Master Thesis

Kedlaya's slope filtration theorem

Submitted to the
Faculty of Mathematics
at the University of Duisburg-Essen

by

Alexander Niclas Kutzim
Matr.-Nr.: 3031810

Supervised by: Prof. Dr. Jan Kohlhaase
Second examiner: Prof. Dr. Jochen Heinloth

March 7, 2021

Contents

Introduction

Given a finite extension K of the field $\mathbb{Q}_p$ of p-adic numbers, it is a fundamental goal in number theory to understand the absolute Galois group of K. One studies this group using certain representations. Around 1990, Fontaine devised a strategy to compare these to objects of semilinear algebra, the so-called étale (φ, Γ)-modules. We will forget about the group Γ until the last section and talk instead about φ-modules. The goal of this thesis is to provide a mostly complete proof of Kedlaya's slope filtration theorem and talk a bit about its applications and the general importance of slope filtrations. Put briefly, the slope filtration theorem is a classification result for φ-modules over the Robba ring similar to the Dieudonné-Manin classification of isocrystals (cf. [19, Satz 8.21]). The main reference is Kedlaya's 2008 article [16] and we will use the section titles from that article.

In the first two sections, we explain all the objects of interest, starting in Section 1 with the Robba ring $\mathcal{R}$. It is the ring of infinite Laurent series over some complete discretely valued field which converge on suitable annuli of outer radius 1. We will be especially interested in two subrings of $\mathcal{R}$, consisting of series with bounded and integral coefficients, respectively. All three rings have many good algebraic properties which are proven very thoroughly in [19, §§9-10]. On the Robba ring, we will be interested in relative Frobenius lifts. These are special endomorphisms of $\mathcal{R}$ which (almost) act as the q-power map on the series variable and act as an arbitrary isometry on the coefficients. Here $q > 1$ is some fixed integer.

We then look at more general φ-rings, that is, rings R equipped with an endomorphism φ. A φ-module over R is then a finite free module M over R together with an R-linear isomorphism $\varphi^*M \longrightarrow M$. We can describe and study such objects in terms of representing matrices as in linear algebra. Modelled after the theory of vector bundles on a curve, we define a notion of semistability for φ-modules and prove general properties. A formal consequence of the formalism of slopes will be the existence of a canonical semistable filtration, the HN filtration. We conclude our preliminary study of φ-modules by introducing the properties we are really interested in, namely étale and pure φ-modules.

In the third section, we state the slope filtration theorem: Any semistable φ-module over the Robba ring is pure. This means that the HN filtration has much more structure than one might have expected from its definition and only because of the slope filtration theorem does it become a useful tool for studying φ-modules over $\mathcal{R}$. The theorem has found various applications in p-adic Hodge theory and encouraged a lot of progress in the theory of (φ, Γ)-modules, for example due to Berger. We will come back to this in the last section and only make a few immediate observations here. To conclude this section, we outline how the proof of the slope filtration theorem will be carried out in Sections 4 and 5.

In Section 4, we construct a larger ring using Hahn series for which one can prove the slope filtration theorem by explicit calculations. This is where most of the work comes in. It then remains to pass back to $\mathcal{R}$ which will make use of faithfully flat descent. We will need to adapt the theory of Newton polygons to twisted polynomials to solve φ-equations and construct a suitable field of coefficients for this "extended Robba ring".

In the final section, we switch to the side of Galois representations. We explain Fontaine's strategy to attach Galois representations to (φ, Γ)-modules and vice versa and outline some work of Berger leading to a proof of a conjecture made by Fontaine. In the last subsection, we talk a bit about 2-dimensional trianguline representations. Roughly speaking, these representations are connected to upper triangular matrices, explaining the terminology. We comment on the appearance of trianguline representations in the p-adic Langlands program and outline the strategy used in the proof of the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. The whole section is more of a survey, but the impact the slope filtration theorem has had should become apparent.

Notations and conventions

We make use of the usual conventions regarding $+\infty$ and $-\infty$. Applying a ring homomorphism to a vector or matrix means applying the morphism to each entry of that vector or matrix. If (K, v_K) is a (discretely) valued field, we denote by $\mathfrak{o}_K$ its valuation subring with maximal ideal $\mathfrak{m}_K$ and by κ_K its residue field. We drop K from the notation whenever it is clear from context. All rings are assumed to be unital and (with the exception of twisted (Laurent) polynomial rings) commutative.

1 The Robba ring

The Robba ring $\mathcal{R}$ will be the main object of interest for us. We collect here some properties of it and refer to Schneider's lecture notes [19] for proofs. Note that Schneider's notation differs slightly from that of [16].

1.1 Basic properties

We collect some basic properties of the Robba ring and two of its subrings. Let (K, v_K) be a complete discretely valued field and let $\pi \in K$ be some uniformizer. Fix some absolute value $|\cdot|$ associated to v_K and extend it to an absolute value $|\cdot|$ on a fixed algebraic closure $\overline{K}$ of K. The normalization is not important. We start with some more notation.

Notation. *We will need to talk about open or closed circles, discs and annuli in $\overline{K}$ or K. Given a real number $\alpha > 0$, the circle $|t| = \alpha$ is the set of elements of K (or $\overline{K}$; for our purposes it does not matter) of absolute value α. Similarly, the annulus $\alpha \le |t| < 1$ is the set of elements of absolute value at least α and strictly less than 1. Open or closed discs and annuli where the outer circle is included are defined the same way.*

Definition 1.1. *We write $K(\!(t, t^{-1})\!)$ for the K-vector space of all infinite Laurent series $f = \sum_{i\in\mathbb{Z}} a_i t^i$ with coefficients in K. If $r > 0$ is a real number, then we say that f converges on the annulus $e^{-r} \le |t| < 1$ if for all $\rho \in [e^{-r}, 1)$,*

$$\lim_{i\to+\infty} |a_i|\rho^i = 0 \quad \textit{and} \quad \lim_{i\to-\infty} |a_i|\rho^i = 0.$$

By convention, we also allow $r = +\infty$. In this case, the condition on the right should be read as $a_i = 0$ for all $i < 0$ and we say that f converges on the entire open unit disc. The set of all infinite Laurent series converging on $e^{-r} \le |t| < 1$ is denoted $\mathcal{R}^r$ and their union $\mathcal{R}_K = \bigcup_{r>0} \mathcal{R}^r$ is called the Robba ring (over K). We omit K if it can be inferred from the context.

Remark 1.2. *$\mathcal{R}$ is a K-vector space which becomes a K-algebra by the formula*

$$\Big(\sum_{i\in\mathbb{Z}} a_i t^i\Big) \cdot \Big(\sum_{j\in\mathbb{Z}} b_j t^j\Big) = \sum_{k\in\mathbb{Z}} \Big(\sum_{i+j=k} a_i b_j\Big) t^k.$$

One can show that $\mathcal{R}$ is an integral domain (cf. [19, Lemma 9.2]).

Remark 1.3. *An element $f = \sum_{i\in\mathbb{Z}} a_i t^i \in \mathcal{R}^r$ can be thought of as a function from the annulus $e^{-r} \le |t| < 1$ in $\overline{K}$ to $\overline{K}$. Namely, the convergence condition ensures that in this range the assignment $z \mapsto f(z) = \sum_{i\in\mathbb{Z}} a_i z^i$ gives a well-defined map (note that $f(z)$ converges in $K[z] \subseteq \overline{K}$ which is complete). Such a map is called a (rigid) analytic function.*

Example 1.4. *If K has characteristic 0, then the logarithm*

$$\log(1 + t) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} t^i$$

is an element of $\mathcal{R}$ which converges on the entire open unit disc $0 \le |t| < 1$ (cf. [17, Remark 16.2.1]).

Two subrings of $\mathcal{R}$ are of particular importance.

Definition 1.5. *We write $\mathcal{R}^{\mathrm{b}}$ and $\mathcal{R}^{\mathrm{int}}$ for the subrings of $\mathcal{R}$ consisting of series with bounded and integral coefficients, respectively. That is, a series $\sum_{i\in\mathbb{Z}} a_i t^i \in \mathcal{R}$ lies in $\mathcal{R}^{\mathrm{b}}$ if and only if the set $\{|a_i|\}_{i\in\mathbb{Z}}$ is bounded. It is an element of $\mathcal{R}^{\mathrm{int}}$ if and only if $|a_i| \le 1$ for all $i \in \mathbb{Z}$. $\mathcal{R}^{\mathrm{b}}$ is sometimes called the bounded Robba ring and its elements are correspondingly called bounded elements.*

Theorem 1.6. *$\mathcal{R}^{\mathrm{b}}$ is a field and $\mathcal{R}^\times = (\mathcal{R}^{\mathrm{b}})^\times = \mathcal{R}^{\mathrm{b}} \setminus \{0\}$.*

Proof. See [19, Satz 10.2] and op. cit. [Bemerkung 10.3]. □

The valuation $w(\sum_{i\in\mathbb{Z}} a_i t^i) = \min_{i\in\mathbb{Z}} v_K(a_i)$ makes $\mathcal{R}^{\mathrm{b}}$ a discretely valued field. Clearly, $\mathcal{R}^{\mathrm{int}}$ is the corresponding valuation subring and its maximal ideal is $\mathfrak{m}_K\mathcal{R}^{\mathrm{int}}$. We have the following.

Theorem 1.7. *$\mathcal{R}^{\mathrm{int}}$ is a henselian DVR with residue field $\kappa(\!(t)\!)$. It is not $\mathfrak{m}_K$-adically complete.*

Proof. See [19, Satz 10.6] and the discussion after op. cit. [Bemerkung 10.3]. □

Definition 1.8. *We denote by $\mathcal{E}^{\mathrm{int}}$ and $\mathcal{E}$ the ($\mathfrak{m}_K$-adic) completions of $\mathcal{R}^{\mathrm{int}}$ and $\mathcal{R}^{\mathrm{b}}$, respectively.*

One can describe explicitly what elements of $\mathcal{E}^{\mathrm{int}}$ and $\mathcal{E}$ look like.

Lemma 1.9. *$\mathcal{E}$ is the field of all Laurent series $\sum_{i\in\mathbb{Z}} a_i t^i$ such that the set $\{|a_i|\}_{i\in\mathbb{Z}}$ is bounded and such that $\lim_{i\to-\infty} v_K(a_i) = \infty$. The discrete valuation w on $\mathcal{E}$ is given by the same formula as on $\mathcal{R}^{\mathrm{b}}$ and the corresponding valuation ring is $\mathcal{E}^{\mathrm{int}}$.*

Proof. See [19, Lemma 10.4]. □

Next, we define the r-norms. These will be extremely important.

Definition 1.10. *For $r > 0$, let $|\cdot|_r$ be the (multiplicative) supremum norm on the circle $|t| = e^{-r}$ as applied to elements of $\mathcal{R}^r$. By the maximum principle (cf. [19, Lemma 9.3]), we have*

$$\left|\sum_{i\in\mathbb{Z}} a_i t^i\right|_r = \sup_{i\in\mathbb{Z}}\{|a_i|e^{-ri}\}.$$

The norm $|\cdot|_r$ is called the r-norm (or e^{-r}-Gauss norm). $\mathcal{R}^r$ is complete with respect to $|\cdot|_r$ (cf. [19, Übungsaufgabe 9.15]). We generalize the definition to vectors and matrices over $\mathcal{R}^r$ by taking the maximum over entries.

Remark 1.11. *If f is represented by an ordinary power series (that is, no negative powers of t appear), then $|f|_r \le |f|_s$ for all $0 < s \le r$. Hence the supremum of f over the entire disc $|t| \le e^{-s}$ is achieved at the boundary circle $|t| = e^{-s}$. Analogously, $|f|_r \ge |f|_s$ for all $0 < s \le r$ if no positive powers of t appear. Both combined imply that if f converges on $e^{-r} \le |t| \le e^{-s}$, then the supremum of f on this closed annulus is $\max\{|f|_r, |f|_s\}$. That is, the supremum of f over a closed annulus is achieved on the boundary circles. In particular, f is bounded on any closed annulus on which it converges.*

For $r > 0$, we have on $\mathcal{R}^r$ the s-norm $|\cdot|_s$ for any $s \in (0, r]$. This defines what is called a Fréchet topology on $\mathcal{R}^r$. We equip $\mathcal{R}$ with the direct limit topology which is sometimes called the LF topology. A sequence in $\mathcal{R}$ converges w.r.t. the LF topology if it is contained in some $\mathcal{R}^r$ and converges there for the Fréchet topology (i.e. if it converges w.r.t. $|\cdot|_s$ for any $s \in (0, r]$). This does not depend on the choice of $r > 0$ since the inclusion $\mathcal{R}^r \longrightarrow \mathcal{R}^{r'}$ for $r' < r$ is a homeomorphism onto its image equipped with the subspace topology (cf. [17, Definition 16.2.3]). Given $f \in \mathcal{R}^r$, we can detect whether $f \in \mathcal{R}^{\mathrm{b}}$ by looking at the growth of the function $s \longmapsto |f|_s$.

Lemma 1.12. *If $f \in \mathcal{R}$, then $f \in \mathcal{R}^{\mathrm{int}}$ if and only if there exists $m \in \mathbb{Z}$ and $r > 0$ such that $t^m f$ is bounded by 1 on the annulus $e^{-r} \le |t| < 1$, that is, $|t^m f|_s \le 1$ for all $s \in (0, r]$. The same then holds for any integer $m' \ge m$.*

Proof. Write $f = \sum_{i\in\mathbb{Z}} a_i t^i$. We first assume that $t^m f$ is bounded by 1 on some annulus $e^{-r} \le |t| < 1$. Clearly, $f \in \mathcal{R}^{\text{int}}$ if and only if $t^m f \in \mathcal{R}^{\text{int}}$ so we may assume that $m = 0$. Now $\sup_{i\in\mathbb{Z}}\{|a_i|e^{-si}\} = |f|_s \le 1$ implies that $|a_i|e^{-si} \le 1$ for all i. In particular, $|a_i| \le 1$ for all $i \le 0$. We come to the same conclusion for $i > 0$ by letting $s \longrightarrow 0$ as then e^{-si} tends to 1 from below. It follows that $|a_i| \le 1$ for all i so $f \in \mathcal{R}^{\text{int}}$. Conversely, assume that $f \in \mathcal{R}^{\text{int}} \cap \mathcal{R}^r$. We then have $|a_i|e^{-si} \le 1$ for all $i \ge 0$ and all $s \in (0, r]$ since $e^{-si} \le 1$ in this range. By convergence, there can only be finitely many $i < 0$ with $|a_i|e^{-ri} > 1$. Choose $m \ge 0$ large enough so that $i + m \ge 0$ for all those i. Then $|a_i|e^{-r(i+m)} \le 1$ for all $i < 0$. For $i < 0$, there are now two possibilities. Either $i + m < 0$ in which case $|a_i|e^{-s(i+m)} \le |a_i|e^{-r(i+m)} \le 1$ for any $s \in (0, r]$, or $i + m \ge 0$ in which case $|a_i|e^{-s(i+m)} \le |a_i| \le 1$ for any $s \in (0, r]$. It follows that $|a_i|e^{-s(i+m)} \le 1$ for all $i < 0$ and all $s \in (0, r]$ and the same is true for all $i \ge 0$. Hence $|a_i|e^{-s(i+m)} \le 1$ for all $i \in \mathbb{Z}$ and all $s \in (0, r]$ so that $t^m f$ is bounded by 1 on $e^{-r} \le |t| < 1$, as desired. The last remark is clear. □

Corollary 1.13. *If $f \in \mathcal{R}$, then $f \in \mathcal{R}^{\text{b}}$ if and only if there is $r > 0$ such that f is bounded on the annulus $e^{-r} \le |t| < 1$.*

Proof. This follows from the lemma since $f \in \mathcal{R}^{\text{b}}$ if and only if $\pi^m f \in \mathcal{R}^{\text{int}}$ for some $m \ge 0$. □

Remark 1.14. *Both results hold verbatim for vectors and matrices with entries in $\mathcal{R}^r$.*

We have seen that $\mathcal{R}^{\text{b}}$ and $\mathcal{R}^{\text{int}}$ have very good algebraic properties, being a field and a DVR, respectively. $\mathcal{R}$ itself on the other hand is not even noetherian (cf. [17, Exercise (5)]), but we do have the following.

Theorem 1.15. *$\mathcal{R}$ is a Bézout domain, i.e. every finitely generated ideal of $\mathcal{R}$ is principal.*

Proof. See [19, Satz 10.1]. □

1.2 Relative Frobenius lifts

Choose an arbitrary integer $q > 1$. It will be fixed until the very last section.

Definition 1.16. *A relative (q-power) Frobenius lift on the Robba ring is an endomorphism φ of $\mathcal{R}$ of the form $\sum_{i\in\mathbb{Z}} a_i t^i \mapsto \sum_{i\in\mathbb{Z}} \varphi_K(a_i)u^i$, where $\varphi_K : K \longrightarrow K$ is some isometry and $u \in \mathcal{R}^{\text{int}}$ is such that $u - t^q \in \mathfrak{m}_K\mathcal{R}^{\text{int}}$. If κ has characteristic $p > 0$ and q is a power of p, then we define an absolute (q-power) Frobenius lift to be a relative Frobenius lift in which φ_K is a lift of the q-power map on κ.*

Before verifying that a relative Frobenius lift actually gives a well-defined map $\mathcal{R} \longrightarrow \mathcal{R}$, we look at a simple example. An important step in the proof of the slope filtration theorem will be to reduce to this situation.

