Motivic Linking

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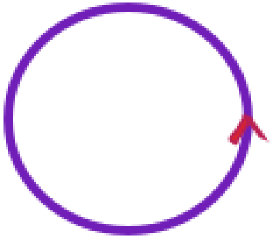

Figure: The unknot

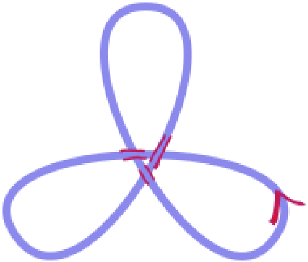

Figure: The trefoil knot

Knot theory in a nutshell

Topological objects of interest are knots and links.

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- An **oriented knot** is a knot with a "continuous" local trivialization of its tangent bundle, or equivalently of its normal bundle (the ambient space being oriented). There are two orientation classes.
- A **link** is a finite union of disjoint knots. A link is **oriented** if all its components (i.e. its knots) are oriented.

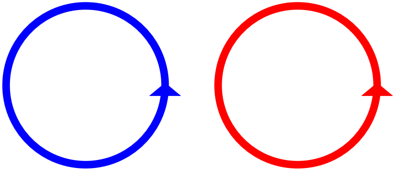


Figure: The unlink with two components (linking number $= 0$)

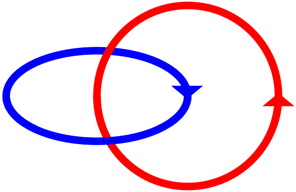


Figure: The Hopf link (linking number $= 1$)

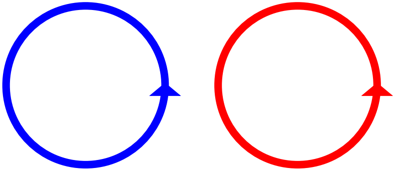


Figure: The unlink with two components (linking number $= 0$)

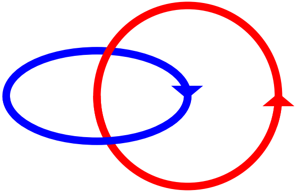


Figure: The Hopf link (linking number $= 1$)

The **linking number** of an oriented link with two components is the number of times one of the components turns around the other component (the sign indicating the direction).

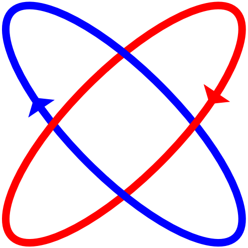


Figure: The Solomon link (linking number $= 2$)

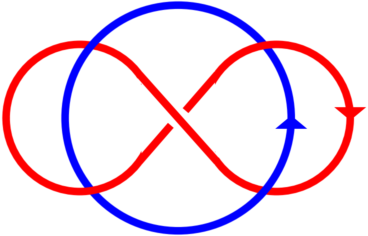


Figure: The Whitehead link (linking number $= 0$)

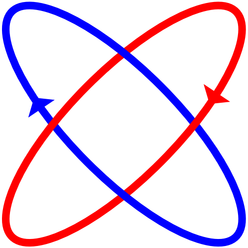


Figure: The Solomon link (linking number = 2)

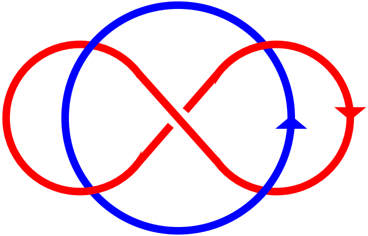
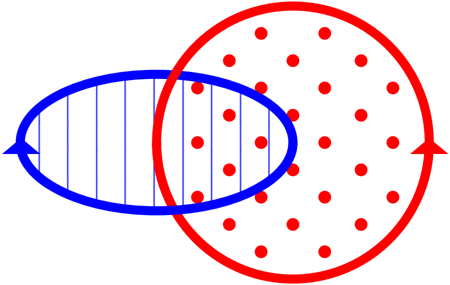


Figure: The Whitehead link (linking number = 0)

The linking number is a complete invariant of oriented links with two components for link homotopy (i.e. $L = K_1 \sqcup K_2$ and $L' = K_1' \sqcup K_2'$ are link homotopic if and only if they have the same linking number).

