future work 22 (Examples in other cases)**.** We also plan to study examples in several other interesting cases than $\mathbb{A}^2_F \setminus \{0\} \sqcup \mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$ (see the bullets in the beginning of Chapter 6). In addition to these cases, there are also cases in which we would want to compute the ambient quadratic linking degree or the quadratic linking degree couple but for which we do not yet have an intersection formula (see Future works 1 and 18) or for which we do not yet have an explicit isomorphism between the Rost-Schmid group in question and $\mathrm{W}(F)$, $\mathrm{GW}(F)$ or $K_1^{\mathrm{MW}}(F)$ (see Future works 2 and 3).

Bibliography

[Ada94] Colin C. Adams. *The knot book*. W. H. Freeman and Company, New York, 1994. An elementary introduction to the mathematical theory of knots.

[ADF16] Aravind Asok, Brent Doran, and Jean Fasel. Smooth Models of Motivic Spheres and the Clutching Construction. *International Mathematics Research Notices*, 2017(6):1890–1925, 06 2016.

[AF14] Aravind Asok and Jean Fasel. Algebraic vector bundles on spheres. *Journal of Topology*, 7(3):894–926, 2014.

[AF22] Aravind Asok and Jean Fasel. Euler class groups and motivic stable cohomotopy. *Journal of the European Mathematical Society*, 24:2775–2822, 2022.

[AK81] Selman Akbulut and Henry King. All knots are algebraic. *Commentarii Mathematici Helvetici*, 56:339–351, 1981.

[BFS14] Michael Bush, Katelyn French, and Joseph Smith. Total Linking Numbers of Torus Links and Klein Links. *Rose–Hulman Undergraduate Mathematics Journal*, 15, 2014.

[BM60] Armand Borel and John C. Moore. Homology theory for locally compact spaces. *Michigan Math. J.*, 7:137–159, 1960.

[BM00] Jean Barge and Fabien Morel. Groupe de Chow des cycles orientés et classe d'Euler des fibrés vectoriels. *Comptes Rendus De L'Académie Des Sciences Serie I-mathematique - C R ACAD SCI SER I MATH*, 330:287–290, 02 2000.

[Bod22] Benjamin Bode. All links are semiholomorphic, 2022. `https://arxiv.org/abs/2211.12329`.

[BOS02] Marco Baiesi, Enzo Orlandini, and A. Stella. Topological and geometrical entanglement in a model of circular DNA undergoing denaturation. *Physics of Condensed Matter*, 28:467–473, 08 2002.

[Bre97] Glen E. Bredon. *Topology and geometry*, volume 139 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997. Corrected third printing of the 1993 original.

[Cro04] Peter R. Cromwell. *Knots and links*. Cambridge University Press, Cambridge, 2004.

[DDØ22] Frédéric Déglise, Adrien Dubouloz, and Paul Arne Østvær. Punctured tubular neighborhoods and stable homotopy at infinity, 2022. `https://arxiv.org/abs/2206.01564`.

[Dég23] Frédéric Déglise. Notes on Milnor-Witt K-theory, 2023. `https://arxiv.org/abs/2305.18609`.

[DF14] Adrien Dubouloz and David Finston. On exotic affine 3-spheres. *Journal of Algebraic Geometry*, 23:445–469, 01 2014.

[DFJ22] Frédéric Déglise, Niels Feld, and Fangzhou Jin. Perverse homotopy heart and MW-modules, 2022. `https://arxiv.org/abs/2210.14832`.

[Dol95] Albrecht Dold. *Lectures on algebraic topology*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1972 edition.

[dSP11] Clément de Seguins Pazzis. *Invitation aux formes quadratiques*. Mathématiques en devenir. Calvage et Mounet, Paris, 2011.

[Dur99] Alan Durfee. Singularities. In *History of topology*, pages 417–434. North-Holland, Amsterdam, 1999.

[EH16] David Eisenbud and Joe Harris. *3264 and All That: A Second Course in Algebraic Geometry*. Cambridge University Press, 2016.

[EKM08] Richard Elman, Nikita Karpenko, and Alexander Merkurjev. *The algebraic and geometric theory of quadratic forms*, volume 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

[Fas13] Jean Fasel. The projective bundle theorem for Ij-cohomology. *Journal of K-Theory*, 11(2):413–464, 2013.

[Fas20] Jean Fasel. Lectures on Chow-Witt groups. In Federico Binda, Marc Levine, Manh Toan Nguyen, and Oliver Röndigs, editors, *Motivic homotopy theory and refined enumerative geometry*, volume 745, pages 83–122. American Mathematical Soc., 2020.

[Fel20] Niels Feld. Milnor-Witt cycle modules. *Journal of Pure and Applied Algebra*, 224(7):106298, 2020.