Example 1.17. *The substitution $\varphi : (t \mapsto t^q)$ is a relative Frobenius lift. Here φ_K is the identity on K and $u = t^q$. Note that φ does not preserve $\mathcal{R}^r$ since it maps $\mathcal{R}^r$ into $\mathcal{R}^{r/q}$, but it does define a map on the whole Robba ring. Note also that $|\varphi(f)|_{r/q} = |f|_r$ for any $r > 0$.*

Lemma 1.18. *Let φ be a relative Frobenius lift on $\mathcal{R}$. There exists $r_0 > 0$ such that for any $r \in (0, r_0]$ and all $f \in \mathcal{R}^r$, we have $\varphi(f) \in \mathcal{R}^{r/q}$. Moreover, $|\varphi(f)|_{r/q} = |f|_r$ for r in this range.*

Proof. Write $\varphi(t) = u = \sum_{i\in\mathbb{Z}} c_i t^i$. Since $u - t^q \in \pi\mathcal{R}^{\text{int}}$, we have $|c_i| \le |\pi|$ for all $i \ne q$ and $|c_q - 1| \le |\pi|$. Note that the latter condition implies that $c_q \in \mathfrak{o}_K^\times$, i.e. $|c_q| = 1$. We prove the following claim.

Claim. *There is $r_0 > 0$ such that $|u|_{r/q} = |t|_r = e^{-r}$ for any $r \in (0, r_0]$. Moreover, $|u - t^q|_{r/q} < |t^q|_{r/q} = e^{-r}$ for r in this range.*

Proof of Claim. Write $u - t^q = \sum_{i\in\mathbb{Z}} b_i t^i$, that is, $b_i = c_i$ for $i \neq q$ and $b_q = c_q - 1$. Then $b_q \in \pi\mathfrak{o}_K$ for all $i \in \mathbb{Z}$ and there is some $s > 0$ such that $\sum_{i\in\mathbb{Z}} b_i t^i \in \mathcal{R}^s$. This means that $\lim_{i\to\pm\infty}|b_i|e^{-ri} = 0$ for all $r \in (0, s]$. In particular, $|b_i|e^{-si} > |\pi|$ only for finitely many $i \in \mathbb{Z}$ all of which have to be negative since $|b_i|e^{-si} \leq |\pi|e^{-si} \leq |\pi|$ for $i \geq 0$. Hence the set $\{i \in \mathbb{Z} : |b_i|e^{-si} > |\pi|\}$ is finite and it becomes smaller if we make s smaller because $|b_i|e^{-s'i} \leq |b_i|e^{-si}$ for all $s' \leq s$ if $i < 0$. We may thus choose $r_0 > 0$ such that $|b_i|e^{-r_0 i} \leq |\pi|$ for all $i < 0$ and the same is of course true for all $i \geq 0$. Making r_0 even smaller, we may also assume that $e^{-r_0} > |\pi|$. It follows that for all $i \geq 0$ and all $r \in (0, r_0]$,

$$|b_i|e^{-ri/q} \leq |\pi| < e^{-r_0} \leq e^{-r}.$$

For $i < 0$ and $r \in (0, r_0]$, we have

$$|b_i|e^{-ri/q} < |b_i|e^{-ri} \leq |b_i|e^{-r_0 i} \leq |\pi| < e^{-r_0} \leq e^{-r}.$$

Altogether,

$$|u|_{r/q} = \sup_{i\in\mathbb{Z}}\{|c_i|e^{-ri/q}\} = \max\{e^{-rq/q}, \sup_{i\neq q}\{|b_i|e^{-ri/q}\}\} = e^{-r}$$

for any $r \in (0, r_0]$. This proves the claim. □

Now let $f = \sum_{i\in\mathbb{Z}} a_i t^i \in \mathcal{R}^r$ for some $r \in (0, r_0]$. To prove that $\varphi(f) \in \mathcal{R}^{r/q}$, it suffices to show that the sequences $(\varphi_K(a_i)u^i)_{i\geq 0}$ and $(\varphi_K(a_{-i}u^{-i})_{i\geq 0}$ converge to zero with respect to the s-norm for $s \in (0, r/q]$. We have, for s in this range,

$$\lim_{i\to\pm\infty}|\varphi_K(a_i)u^i|_s = \lim_{i\to\pm\infty}|a_i|e^{-sqi} = 0$$

because $sq \in (0, r]$. This proves the first part of the lemma. Given $r_0 > 0$ as in the claim, we now verify the compatibility of φ with the r-norms for $r \in (0, r_0]$. We have $|u - t^q|_{r/q} < e^{-r}$. This implies $|\frac{u}{t^q} - 1|_{r/q} < 1$ by multiplicativity of the r-norms and hence $\frac{u}{t^q}$ is a unit in $\mathcal{R}^{r_0}$.

Claim. *If we write $x = \frac{u}{t^q}$, then $|x^i - 1|_{r/q} < 1$ for all $i \in \mathbb{Z}$.*

Proof of Claim. The claim is trivial for $i = 0$. For $i > 0$, we have

$$\begin{aligned}|x^i - 1|_{r/q} = |((x-1)+1)^i - 1|_{r/q} &= \left|\sum_{k=1}^{i}\binom{i}{k}(x-1)^k\right|_{r/q}\\ &\leq \max_{1\leq k\leq i}\{|x-1|_{r/q}^k\} = |x-1|_{r/q} < 1.\end{aligned}$$

We then get the desired result for $i < 0$ as well since $|x^i - 1|_{r/q} = |x^i|_{r/q}|x^{-i} - 1|_{r/q} < |x|_{r/q}^i = 1$. □

The claim implies that for all $r \in (0, r_0]$ and all $i \in \mathbb{Z}$,

$$|u^i - t^{qi}|_{r/q} < |t^{qi}|_{r/q} = e^{-ri}. \tag{1.2.1}$$

If we now write

$$\varphi(f) = \sum_{i\in\mathbb{Z}} \varphi_K(a_i)u^i = \sum_{i\in\mathbb{Z}} \varphi_K(a_i)t^{qi} + \sum_{i\in\mathbb{Z}} \varphi_K(a_i)(u^i - t^{qi}),$$

then the first summand has r/q-norm

$$\left|\sum_{i\in\mathbb{Z}} \varphi_K(a_i)t^{qi}\right|_{r/q} = \sup_{i\in\mathbb{Z}}\{|\varphi_K(a_i)|e^{-(r/q)qi}\} = \sup_{i\in\mathbb{Z}}\{|a_i|e^{-ri}\} = \lfloor f\rfloor_r$$

whereas the second summand has r/q-norm

$$\left|\sum_{i\in\mathbb{Z}} \varphi_K(a_i)(u^i - t^{qi})\right|_{r/q} \le \sup_{i\in\mathbb{Z}}\{|a_i|\cdot|u^i - t^{qi}|_{r/q}\} < \sup_{i\in\mathbb{Z}}\{|a_i|e^{-ri}\} = \lfloor f\rfloor_r.$$

Here we use (1.2.1) in the last line and the fact that the supremum is really a maximum. This implies $|\varphi(f)|_{r/q} = \lfloor f\rfloor_r$ by the strict triangle inequality. □

Remark 1.19. *It follows that a relative Frobenius lift gives a well-defined endomorphism of the Robba ring. The compatibility with the r-norms together with Lemma 1.13 then implies that φ preserves $\mathcal{R}^{\mathrm{b}}$ and $\mathcal{R}^{\mathrm{int}}$. In fact, if we let $r \longrightarrow 0$ then it follows that the discrete valuation w on $\mathcal{R}^{\mathrm{b}}$ is φ-invariant.*

Proposition 1.20. *Let φ be a relative Frobenius lift on $\mathcal{R}$ and let A be a $n\times n$ matrix with entries in $\mathcal{R}^{\mathrm{int}}$. Then the map $\mathbf{v}\mapsto \mathbf{v} - A\varphi(\mathbf{v})$ induces a bijection on $(\mathcal{R}/\mathcal{R}^{\mathrm{b}})^n$.*

Proof. For any $m \in \mathbb{Z}$ the matrix $A' = (t^m/\varphi(t^m))A$ has entries in $\mathcal{R}^{\mathrm{int}}$ as well. We have a commutative square

$$\begin{array}{ccc} (\mathcal{R}/\mathcal{R}^{\mathrm{b}})^n & \xrightarrow{\mathbf{v}\mapsto\mathbf{v}-A\varphi(\mathbf{v})} & (\mathcal{R}/\mathcal{R}^{\mathrm{b}})^n \\ \downarrow{\cdot t^m} & & \downarrow{\cdot t^m} \\ (\mathcal{R}/\mathcal{R}^{\mathrm{b}})^n & \xrightarrow{\mathbf{v}\mapsto\mathbf{v}-A'\varphi(\mathbf{v})} & (\mathcal{R}/\mathcal{R}^{\mathrm{b}})^n \end{array}$$

Clearly, the vertical arrows are bijective so it suffices to show that the lower arrow is bijective. By Lemma 1.12, we may choose $m' \le 0$ such that the entries of $t^{-m'}A$ are bounded by 1 on some annulus $e^{-r} \le |t| < 1$. Since $t^{m'}/\varphi(t^{m'}) = t^{(1-q)m'}$, we then have $|A'|_r = |t^{(1-q)m'}A|_r \le |t^{-m'}A|_r \le 1$. Replacing $A, \mathbf{v}$ by $A', t^m\mathbf{v}$ for $m = -m'$, we may therefore assume that the entries of A are bounded by 1 on the annulus $e^{-r} \le |t| < 1$. After possibly making r smaller, we may also assume that $r \in (0, r_0]$ where $r_0 > 0$ is as in Lemma 1.18.

To show injectivity, we must see that if $\mathbf{w} = \mathbf{v} - A\varphi(\mathbf{v})$ is bounded (i.e. has bounded entries), then so is $\mathbf{v}$. Choose $c > 0$ such that $|\mathbf{w}|_s \le c$ for $0 < s \le r$ and $|\varphi(\mathbf{v})|_s \le c$ for $r/q \le s \le r$. The latter is possible because $\varphi(\mathbf{v})$ is bounded on any closed annulus by Remark 1.11. Then $|\mathbf{v}|_s = |\mathbf{w} + A\varphi(\mathbf{v})|_s \le c$ for any $s \in [r/q, r]$ by the strict triangle inequality and because the entries of A are bounded by 1. Hence $|\varphi(\mathbf{v})|_s = |\mathbf{v}|_{sq} \le c$ for all $s \in \left[r/q^2, r/q\right]$. We may repeat the argument inductively to get $|\varphi(\mathbf{v})|_s \le c$ for all $s \in \left[r/q^{m+1}, r/q^m\right]$ for any $m \ge 0$. Thus, $|\mathbf{v}|_s = |\varphi(\mathbf{v})|_{s/q} \le c$ for $0 < s \le r$. Hence $\mathbf{v}$ is bounded by Lemma 1.13, proving injectivity.

To prove surjectivity, let $\mathbf{w} \in \mathcal{R}^n$. Define the sequence $\{\mathbf{w}_l\}_{l\ge 0}$ as follows. Start by setting $\mathbf{w}_0 = \mathbf{w}$. Given $\mathbf{w}_l$, write $\mathbf{w}_l = \sum_{i\in\mathbb{Z}} \mathbf{w}_{l,i}t^i$. Set $\mathbf{w}_l^+ = \sum_{i>0} \mathbf{w}_{l,i}t^i$ and $\mathbf{w}_l^- = \mathbf{w}_l - \mathbf{w}_l^+$, and define $\mathbf{w}_{l+1} = A\varphi(\mathbf{w}_l^+)$. Since the entries of $t^{-1}\mathbf{w}_l^+$ are analytic on the entire open unit disc, we have

$$e^r|\mathbf{w}_l^+|_r = |t^{-1}\mathbf{w}_l^+|_r \le |t^{-1}\mathbf{w}_l^+|_{r/q} = e^{r/q}|\mathbf{w}_l^+|_{r/q} \le e^{r/q}|\mathbf{w}_l|_{r/q}$$

by Remark 1.11. Since the entries of A are bounded by 1, we deduce that

$$|\mathbf{w}_{l+1}|_{r/q} \le |\varphi(\mathbf{w}_l^+)|_{r/q} = |\mathbf{w}_l^+|_r \le e^{-r+r/q}|\mathbf{w}_l|_{r/q}.$$

Hence the $\mathbf{w}_l$ converge to zero under $|\cdot|_{r/q}$. In particular, the $\mathbf{w}_l^+$ converge to zero under $|\cdot|_{r/q}$ and hence under $|\cdot|_s$ for any $s \geq r/q$ using Remark 1.11 again. On the other hand, we have for $0 < s \leq r/q$,

$$|\mathbf{w}_l^-|_s \leq |\mathbf{w}_l^-|_{r/q} \leq |\mathbf{w}_l|_{r/q}$$

by the final observation in Remark 1.11. It follows that the $\mathbf{w}_l^-$ converge to zero under $|\cdot|_s$ for any $s \in (0, r/q]$. Now define $\mathbf{v} = \sum_{l=0}^{\infty} \mathbf{w}_l^+$. Then $\mathbf{v}$ has entries analytic on the closed disc $|t| \leq e^{-r/q}$ by convergence of $\mathbf{w}_l^+$ to zero under $|\cdot|_s$ for $s \geq r/q$. Moreover,

$$\mathbf{w} - \mathbf{v} + A\varphi(\mathbf{v}) = \mathbf{w}_0 - \sum_{l=0}^{\infty} \mathbf{w}_l^+ + \sum_{l=0}^{\infty} A\varphi(\mathbf{w}_l^+) = \mathbf{w}_0 - \sum_{l=0}^{\infty} \mathbf{w}_l^+ + \sum_{l=0}^{\infty} \mathbf{w}_{l+1} = \sum_{l=0}^{\infty} \mathbf{w}_l^-.$$

The series on the right is convergent and its limit is bounded on $e^{-r/q} \leq |t| < 1$ by the observations made on the $\mathbf{w}_l^-$. In other words, the class of $\mathbf{v}$ is a (potential) preimage of the class of $\mathbf{w}$. It remains to extend $\mathbf{v}$ to an annulus of outer radius 1 as then $\mathbf{v}$ will have entries in $\mathcal{R}$. Since $\varphi(\mathbf{v})$ is analytic on the closed disc $|t| \leq e^{-r/q^2}$ by properties of φ, we can write $\mathbf{v} = \mathbf{w} + A\varphi(\mathbf{v}) - \sum_{l\geq 0} \mathbf{w}_l^-$ and thus extend $\mathbf{v}$ across the annulus $e^{-r/q} \leq |t| \leq e^{-r/q^2}$. Proceeding inductively, we may extend $\mathbf{v}$ to the entire open unit disc. This proves surjectivity. □

1.3 Some properties of Bézout domains

We conclude our overview of the Robba ring with some algebraic properties of Bézout domains. These can be found in [16, Remark 1.1.2 and §3.4] and/or Clark's Commutative Algebra notes [6, §3.9.2], for example.

Lemma 1.21. *Let R be a Bézout domain. The following assertions are true:*

- *Any finite locally free R-module is free.*
- *Any torsion-free R-module is flat.*
- *Any finitely generated R-submodule of a torsion-free R-module is free.*
- *If M is a finite free R-module and N is an R-submodule of M which is saturated, that is, $N = M \cap (N \otimes_R \mathrm{Frac}(R))$, then both N and M/N are finite free.*

Remark 1.22. *The set $N^{\mathrm{sat}} = M \cap (N \otimes_R \mathrm{Frac}(R)) = \{m \in M \,:\, rm \in N$ for some $0 \neq r \in R\}$ is called the saturation of N (in M).*

Proposition 1.23. *Let $R \longrightarrow S$ be an inclusion of domains where R is Bézout. Then S is faithfully flat over R if and only if $S^\times \cap R = R^\times$.*

Proof. Recall that S is flat over R if and only if for each proper ideal I of R, the multiplication map $I \otimes_R S \longrightarrow S$ is injective. In fact, it suffices to check this for finitely generated ideals since every ideal of R is an inductive limit of finitely generated ones. Moreover, if S is flat over R, then it is faithfully flat if and only if for every proper ideal I of R, $I \otimes_R S = IS \neq S$. Again, it suffices to check this for finitely generated ideals.

Now let $I \subsetneq R$ be a finitely generated ideal of R. Since R is Bézout, there is $r \in R \setminus R^\times$ such that $I = rR$. Then $I \otimes_R S = rR \otimes_R S \simeq rS$ whence the map $I \otimes_R S \longrightarrow S$ is just the inclusion $rS \subseteq S$ which is of course injective. We have $rS = S$ if and only if $r \in S^\times$. This gives the claim. □

Lemma 1.24. *Let R be a Bézout ring. If $u_1, \ldots, u_n$ generate the unit ideal, then there exists an invertible $n \times n$ matrix U over R such that $U_{i1} = u_i$ for all $1 \leq i \leq n$.*

Proof. We proof the claim by induction on the number of elements, the case $n = 1$ being trivial. Let d be a generator of the ideal $(u_1, \dots, u_{n-1})$ so that the u_i/d generate the unit ideal. By induction hypothesis there exists an invertible matrix B such that $B_{i1} = u_i/d$ for $1 \leq i \leq n-1$. We extend B to an invertible $n \times n$ matrix by setting $B_{nn} = 1$ and $B_{in} = B_{ni} = 0$ for $1 \leq i \leq n-1$. Since (d, u_n) is the unit ideal, we can find $e, f \in R$ such that $de - fu_n = 1$. Define the matrix

$$C = \begin{pmatrix} d & 0 & \dots & 0 & f \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ u_n & 0 & \dots & 0 & e \end{pmatrix}$$

Then $U = BC$ does the job. ☐

Lemma 1.25. *Let M, N be two modules over a Bézout domain R. Given a presentation $\sum_{i=1}^n y_i \otimes z_i$ of $x \in M \otimes_R N$ and elements $u_1, \dots, u_n \in R$ which generate the unit ideal, there exists another presentation $x = \sum_{j=1}^n y'_j \otimes z'_j$ with $y'_1 = \sum_{i=1}^n u_i y_i$.*

Proof. By the previous lemma, there exists an invertible matrix U over R with $U_{i1} = u_i$ for $1 \leq i \leq n$. Given such U, we calculate

$$\begin{aligned} \sum_i y_i \otimes z_i &= \sum_{i,j,l} U_{ij}(U^{-1})_{jl} y_i \otimes z_l \\ &= \sum_j \Big(\sum_i U_{ij} y_i\Big) \otimes \Big(\sum_l (U^{-1})_{jl} z_l\Big). \end{aligned}$$

Hence $y'_j = \sum_{i=1}^n U_{ij} y_i$ and $z'_j = \sum_{l=1}^n (U^{-1})_{jl} z_l$ are as required. ☐

Corollary 1.26. *Let M, N be two modules over a Bézout domain R. If $\sum_{i=1}^n y_i \otimes z_i$ is a presentation of some $x \in M \otimes_R N$ with n minimal (i.e. there is no presentation of x with fewer summands), then $y_1, \dots, y_n$ are linearly independent over R.*

Proof. Suppose that $y_1, \dots, y_n$ are linearly dependent over R. Choose $u_1, \dots, u_n \in R$ not all zero such that $\sum_{i=1}^n u_i y_i = 0$. Since R is Bézout, the ideal generated by the u_i is a principal ideal, say generated by $u \in R \setminus \{0\}$. Replacing $u_1, \dots, u_n$ by $u_1/u, \dots, u_n/u$ if necessary, we may assume that the u_i generate the unit ideal. By Lemma 1.25, we can then find another presentation $x = \sum_{j=1}^n y'_j \otimes z'_j$ with $y'_1 = \sum_{i=1}^n u_i y_i = 0$, contradicting the minimality of n. Hence the y_i must be linearly independent over R. ☐

2 φ-modules

In this section, we introduce φ-modules and various properties they may have. Such objects appear in a variety of situations, e.g. in the classification of p-divisible groups.