Defining the linking number: Seifert surfaces



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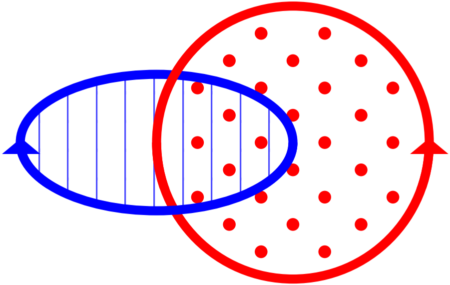
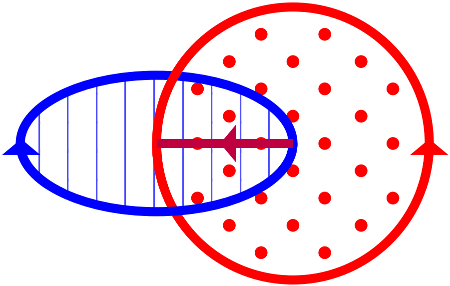


The class S_1 in $H^1(\mathbb{S}^3 \setminus L) \simeq H_2^{\mathrm{BM}}(\mathbb{S}^3, L)$ of Seifert surfaces of the oriented knot K_1 is the unique class that is sent by the boundary map to the (oriented) fundamental class of K_1 in $H^0(K_1) \subset H^0(L)$.

Defining the linking number: intersection of S. surfaces



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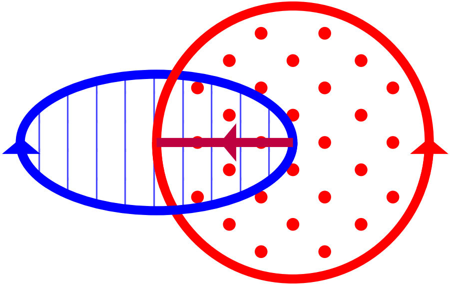
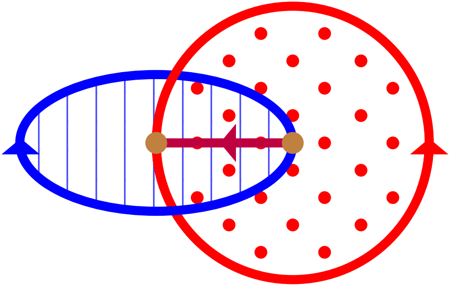


This corresponds to the cup-product $S_1 \cup S_2 \in H^2(\mathbb{S}^3 \setminus L)$.

Defining the linking number: boundary of int. of S. surf.



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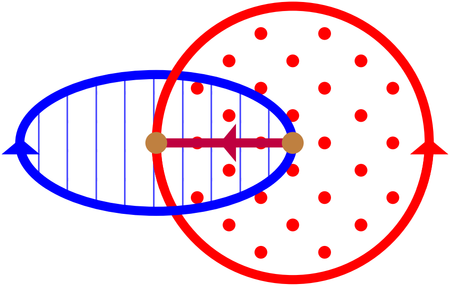

This corresponds to $\partial(S_1 \cup S_2) \in H^1(L) \simeq H^1(K_1) \oplus H^1(K_2)$, which we call the **linking class**.

The linking number

The linking number

The linking number of L is the image of the part of the linking class which is in $H^1(K_1)$ by the composite of the morphism $(i_1)_* : H^1(K_1) \to H^3(\mathbb{S}^3)$ induced by the inclusion $i_1 : K_1 \to \mathbb{S}^3$ and of the "right-hand rule" isomorphism $r : H^3(\mathbb{S}^3) \to \mathbb{Z}$.

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What about the other number?

The image of the part of the linking class which is in $H^1(K_2)$ by the composite of the morphism $(i_2)_* : H^1(K_2) \to H^3(\mathbb{S}^3)$ induced by the inclusion $i_2 : K_2 \to \mathbb{S}^3$ and of the isomorphism $r : H^3(\mathbb{S}^3) \to \mathbb{Z}$ is equal to the opposite of the linking number.

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The absolute value of the linking number does not depend on the orientation of the link.

The linking couple

The linking couple

The linking couple is the image of the linking class by the isomorphism $h_1 \oplus h_2 : H^1(K_1) \oplus H^1(K_2) \to \mathbb{Z} \oplus \mathbb{Z}$ which is induced by the volume forms ω_{K_1} of K_1 and ω_{K_2} of K_2.

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Important fact

The linking couple is equal to $(\pm n, \pm n)$ with n the linking number.

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Important fact

The linking couple is equal to $(\pm n, \pm n)$ with n the linking number.

The absolute value of either component of the linking couple is equal to the absolute value of the linking number.