[Fel21] Niels Feld. *Milnor-Witt sheaves and modules*. PhD thesis, Université Grenoble-Alpes, 2021. https://theses.hal.science/tel-03225375.

[Ful98] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[GW10] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I*. Advanced Lectures in Mathematics. Vieweg + Teubner, Wiesbaden, 2010. Schemes with examples and exercises.

[Hae62] André Haefliger. Knotted $(4k-1)$-spheres in $6k$-space. *Ann. of Math. (2)*, 75:452–466, 1962.

[Har77] Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[Hos05] Jim Hoste. The enumeration and classification of knots and links. In *Handbook of knot theory*, pages 209–232. Elsevier B. V., Amsterdam, 2005.

[HTW98] Jim Hoste, Morwen Thistlethwaite, and Jeff Weeks. The first 1,701,936 knots. *Math. Intelligencer*, 20(4):33–48, 1998.

[HWXZ21] Jens Hornbostel, Matthias Wendt, Heng Xie, and Marcus Zibrowius. The real cycle class map. *Ann. K-Theory*, 6(2):239–317, 2021.

[Lam05] T. Y. Lam. *Introduction to quadratic forms over fields*, volume 67 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2005.

[Lem23] Clémentine Lemarié–Rieusset. The quadratic linking degree, 2023. https://arxiv.org/abs/2210.11048.

[Lic97] W. B. Raymond Lickorish. *An introduction to knot theory*, volume 175 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1997.

[Mas78] William S. Massey. *Homology and cohomology theory*. Monographs and Textbooks in Pure and Applied Mathematics, Vol. 46. Marcel Dekker, Inc., New York-Basel, 1978. An approach based on Alexander-Spanier cochains.

[MH73] John Milnor and Dale Husemoller. *Symmetric bilinear forms*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73. Springer-Verlag, New York-Heidelberg, 1973.

[Mil54] John Milnor. Link groups. *Ann. of Math. (2)*, 59:177–195, 1954.

[Mil69] John Milnor. *Singular Points of Complex Hypersurfaces (AM-61), Volume 61*. Princeton University Press, Princeton, 1969.

[Mil70] John Milnor. Algebraic K-theory and quadratic forms. *Invent. Math.*, 9:318–344, 1970.

[Mor03] Fabien Morel. An introduction to A1-homotopy theory. *ICTP Lecture Notes*, 15:357–441, 01 2003.

[Mor12] Fabien Morel. $\mathbb{A}^1$-*algebraic topology over a field*, volume 2052. Springer, 2012.

[Mur96] Kunio Murasugi. *Knot theory and its applications*. Birkhäuser Boston, Inc., Boston, MA, 1996. Translated from the 1993 Japanese original by Bohdan Kurpita.

[MV99] Fabien Morel and Vladimir Voevodsky. A1-homotopy theory of schemes. *Publications Mathématiques de l'IHÉS*, 90:45–143, 1999. http://www.numdam.org/item/PMIHES_1999__90__45_0.pdf.

[Oga18] Eiji Ogasa. Introduction to high dimensional knots, 2018. https://arxiv.org/abs/1304.6053.

[Ore21] S. Yu. Orevkov. Multivariate signatures of iterated torus links. *Funktsional. Anal. i Prilozhen.*, 55(1):73–92, 2021.

[Rol90] Dale Rolfsen. *Knots and links*, volume 7 of *Mathematics Lecture Series*. Publish or Perish, Inc., Houston, TX, 1990. Corrected reprint of the 1976 original.

[Ros96] Markus Rost. Chow groups with coefficients. *Doc. Math.*, 1(16):319–393, 1996. `https://www.emis.de/journals/DMJDMV/vol-01/16.pdf`.

[Rot88] Joseph J. Rotman. *An introduction to algebraic topology*, volume 119 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1988.

[Sch85] Winfried Scharlau. *Quadratic and Hermitian forms*, volume 270 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1985.

[Sch98] Manfred Schmid. *Wittringhomologie*. PhD thesis, Universität Regensburg, 1998. `https://www.math.uni-bielefeld.de/~rost/schmid.html`.

[ST80] Herbert Seifert and William Threlfall. *Seifert and Threlfall: A Textbook of Topology*, volume 89 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980. Translated from the German edition of 1934 by Michael A. Goldman, With a preface by Joan S. Birman, With "Topology of 3-dimensional fibered spaces" by Seifert, Translated from the German by Wolfgang Heil.

[Sta63] John Stallings. On topologically unknotted spheres. *Ann. of Math. (2)*, 77:490–503, 1963.

[Yan21] Nanjun Yang. Projective bundle theorem in MW-motivic cohomology. *Doc. Math.*, 26:1045–1083, 2021.

[Zee63] E. C. Zeeman. Unknotting combinatorial balls. *Ann. of Math. (2)*, 78:501–526, 1963.

Index of notations

General index