2.1 The category of φ-modules

We begin our study of φ-modules with two equivalent definitions of what a φ-module is and some simple semilinear algebra. It will pay off later to be a bit more precise here than what is perhaps necessary.

Definition 2.1. *A φ-ring (resp. a φ-field) is a ring (resp. a field) R equipped with an endomorphism $\varphi = \varphi_R$. We say that R is inversive if φ is bijective. A morphism of φ-rings $f : R \longrightarrow S$ is a φ-equivariant ring homomorphism $f : R \longrightarrow S$, that is, $f \circ \varphi_R = \varphi_S \circ f$.*

Starting now, every ring will be a φ-ring.

Notation. *By abuse of notation, we write φ instead of φ_R even when there are multiple rings. Since all ring homomorphisms φ will be compatible this should not cause any confusion.*

Definition 2.2. *Let M be a finite free R-module equipped with a φ-semilinear map $\varphi_M : M \longrightarrow M$, that is, φ_M is additive and satisfies $\varphi_M(rm) = \varphi(r)\varphi_M(m)$ for all $r \in R$ and all $m \in M$. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be some R-basis of M. The matrix $A = (A_{ij})_{i,j} \in R^{n\times n}$ satisfying $\varphi_M(\mathbf{e}_j) = \sum_{i=1}^n A_{ij}\mathbf{e}_i$ for all $1 \leq j \leq n$ is called the representing matrix of φ_M (w.r.t. $\mathbf{e}_1, \dots, \mathbf{e}_n$).*

Remark 2.3. *If we use the basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ to identify M with R^n, then $\varphi_M(\mathbf{v}) = A\varphi(\mathbf{v})$, that is, the action of φ on M is given by multiplication with the matrix A times the componentwise action of φ. For this reason, we often call φ_M the φ-action on M.*

Remark 2.4. *If $\mathbf{e}'_1, \dots, \mathbf{e}'_n$ is another R-basis of M and $U = (U_{ij})_{i,j}$ denotes the change-of-base matrix (i.e. $\mathbf{e}'_j = \sum_{i=1}^n U_{ij}\mathbf{e}_i$ for all j), then the representing matrix of φ_M w.r.t. $\mathbf{e}'_1, \dots, \mathbf{e}'_n$ is $U^{-1}A\varphi(U)$. In particular, the representing matrix of φ_M w.r.t. some basis is invertible over R if and only if this is true for any basis.*

By the previous remark, the following definition is reasonable.

Definition 2.5. *A φ-module over R is a finite free R-module equipped with a φ-semilinear map $\varphi_M : M \longrightarrow M$ whose representing matrix w.r.t. some (hence any) R-basis is invertible over R.*

We will primarily use the above definition of a φ-module, but in some instances it is convenient to use another definition. Both definitions are easily seen to be equivalent.

Definition 2.6. *A φ-module over R is a finite free R-module equipped with an R-linear isomorphism $\varphi_M^{\mathrm{lin}} : \varphi^*M = R \otimes_{R,\varphi} M \longrightarrow M$. We recover φ_M from φ_M^{lin} by precomposing with the canonical map $M \longrightarrow \varphi^*M$, $m \longmapsto 1 \otimes m$ and we obtain φ_M^{lin} from φ_M via $\varphi_M^{\mathrm{lin}}(r \otimes m) = r\varphi_M(m)$.*

Example 2.7. *R equipped with $\varphi : R \longrightarrow R$ is a φ-module of rank 1 over R. Similarly, R^n with the componentwise action of φ is a φ-module of rank n over R. It is called the trivial φ-module (of rank n) over R. A φ-module M over R is called trivial if it is isomorphic to the trivial φ-module over R. This means that M should admit a basis which is invariant under the φ-action, i.e. a basis such that the representing matrix of φ_M is the $n \times n$ identity matrix E_n. Such a basis will be called a φ-invariant basis.*

Definition 2.8. *A morphism of φ-modules $f : M \longrightarrow N$ is an R-linear map $f : M \longrightarrow N$ commuting with the φ-actions, i.e. $f \circ \varphi_M = \varphi_N \circ f$. The set of morphisms of φ-modules from M to N is denoted $\mathrm{Hom}_{R,\varphi}(M, N)$. We write $\mathrm{Hom}(M, N)$ if there is no confusion about R or φ and reserve $\mathrm{Hom}_R(M, N)$ for the set of all R-linear maps from M to N.*

We obtain the category of φ-modules over R. This allows us to talk about categorical notions such as φ-submodules or irreducible φ-modules. We look at some natural constructions with φ-modules.

Definition 2.9. *Given two φ-modules M, N over R, we equip the tensor product $M \otimes_R N$ with the φ-action $\varphi_{M\otimes_R N} = \varphi_M \otimes \varphi_N$. In particular, if $R \longrightarrow S$ is a morphism of φ-rings, then we can perform a base change from φ-modules over R to φ-modules over S. Having defined the tensor product of two φ-modules, one can also define symmetric and exterior powers of φ-modules. Note that these are free by construction so we really get φ-modules.*

Notation. *If R is clear from context, we write $M \otimes N$ instead of $M \otimes_R N$.*

Definition 2.10. *Given a φ-module M over R, there is a unique way to equip the dual $M^\vee = \mathrm{Hom}_R(M, R)$ with the structure of a φ-module such that $\varphi_{M^\vee}(f)(\varphi_M(m)) = f(m)$ for any $f \in M^\vee$ and all $m \in M$. Alternatively, the isomorphism $\varphi_M^{\mathrm{lin}} : \varphi^* M \longrightarrow M$ induces an isomorphism $(\varphi_M^{\mathrm{lin}})^\vee : M^\vee \longrightarrow (\varphi^* M)^\vee$. Taking the inverse and precomposing with the isomorphism $\varphi^* M^\vee \simeq (\varphi^* M)^\vee$ gives the desired structure of a φ-module on $M^\vee$.*

Remark 2.11. *If R is inversive and M is a φ-module over R, then φ_M is bijective and $\varphi_{M^\vee}$ is given by $\varphi_{M^\vee}(f) = \varphi \circ f \circ \varphi_M^{-1}$ for $f \in M^\vee$.*

One would also like to take kernels, images or quotients in the category of φ-modules, but these need not be free in general. Thankfully, we will be working mostly over Bézout domains where we can take quotients by saturated submodules (cf. Lemma 1.21). Hence, if R is Bézout, kernels of morphisms of φ-modules are φ-submodules of the source because they are clearly saturated. This implies that images of morphisms of φ-modules are naturally φ-modules as well. Next, we talk about morphisms and extensions of φ-modules. As it turns out, the following definition will be crucial.

Definition 2.12. *If M is a φ-module over R, set*

$$H^0(M) = \ker(\varphi_M - 1) \quad \text{and} \quad H^1(M) = \mathrm{coker}(\varphi_M - 1).$$

Lemma 2.13. *If M, N are φ-modules, then*

$$\mathrm{Hom}(M, N) = H^0(M^\vee \otimes N) \quad \text{and} \quad \mathrm{Ext}_{R,\varphi}(M, N) = H^1(M^\vee \otimes N).$$

Here $\mathrm{Ext}_{R,\varphi}(M, N)$ denotes the group of isomorphism classes of extensions of φ-modules

$$0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0$$

where two such extensions are equivalent if there is an isomorphism of the middle terms inducing the identities on the outer terms. We write $\mathrm{Ext}(M, N)$ if there is no ambiguity about R or φ.

Proof. We prove the lemma only under the assumption that φ is bijective since we will only need it in this case. If φ is bijective, then the natural isomorphism of R-modules $M^\vee \otimes N \simeq \mathrm{Hom}_R(M, N)$ is an isomorphism of φ-modules with the φ-action on $H = \mathrm{Hom}_R(M, N)$ given by $\varphi_H(f) = \varphi_N \circ f \circ \varphi_M^{-1}$ for $f \in H$. An element of $\ker(\varphi_H - 1)$ is thus precisely an R-linear map $f : M \longrightarrow N$ which satisfies $f = \varphi_N \circ f \circ \varphi_M^{-1}$. This is just another way of saying that f is a morphism of φ-modules, proving the claim about H^0. For the claim about H^1, we follow the proof of [12, Proposition 2.4]. Given $h \in H$, we set $E_h = N \oplus M$ with the φ-action given by

$$\varphi_{E_h} = \begin{pmatrix} \varphi_N & h \circ \varphi_M \\ 0 & \varphi_M \end{pmatrix}$$

i.e. $\varphi_{E_h}(n, m) = (\varphi_N(n) + h \circ \varphi_M(m), \varphi_M(m))$. This gives rise to an extension of φ-modules

$$0 \longrightarrow N \longrightarrow E_h \longrightarrow M \longrightarrow 0$$

Since φ-modules are free, any short exact sequence of φ-modules splits as a short exact sequence of R-modules and hence is isomorphic to one of the above form. It remains to see when the extensions associated to $h, h' \in H$ are isomorphic. This is the case if and only if

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \varphi_N & h \circ \varphi_M \\ 0 & \varphi_M \end{pmatrix} \cdot \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \varphi_N & h' \circ \varphi_M \\ 0 & \varphi_M \end{pmatrix}$$

for some $k \in H$. Writing this out gives

$$h \circ \varphi_M + k \circ \varphi_M - \varphi_N \circ k = h' \circ \varphi_M \iff h - h' = \varphi_N \circ k \circ \varphi_M^{-1} - k = (\varphi_H - 1)(k).$$

Hence the extensions associated to h and h' are isomorphic if and only if the classes of h and h' in $H^1(H)$ agree, as desired. □

2.2 Twisted polynomial rings

An R-module M equipped with an R-linear map $M \longrightarrow M$ can be viewed as a module over the polynomial ring $R[T]$. We can think of φ-modules in a similar way, but one has to be a bit careful since our definition of a φ-module includes the condition that it is finite free.

Definition 2.14. *The twisted polynomial ring $R\{T\}$ over R is the set of polynomials in the variable T with coefficients in R, equipped with the unique structure of a (non-commutative) ring such that $Ta = \varphi(a)T$ for all $a \in R$. If R is inversive, we can impose a similar condition for φ^{-1} and T^{-1} and define the twisted Laurent polynomial ring $R\{T^{\pm}\}$. We may also view this as a localization of $R\{T\}$ in the non-commutative setting. Since we will work only with twisted polynomials, we will drop the descriptor "twisted" most of the time.*

Remark 2.15. *If R is a field then all left ideals of $R\{T\}$ (and $R\{T^{\pm}\}$, if R is inversive) are principal. This is because we have a Euclidean algorithm as for $R[T]$ (cf. [18, Theorem 6]).*

We can now view a φ-module M over R as an $R\{T\}$-module by letting T act as φ_M. If R is inversive, we see that a φ-module over R is the same as a left $R\{T^{\pm}\}$-module which is finite free over R.

Lemma 2.16. *If R is an inversive φ-field then any irreducible φ-module M over R is isomorphic to $R\{T^{\pm}\}/R\{T^{\pm}\}P$ for some irreducible Laurent polynomial $P \in R\{T^{\pm}\}$.*

Proof. Let $m \in M$ be nonzero and write $n = \mathrm{rk}(M)$. The R-submodule of M generated by $m, \varphi_M(m), \dots, \varphi_M^{n-1}(m)$ is φ-stable and must be all of M because M is irreducible. Consider the map $R\{T^{\pm}\} \longrightarrow M$ sending $\sum_{i\in\mathbb{Z}} a_i T^i$ to $\sum_{i\in\mathbb{Z}} a_i \varphi_M^i(m)$. It is surjective by the previous observation. The kernel of this map is a left ideal of $R\{T^{\pm}\}$ and hence is a principal ideal generated by some $P \in R\{T^{\pm}\}$. Thus, $M = R\{T^{\pm}\}/R\{T^{\pm}\}P$ with the φ-action given by multiplication with T from the left. To conclude, we note that any nontrivial factorization of P would give rise to a nontrivial φ-submodule of M. Since M is irreducible, this forces P to be irreducible. $\square$

Remark 2.17. *Since T is a unit in $R\{T^{\pm}\}$, we can always achieve that $P \in R\{T\}$.*

Definition 2.18. *Let $a \geq 1$ be an integer. View φ-modules as left modules over $R\{T\}$ and φ^a-modules as left modules over $R\{T^a\}$. Define the a-pushforward functor $[a]_*$ (resp. the a-pullback functor $[a]^*$) to be the restriction of scalars (resp. the extension of scalars) along the inclusion $R\{T^a\} \longrightarrow R\{T\}$.*

Remark 2.19.

- *If M is a φ-module, then the φ^a-module $[a]_*M$ is the R-module M equipped with the φ^a-semilinear map φ_M^a. Note that if A is the representing matrix of φ_M, then the representing matrix of φ_M^a is $A\varphi(A)\cdots\varphi^{a-1}(A)$. It will be convenient to define the twisted powers $A^{\{n\}}$ of A by the recurrence*

$$A^{\{0\}} = E_n, \quad A^{\{n+1\}} = A\varphi(A^{\{n\}}).$$

Then the representing matrix of φ_M^a is $A^{\{a\}}$. Note that the above recurrence can be viewed as a two-way recurrence if φ is invertible. Hence we can also define twisted powers with negative exponent in this case.

- *If N is a φ^a-module, then we can explain the a-pullback functor as follows. Recall that $R\{T\}$ is a free $R\{T^a\}$-module of rank a with basis $1, T, \dots, T^{a-1}$. Now $[a]_*N = R\{T\} \otimes_{R\{T^a\}} N$ is a φ-module via multiplication with T from the left on $R\{T\}$. Writing $R\{T\} = R\{T^a\} \oplus TR\{T^a\} \oplus \cdots \oplus T^{a-1}R\{T^a\}$, we can identify $[a]^*N$ with N^a, the direct sum of a copies of N. Since φ_N is multiplication by T^a, we see that under this identification the map $\varphi_{[a]^*N}$ is given by $(n_0, \dots, n_{a-1}) \mapsto (\varphi_N(n_{a-1}), n_0, \dots, n_{a-2})$.*

We collect some properties of the functors $[a]_*$ and $[a]^*$ in the following lemma. Most of these are to be expected.

Lemma 2.20. *Let M be a φ-module and N a φ^a-module over R. The following assertions are true:*

1. *$[a]^*$ and $[a]_*$ are adjoint (in both orders).*
2. *Both functors are exact and commute with duals.*
3. *$[a]_*$ commutes with tensor products over R, but $[a]^*$ does not.*
4. *$M \otimes_R [a]^*N \simeq [a]^*([a]_*M \otimes_R N)$ ("projection formula").*
5. *$\mathrm{rk}([a]_*M) = \mathrm{rk}(M)$ and $\mathrm{rk}([a]^*N) = a\,\mathrm{rk}(N)$.*
6. *$[a]_*[a]^*N \simeq N \oplus \varphi^*N \oplus \cdots \oplus (\varphi^{a-1})^*N$.*

Proof. We only prove (vi) because it is maybe the least obvious and the φ-action on the right deserves some explanation. We identify $[a]^*N$ with N^a as in the above remark. That is, φ_{N^a} is given by the map $(n_0, \dots, n_{a-1}) \mapsto (\varphi_N(n_{a-1}), n_0, \dots, n_{a-2})$. Applying $[a]_*$, we see that $[a]_*[a]^*N$ is isomorphic to N^a equipped with the φ^a-semilinear map $\varphi^a_{N^a} : (n_0, \dots, n_{a-1}) \mapsto (\varphi_N(n_0), \dots, \varphi_N(n_{a-1}))$. Finally, note that in the isomorphism $[a]^*N \simeq N^a$, each summand N is identified with $(\varphi^i)^*N$, and the φ-action given by $(\varphi^i)^*\varphi_N$. Hence the φ-action on the right is really φ_N in each component. Putting everything together gives the claim. □

Lemma 2.21. *If N is a φ^a-module for some integer $a \geq 1$, then*

$$H^i(N) \simeq H^i([a]^*N), \quad i = 0, 1.$$

Proof. Again, we identify $[a]^*N$ with N^a and the φ-action given by the map $(n_0, \dots, n_{a-1}) \mapsto (\varphi_N(n_{a-1}), n_0, \dots, n_{a-2})$. If $\varphi_N(n) = n$ then $\varphi_{[a]^*N}(n, \dots, n) = (n, \dots, n)$ and it is also easy to see that any element fixed by $\varphi_{[a]^*N}$ must be of the form $(n, \dots, n)$ for some $n \in N$ by applying $\varphi_{[a]^*N}$ repeatedly. This proves the assertion for $i = 0$. Consider the morphism $[a]^*N \ni (n_0, \dots, n_{a-1}) \mapsto \sum_{i=0}^{a-1} n_i + \mathrm{im}(\varphi_N - 1) \in H^1(N)$. It is clearly surjective and one checks that its kernel is exactly $\mathrm{im}(\varphi_{[a]^*N} - 1)$. This gives the desired result for $i = 1$. □

2.3 Semistability and the HN filtration

We introduce a notion of semistability for φ-modules. A formal consequence will be the existence of a canonical filtration, the HN filtration. Most of this can be done axiomatically as in [1, §1.3], but there is no real benefit to introducing the category theory required for it. For the rest of this section, we make the following hypothesis which comes in a weak and a strong form. Unless otherwise stated, we only assume that the weak form holds.