Generalisation of the linking number

Linking of spheres

The linking class, the linking number and the linking couple can be defined in a similar manner to study the linking of a p-dimensional sphere $\mathbb{S}^p$ and a q-dimensional sphere $\mathbb{S}^q$ in a $(p+q+1)$-dimensional sphere $\mathbb{S}^{p+q+1}$. We are particularly interested in the case $p = q \geq 1$.

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The linking number is studied in these cases but also for other manifolds (see *Lehrbuch der Topologie* or its translation *A textbook of topology* by Seifert and Threlfall for a general definition).

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Linking of circles in the real projective 3-space

We are particularly interested in the linking of circles in $\mathbb{RP}^3$ (see for instance *Classification of links in* $\mathbb{RP}^3$ *with at most six crossings* by Julia Drobotukhina (now Julia Viro)).

Contents

Oriented links in algebraic geometry

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Link with two components of type $(\mathbb{A}^2_F \setminus \{0\}, \mathbb{A}^2_F \setminus \{0\}, \mathbb{A}^4_F \setminus \{0\})$

A link with two components of type $(\mathbb{A}^2_F \setminus \{0\}, \mathbb{A}^2_F \setminus \{0\}, \mathbb{A}^4_F \setminus \{0\})$ is a couple of closed immersions $\varphi_i : \mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$ with disjoint images Z_i (where $i \in \{1, 2\}$). The morphisms φ_1, φ_2 are called parametrisations of Z_1, Z_2 respectively.

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An oriented link with two components of type $(\mathbb{A}^2_F \setminus \{0\}, \mathbb{A}^2_F \setminus \{0\}, \mathbb{A}^4_F \setminus \{0\})$ is a link with two components $(\varphi_1 : \mathbb{A}^2_F \setminus \{0\} \to Z_1, \varphi_2 : \mathbb{A}^2_F \setminus \{0\} \to Z_2)$ together with an orientation class $\overline{o_1}$ of Z_1 and an orientation class $\overline{o_2}$ of Z_2.

An orientation o_i of Z_i is an isomorphism from the determinant (i.e. the maximal exterior power) of the normal sheaf $\mathcal{N}_{Z_i/\mathbb{A}^4_F \setminus \{0\}}$ of Z_i in $\mathbb{A}^4_F \setminus \{0\}$ to the tensor product of an invertible $\mathcal{O}_{Z_i}$-module $\mathcal{L}_i$ with itself:

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More concretely

In our examples, an orientation of a knot will be given by the choice of a first polynomial equation f and a second polynomial equation g such that the knot corresponds to $\{f = 0, g = 0\}$.

Orientation classes

Two orientations $o_i : \nu_{Z_i} \to \mathcal{L}_i \otimes \mathcal{L}_i$ and $o_i' : \nu_{Z_i} \to \mathcal{L}_i' \otimes \mathcal{L}_i'$ of Z_i represent the same orientation class of Z_i if there exists an isomorphism $\psi : \mathcal{L}_i \simeq \mathcal{L}_i'$ such that $(\psi \otimes \psi) \circ o_i = o_i'$.

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Let $i \in \{1, 2\}$. The orientation classes of Z_i are parametrized by the elements of $F^*/(F^*)^2$ (where $(F^*)^2 = \{a \in F^*, \exists b \in F^*, a = b^2\}$).

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If $F = \mathbb{C}$ then $F^*/(F^*)^2$ has one element.
If $F = \mathbb{Q}$ then $F^*/(F^*)^2$ has infinitely many elements (the classes of the integers without square factors).

The Hopf link in algebraic geometry

We fix coordinates x, y, z, t for $\mathbb{A}^4_F$ and u, v for $\mathbb{A}^2_F$ once and for all.

- The image of the Hopf link:

$$\{x = 0, y = 0\} \sqcup \{z = 0, t = 0\} \subset \mathbb{A}^4_F \setminus \{0\}$$

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$$\varphi_1 : (x, y, z, t) \leftrightarrow (0, 0, u, v), \varphi_2 : (x, y, z, t) \leftrightarrow (u, v, 0, 0)$$

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- The orientation of the Hopf link:

$$o_1 : \overline{x}^* \wedge \overline{y}^* \mapsto 1 \otimes 1, o_2 : \overline{z}^* \wedge \overline{t}^* \mapsto 1 \otimes 1$$

A variant of the Hopf link

- The image is the same as the image of the Hopf link:

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- The orientation is different:

$$o_1 : \overline{x - y}^* \wedge \overline{y}^* \mapsto 1 \otimes 1, o_2 : \overline{z}^* \wedge \overline{at}^* \mapsto 1 \otimes 1$$

The singular complex and the Rost-Schmid complex

Classical algebraic topology

Each topological space X has a singular cochain complex:

$$\dots \longrightarrow \mathcal{C}^i(X) \longrightarrow \mathcal{C}^{i+1}(X) \longrightarrow \dots$$

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Motivic algebraic topology

Each smooth F-scheme X has a Rost-Schmid complex *for each integer $j \in \mathbb{Z}$ and invertible $\mathcal{O}_X$-module $\mathcal{L}$*:

$$\begin{array}{c} \dots \longrightarrow \bigoplus_{p \in X^{(i)}} K^{\mathrm{MW}}_{j-i}(\kappa(p)) \otimes_{\mathbb{Z}[\kappa(p)^*]} \mathbb{Z}[(\nu_p \otimes \mathcal{L}_{|p}) \setminus \{0\}] \\ \downarrow \\ \bigoplus_{q \in X^{(i+1)}} K^{\mathrm{MW}}_{j-i-1}(\kappa(q)) \otimes_{\mathbb{Z}[\kappa(q)^*]} \mathbb{Z}[(\nu_q \otimes \mathcal{L}_{|q}) \setminus \{0\}] \longrightarrow \dots \end{array}$$

The singular cohomology ring and the Rost-Schmid ring

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The i-th cohomology group $H^i(X)$ of X is the i-th cohomology group of the singular cochain complex of X.

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Motivic algebraic topology

The i-th Rost-Schmid group $H^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L}\})$ of X with respect to j and $\mathcal{L}$ is the i-th cohomology group of the Rost-Schmid complex of X w.r.t. j and $\mathcal{L}$. We denote $H^i(X, \underline{K}_j^{\mathrm{MW}}) := H^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{O}_X\})$.

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Classical algebraic topology

Let (Z, i, X, j, U) be a boundary triple. We have the following long exact sequence (where ∂ is the boundary map):

$$\dots \longrightarrow H^n(Z) \xrightarrow{i_*} H^{n+d_X-d_Z}(X) \xrightarrow{j^*} H^{n+d_X-d_Z}(U) \xrightarrow{\partial} H^{n+1}(Z) \longrightarrow$$

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Motivic algebraic topology

Let (Z, i, X, j, U) be a boundary triple. We have the localization long exact sequence (where ∂ is the boundary map):

$$\ldots \longrightarrow H^n(Z, \underline{K}^{\mathrm{MW}}_m\{\nu_Z\}) \xrightarrow{i_*} H^{n+d_X-d_Z}(X, \underline{K}^{\mathrm{MW}}_{m+d_X-d_Z}) \xrightarrow{j^*}$$

$$\xrightarrow{j^*} H^{n+d_X-d_Z}(U, \underline{K}^{\mathrm{MW}}_{m+d_X-d_Z}) \xrightarrow{\partial} H^{n+1}(Z, \underline{K}^{\mathrm{MW}}_m\{\nu_Z\}) \longrightarrow \ldots$$

Classical algebraic topology

Let $n \geq 2$ and $i \geq 0$ be integers. The singular cohomology group

$$H^i(\mathbb{S}^{n-1}) \text{ is isomorphic to } \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} & \text{if } i = n-1 \\ 0 & \text{otherwise} \end{cases}.$$

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Motivic algebraic topology

Let $n \geq 2$, $i \geq 0, j \in \mathbb{Z}$ be integers. The Rost-Schmid group

$$H^i(\mathbb{A}^n_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_j) \text{ is isomorphic to } \begin{cases} K^{\mathrm{MW}}_j(F) & \text{if } i = 0 \\ K^{\mathrm{MW}}_{j-n}(F) & \text{if } i = n-1 \\ 0 & \text{otherwise} \end{cases}.$$

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In particular, $H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_0) \simeq K^{\mathrm{MW}}_{-2}(F) \simeq \mathrm{W}(F)$ and $H^3(\mathbb{A}^4_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_2) \simeq K^{\mathrm{MW}}_{-2}(F) \simeq \mathrm{W}(F)$. These iso. are not canonical.