Hypothesis 2.22. *Let $R^{\mathrm{int}} \subseteq R^{\mathrm{b}} \subseteq R$ be inclusions of integral domains where R is Bézout and R^{b} is a discretely valued field with valuation subring R^{int}. Also assume that $R^\times \subseteq R^{\mathrm{b}}$. Let φ be an endomorphism of R which preserves R^{b} and R^{int}. Assume that the discrete valuation $w : R^{\mathrm{b}} \longrightarrow \mathbb{Z} \cup \{+\infty\}$ is φ-invariant and that $w(R^\times) = \mathbb{Z}$. Finally, assume that for any $n \times n$ matrix A over R^{int} the map $\mathbf{v} \mapsto \mathbf{v} - A\varphi(\mathbf{v})$ on R^n induces an injection (weak form) or a bijection (strong form) on $(R/R^{\mathrm{b}})^n$.*

Remark 2.23. *In [16, Hypothesis 1.4.1] the rings R^{int}, R^{b} and R are only assumed to be Bézout, but this may not be sufficient in some places. With the above assumptions we should not run into any problems.*

Lemma 2.24. *Let $a \geq 1$ be an integer. Then Hypothesis 2.22 also holds for φ^a.*

Proof. We only need to check the last assertion. Given an $n \times n$ matrix A over R^{int}, apply the assertion for φ to the $na \times na$ matrix

$$\bar{A} = \begin{pmatrix} 0 & 0 & \dots & 0 & A \\ E_n & 0 & & & 0 \\ 0 & E_n & \ddots & & \\ \vdots & & \ddots & 0 & \\ 0 & \dots & 0 & E_n & 0 \end{pmatrix}$$

It follows that the map

$$(\mathbf{v}_0, \dots, \mathbf{v}_{a-1}) \longmapsto (\mathbf{v}_0 - A\varphi(\mathbf{v}_{a-1}), \mathbf{v}_1 - \varphi(\mathbf{v}_0), \dots, \mathbf{v}_{a-1} - \varphi(\mathbf{v}_{a-2})) \tag{2.3.1}$$

is injective or bijective on $(R/R^{\text{b}})^{na}$. Let $\mathbf{v} \in R^n$ with $\mathbf{v} - A\varphi^a(\mathbf{v}) \in (R^{\text{b}})^n$ and set $\mathbf{v}_i = \varphi^i(\mathbf{v})$ for $0 \leq i \leq a-1$. We have $\mathbf{v}_0 - A\varphi(\mathbf{v}_{a-1}) = \mathbf{v} - A\varphi^a(\mathbf{v}) \in (R^{\text{b}})^n$, and

$$\mathbf{v}_1 - \varphi(\mathbf{v}_0) = \mathbf{v}_2 - \varphi(\mathbf{v}_1) = \dots = \mathbf{v}_{a-1} - \varphi(\mathbf{v}_{a-2}) = 0$$

by definition of $\mathbf{v}_i$. But then the injectivity of (2.3.1) implies that $\mathbf{v} = \mathbf{v}_0 \in (R^{\text{b}})^n$. If $\mathbf{w} \in R^n$ and (2.3.1) is surjective, then we find $\mathbf{v}_0, \dots, \mathbf{v}_{a-1} \in R^n$ with

$$\mathbf{w} \equiv \mathbf{v}_0 - A\varphi(\mathbf{v}_{a-1}) \mod (R^{\text{b}})^n$$

and $\mathbf{v}_i - \varphi(\mathbf{v}_{i-1}) \equiv 0 \mod (R^{\text{b}})^n$ for $1 \leq i \leq a-1$. The last condition implies that $\mathbf{v}_{a-1} \equiv \varphi^{a-1}(\mathbf{v}_0) \mod (R^{\text{b}})^n$, that is,

$$\mathbf{w} \equiv \mathbf{v}_0 - A\varphi(\mathbf{v}_{a-1}) \equiv \mathbf{v}_0 - A\varphi^a(\mathbf{v}_0) \mod (R^{\text{b}})^n.$$

Hence $\mathbf{v}_0$ is a preimage of $\mathbf{w}$ modulo $(R^{\text{b}})^n$. □

Remark 2.25. *If N is a φ^a-module and A is the representing matrix of φ_N w.r.t. some basis, then the corresponding representing matrix of $\varphi_{[a]^*N}$ is the matrix $\bar{A}$ above (see also Remark 2.19).*

Example 2.26. *As the notation suggests, Hypothesis 2.22 holds for $(R^{\text{int}}, R^{\text{b}}, R) = (\mathcal{R}^{\text{int}}, \mathcal{R}^{\text{b}}, \mathcal{R})$, the Robba ring and variants, if we take as φ any relative Frobenius lift and as w the discrete valuation on $\mathcal{R}^{\text{b}}$ (cf. Section 1). The hypothesis is also satisfied for $(R^{\text{int}}, R^{\text{b}}, R) = (\mathfrak{o}_K, K, K)$, the discretely valued field from Section 1. Here $\varphi = \varphi_K$ and $w = v_K$.*

We now define the slope of a φ-module.

Definition 2.27. *Let M be a φ-module over R and let A be the representing matrix of φ_M w.r.t. some R-basis. The determinant of A is a unit in R and hence an element of R^{b} so we can apply the valuation w to it. Since w is φ-invariant, the quantity* $\deg(M) = w(\det(A))$ *is independent of the choice of basis (cf. Remark 2.4). It is called the degree of M. If M is nonzero, then we define the slope of M as* $\mu(M) = \deg(M)/\operatorname{rk}(M)$.

Remark 2.28. *Since we will often only be interested in the slope or the degree of a φ-module M, we will sometimes speak of "the determinant* $\det(\varphi_M)$ *of φ_M" or "the representing matrix A of φ_M" although both may well depend on the chosen basis.*

Example 2.29.
- *The trivial φ-module R has degree and slope 0. Given $n \in \mathbb{Z}$, there is $\lambda \in R^\times$ with $w(\lambda) = n$. The φ-module $R(n)$ is the R-module R equipped with the φ-semilinear map $\lambda\varphi : R \longrightarrow R$. It has degree and slope n.*

- *If $M = R\{T\}/R\{T\}P$ for some polynomial $P(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_0 \in R\{T\}$, then the degree of M is $w(a_0)$. This is because the representing matrix of φ_M (i.e. multiplication by T) w.r.t. the basis $1, T, \ldots, T^{n-1}$ is a companion matrix of P and hence has determinant $\pm a_0$.*

Before giving the definition of semistability, we prove some useful properties of the degree and slope.

Lemma 2.30. *The following assertions are true:*

1. *If $0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$ is a short exact sequence of φ-modules, then we have $\deg(M) = \deg(M_1) + \deg(M_2)$.*
2. *If M, N are two φ-modules, then $\mu(M \otimes N) = \mu(M) + \mu(N)$.*
3. *If M is a φ-module of rank n, then $\mu(\wedge^i M) = i\mu(M)$ for $0 \leq i \leq n$.*
4. *If M is a φ-module, then $\deg(M^\vee) = -\deg(M)$ and hence $\mu(M^\vee) = -\mu(M)$.*
5. *If M is a φ-module, then $\mu([a]_*M) = a\mu(M)$.*
6. *If N is a φ^a-module, then $\mu([a]^*N) = a^{-1}\mu(N)$.*

Proof. All assertions can be checked easily, but let us give some short arguments.
(1.) The representing matrix of φ_M is a block diagonal matrix with one block the representing matrix of φ_{M_1} and one block the representing matrix of φ_{M_2}. The claim now follows from properties of the determinant and of the valuation w.
(2.) Let $\mathrm{rk}(M) = m$ and $\mathrm{rk}(N) = n$ so that $\mathrm{rk}(M \otimes N) = mn$, and write $\varphi_M \otimes \varphi_N = (\varphi_M \otimes 1) \circ (1 \otimes \varphi_N)$. The determinant of the first morphism is $\det(\varphi_M)^n$ and that of the second is $\det(\varphi_N)^m$. Applying w and dividing by mn yields the claim.
(3.) $\wedge^n M$ is a φ-module of rank 1 and the map $\wedge^n M \longrightarrow \wedge^n M$ induced by φ_M is multiplication by $\det(\varphi_M)$. This gives the claim for $i = n$. The rest is a generalization of this special case using the isomorphism in [1, Proposition 2.2.2].
(4.) Let A be the representing matrix of φ_M w.r.t. some basis. By definition of the φ-module structure on $M^\vee$, the representing matrix of $\varphi_{M^\vee}$ w.r.t. the dual basis is the transpose of A^{-1}. In particular,

$$\deg(M^\vee) = w(\det(\varphi_{M^\vee})) = w(\det(A^{-1})) = -w(\det(A)) = -\deg(M).$$

(5.) $[a]_*M$ has the same rank as M and $\varphi_{[a]_*M} = \varphi_M^a$ has determinant $\det(\varphi_M)^a$.
(6.) $[a]^*N$ has rank $a\,\mathrm{rk}(N)$ and degree $\deg([a]^*N) = \deg(N)$ (see Remark 2.25). ☐

Remark 2.31.

- *Item (1.) implies that $\mu(M)$ is a weighted average of $\mu(M_1)$ and $\mu(M_2)$, provided that M_1 and M_2 are nonzero. More precisely,*

$$\mu(M) = \mu(M_1)\frac{\mathrm{rk}(M_1)}{\mathrm{rk}(M)} + \mu(M_2)\frac{\mathrm{rk}(M_2)}{\mathrm{rk}(M)}.$$

- *A short exact sequence as in (1.) is really a commutative diagram with exact rows*

$$\begin{tikzcd} 0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\varphi"] & M \arrow[r] \arrow[d, "\varphi"] & M_2 \arrow[r] \arrow[d, "\varphi"] & 0 \\ 0 \arrow[r] & M_1 \arrow[r] & M \arrow[r] & M_2 \arrow[r] & 0 \end{tikzcd}$$

  Note that the diagram remains commutative if we replace the vertical arrows by $\varphi - 1$ so that the snake lemma gives an exact sequence relating H^0 and H^1 of M, M_1, M_2.

- *By a similar argument as for item (2.), we see that base change of φ-modules preserves slopes, at least when this makes sense.*

As for vector bundles on a curve, we can now define semistability for φ-modules. Note however the opposite choice of sign in our definition of the slope. This causes all inequalities to be flipped!

Definition 2.32. *A φ-module M is called semistable (resp. stable) if we have $\mu(N) \geq \mu(M)$ (resp. $\mu(N) > \mu(M)$) for any nonzero φ-submodule $N \subsetneq M$.*

Both properties are preserved under twisting, i.e. tensoring with a φ-module of rank 1.

Lemma 2.33. *Let M, N be φ-modules where N has rank 1. Then M is semistable (resp. stable) if and only if $M \otimes N$ is.*

Proof. This follows directly from Lemma 2.30 and the exactness of twisting. For the converse, twist by $N^\vee$ instead. □

Our next goal is to construct the HN filtration. We start with some preliminary results.

Proposition 2.34. *Any φ-module of rank 1 is stable.*

Proof. If M is a φ-module of rank 1, then we may twist by $M^\vee$ and invoke Lemma 2.30 and 2.33 to reduce to the case that M has degree 0. In other words, it suffices to show that $M = R$ is stable. Let $N \subseteq M$ be a nonzero φ-submodule of M. Since N is finitely generated and R is Bézout, N must be of the form $N = Rx$ for some nonzero $x \in R$. Let $\lambda \in R^\times \subseteq R^{\mathrm{b}}$ with $\varphi(x) = \lambda x$. By definition, $\mu(N) = w(\lambda)$. Assume that $\mu(N) \leq 0$, so that $\lambda^{-1} \in R^{\mathrm{int}}$. Then the injectivity of $\mathbf{v} \mapsto \mathbf{v} - \lambda^{-1}\varphi(\mathbf{v})$ on R/R^b (Hypothesis 2.22) implies that $x \in R^{\mathrm{b}}$ because $x - \lambda^{-1}\varphi(x) = 0 \in R^{\mathrm{b}}$. But then $x \in (R^{\mathrm{b}})^\times \subseteq R^\times$ whence $N = M$ and $\mu(N) = \mu(M) = 0$. This shows that $\mu(N) > 0 = \mu(M)$ unless $N = M$, as desired. □

Corollary 2.35. *If $N \subseteq M$ is an inclusion of φ-modules of the same rank then $\mu(N) \geq \mu(M)$ with equality if and only if $N = M$.*

Proof. Let $n = \mathrm{rk}(M) = \mathrm{rk}(N)$. It suffices to show that $\deg(N) \geq \deg(M)$. We have an inclusion of φ-modules of rank 1, $\wedge^n N \subseteq \wedge^n M$ where the slopes are the degrees of N and M, respectively (cf. Lemma 2.30). Hence the claim follows from the previous proposition. □

Lemma 2.36. *Let M be a φ-module. Then the slopes of nonzero φ-submodules of M are bounded below.*

Proof. We proceed by induction on $n = \mathrm{rk}(M)$. The case $n = 1$ follows from Proposition 2.34 so we are left with the induction step. The corollary above gives the claim for φ-submodules of full rank. If M has no nonzero φ-submodules of lower rank, then we are done. Otherwise, let N be a φ-submodule of rank $m < n$. Replacing N by its saturation (which has rank m as well), we may assume that N is saturated so that M/N is also a φ-module. By induction hypothesis, the slopes of nonzero φ-submodules of N and M/N are bounded below. If P is any nonzero φ-submodule of M, then we have a short exact sequence

$$0 \longrightarrow N \cap P \longrightarrow P \longrightarrow P/(N \cap P) \longrightarrow 0$$

where the outer terms are φ-submodules of N and M/N, respectively. If one of these vanishes, $\mu(P)$ is simply the slope of the other term, hence is bounded below. If not, then both $\mu(N \cap P)$ and $\mu(P/(N \cap P))$ are bounded below. Being a weighted average of the two, $\mu(P)$ is bounded below as well. □

Lemma 2.37. *Let M be a nonzero φ-module over R. Then there exists a largest φ-submodule (w.r.t. inclusion) of M of least slope. It is semistable and saturated.*

Proof. By the previous lemma, the slopes of nonzero φ-submodules of M are bounded below. Since the denominators of possible slopes of φ-submodules of M are bounded above by $\mathrm{rk}(M)$, this implies the existence of a least slope s. By minimality, any φ-submodule of M of slope s must be semistable. To prove the lemma, it remains to check that the set of φ-modules of slope s is stable under taking sums and saturations. Let N be a φ-submodule of slope s with saturation N^{sat}. This is a saturated φ-submodule of M of the same rank as N so we must have $\mu(N) \geq \mu(N^{\mathrm{sat}})$ by Corollary 2.35. The reverse inequality holds by minimality of s whence $\mu(N^{\mathrm{sat}}) = s$ and $N = N^{\mathrm{sat}}$ by Corollary 2.35 once more. Next, let N_1, N_2 be two saturated φ-submodules of M of slope s. We wish to show that the slope of $N_1 + N_2$ (which is indeed a φ-module) is also s. We consider the following two short exact sequences of φ-modules

$$0 \longrightarrow N_1 \cap N_2 \longrightarrow N_1 \longrightarrow (N_1 + N_2)/N_2 \longrightarrow 0$$

and

$$0 \longrightarrow N_2 \longrightarrow N_1 + N_2 \longrightarrow (N_1 + N_2)/N_2 \longrightarrow 0$$

This gives

$$\begin{aligned}
\mu(N_1 + N_2) &= \mu(N_2)\frac{\mathrm{rk}(N_2)}{\mathrm{rk}(N_1 + N_2)} + \mu((N_1 + N_2)/N_2)\frac{\mathrm{rk}((N_1 + N_2)/N_2)}{\mathrm{rk}(N_1 + N_2)} \\
&= \mu(N_2)\frac{\mathrm{rk}(N_2)}{\mathrm{rk}(N_1 + N_2)} + \left(\mu(N_1) - \mu(N_1 \cap N_2)\frac{\mathrm{rk}(N_1 \cap N_2)}{\mathrm{rk}(N_1)}\right)\frac{\mathrm{rk}(N_1)}{\mathrm{rk}(N_1 + N_2)} \\
&\leq s\frac{\mathrm{rk}(N_2)}{\mathrm{rk}(N_1 + N_2)} + \left(s - s\frac{\mathrm{rk}(N_1 \cap N_2)}{\mathrm{rk}(N_1)}\right)\frac{\mathrm{rk}(N_1)}{\mathrm{rk}(N_1 + N_2)} \\
&= s.
\end{aligned}$$

Hence $\mu(N_1 + N_2) = s$ because the reverse inequality holds by minimality of s. □

Remark 2.38. *Clearly, M is semistable if and only if the φ-submodule in the lemma is M itself.*

Lemma 2.39. *Let M, N be two semistable φ-modules with $\mu(M) < \mu(N)$. Then* $\mathrm{Hom}(M, N) = 0$.

Proof. Assume that $f : M \longrightarrow N$ is nonzero. Then $\mathrm{im}(f) \subseteq N$ is a nonzero φ-submodule whence $\mu(\mathrm{im}(f)) \geq \mu(N)$ by semistability of N. If f were injective, we would get the contradiction

$$\mu(N) > \mu(M) = \mu(\mathrm{im}(f)) \geq \mu(N).$$

Thus, $\ker(f) \subseteq M$ is a nonzero φ-submodule which forces $\mu(\ker(f)) \geq \mu(M)$ because M is also semistable. Consider the short exact sequence of φ-modules

$$0 \longrightarrow \ker(f) \longrightarrow M \longrightarrow \mathrm{im}(f) \longrightarrow 0$$

This leads to the contradiction

$$\begin{aligned}
\mu(M) &= \mu(\ker(f))\frac{\mathrm{rk}(\ker(f))}{\mathrm{rk}(M)} + \mu(\mathrm{im}(f))\frac{\mathrm{rk}(\mathrm{im}(f))}{\mathrm{rk}(M)} \\
&\geq \mu(M)\frac{\mathrm{rk}(\ker(f))}{\mathrm{rk}(M)} + \mu(N)\frac{\mathrm{rk}(\mathrm{im}(f))}{\mathrm{rk}(M)} \\
&> \mu(M).
\end{aligned}$$

We conclude that any morphism of φ-modules from M to N must vanish. □

Remark 2.40. *One can generalize this result quite a bit as in [16, Proposition 1.4.18] by introducing slope polygons. We will not need this explicitly, but a similar argument will be used later.*

Corollary 2.41. *Let M be a φ-module over R. Then for any integer $a \geq 1$, M is semistable if and only if $[a]_*M$ is semistable.*

Proof. Assume that $[a]_*M$ is semistable and let $M' \subseteq M$ be a nonzero φ-submodule. By exactness of the a-pushforward, $[a]_*M' \subseteq [a]_*M$ is a φ^a-submodule. It follows from Lemma 2.30 and the semistability of $[a]_*M$ that $a\mu(M') = \mu([a]_*M') \leq \mu([a]_*M) = a\mu(M)$ so M is also semistable. Conversely, if $[a]_*M$ is not semistable, then its largest φ^a-submodule of least slope is a φ^a-submodule M_1 of lower rank. Write $\mu(M_1) = s$. Note that $\varphi_M(M_1) \subseteq M$ is a φ^a-submodule of M which has the same slope as M_1. By the uniqueness in Lemma 2.37, φ_M must therefore preserve M_1, i.e. M_1 is even a φ-submodule of M. But if we consider M_1 as a φ-module, then its slope is $s/a < \mu([a]_*M)/a = \mu(M)$. Hence M is not semistable either. ☐

Proposition 2.42. *Every nonzero φ-module M over R admits a unique filtration $0 = M_0 \subset M_1 \subset \ldots \subset M_l = M$ by saturated φ-submodules such that M_i/M_{i-1} is semistable for $1 \leq i \leq l$, and*

$$\mu(M_1) < \mu(M_2/M_1) < \cdots < \mu(M/M_{l-1}).$$

It is called the Harder-Narasimhan filtration (HN filtration, for short) of M. One calls the quantities $\mu(M_i/M_{i-1})$ the HN slopes of M.

Proof. The proposition is really a formal consequence of Lemma 2.37. Namely, start by letting M_1 be the φ-submodule constructed there. If $M_1 = M$, we are done. Otherwise, let M_2 be the preimage in M of the largest φ-submodule of least slope of M/M_1. It suffices to check that M_2/M_1 is semistable with $\mu(M_1) < \mu(M_2/M_1)$ as the desired filtration can then be constructed by repeating the process, proving existence. Note that this process must terminate after finitely many steps since the ranks of the φ-submodules must increase strictly every time. By construction, M_2/M_1 is isomorphic to the largest φ-submodule of least slope of M/M_1 and hence is semistable. We have a short exact sequence of φ-modules

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_2/M_1 \longrightarrow 0$$

Note that we cannot have $\mu(M_2) \leq \mu(M_1)$ as M_1 is strictly contained in M_2 and the former is the largest φ-submodule of least slope of M. So $\mu(M_2) > \mu(M_1)$ and we get the desired inequality

$$\mu(M_2/M_1) = \mu(M_2)\frac{\mathrm{rk}(M_2)}{\mathrm{rk}(M_2/M_1)} - \mu(M_1)\frac{\mathrm{rk}(M_1)}{\mathrm{rk}(M_2/M_1)} > \mu(M_1).$$

Next, we show uniqueness. Assume we are given two HN filtrations of M,

$$0 = M_0 \subset M_1 \subset \ldots \subset M_l = M \quad \text{and} \quad 0 = M_0' \subset M_1' \subset \ldots \subset M_m' = M.$$

We may assume $\mu(M_1) \leq \mu(M_1')$. By Lemma 2.39, the morphism $M_1 \longrightarrow M \longrightarrow M/M_{m-1}'$ must be the zero map. Hence $M_1 \subseteq M_{m-1}'$. By the same argument, $M_1 \longrightarrow M_{m-1}' \longrightarrow M_{m-1}'/M_{m-2}'$ is also the zero map and inductively we deduce $M_1 \subseteq M_1'$. In particular, $\mu(M_1) \geq \mu(M_1')$ because M_1' is semistable. By a symmetric argument, we then get $M_1 = M_1'$ and hence $\mu(M_1) = \mu(M_1')$. We repeat the whole argument for M_2/M_1 and M_2'/M_1' to see that $M_2 = M_2'$. It follows by induction that $l = m$ and $M_i = M_i'$ for all i. This finishes the proof. ☐

Lemma 2.43. *If M is semistable of slope s and $N \subsetneq M$ is a saturated φ-submodule of slope s, then M/N is also semistable of slope s.*

Proof. Note that N is necessarily semistable and that $\mu(M/N) = s$ by the remark after Lemma 2.30. If we can show that the slope of the largest φ-submodule of least slope of M/N is at least s, then it must be M/N and we are done. Denote the preimage in M of that submodule by M'. Then M' is a φ-submodule of M containing N and in particular is nonzero. We have

$$\mu(M'/N) = \mu(M')\frac{\mathrm{rk}(M')}{\mathrm{rk}(M'/N)} - \mu(N)\frac{\mathrm{rk}(N)}{\mathrm{rk}(M'/N)} \geq s$$

using that $\mu(N) = s$ and $\mu(M') \geq s$ by semistability of M. Now M'/N is isomorphic to the largest φ-submodule of least slope of M/N, finishing the proof. □

2.4 Étale and pure φ-modules

Having introduced the auxiliary notion of a semistable φ-module, we now turn to the properties of φ-modules we are really interested in, namely étale and pure φ-modules. These can be expressed in terms of representing matrices.

Definition 2.44. *A φ-module M over R or R^{b} is called étale if it can be obtained by base extension from a φ-module over R^{int}. That is, there should exist a finite free R^{int}-submodule $N \subseteq M$ whose base extension to R or R^{b} is M and such that φ_M induces an isomorphism $\varphi^*N \longrightarrow N$. We call N an étale lattice of M. It need not be unique.*

Remark 2.45.

- *Since N is a φ-module over R^{int}, the representing matrix of φ_N w.r.t. any R^{int}-basis is invertible over R^{int}. It follows that M admits an R-basis such that the representing matrix of φ_M is invertible over R^{int}. In particular, M has degree and slope 0. Conversely, any φ-module of rank 1 and slope 0 is étale since it must contain a φ-invariant generator and hence must be isomorphic to R or R^{b}.*
- *The dual of an étale φ-module M is again étale, because we can take as an étale lattice for $M^\vee$ the R^{int}-dual of an étale lattice of M.*
- *If M is étale, then so is $\wedge^i M$ for $0 \leq i \leq \mathrm{rk}(M)$.*

Definition 2.46. *An isogeny φ-module over R^{int} is a finite free R^{int}-module M, equipped with an injection $\varphi^*M \longrightarrow M$ whose cokernel is annihilated by some power of a uniformizer of R^{int}. Since any uniformizer is a unit in R, such an object becomes a φ-module after tensoring with R or R^{b}.*

Proposition 2.47. *Let M be an isogeny φ-module over R^{int}. Then the natural maps $H^i(M \otimes_{R^{\mathrm{int}}} R^{\mathrm{b}}) \longrightarrow H^i(M \otimes_{R^{\mathrm{int}}} R)$ for $i = 0$ (under weak Hypothesis 2.22) or $i = 0, 1$ (under strong Hypothesis 2.22) are bijective.*

Proof. Choosing a basis, we can write $M = (R^{\mathrm{int}})^n$, $M \otimes_{R^{\mathrm{int}}} R^{\mathrm{b}} = (R^{\mathrm{b}})^n$, $M \otimes_{R^{\mathrm{int}}} R = R^n$ and $\varphi_M = A\varphi$. Assume that the weak form of Hypothesis 2.22 holds. We need to show that the inclusion $H^0((R^{\mathrm{b}})^n) \hookrightarrow H^0(R^n)$ is bijective. The injectivity is clear. If $\mathbf{v} \in H^0(R^n)$, then $\mathbf{v} = \varphi_M(\mathbf{v}) = A\varphi(\mathbf{v})$, i.e. $\mathbf{v} + (R^{\mathrm{b}})^n$ lies in the kernel of $(R/R^{\mathrm{b}})^n \longrightarrow (R/R^{\mathrm{b}})^n, \mathbf{v} \mapsto \mathbf{v} - A\varphi(\mathbf{v})$. But then we must have $\mathbf{v} \in (R^{\mathrm{b}})^n$ by the injectivity in Hypothesis 2.22. This proves the surjectivity.

Now assume that the strong form of Hypothesis 2.22 holds. We need to show that the natural map $H^1((R^{\mathrm{b}})^n) \longrightarrow H^1(R^n)$ is bijective. If the class of $\mathbf{v} \in (R^{\mathrm{b}})^n$ in $H^1(R^n)$ is zero, then $\mathbf{v} = \mathbf{w} - A\varphi(\mathbf{w})$ for some $\mathbf{w} \in R^n$. Since the class of $\mathbf{v}$ in $(R/R^{\mathrm{b}})^n$ is trivial and $(R/R^{\mathrm{b}})^n \longrightarrow (R/R^{\mathrm{b}})^n$ is injective, we must in fact have $\mathbf{w} \in (R^{\mathrm{b}})^n$. Hence the class of $\mathbf{v}$ in $H^1((R^{\mathrm{b}})^n)$ is zero, proving injectivity. To prove surjectivity, let $\mathbf{v} \in R^n$. Since $(R/R^{\mathrm{b}})^n \longrightarrow (R/R^{\mathrm{b}})^n$ is surjective, we find $\mathbf{v}' \in R^n$ such that $\mathbf{v} + (R^{\mathrm{b}})^n = \mathbf{v}' - A\varphi(\mathbf{v}') + (R^{\mathrm{b}})^n$. Since $(R/R^{\mathrm{b}})^n \longrightarrow (R/R^{\mathrm{b}})^n$ is injective, this forces $\mathbf{w} = \mathbf{v} - \mathbf{v}' + A\varphi(\mathbf{v}') \in (R^{\mathrm{b}})^n$. Hence the class of $\mathbf{w}$ in $H^1((R^{\mathrm{b}})^n)$ is a preimage of the class of $\mathbf{v}$. This finishes the proof. □

Proposition 2.48. *The base change functor from étale φ-modules over R^{b} to étale φ-modules over R is an equivalence of categories.*

Proof. Given an étale φ-module M over R, choose an étale lattice M_0 of M. The φ-module $M_0 \otimes_{R^{\mathrm{int}}} R^{\mathrm{b}}$ over R^{b} is visibly étale and base changes to M. This proves essential surjectivity.

To check full faithfulness, we need to see that if M_1, M_2 are two étale φ-modules over R^{b}, then

$$\mathrm{Hom}_{R^{\mathrm{b}},\varphi}(M_1, M_2) = \mathrm{Hom}_{R,\varphi}(M \otimes_{R^{\mathrm{b}}} R, N \otimes_{R^{\mathrm{b}}} R).$$

We use Lemma 2.13 to rewrite these sets in terms of H^0 and thus need to show that the natural map

$$H^0(M_1^\vee \otimes_{R^{\mathrm{b}}} M_2) \longrightarrow H^0(M_1^\vee \otimes_{R^{\mathrm{b}}} M_2 \otimes_{R^{\mathrm{b}}} R)$$

is a bijection. This is true by the same argument as in the proof of Proposition 2.47. $\square$

Remark 2.49.

- *As a consequence of the proposition, many of the results to follow are true both for R and R^{b}. To simplify statements slightly, we write S whenever we mean either one of R or R^{b}. For example, we may say that the category of étale φ-modules over S is a full subcategory of the category of all φ-modules over S.*

- *Note that we only used the weak form of Hypothesis 2.22 to prove full faithfulness.*

Proposition 2.50. *Let $0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$ be a short exact sequence of φ-modules over R. If any two of them are étale (except possibly M_1, M_2 under weak Hypothesis 2.22), then so is the third.*

Proof. Suppose that M and M_2 are étale. By Proposition 2.48, the φ-modules M, M_2 and the morphism $M \longrightarrow M_2$ are obtained by base change from φ-modules $M^{\mathrm{b}}, M_2^{\mathrm{b}}$ over R^{b} and a morphism $M^{\mathrm{b}} \longrightarrow M_2^{\mathrm{b}}$, respectively. Choose an étale lattice M_0^{b} of M^{b}. The kernel of the map $M_0^{\mathrm{b}} \longrightarrow M_2^{\mathrm{b}}$ is a finitely generated φ-stable R^{int}-submodule of M (NB: it is saturated; now use Lemma 1.21), hence is a φ-module over R^{int} by Lemma 2.51 below. This gives an étale lattice for M_1. If M and M_1 are étale then we consider the short exact sequence obtained by dualizing the one of the proposition. Since $M^\vee$ and $M_1^\vee$ are étale, the case already treated shows that $M_2^\vee$ is étale. Thus, $M_2 = (M_2^\vee)^\vee$ is also étale.

Finally, suppose that M_1 and M_2 are étale and that the strong form of Hypothesis 2.22 holds. By Proposition 2.48, M_1, M_2 are obtained by base change from étale φ-modules $M_1^{\mathrm{b}}, M_2^{\mathrm{b}}$ over R^{b}. Additionally, the extension of M_2 by M_1 is obtained from an extension of M_2^{b} by M_1^{b} by Lemma 2.13 and Proposition 2.47 (with $i = 1$). So we have a short exact sequence of φ-modules over R^{b}

$$0 \longrightarrow M_1^{\mathrm{b}} \longrightarrow M^{\mathrm{b}} \longrightarrow M_2^{\mathrm{b}} \longrightarrow 0$$

which base changes to the one of the proposition. The morphisms $M_1^{\mathrm{b}} \longrightarrow M^{\mathrm{b}}$ and $M^{\mathrm{b}} \longrightarrow M_2^{\mathrm{b}}$ are given by multiplication with a matrix with entries in R^{b}. Multiplying by a sufficiently large power of a uniformizer, we can achieve that all entries are in R^{int}. We then get an étale lattice for M^{b} by lifting one of M_2^{b} (i.e. taking preimages of a basis) and adding one of M_1^{b} and invoking Lemma 2.51 again. $\square$

Lemma 2.51. *Let M be an étale φ-module over R^{b}. Then any finitely generated φ_M-stable R^{int}-submodule of M is a φ-module over R^{int}.*

Proof. Let M_0 be an étale lattice of M and let N be a finitely generated φ-stable R^{int}-submodule of M. Write generators of N as linear combinations (with coefficients in R^{b}) of a basis of M_0. Multiplying by some power of a uniformizer, we may assume that all coefficients lie in R^{int} so that $N \subseteq M_0$. Then N is a finitely generated R^{int}-submodule of the finite free R^{int}-module M_0, hence is finite free itself because R^{int} is Bézout. Now note that φ makes R^{int} a torsion-free R^{int}-module. Since torsion-free modules over Bézout domains are flat, we deduce that $\varphi^*N \longrightarrow \varphi^*M_0$ is injective whence $\varphi^*N \longrightarrow N$ (note that N is φ_M-stable) is also injective. Thus, $\varphi^*N \longrightarrow N$ is an injection of R^{int}-modules of the same rank so its cokernel is torsion. Since R^{int} is a DVR, it follows that it is annihilated by some power of a uniformizer. To summarize, we at least know that N is an isogeny φ-module over R^{int}.

It remains to see that the determinant of φ_N has valuation 0. After possibly replacing M, N by $\wedge^{\text{rk}(N)}M, \wedge^{\text{rk}(N)}N$ (NB: $\wedge^{\text{rk}(N)}M$ is also étale), we may reduce to the case that $\text{rk}(N) = 1$. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be an R^{int}-basis of M_0, let A be the corresponding representing matrix and let $\mathbf{v}$ be a generator of N. Write $\mathbf{v} = \sum_{j=1}^n c_j\mathbf{e}_j$ so that $\varphi_M(\mathbf{v}) = \sum_{i=1}^n \sum_{j=1}^n \varphi(c_j)A_{ij}\mathbf{e}_i = \sum_{i=1}^n d_i\mathbf{e}_i$. On the other hand, $\varphi_M(\mathbf{v}) = r\mathbf{v} = \sum_{i=1}^n rc_i\mathbf{e}_i$ for some $r \in R^{\text{int}}$ since N is φ_M-stable. The equality

$$\sum_{i=1}^n rc_i\mathbf{e}_i = \varphi_M(\mathbf{v}) = \sum_{i=1}^n d_i\mathbf{e}_i$$

yields $r = c_i/d_i$ for any i. In particular, $w(c_i) \geq w(d_i)$ for any i bacause $w(r) \geq 0$. Let $w(c_k) = \min_{1 \leq i \leq n}\{w(c_i)\}$. For any i, we have

$$w(d_i) = w(\sum_{j=1}^n \varphi(c_j)A_{ij}) \geq \min_{1\leq j\leq n}\{w(\varphi(c_j) + w(A_{ij})\} \geq \min_{1\leq j\leq n}\{w(c_j)\} = w(c_k).$$

Altogether, $w(c_k) \geq w(d_k) \geq w(c_k)$ and hence $w(r) = w(c_k) - w(d_k) = 0$. □

We now introduce the more general notion of pure φ-modules.