Notations

- Let $L = K_1 \sqcup K_2$ be an oriented link (in knot theory).
- Let $\mathscr{L}$ be an oriented link with two components (in motivic knot theory), i.e. a couple of closed immersions $\varphi_i : \mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$ with disjoint images Z_i and orientation classes $\overline{o_i}$ (with $i \in \{1, 2\}$).
- We denote $Z := Z_1 \sqcup Z_2$ and $\nu_Z := \det(\mathcal{N}_{Z/\mathbb{A}^4_F \setminus \{0\}})$.

Oriented fundamental classes and Seifert classes

Let $i \in \{1, 2\}$.

Knot theory

The class S_i in $H^1(\mathbb{S}^3 \setminus L)$ of Seifert surfaces of the oriented knot K_i is the unique class that is sent by the boundary map to the (oriented) fundamental class of K_i in $H^0(K_i) \subset H^0(L)$.

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Motivic knot theory

We define the oriented fundamental class $[o_i]$ as the unique class in $H^0(Z_i, \underline{K}^{\mathrm{MW}}_{-1}\{\nu_{Z_i}\})$ that is sent by $\widetilde{o_i}$ to the class of η in $H^0(Z_i, \underline{K}^{\mathrm{MW}}_{-1})$, then we define the Seifert class $\mathcal{S}_i$ as the unique class in $H^1(X \setminus Z, \underline{K}^{\mathrm{MW}}_1)$ that is sent by the boundary map ∂ to the oriented fundamental class $[o_i] \in H^0(Z, \underline{K}^{\mathrm{MW}}_{-1}\{\nu_Z\})$.

The quadratic linking class

Knot theory

The linking class of L is the image of the cup-product $S_1 \cup S_2 \in H^2(\mathbb{S}^3 \setminus L)$ by the boundary map $\partial : H^2(\mathbb{S}^3 \setminus L) \to H^1(L)$.

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Motivic knot theory

We define the quadratic linking class of $\mathscr{L}$ as the image of the intersection product $\mathcal{S}_1 \cdot \mathcal{S}_2 \in H^2(X \setminus Z, \underline{K}_2^{\mathrm{MW}})$ by the boundary map $\partial : H^2(X \setminus Z, \underline{K}_2^{\mathrm{MW}}) \to H^1(Z, \underline{K}_0^{\mathrm{MW}}\{\nu_Z\})$.

The ambient quadratic linking degree

Knot theory: the linking number

The linking number of the oriented link $L = K_1 \sqcup K_2$ is the image of the part of the linking class of L which is in $H^1(K_1)$ by the composite of the morphism $(i_1)_* : H^1(K_1) \to H^3(\mathbb{S}^3)$ which is induced by the inclusion $i_1 : K_1 \to \mathbb{S}^3$ and of the isomorphism $r : H^3(\mathbb{S}^3) \to \mathbb{Z}$ which corresponds to the "right-hand rule".

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Motivic knot theory: the ambient quadratic linking degree

We define the *ambient quadratic linking degree* as the image of the part of the quadratic linking class which lives over Z_1 by the composite of the morphism $(i_1)_* : H^1(Z_1, \underline{K}_0^{\mathrm{MW}}\{\nu_{Z_1}\}) \to H^3(\mathbb{A}^4_F \setminus \{0\}, \underline{K}_2^{\mathrm{MW}})$ induced by the inclusion $i_1 : Z_1 \to \mathbb{A}^4_F \setminus \{0\}$ and of an isomorphism between $H^3(\mathbb{A}^4_F \setminus \{0\}, \underline{K}_2^{\mathrm{MW}})$ and $\mathrm{W}(F)$ which has been fixed once and for all (thanks to the coordinates x, y, z, t).

The quadratic linking degree couple

The linking couple

The linking couple is the image of the linking class by the isomorphism $h_1 \oplus h_2 : H^1(K_1) \oplus H^1(K_2) \to \mathbb{Z} \oplus \mathbb{Z}$ which is induced by the volume forms ω_{K_1} of K_1 and ω_{K_2} of K_2.

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Motivic knot theory

We define the quadratic linking degree couple of $\mathscr{L}$ as the image of the quadratic linking class of $\mathscr{L}$ by the isomorphism
$H^1(Z, \underline{K}_0^{\mathrm{MW}}\{\nu_Z\}) \to H^1(Z, \underline{K}_0^{\mathrm{MW}}) \to$
$H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}_0^{\mathrm{MW}}) \oplus H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}_0^{\mathrm{MW}}) \to \mathrm{W}(F) \oplus \mathrm{W}(F)$.