Definition 2.52. *Let M be a φ-module over S of slope $s = c/d$ where c, d are coprime integers with $d > 0$. M is called pure (or isoclinic; or isocline) of slope s if for some φ^d-module N of rank 1 and degree $-c$ over S, the φ^d-module $([d]_*M) \otimes_S N$ is étale.*

We make some immediate observations.

Remark 2.53.

- *If we can find one N as in the definition, then any other will work as well. This follows from the fact that modules of rank 1 and slope 0 are étale.*
- *Any φ-module of rank 1 is pure since $d = 1$ and we can take $N = M^\vee$.*
- *As for étale φ-modules, the dual of a pure φ-module of slope s is pure of slope $-s$.*
- *A φ-module is pure of slope 0 if and only if it is étale since we must have $c = 0$ and $d = 1$ in this case. More precisely, if M is pure of slope 0 then $M \otimes_S N$ is étale for some N which is itself étale by Remark 2.45. Therefore, also $M \simeq (M \otimes_S N) \otimes_S N^\vee$ is étale.*

We look at the most basic example of a pure φ-module of slope c/d.

Example 2.54. *Let π be a uniformizer of R^{int}. For coprime integers $c, d \in \mathbb{Z}$ with $d > 0$, we define the φ-module $M_{\pi,c,d} = M_{c,d} = [d]_*R(c)$ to be the free module of rank d over R with basis $\mathbf{e}_1, \dots, \mathbf{e}_d$ and φ-action given by*

$$\varphi_{M_{c,d}}(\mathbf{e}_1) = \mathbf{e}_2, \dots, \varphi_{M_{c,d}}(\mathbf{e}_{d-1}) = \mathbf{e}_d, \varphi_{M_{c,d}}(\mathbf{e}_d) = \pi^c\mathbf{e}_1.$$

It is easily seen that $\varphi^d_{M_{c,d}}$ acts on the basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ by a diagonal matrix where each entry is the image of π^c under some power of φ. It follows that the representing matrix of $\pi^{-c}\varphi^d_{M_{c,d}}$ is invertible over R^{int} whence $M_{c,d}$ is pure of slope c/d. A φ-module isomorphic to $M_{c,d}$ is called a standard pure φ-module.

Lemma 2.55. *Let M be a φ-module over S, and let $a \geq 1$ be an integer. Then M is pure of slope s if and only if $[a]_*M$ is pure of slope as.*

Proof. We first treat the case that $s = 0$. Here we have to prove that M is étale if and only if $[a]_*M$ is. Assume that M is étale. Since $[a]_*$ is exact and commutes with tensor products, we can obtain an étale lattice for $[a]_*M$ by applying $[a]_*$ to one of M. It follows that $[a]_*M$ is étale. Conversely, if $[a]_*M$ is étale, then for any $i \geq 0$ we have an isomorphism $((\varphi^a)^{i+1})^*[a]_*M \longrightarrow ((\varphi^a)^i)^*)[a]_*M$ by applying $(\varphi^a)^*$ to the isomorphism $(\varphi^a)^*[a]_*M \longrightarrow [a]_*M$ repeatedly. By Proposition 2.48, these isomorphisms all descend to R^{b}. In other words, we may work over R^{b}. Let N be an étale lattice of $[a]_*M$ and let N' be the R^{int}-span of $N, \varphi_M(N), \ldots, \varphi_M^{a-1}(N)$. By assumption, φ_M^a acts on N by a matrix with coefficients in R^{int}. It follows that N' is φ_M-stable and it is also visibly finitely generated hence finite free over R^{int}. Thus, N' is an étale lattice for M.

In the general case, we write $s = c/d$ in lowest terms and set $b = \gcd(a, d)$. Then, again in lowest terms, $as = (ac/b)/(d/b)$. Let N be a φ^d-module of rank 1 and degree $-c$. Then $[a/b]_*N$ has rank 1 and degree $-ac/b$. The following are equivalent:

- M is pure of slope s
- $([d]_*M) \otimes_S N$ is étale
- $[a/b]_*(([d]_*M) \otimes_S N) \simeq ([ad/b]_*M) \otimes_S ([a/b]_*N) \simeq ([d/b]_*([a]_*M)) \otimes_S ([a/b]_*N)$ is étale
- $[a]_*M$ is pure of slope as

The first and last two are equivalent by definition and the second and third are equivalent by the slope 0 case treated before. This yields the claim. ☐

Remark 2.56.
- *By the lemma, it is equivalent to impose the condition in the definition of a pure φ-module for any pair of integers $c, d \in \mathbb{Z}$ with $s = c/d$ and $d > 0$.*
- *As for étale φ-modules, we see that if M is pure of slope s, then $\wedge^i M$ is pure of slope is for any $0 \leq i \leq \text{rk}(M)$.*

Corollary 2.57. *If M, N are pure φ-modules of slopes s_1, s_2 over S, then $M \otimes_S N$ is pure of slope $s_1 + s_2$.*

Proof. First of all, the slope of $M \otimes_S N$ is $s_1 + s_2$ by Lemma 2.30. Since $[a]_*$ commutes with tensor products over S, we may reduce to the case $s_1, s_2 \in \mathbb{Z}$ by choosing $a = \text{rk}(M)\,\text{rk}(N)$ and invoking the lemma above. We may then twist by φ-modules of degrees $-s_1$ and $-s_2$ to reduce to the case that $s_1 = s_2 = 0$, i.e. it suffices to show that if M and N are étale then so is $M \otimes_S N$. Since φ-modules over R^{int} admit tensor products, we may take as an étale lattice for $M \otimes_S N$ the tensor product over R^{int} of an étale lattice of M and of N. ☐

Theorem 2.58. *For any rational number s, the base change functor from pure φ-modules of slope s over R^{b} to pure φ-modules of slope s over R is an equivalence of categories.*

Proof. If M, N are pure of slope s, then $M^\vee \otimes_{R^{\text{b}}} N$ is pure of slope 0 by the above corollary, hence is étale. Therefore the proof of Theorem 2.48 works unchanged. ☐

Theorem 2.59. *Let $0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$ be a short exact sequence of φ-modules over R. If any two of them are pure of slope s (except possibly M_1, M_2 in the case of weak Hypothesis 2.22), then so is the third.*

Proof. $[a]_*$ and twisting are both exact functors so we may reduce to the case $s = 0$. The theorem then follows from Proposition 2.50. □

Lemma 2.60. *Let M be a pure φ-module over R with $\mu(M) > 0$. Then $H^0(M) = 0$.*

Proof. Note that for any $a \geq 1$, we have $H^0(M) \subseteq H^0([a]_*M)$ because if $\varphi_M(m) = m$ then certainly $\varphi_M^a(m) = m$. We may thus replace M by $[a]_*M$ for $a = \mathrm{rk}(M)$ to reduce to the case where $\mu(M) \in \mathbb{Z}_{>0}$. By Theorem 2.58, there exists a pure φ-module M_0 over R^{b} with $M = M_0 \otimes_{R^{\mathrm{b}}} R$. We have $H^0(M_0) = H^0(M)$ by Proposition 2.47.

Since M is pure, $M \otimes_S N$ has a representing matrix in R^{int} for some N of rank 1 and slope $-\mu(M) < 0$. Let $f_1, \dots, f_n$ be a corresponding basis of $M \otimes_S N$, denote by e a generator of N and by $e^\vee$ the generator of $N^\vee$ dual to e. Then all entries of the representing matrix A of φ_M w.r.t. the basis $\mathbf{e}_i = f_i \otimes e^\vee$ of $M \otimes_S N \otimes_S N^\vee \simeq M$ have valuation at least $\mu(M)$ (because φ acts on the generator $\mathbf{e}^\vee$ of $N^\vee$ by an element of valuation $\mu(M)$). Now if $\mathbf{v} = \sum_{i=1}^n c_i \mathbf{e}_i \in H^0(M)$ is nonzero, then writing out $\mathbf{v} = \varphi_M(\mathbf{v})$ in terms of the $\mathbf{e}_i$ gives $c_i = \sum_{j=1}^n A_{ij}\varphi(c_j)$ for all i. Applying w yields, for any i,

$$w(c_i) = w(\sum_{j=1}^{n} A_{ij}\varphi(c_j)) \geq \min_j\{w(A_{ij}) + w(c_j)\} > \min_j\{w(c_j)\}$$

which is absurd. Hence $H^0(M) = H^0(M_0) = 0$. □

Corollary 2.61. *Let M, N be pure φ-modules over R with $\mu(M) < \mu(N)$. Then* $\mathrm{Hom}(M, N) = 0$.

Proof. Since $\mathrm{Hom}(M, N) = H^0(M^\vee \otimes_R N)$ with $M^\vee \otimes_R N$ pure of slope $\mu(N) - \mu(M) > 0$ this follows directly from the lemma. □

Theorem 2.62. *Let M be a pure φ-module of slope s over R. We have:*

1. *M is semistable.*
2. *If N is a φ-submodule of M with $\mu(N) = s$, then N is saturated and both N and M/N are pure of slope s.*

Proof. (1.) Let N be any nonzero φ-submodule of M. We wish to show that $\mu(N) \geq s$. Since $\wedge^i M$ is pure of slope is for any $0 \leq i \leq \mathrm{rk}(M)$, we may replace M and N by $\wedge^{\mathrm{rk}(N)}M$ and $\wedge^{\mathrm{rk}(N)}N$, respectively, and thus assume that N is a φ-submodule of rank 1. We then apply $[\mathrm{rk}(M)]_*$ and invoke Lemma 2.55 to reduce to the case that $s = \mu(M) \in \mathbb{Z}$. Finally, we twist by $N^\vee$ and invoke Lemma 2.33 to reduce to the case that N is isomorphic to R as a φ-module. Since $1 \in R$ is φ-invariant, we then have $0 \neq H^0(N) \subseteq H^0(M)$. To avoid contradicting Lemma 2.60, we must have $s \leq 0 = \mu(N)$, as required. Hence M is semistable.

(2.) That N is saturated was shown in the course of the proof of Lemma 2.37. As usual, we reduce to the case that $s = 0$ by applying $[a]_*$ and twisting. In this case, M is étale and N has slope 0 and we need to show that N and M/N are both étale also. Let M_0 be an étale lattice of M. The kernel of $M_0 \longrightarrow M \longrightarrow M/N$ is a φ_M-stable R^{int}-submodule of M. It is saturated and therefore finite free, hence is a φ-module over R^{int} by Lemma 2.51. In particular, the image P_0 of $M_0 \longrightarrow M/N$ is a φ-module over R^{int}. Let P be the R-span of P_0. Then P_0 is an étale lattice for P whence the latter is an étale φ-submodule of M/N. Note that P has full rank by construction.

Now $\mu(N) = \mu(M) = 0$ by assumption so that $\mu(M/N) = 0$ by Lemma 2.30. Since P is étale it has slope 0 as well and we must have $M/N = P$ by Corollary 2.35. Hence M/N is étale as then must be N by Proposition 2.50. □

Corollary 2.63. *If M, N are φ-modules over R, then $M \oplus N$ is pure of slope s if and only if M and N are both pure of slope s.*

Proof. It is worth noting that here we equip $M \oplus N$ with the φ-linear map $\varphi_M \oplus \varphi_N$. If M and N are both pure of slope s, then so is $M \oplus N$. This is because both $[a]_*$ and tensor product commute with direct sums. In particular, we don't need to use Theorem 2.59 which would require the strong form of Hypothesis 2.22. Conversely, if $M \oplus N$ is pure of slope s, then it is semistable by Theorem 2.62 (1.) so M and N each have slope at least s. Since $\mu(M \oplus N)$ is a weighted average of $\mu(M)$ and $\mu(N)$ by Lemma 2.30, this is only possible if $\mu(M) = \mu(N) = s$. Hence M and N are both pure of slope s by Theorem 2.62 (2.). □

We conclude this section with an analogue of Lemma 2.55 for $[a]^*$.

Corollary 2.64. *Let N be a φ^a-module over R. Then N is pure of slope s if and only if $[a]^*N$ is pure of slope s/a.*

Proof. By Lemma 2.55, $[a]^*N$ is pure of slope s/a if and only if $[a]_*[a]^*N$ is pure of slope s. By Lemma 2.20, we have

$$[a]_*[a]^*N \simeq \bigoplus_{i=0}^{a-1} (\varphi^i)^*N.$$

If N is pure of slope s, then clearly so is $(\varphi^i)^*N$ for $i = 0, \ldots, a-1$. Thus, $[a]_*[a]^*N$ is a direct sum of φ-modules which are pure of slope s and hence is pure of slope s itself by the previous corollary. Conversely, if $[a]_*[a]^*N$ is pure of slope s, then the above decomposition shows that N is a direct summand of a pure φ^a-module of slope s and is therefore itself pure of slope s, again by the previous corollary. □

3 The slope filtration theorem

We state the slope filtration theorem here as a forward reference and outline how the proof will be done in the following sections.

Theorem 3.1 (Slope filtration theorem)**.** *Every semistable φ-module over the Robba ring is pure. In particular, any φ-module M over $\mathcal{R}$ admits a unique filtration $0 = M_0 \subset M_1 \subset \ldots \subset M_l = M$ by saturated φ-submodules whose successive quotients are all pure of strictly increasing slopes.*

Proof. The proof is divided into two steps. In the first step, we construct a φ-ring $\tilde{\mathcal{R}}$ and a φ-equivariant embedding $\psi : \mathcal{R} \longrightarrow \tilde{\mathcal{R}}$. We then prove that semistable φ-modules over $\tilde{\mathcal{R}}$ are pure (cf. Theorem 4.29). In the second step, we establish the following two facts using faithfully flat descent:

- If M is a semistable φ-module over $\mathcal{R}$, then $M \otimes \tilde{\mathcal{R}}$ is also semistable (cf. Theorem 5.19);
- If $M \otimes \tilde{\mathcal{R}}$ is pure, then so is M (cf. Theorem 5.21).

Everything else follows from the existence of the HN filtration. □

Remark 3.2. *By the slope filtration theorem, the tensor product of semistable φ-modules over $\mathcal{R}$ is semistable again. Indeed, the tensor product of pure φ-modules is pure (cf. Corollary 2.57) and hence semistable (cf. Theorem 2.62). This implies that the HN filtration is $\otimes$-multiplicative (cf. [1, §2.3] which in turn implies that the HN filtration of the tensor product of two φ-modules is given by the product filtration (op. cit. [Theorem 2.3.3]). Note that the analogous statement for vector bundles on smooth varieties in characteristic zero is also true and is nontrivial as well (see the introduction of [16, §1.7]).*

Remark 3.3. *The uniqueness of the HN filtration implies that it inherits any additional group action that M may carry. For example, the φ-submodules appearing in the HN filtration of a (φ, Γ)-module are all (φ, Γ)-submodules (see Section 6).*

Remark 3.4. *The slope filtration theorem does not assert that the HN filtration splits. Of course, we shouldn't expect such a strong statement to begin with. However, a slope filtration gives a replacement for a direct sum decomposition which is good enough for many purposes because of the formal properties of slopes.*

4 Classification over an extended Robba ring

The aim of this section is to construct the ring $\tilde{\mathcal{R}}$ appearing in the proof of Theorem 3.1 and prove that semistable φ-modules over it are pure. We may think of $\tilde{\mathcal{R}}$ as a very rough analogue of an "algebraic closure" of $\mathcal{R}$. Our first task will be to explain what this should mean.

4.1 Difference-closed fields

We give a brief overview of difference-closed fields. This is an analogue of algebraic closure for φ-fields in the sense that we can solve polynomial φ-equations. It will however be more convenient to work with a different characterization (cf. [17, Lemma 14.3.3]).

Definition 4.1. *A φ-field F is called weakly difference-closed if every φ-module over F is trivial. We say that F is strongly difference closed if it is weakly difference closed and inversive.*

Remark 4.2. *Note that being weakly difference-closed includes the property that short exact sequences of φ-modules over F always split.*

For the rest of this section, we make the following hypothesis. A field satisfying these conditions will be constructed in the next section.

Hypothesis 4.3. *Throughout this section, assume that φ is a relative Frobenius lift on $\mathcal{R}$ such that φ_K is an automorphism of K. Also assume that $\kappa = \kappa_K$ is weakly difference-closed. This implies that any extension of φ-modules over κ splits so that H^1 vanishes for any φ-module over κ (cf. Lemma 2.13).*

Remark 4.4.

- *Keep in mind that φ_K is also isometric so that it reduces to an automorphism of κ. In other words, κ is automatically strongly difference-closed.*
- *Using Lemma 2.21, we see that the hypothesis still holds if we replace φ by φ^a for some integer $a \geq 1$.*

Lemma 4.5. *In Hypothesis 4.3, it is equivalent to ask that all étale φ-modules over K be trivial. This implies that any extension of étale φ-modules over K splits since such extensions are themselves étale by Proposition 2.50. In particular, H^1 vanishes for any étale φ-module over K.*

Proof. If φ-modules over κ are trivial, then one can adjust the proof of [19, Satz 8.20] slightly to show that étale φ-modules over K are trivial. For the basis property one uses Nakayama's lemma and instead of the residue field being algebraically closed one uses that $\varphi_\kappa - 1$ is surjective (cf. the proof of Corollary 4.10 for the surjectivity). Conversely, if étale φ-modules over K are trivial, then we can just lift a given φ-module over κ to a φ-module over $\mathfrak{o}_K$. This gives rise to an étale φ-module over K which must admit a φ-invariant basis by assumption. Hence the φ-module over κ we started with is trivial. $\square$

4.2 The extended Robba ring

We now construct the ring $\tilde{\mathcal{R}}$ using Hahn series. By a well-ordered set we mean a set equipped with a total order such that any non-empty subset contains a least element. This is equivalent to the requirement that any strictly decreasing sequence of elements of the set terminates after finitely many steps.

Definition 4.6. *Let R be any ring. The ring of Hahn series over R (with value group $\mathbb{Q}$), denoted $R(\!(u^{\mathbb{Q}})\!)$, is the set of formal series $\sum_{i\in\mathbb{Q}} a_i u^i$ in some variable u over R with well-ordered support, that is, $\{ i \in \mathbb{Q} \,:\, a_i \neq 0\}$ is well-ordered. $R(\!(u^{\mathbb{Q}})\!)$ becomes a ring by defining addition coefficientwise and multiplication by the convolution product*

$$(\sum_{i\in\mathbb{Q}} x_i u^i) \cdot (\sum_{j\in\mathbb{Q}} y_j u^j) = \sum_{k\in\mathbb{Q}} (\sum_{i+j=k} x_i y_j) u^k.$$

This is well-defined because

- *the set of $k \in \mathbb{Q}$ admitting at least one representation as $i + j$ where $x_i y_j \neq 0$ is well-ordered;*
- *for each $k \in \mathbb{Q}$, the number of representations of k as $i + j$ where $x_i y_j \neq 0$ is finite.*

The ring $R(\!(u^{\mathbb{Q}})\!)$ is a field if R is. See [17, Example 1.5.8].

Remark 4.7. *If $\sum_{i\in\mathbb{Q}} a_i u^i \in R(\!(u^{\mathbb{Q}})\!)$ is nonzero, then its support is non-empty and hence contains a least element. That is, $R(\!(u^{\mathbb{Q}})\!)$ comes equipped with a u-adic valuation v sending $\sum_{i\in\mathbb{Q}} a_i u^i$ to the smallest $i \in \mathbb{Q}$ with $a_i \neq 0$. Note that $R(\!(u^{\mathbb{Q}})\!)$ is complete w.r.t. the u-adic valuation (the argument is the same as for the ring of Laurent series). If R is a field, then the subring $R(\!(u^{\mathbb{Q}_+})\!)$ of $R(\!(u^{\mathbb{Q}})\!)$ of series with support in $[0, +\infty)$ is a local ring with maximal ideal generated by u and residue field R.*

Lemma 4.8. *Let φ be an automorphism of $R(\!(u^{\mathbb{Q}})\!)$ of the form $\sum_{i\in\mathbb{Q}} a_i u^i \mapsto \sum_{i\in\mathbb{Q}} \varphi_R(a_i) u^{qi}$ where φ_R is some automorphism of R. Then the map $1 - \varphi$ is bijective on the set of series with constant term zero.*

Proof. Note first that if $x \in R(\!(u^{\mathbb{Q}})\!)$ has constant term zero, then so does $x - \varphi(x)$. If x is nonzero, then $v(x) = j$ for some $j \in \mathbb{Q} \setminus \{0\}$. We have $v(\varphi(x)) = qj = qv(x)$ whence $v(x) \neq v(\varphi(x))$. It follows from the strict triangle inequality that $v(x - \varphi(x)) = \min\{qv(x), v(x)\} \neq 0$. In other words, no series with constant term zero is mapped to zero so $1 - \varphi$ is injective.

Given $x = \sum_{i\in\mathbb{Q}} x_i u^i \in R(\!(u^{\mathbb{Q}})\!)$, set

$$y_+ = \sum_{j=0}^{\infty} \sum_{i>0} \varphi_R^j(x_i) u^{iq^j}, \quad y_- = \sum_{i<0} (\sum_{j=0}^{\infty} -\varphi_R^{-j-1}(x_{iq^{j+1}})) u^i$$

Note that the sum over j is finite for each i in the definition of y_- since there are no infinite decreasing sequences in the support of x. Hence both sums give well-defined elements of

$R(\!(u^{\mathbb{Q}})\!)$. Note also that applying φ to either series just causes the sum over j to start from $j = 1$ instead of $j = 0$. Set $y = y_+ + y_-$. This series has constant term zero by construction. Using the observation made before, we calculate

$$\begin{aligned} y - \varphi(y) &= y_+ - \varphi(y_+) + y_- - \varphi(y_-) \\ &= -\sum_{i>0} \varphi_R^0(x_i) u^{iq^0} - \sum_{i<0} -\varphi_R^0(x_{iq^0}) u^i \\ &= x - x_0. \end{aligned}$$

This proves surjectivity. ☐

Remark 4.9. *Obviously, the assertion of the lemma holds also for $\varphi - 1$, but note that it also holds for $1 - c\varphi$ for any $c \in R^\times$. Indeed, we only need to replace φ_R by $c\varphi_R$ everywhere.*

Corollary 4.10. *For any $c \in \kappa^\times$, the map $1 - c\varphi$ on $\kappa(\!(u^{\mathbb{Q}})\!)$ is surjective.*

Proof. Since φ-modules over κ are trivial by Hypothesis 4.3, there is $a \in \kappa^\times$ which is invariant under $c^{-1}\varphi_\kappa$. This amounts to $\varphi_\kappa(a) = ca$. We can therefore always write

$$(1 - c\varphi)(x) = a^{-1}(ax - \varphi(ax))$$

so it suffices to prove the corollary in the case that $c = 1$. Now the surjectivity of $1 - \varphi$ on the series with constant term zero follows from Lemma 4.8. It remains to show that we can also find a preimage of the constant term, i.e. that $1 - \varphi_\kappa$ is surjective on κ. This follows again from Hypothesis 4.3 since $\mathrm{Ext}(\kappa, \kappa)$ is trivial and can be identified with the cokernel of $1 - \varphi_\kappa$ by Lemma 2.13. ☐

Remark 4.11. *The corollary implies that any extension of trivial φ-modules over $\kappa(\!(u^{\mathbb{Q}})\!)$ splits since it implies that H^1 vanishes for any trivial φ-module over $\kappa(\!(u^{\mathbb{Q}})\!)$. In other words, any extension of trivial φ-modules over $\kappa(\!(u^{\mathbb{Q}})\!)$ is trivial.*

We are now ready to define the extended Robba ring $\tilde{\mathcal{R}}$. We use the definition given in [17, Definition 16.5.6] instead of the one in [16, Definition 2.2.4] since the former includes an important erratum.

Definition 4.12. *If $r > 0$ is a real number, we denote by $\tilde{\mathcal{R}}^r$ the set of formal series $\sum_{i\in\mathbb{Q}} a_i u^i$ with coefficients in K, satisfying the following conditions:*

1. *For all $s \in (0, r]$, we have $\lim_{i\to\pm\infty} |a_i| e^{-si} = 0$*
2. *For all $s \in (0, r]$ and all $c > 0$, the set of $i \in \mathbb{Q}$ such that $|a_i| e^{-si} \geq c$ is well-ordered.*

The union $\tilde{\mathcal{R}} = \bigcup_{r>0} \tilde{\mathcal{R}}^r$ is called the extended Robba ring (over K). Addition on $\tilde{\mathcal{R}}$ is defined coefficientwise and multiplication by the usual formula. One sees that this is well-defined by combining the arguments for $\mathcal{R}$ and the ring of Hahn series with coefficients in K. We denote by $\tilde{\mathcal{R}}^{\mathrm{b}}$ and $\tilde{\mathcal{R}}^{\mathrm{int}}$ the subrings of series with bounded and integral coefficients, respectively. On $\tilde{\mathcal{R}}^r$, we define the r-norm $|\cdot|_r$ by the usual formula

$$\left| \sum_{i\in\mathbb{Q}} a_i u^i \right|_r = \sup_{i\in\mathbb{Q}} \{|a_i| e^{-ri}\}.$$

This definition makes sense by (2.).

$\tilde{\mathcal{R}}$ can also be constructed as follows. Let $K(\!(u^{\mathbb{Q}})\!)$ be the field of Hahn series with coefficients in K. On the subring of series with bounded coefficients, we can define for any $r > 0$ the r-norm $|\cdot|_r$ by the above formula. We can then take the completion with respect to $|\cdot|_r$. The union over $r > 0$ of these completions gives the ring $\tilde{\mathcal{R}}$ (see the discussion after [17, Definition 16.5.6]).

Lemma 4.13. *If $f \in \tilde{\mathcal{R}}$, then $f \in \tilde{\mathcal{R}}^{\mathrm{int}}$ if and only if there exists $j \in \mathbb{Q}$ and $r > 0$ such that $u^j f$ is bounded by 1 under $|\cdot|_s$ for all $s \in (0, r]$. More generally, $f \in \tilde{\mathcal{R}}^{\mathrm{b}}$ if and only if there exists some $r > 0$ such that f is uniformly bounded under $|\cdot|_s$ for all $s \in (0, r]$.*

Proof. One only has to adjust the proof of Lemma 1.12 slightly. Instead of noting that convergence implies that $|a_i|e^{-ri} > 1$ only for finitely many i, one uses that the set of such i is well-ordered and hence contains a minimal element (if it is nonempty). Everything else can be copied. $\square$

Lemma 4.14. *If $f \in \tilde{\mathcal{R}}^r$ and $r' \leq r$, then f is uniformly bounded under $|\cdot|_s$ for all $s \in [r', r]$.*

Proof. This is the analog of Remark 1.11. One needs to make the same adjustments as in the previous lemma. $\square$

Lemma 4.15. *$\tilde{\mathcal{R}}$ is a Bézout domain. Its units are the nonzero elements of the discretely valued field $\tilde{\mathcal{R}}^{\mathrm{b}}$. The corresponding valuation subring is $\tilde{\mathcal{R}}^{\mathrm{int}}$. It is henselian with residue field $\kappa(\!(u^{\mathbb{Q}})\!)$.*

Proof. See the discussion after [17, Definition 16.5.6]. The discrete valuation w on $\tilde{\mathcal{R}}^{\mathrm{b}}$ is given by the same formula as on $\mathcal{R}^{\mathrm{b}}$. $\square$

Just like $\mathcal{R}^{\mathrm{int}}$, the discrete valuation ring $\tilde{\mathcal{R}}^{\mathrm{int}}$ is not π-adically complete. We denote the π-adic completions of $\tilde{\mathcal{R}}^{\mathrm{b}}$ and $\tilde{\mathcal{R}}^{\mathrm{int}}$ by $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{E}}^{\mathrm{int}}$, respectively.

Definition 4.16. *We equip $\tilde{\mathcal{R}}$ with the endomorphism*

$$\varphi(\sum_{i\in\mathbb{Q}} a_i u^i) = \sum_{i\in\mathbb{Q}} \varphi_K(a_i) u^{qi}.$$

It is an automorphism because φ_K is bijective by Hypothesis 4.3. Moreover, it restricts to an automorphism of $\tilde{\mathcal{R}}^{\mathrm{b}}$ and $\tilde{\mathcal{R}}^{\mathrm{int}}$ since φ_K is an isometry.

Remark 4.17. *The fact that $\varphi : \tilde{\mathcal{R}} \longrightarrow \tilde{\mathcal{R}}$ is an automorphism makes the study of φ-modules over $\tilde{\mathcal{R}}$ much easier than that of φ-modules over $\mathcal{R}$. We also see directly that the discrete valuation w on $\tilde{\mathcal{R}}$ is φ-invariant and that $|\varphi(f)|_{r/q} = |f|_r$ for any $r > 0$ and any $f \in \tilde{\mathcal{R}}^r$.*

If we want any chance of deducing results for $\mathcal{R}$ from results for $\tilde{\mathcal{R}}$, we need a φ-equivariant embedding $\mathcal{R} \longrightarrow \tilde{\mathcal{R}}$. This is the content of the following lemma.

Lemma 4.18. *There exists a φ-equivariant embedding $\psi : \mathcal{R} \longrightarrow \tilde{\mathcal{R}}$. Moreover, for $r_0 > 0$ as in Remark 1.18, it preserves r-norms for all $r \in (0, r_0)$.*

Proof. If $\varphi(t) = t^q$, then we can set $\psi(t) = u$ and we are done. In general, we must work a bit harder. We inductively construct morphisms $\psi_l : \mathcal{R} \longrightarrow \tilde{\mathcal{R}}$ for $l \geq 1$, each of the form $\psi_l(\sum_{i\in\mathbb{Z}} a_i t^i) = \sum_{i\in\mathbb{Z}} a_i u_l^i$ for some $u_l \in \tilde{\mathcal{R}}^{\mathrm{int}}$ such that the following conditions are satisfied:

1. $|u_l|_r = |t|_r$ for $r \in (0, r_0)$.
2. $\psi_l(\varphi(x)) \equiv \varphi(\psi_l(x)) \mod \pi^l \tilde{\mathcal{R}}^{\mathrm{int}}$ for all $x \in \mathcal{R}^{\mathrm{int}}$. It suffices to check this for $x = t$.

If in addition the ψ_l are compatible with the transition maps, that is, $u_{l+1} \equiv u_l \mod \pi^l \tilde{\mathcal{R}}^{\text{int}}$ for any $l \geq 1$, then we get an induced map $\psi : \mathcal{E}^{\text{int}} \longrightarrow \tilde{\mathcal{E}}^{\text{int}}$ between the projective limits which is φ-equivariant by (2.). Condition (1.) implies that ψ preserves r-norms for $r \in (0, r_0)$ (compare with the proof of Lemma 1.18) so ψ maps $\mathcal{R}^{\text{int}}$ into $\tilde{\mathcal{R}}^{\text{int}}$. By localizing, we obtain a morphism $\mathcal{R}^{\text{b}} \longrightarrow \tilde{\mathcal{R}}^{\text{b}}$ preserving r-norms which then extends to $\psi : \mathcal{R} \longrightarrow \tilde{\mathcal{R}}$ with all the desired properties by continuity. Hence it remains to construct the ψ_l.

Start with $u_1 = u$. Then condition (1.) is satisfied for all $r > 0$. Since $\varphi(t) \equiv t^q \mod \pi\mathcal{R}^{\text{int}}$, we have

$$\psi_1(\varphi(t)) \equiv \psi_1(t^q) = u^q = \varphi(u) = \varphi(\psi_1(t)) \mod \pi\tilde{\mathcal{R}}^{\text{int}}.$$

Given ψ_l, u_l, we need to construct u_{l+1} with the desired properties. Our first goal is to construct $\Delta \in \tilde{\mathcal{R}}^{\text{int}}$ with

$$\varphi(\pi^l\Delta/u) - q\pi^l\Delta/u = (\psi_l(\varphi(t)) - \varphi(u_l))/u^q. \tag{4.2.1}$$

Note that $u_l = \psi_l(t)$ so the right side of this equation, which we will denote by Δ', is an element of $\pi^l\tilde{\mathcal{R}}^{\text{int}}$ by induction hypothesis. Write $\Delta' = \sum_{i\in\mathbb{Q}} a_i u^i$. We first assume that $q \neq 0$ in κ. In this case, $q \in \mathfrak{o}_K^\times$ and we can adjust the construction in the proof of Lemma 4.8 as follows. Define the twisted powers $q^{\{n\}}$, $n \in \mathbb{Z}$, of q by the two-way recurrence

$$q^{\{0\}} = 1, \quad q^{\{n+1\}} = \varphi_K(q^{\{n\}})q^{-1}.$$

The definition is made so that $(q^{-1}\varphi_K)^n = q^{\{n\}}\varphi_K^n$ for any $n \in \mathbb{Z}$. If we now set $y' = y_+ + y_-$ where

$$y_+ = \sum_{j=0}^{\infty}\sum_{i>0} q^{\{j\}}\varphi_K^j(a_i)u^{iq^j}, \quad y_- = \sum_{i<0}(\sum_{j=0}^{\infty} q^{\{-j-1\}}\varphi_K^{-j-1}(a_{iq^{j+1}}))u^i,$$

then it is easily checked that $\varphi(y') - qy' = \sum_{i\neq 0} a_i u^i$ and that y' defines an element of $\tilde{\mathcal{R}}^{\text{int}}$. In fact, it is an element of $\pi^l\tilde{\mathcal{R}}^{\text{int}}$ since $a_i \in \pi^l\mathfrak{o}_K$ and $q^{-1}\varphi_K$ is isometric. Hence it remains to construct a preimage of a_0 under $\varphi_K - q$. By Hypothesis 4.3, the étale φ-module $(K, q^{-1}\varphi_K : K \longrightarrow K)$ is trivial. It follows that the map $q^{-1}\varphi_K$ is surjective on $\mathfrak{o}_K$, i.e. there is y_0 such that $q^{-1}\varphi_K(y_0) - y_0 = q^{-1}a_0$. Let $y = y' + y_0$. Then $y \in \pi^l\tilde{\mathcal{R}}^{\text{int}}$ and $\Delta = \pi^{-l}yu$ satisfies (4.2.1).