This isomorphism between $H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}_0^{\mathrm{MW}})$ and $\mathrm{W}(F)$ has been fixed once and for all (thanks to the coordinates u, v).

The Hopf link

Recall that we fixed coordinates x, y, z, t for $\mathbb{A}^4_F$ and u, v for $\mathbb{A}^2_F$.

- The image of the Hopf link:

$$\{x = 0, y = 0\} \sqcup \{z = 0, t = 0\} \subset \mathbb{A}^4_F \setminus \{0\}$$

- The parametrisation of the Hopf link:

$$\varphi_1 : (x, y, z, t) \leftrightarrow (0, 0, u, v), \varphi_2 : (x, y, z, t) \leftrightarrow (u, v, 0, 0)$$

- The orientation of the Hopf link:

$$o_1 : \overline{x}^* \wedge \overline{y}^* \mapsto 1, o_2 : \overline{z}^* \wedge \overline{t}^* \mapsto 1$$

The (amb.) quadratic linking degree (cpl.) of the Hopf link

Or. fund. cl.	$\eta \otimes (\overline{x}^* \wedge \overline{y}^*)$	$\mid$	$\eta \otimes (\overline{z}^* \wedge \overline{t}^*)$
Seifert cl.	$\langle x \rangle \otimes \overline{y}^*$	$\mid$	$\langle z \rangle \otimes \overline{t}^*$
Apply int. prod.	$\langle xz \rangle \otimes (\overline{t}^* \wedge \overline{y}^*)$		
Quad. lk. class	$-\langle z \rangle \eta \otimes (\overline{t}^* \wedge \overline{x}^* \wedge \overline{y}^*)$	$\oplus$	$\langle x \rangle \eta \otimes (\overline{y}^* \wedge \overline{z}^* \wedge \overline{t}^*)$
Apply $(i_1)_*$	$-\langle z \rangle \eta \otimes (\overline{t}^* \wedge \overline{x}^* \wedge \overline{y}^*)$		
Apply ∂	$-\eta^2 \otimes (\overline{x}^* \wedge \overline{y}^* \wedge \overline{z}^* \wedge \overline{t}^*)$		
Amb. qld.	-1		
Quad. lk. class	$-\langle z \rangle \eta \otimes (\overline{t}^* \wedge \overline{x}^* \wedge \overline{y}^*)$	$\oplus$	$\langle x \rangle \eta \otimes (\overline{y}^* \wedge \overline{z}^* \wedge \overline{t}^*)$
Apply $\widetilde{o_1} \oplus \widetilde{o_2}$	$-\langle z \rangle \eta \otimes \overline{t}^*$	$\oplus$	$\langle x \rangle \eta \otimes \overline{y}^*$
Apply $\varphi_1^* \oplus \varphi_2^*$	$-\langle u \rangle \eta \otimes \overline{v}^*$	$\oplus$	$\langle u \rangle \eta \otimes \overline{v}^*$
Apply $\partial \oplus \partial$	$-\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$	$\oplus$	$\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$
Qld. couple	-1	$\oplus$	1

A variant of the Hopf link

- The image is the same as the Hopf link's image:

$$\{x = y, y = 0\} \sqcup \{z = 0, a \times t = 0\} \subset \mathbb{A}^4_F \setminus \{0\} \text{ with } a \in F^*$$

- The parametrisation is the same:

$$\varphi_1 : (x, y, z, t) \leftrightarrow (0, 0, u, v), \varphi_2 : (x, y, z, t) \leftrightarrow (u, v, 0, 0)$$

- The orientation is different:

$$o_1 : \overline{x - y}^* \wedge \overline{y}^* \mapsto 1, o_2 : \overline{z}^* \wedge \overline{at}^* \mapsto 1$$

The quadratic linking degree of a variant of the Hopf link

$[o_1^{var}] = \eta \otimes \overline{x-y}^* \wedge \overline{y}^* = [o_1^{Hopf}]$ $[o_2^{var}] = \eta \otimes \overline{z}^* \wedge \overline{at}^* = \langle a \rangle [o_2^{Hopf}]$

since $\begin{pmatrix} x-y \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ since $\begin{pmatrix} z \\ at \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix}$

$\mathcal{S}_1^{var} = \mathcal{S}_1^{Hopf}$ $\mathcal{S}_2^{var} = \langle a \rangle \mathcal{S}_2^{Hopf}$

$$\mathcal{S}_1^{var} \cdot \mathcal{S}_2^{var} = \langle a \rangle \mathcal{S}_1^{Hopf} \cdot \mathcal{S}_2^{Hopf}$$
$$\partial(\mathcal{S}_1^{var} \cdot \mathcal{S}_2^{var}) = \langle a \rangle \partial(\mathcal{S}_1^{Hopf} \cdot \mathcal{S}_2^{Hopf})$$

The ambient quadratic linking degree of the variant is $-\langle a \rangle$.
The quadratic linking degree couple of the variant is $(-\langle a \rangle, 1)$.

Another Hopf link

From now on, F is a perfect field of characteristic different from 2. Recall that we fixed coordinates x, y, z, t for $\mathbb{A}^4_F$ and u, v for $\mathbb{A}^2_F$.

- The image is different from the Hopf link we saw before:

$$\{z = x, t = y\} \sqcup \{z = -x, t = -y\} \subset \mathbb{A}^4_F \setminus \{0\}$$

  But the change of coordinates $x' = z - x$, $y' = t - y$, $z' = z + x$, $t' = t + y$ would give $\{x' = 0, y' = 0\} \sqcup \{z' = 0, t' = 0\} \subset \mathbb{A}^4_F \setminus \{0\}$.

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- This Hopf link is an analogue of the Hopf link in knot theory! In knot theory, the Hopf link is given by $\{z = x, t = y\} \sqcup \{z = -x, t = -y\}$ in $\mathbb{S}^3_\varepsilon = \{(x, y, z, t) \in \mathbb{R}^4, x^2 + y^2 + z^2 + t^2 = \varepsilon^2\}$ for ε small enough and has linking number 1.

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- If we change its orientations and its parametrisations then we get $(\langle b \rangle, \langle c \rangle) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$ with $b, c \in F^*$.

The Solomon link

- In knot theory, the Solomon link is given by $\{z = x^2 - y^2, t = 2xy\} \sqcup \{z = -x^2 + y^2, t = -2xy\}$ in $\mathbb{S}^3_\varepsilon$ for ε small enough and has linking number 2.

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- If we change its orientations and its parametrisations then we get $(\langle b \rangle + \langle b \rangle, \langle c \rangle + \langle c \rangle) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$ with $b, c \in F^*$.
- We want to compute quantities from the ambient quadratic linking degree or from the quadratic linking degree couple which are invariant by changes of orientations and by changes of parametrisations of the oriented link. We call these *invariants of the quadratic linking degree*.

Proposition

Let $\mathscr{L}$ be an oriented link with two components of ambient quadratic linking degree $\alpha \in \mathrm{W}(F)$ and of quadratic linking degree couple $(\beta, \gamma) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$. If $\mathscr{L}'$ is obtained from $\mathscr{L}$ by changing orientations and parametrisations then the ambient quadratic linking degree of $\mathscr{L}'$ is equal to $\langle a \rangle \alpha$ for some $a \in F^*$ and the quadratic linking degree couple of $\mathscr{L}'$ is equal to $(\langle b \rangle \beta, \langle c \rangle \gamma)$ for some $b, c \in F^*$.

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Case $F = \mathbb{R}$

The absolute value of an element of $\mathrm{W}(\mathbb{R}) \simeq \mathbb{Z}$ is invariant by multiplication by $\langle a \rangle$ for all $a \in \mathbb{R}^*$. This gives an invariant of the qld.

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General case

The rank modulo 2 is invariant by multiplication by $\langle a \rangle$ for all $a \in F^*$.

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- Etc. for Σ_{2m} with $m \in \mathbb{N}$

Contents

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For instance, $Z_1 \simeq \mathbb{A}^2_F \setminus \{0\} \simeq Z_2$ in $X = \mathbb{A}^4_F \setminus \{0\}$ with any $j_1 \leq 0$ and $j_2 \leq 0$ (what we considered before, with $j_1 = -1$ and $j_2 = -1$).

Oriented fundamental classes and Seifert classes

Let $i \in \{1, 2\}$.

Oriented fundamental class

We define the oriented fundamental class $[o_i]_{j_i}$ with respect to $j_i \leq 0$ as the unique class in $H^0(Z_i, \underline{K}^{MW}_{j_i}\{\nu_{Z_i}\})$ that is sent by $\widetilde{o}_i$ to the class of η^{-j_i} in $H^0(Z_i, \underline{K}^{MW}_{j_i})$.

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Seifert class

We define the Seifert class $\mathcal{S}_{o_i, j_i}$ with respect to j_i as the unique class in $H^{c-1}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_i+c})$ that is sent by the boundary map ∂ to the oriented fundamental class $[o_i]_{j_i} \in H^0(Z, \underline{K}^{\mathrm{MW}}_{j_i}\{\nu_Z\})$.

The quadratic linking class

The quadratic linking class

We define the quadratic linking class with respect to (j_1, j_2) as the image of the intersection product $\mathcal{S}_{o_1,j_1} \cdot \mathcal{S}_{o_2,j_2}$ by the boundary map $\partial : H^{2c-2}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2c}) \to H^{c-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\})$.

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- $Q_{2n} := \mathrm{Spec}(F[x_1, \ldots, x_n, y_1, \ldots, y_n, z]/(\sum_{i=1}^{n} x_i y_i - z(1+z)))$
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In the cases $Q_n \sqcup Q_n \to Q_{n+\lfloor \frac{n}{2} \rfloor+1} = X$ with $n \in \{2, 3, 4\}$, the only conditions which are not verified are the ones which are there to ensure the existence of Seifert classes ($H^c(X, \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$ and $H^c(X, \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$).

In these settings, the ambient quadratic linking degree is in $\mathrm{W}(F)$ or in $\mathrm{GW}(F)$ and each component of the quadratic linking degree couple is either in the zero group, in $\mathrm{W}(F)$, in $\mathrm{GW}(F)$ or in $K_1^{\mathrm{MW}}(F)$.

In the case of $\mathrm{GW}(F)$, we have refinements of the invariants of the quadratic linking degree we discussed before:

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- $\Sigma_k(\sum_{i=1}^{n} \varepsilon_i \langle a_i \rangle) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} (\prod_{1 \leq l \leq k} \varepsilon_{i_l}) \langle \prod_{1 \leq j \leq k} a_{i_j} \rangle$ with $k \geq 2$ even, where $\Sigma_k : \mathrm{GW}(F) \to \mathrm{GW}(F)$.

Examples of $Q_2 \sqcup Q_2 \to \mathbb{A}^4_F \setminus \{0\}$ $(j_1 = -1 = j_2)$

Assume $F \neq \mathbb{Z}/2\mathbb{Z}$. Let $a \neq b \in F^*$. $Z_1 = \{xy = z(z+1), t = a\}$ and $Z_2 = \{xy = z(z+1), t = b\}$ are of ambient qld 0 and of qld couple $(0, 0)$.

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Assume the characteristic of F to be different from 2 and 3. $Z_1 = \{xy = z(z+1), t = 1\}$ and $Z_2 = \{xy = t(t+1), z = 2\}$ (with the orientation classes and parametrisations which you can guess) are of ambient qld 0 and of qld couple $(-1, -1) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$.

Examples of $Q_2 \sqcup Q_2 \to Q_4$ $(j_1 = -1 = j_2)$

For both examples, assume F of characteristic different from 2. Recall that $Q_4 = \mathrm{Spec}(F[x_1, y_1, x_2, y_2, z]/(x_1 y_1 + x_2 y_2 - z(z+1)))$.

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$Z_1 = \{x_1y_1 = (z-1)z, y_2 = 1\}$ and $Z_2 = \{x_1y_1 = (z+1)(z+2), x_2 = 1\}$ (with the orientation classes and parametrisations which you can guess) are of qld couple $(\langle 2\rangle, \langle 2\rangle) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$.

Other interesting setting

$\mathbb{P}^n_F \sqcup \mathbb{P}^n_F \to \mathbb{P}^{2n+1}_F$ with $n \geq 1$ odd (and ambient qlc ✓), with $j_1, j_2 \leq -2$.

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Example for $\mathbb{P}^1_F \sqcup \mathbb{P}^1_F \to \mathbb{P}^3_F$: the Hopf link, with $Z_1 := \{x = 0, y = 0\}$ and $Z_2 := \{z = 0, t = 0\}$, whose qld couple (with the orientation classes and parametrisations which you can guess) is $(-1, 1) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$.

See my webpage https://www.esaga.uni-due.de/clementine.lemarie-rieusset/ for more information (I will put links to pdfs (in the blue bar at the top of the page) next week).

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Thanks for your attention!