Next, we construct Δ in the case that $q = 0$ in κ_K, that is, $q \in \pi\mathfrak{o}_K$. In this case we cannot quite make the same construction as above because $q^{-1} \notin \mathfrak{o}_K$. We can however make a similar construction coefficientwise. Define the twisted powers $q^{\{n\}}$ of q as before. However, this time we only need them for $n \geq 0$. Note that $v_K(q^{\{n\}}) = v_K(q^n) = nv_K(q)$ so that $q^{\{n\}} \in \pi^n\mathfrak{o}_K$. If we set $b_i = \sum_{j=0}^{\infty} q^{\{j\}}\varphi_K^{-j}(a_{iq^j})$ for $i \in \mathbb{Q}$, then the series converges π-adically for any i by the previous observation and because φ_K^{-1} is an isometry. In fact, $b_i \in \pi^l\mathfrak{o}_K$ for all i since this is true for all the a_i. Write $\pi^l\Delta/u = \sum_{i\in\mathbb{Q}} b_i u^i$. Then $\Delta \in \pi^l\tilde{\mathcal{R}}^{\text{int}}$ and one checks directly that this solves equation (4.2.1). This finishes the construction of Δ.

For any $r \in (0, r_0)$, we have

$$|\varphi(u_l)|_{r/q} = |u_l|_r = |t|_r = e^{-r} \quad \text{and} \quad |\psi_l(\varphi(t))|_{r/q} = |\varphi(t)|_{r/q} = |t|_r = e^{-r}.$$

On the right we use that ψ_l preserves r-norms as noted in the beginning. It follows from the triangle inequality that Δ' has r/q-norm at most 1, using that $|u^q|_{r/q} = e^{-r}$. If we write $\pi^l\Delta/u = \sum_{i\in\mathbb{Q}} b_i u^i$, then $|b_i|e^{-ri} \leq 1$ for any $i \geq 0$ since $\Delta \in \tilde{\mathcal{R}}^{\text{int}}$. By (4.2.1) and the triangle inequlity, we have

$$|\pi^l\Delta/u|_r = |\varphi(\pi^l\Delta/u)|_{r/q} = |\Delta' + q\pi^l\Delta/u|_{r/q} \leq \max\{|\Delta'|_{r/q}, |q\pi^l\Delta|_{r/q}\} \leq \max\{1, |\pi^l\Delta/u|_{r/q}\}.$$

If we had $|\pi^l\Delta/u|_{r/q} > 1$, then $\sup_{i\in\mathbb{Q}}\{|b_i|e^{-ri/q}\}$ would have to be achieved for some $i < 0$ since $|b_i| \leq 1$. However, for $i < 0$, we have $e^{-ri} > e^{-ri/q}$ which would lead to the contradiction

$$|\pi^l\Delta/u|_r \leq |\pi^l\Delta/u|_{r/q} < |\pi^l\Delta/u|_r.$$

It follows that $\pi^l\Delta/u_r \leq 1$ or, equivalently, $|\pi^l\Delta|_r \leq e^{-r}$. Now set $u_{l+1} = u_l + \pi^l\Delta$. Then $|u_{l+1}|_r \leq \max\{|u_l|_r, |\pi^l\Delta|_r\}$ is at most e^{-r}. On the other hand, we add in each step of the iteration an element of $\pi\tilde{\mathcal{R}}^{\mathrm{int}}$ so the coefficient of u in u_l always remains a unit in $\mathfrak{o}_K$. Hence $|u_{l+1}|_r \geq |u|_r = e^{-r}$. Combined with the previous observation this means u_{l+1} satisfies (1.). For (2.), note that

$$\begin{aligned}\varphi(\psi_{l+1}(t)) &= \varphi(u_{l+1}) = \varphi(u_l) + \varphi(\pi^l\Delta)\\ &= \psi_l(\varphi(t)) + q\pi^l\Delta u^{q-1}\\ &\equiv \psi_{l+1}(\varphi(t)) \mod \pi^{l+1}\tilde{\mathcal{R}}^{\mathrm{int}}.\end{aligned}$$

In the second line, we multiply both sides of (4.2.1) by $u^q = \varphi(u)$ and reorder. The third line requires a bit more work. Write $\varphi(t) = \sum_{i\in\mathbb{Z}} c_i t^i$ with $c_i \in \pi\mathfrak{o}_K$ for all $i \neq q$ and $c_q \in 1 + \mathfrak{m}_K$. Since $u_{l+1} = u_l + \pi^l\Delta$, we see that $c_i u_{l+1}^i \equiv c_i u_l^i \mod \pi^{l+1}\tilde{\mathcal{R}}^{\mathrm{int}}$ for $i \neq q$. For $i = q$, we have

$$c_q u_{l+1}^q = c_q \sum_{k=0}^{q} \binom{q}{k} u_l^k (\pi^l\Delta)^{q-k} \equiv c_q u_l^q + q\pi^l\Delta u^{q-1} \mod \pi^{l+1}\tilde{\mathcal{R}}^{\mathrm{int}},$$

noting that $c_q \equiv 1 \mod \pi$. This proves property (2.) and finishes the construction of the morphisms ψ_l. Since $u_{l+1} \equiv u_l \mod \pi^l\tilde{\mathcal{R}}^{\mathrm{int}}$ they are compatible with the transition maps, finishing the proof. □

Remark 4.19. *Note that by construction ψ maps $\mathcal{R}^{\mathrm{b}}$ and $\mathcal{R}^{\mathrm{int}}$ into $\tilde{\mathcal{R}}^{\mathrm{b}}$ and $\tilde{\mathcal{R}}^{\mathrm{int}}$, respectively. In particular, if M is a pure φ-module of some slope over $\mathcal{R}$, then $M \otimes \tilde{\mathcal{R}}$ is a pure φ-module over $\tilde{\mathcal{R}}$ of the same slope since the representing matrix of the φ-action does not change.*

Lemma 4.20. *The elements of $\tilde{\mathcal{R}}$ which are fixed by φ all belong to K.*

Proof. Assume that $x = \sum_{i\in\mathbb{Q}} a_i u^i \in \tilde{\mathcal{R}}$ is fixed by φ and that $a_i \neq 0$ for some $i \neq 0$. Then x is fixed by φ^n for any $n \in \mathbb{Z}$ and it follows that $a_{iq^n} = \varphi_K^n(a_i)$ for all $n \in \mathbb{Z}$. But this would constitute a strictly decreasing sequence of elements of the support of x which does not terminate, contradicting Definition 4.12. □

We have already seen that $\tilde{\mathcal{R}}$ satifies all assertions of Hypothesis 2.22, except possibly the last one. Just like the other assertions, this is more or less a formality.

Proposition 4.21. *Let A be an $n \times n$ matrix with entries in $\tilde{\mathcal{R}}^{\mathrm{int}}$. Then the map $\mathbf{v} \mapsto \mathbf{v} - A\varphi(\mathbf{v})$ induces a bijection on $(\tilde{\mathcal{R}}/\tilde{\mathcal{R}}^{\mathrm{b}})^n$.*

Proof. Making some small adjustments and using Lemma 4.14 and 4.13 instead of Remark 1.11 and Lemma 1.12, the proof of Proposition 1.20 can be copied. □

4.3 The main result

We prove that semistable φ-modules over $\tilde{\mathcal{R}}$ are pure. To do this, we need some auxiliary results which we state here and prove in the following subsections.

Proposition 4.22. *Let M, N be pure φ-modules over $\tilde{\mathcal{R}}$ obtained by base change from K, with $\mu(M) > \mu(N)$. Then* $\mathrm{Hom}(M, N) \neq 0$.

Proof. The assumptions ensure that $M^\vee \otimes N$ is a pure φ-module obtained by base change from K, with negative slope. Reformulating the proposition using H^0, it is therefore equivalent to show that if M is a pure φ-module over $\tilde{\mathcal{R}}$ of slope $\mu(M) < 0$ which is obtained by base change from K, then $H^0(M) \neq 0$. Write $M = M_0 \otimes_K \tilde{\mathcal{R}}$ for some pure φ-module M_0 over K. Write

$\mu(M) = \mu(M_0) = c/d$ with $c, d \in \mathbb{Z}$ and $d > 0$. Since M_0 is pure, there exists a K-basis such that the representing matrix A of φ_{M_0} has entries in $\pi^{-c}\mathfrak{o}_K$ and $\pi^c A$ is invertible over $\mathfrak{o}_K$. This allows us to view M as a space of column vectors with entries in $\tilde{\mathcal{R}}$ and φ-action given by the matrix A. Let $A^{\{n\}}$, $n \in \mathbb{Z}$, denote as usual the twisted powers of A. Choose any nonzero element $\mathbf{w} \in M_0$ and any $i > 0$. The sum

$$\mathbf{v} = \sum_{n\in\mathbb{Z}} \varphi_M^n(u^i\mathbf{w}) = \sum_{n\in\mathbb{Z}} u^{iq^n} A^{\{n\}} \varphi_K^n(\mathbf{w})$$

converges in M because clearly no negative powers of u appear and

$$|u^{iq^n} A^{\{n\}} \varphi_K^n(\mathbf{w})|_r \leq e^{-riq^n}|A|^n|\mathbf{w}| \leq e^{-riq^n}|\pi|^{-nc}|\mathbf{w}| \longrightarrow 0, \quad n \longrightarrow \infty,$$

for any $r > 0$. Hence $\mathbf{v}$ is a nonzero element of M which satisfies $\varphi_M(\mathbf{v}) = \sum_{n\in\mathbb{Z}} \varphi_M^{n+1}(u^i\mathbf{w}) = \mathbf{v}$, that is, $\mathbf{v} \in H^0(M)$. □

Definition 4.23. *Fix a uniformizer $\pi \in K$. Let $\tilde{\mathcal{R}}(n)$ be the (pure) φ-module of rank 1 and degree $n \in \mathbb{Z}$ over $\tilde{\mathcal{R}}$ obtained by equipping $\tilde{\mathcal{R}}$ with the φ-semilinear map $\pi^n\varphi : \tilde{\mathcal{R}} \longrightarrow \tilde{\mathcal{R}}$. We will use $\tilde{\mathcal{R}}(n)$ for twisting and write $M(n) = M \otimes \tilde{\mathcal{R}}(n)$.*

Proposition 4.24. *Let M be a nonzero φ-module over $\tilde{\mathcal{R}}$. Then $H^0(M(-n)) \neq 0$ and $H^1(M(-n)) = 0$ for $n \gg 0$.*

Remark 4.25. *Note that $H^0(M(-n)) = \mathrm{Hom}(\tilde{\mathcal{R}}(n), M)$. In other words, the proposition ensures the existence of a nonzero morphism of φ-modules $\tilde{\mathcal{R}}(n) \longrightarrow M$ which is necessarily injective.*

Proposition 4.26. *For any $s \in \mathbb{Q}$, the base change functor induces an equivalence of categories between the categories of pure φ-modules of slope s over K and over $\tilde{\mathcal{R}}$.*

Remark 4.27. *The case that $s = 0$ implies that étale φ-modules over $\tilde{\mathcal{R}}$ are trivial.*

Proposition 4.28. *Let N' be a pure φ^n-module over $\tilde{\mathcal{R}}$ of rank 1 and degree 1, let P a pure φ-module over $\tilde{\mathcal{R}}$ of rank 1 and degree -1, and assume that*

$$0 \longrightarrow [n]^*N' \longrightarrow M \longrightarrow P \longrightarrow 0$$

is a short exact sequence of φ-modules. Then $H^0(M) \neq 0$.

These assemble to give the slope filtration theorem for $\tilde{\mathcal{R}}$.

Theorem 4.29. *Any semistable φ-module over $\tilde{\mathcal{R}}$ is pure.*

Proof. We proceed by induction on the rank. Since any φ-module of rank 1 is pure by Remark 2.53, we are left with the induction step. Now assume that for $n \geq 1$ and any $a \geq 1$ every semistable φ^a of rank at most n is pure. Let M be a semistable φ^a-module of rank $n+1$ and slope c/d over $\tilde{\mathcal{R}}$. We need to show that M is pure. By Lemma 2.55, this is the case if and only if $[d]_*M$ is pure. Now $[d]_*M$ is semistable by Corollary 2.41 so that we may replace M by $[d]_*M$ and assume that $\mu(M) \in \mathbb{Z}$. Since twisting preserves and reflects both purity and semistability (cf. Corollary 2.57 and Lemma 2.33), we may then reduce to the case that $\mu(M) = 0$. To summarize, we are now given a semistable φ^a-module M of rank $n+1$ and slope 0 and we need to show that M is étale. We assume hereafter that M is a φ-module to simplify notation. Note again that Hypothesis 4.3 is not disturbed if we replace φ by a power. Before proceeding, we prove the following claim. It will be crucial in the argument.

Claim. *Suppose that Theorem 4.29 holds for all φ-modules of rank at most n. If M is a pure φ-module over $\tilde{\mathcal{R}}$ and N is an arbitrary φ-module over $\tilde{\mathcal{R}}$ with $\mu(N) \leq \mu(M)$ and $\mathrm{rk}(N) \leq n$, then* $\mathrm{Hom}(M, N) \neq 0$. *In particular, if M has rank 1, there is an injection of M into N.*

Proof of Claim. We may assume without loss of generality that N is semistable by replacing it with the first step of its HN filtration, if necessary. In this case, N is pure since Theorem 4.29 holds for φ-modules of rank at most n. By Proposition 4.26, both M and N are obtained by base change from K. If $\mu(M) > \mu(N)$, then the claim follows from Proposition 4.22. If $\mu(M) = \mu(N)$, then $M^\vee \otimes N$ is an étale φ-module over $\tilde{\mathcal{R}}$, hence is trivial. It follows that $0 \neq H^0(M^\vee \otimes N) = \mathrm{Hom}(M, N)$. Finally, if M has rank 1, then any nonzero $\tilde{\mathcal{R}}$-linear map $M \longrightarrow N$ is automatically injective because N is torsion-free. □

Write $M' = [n]_* M$. Then M' is a semistable φ^n-module of rank $n+1$ and slope 0 (Corollary 2.41). By the remark after Proposition 4.24, there exists $c \geq 0$ such that M' admits a pure φ^n-submodule N' of rank 1 and slope c. Choose c minimal with this property. Note that the saturation of N' in M' is also of rank 1 and slope c so that we may assume N' to be saturated. We show that in fact $c = 0$, i.e. that N' is étale.

Suppose that $c \geq 2$. We have $\mu(M'/N') = -c/n < 0 \leq c - 2 = \mu(\tilde{\mathcal{R}}(c-2))$. By the claim above, M'/N' has a φ^n-submodule which is isomorphic to $\tilde{\mathcal{R}}(c-2)$ (the φ^n-module of rank 1 and slope $c - 2$ defined as above with φ replaced by φ^n). Denote the preimage in M' of that submodule by Q'. By tensoring with $\tilde{\mathcal{R}}(1-c)$, we obtain from the short exact sequence

$$0 \longrightarrow N' \longrightarrow Q' \longrightarrow \tilde{\mathcal{R}}(c-2) \longrightarrow 0$$

the short exact sequence

$$0 \longrightarrow N'(1-c) \longrightarrow Q'(1-c) \longrightarrow \tilde{\mathcal{R}}(-1) \longrightarrow 0$$

Applying Proposition 4.28 in the case $n = 1$, we see that $H^0(Q'(1-c)) \neq 0$. But then $0 \neq H^0(Q'(1-c)) \subseteq H^0(M'(1-c))$ so M' would admit a pure φ^n-submodule of rank 1 and slope $c-1$, contradicting the minimality of c.

Next, suppose that $c = 1$ and write $N = [n]^* N'$. By Corollary 2.64, N is pure of slope $c/n = 1/n$. The adjunction between $[n]^*$ and $[n]_*$ converts the inclusion $N' \hookrightarrow M'$ into a nonzero map $f : N \longrightarrow M$. Since N is semistable (Theorem 2.62 (1.)), we have $\mu(\mathrm{im}(f)) \leq 1/n$. Note that the denominator of $\mu(\mathrm{im}(f))$ is at most n because $\mathrm{rk}(N) = n$. Hence we must either have $\mu(\mathrm{im}(f)) \leq 0$ or $\mu(\mathrm{im}(f)) = 1/n$. In the first case, $H^0(\mathrm{im}(f)) = \mathrm{Hom}(\tilde{\mathcal{R}}, \mathrm{im}(f)) \neq 0$ by the claim. In particular, $0 \neq H^0(M) \subseteq H^0(M')$. In the second case, $\mathrm{im}(f)$ has rank n so that f must be injective. This leads to a short exact sequence

$$0 \longrightarrow N \xrightarrow{f} M \longrightarrow P \longrightarrow 0$$

By Theorem 2.59, P is pure of rank $\mathrm{rk}(M) - \mathrm{rk}(N) = 1$ and slope -1 (cf. Lemma 2.30). Applying Proposition 4.28, we again deduce that $0 \neq H^0(M) \subseteq H^0(M')$. This means that in either case we have $H^0(M') \neq 0$ so that M' admits a pure φ^n-submodule of rank 1 and slope 0, contradicting the minimality of c.

We conclude that $c = 0$, that is, M' admits an étale φ^n-submodule N' of rank 1. The quotient M'/N' is semistable of slope 0 (Lemma 2.43), hence étale by induction hypothesis. But then M' must be étale as well by Theorem 2.59. We conclude by invoking Lemma 2.55 once more. □

With some additional work one can deduce the existence of a decomposition of Dieudonné-Manin type for φ-modules over $\tilde{\mathcal{R}}$. We will not need this explicitly and refer to [17, Corollary 16.5.8] for a proof.

Corollary 4.30. *Every φ-module over $\tilde{\mathcal{R}}$ can be split (non-uniquely) into a direct sum of standard pure φ-modules of various slopes (see [17, Definition 14.6.1] or Example 2.54).* □

4.4 Construction of fixed vectors

We prove Proposition 4.24: If M is a nonzero φ-module over $\tilde{\mathcal{R}}$, then $H^0(M(-n)) \neq 0$ and $H^1(M(-n)) = 0$ for $n \gg 0$.

Proof of Proposition 4.24. As usual, we view M as a space of column vectors with φ_M given by multiplication with an invertible matrix A over $\tilde{\mathcal{R}}$ times the componentwise action of φ. In particular, φ_M^{-1} is given by multiplication with $\varphi^{-1}(A^{-1})$ times the componentwise action of φ^{-1}. We may choose $r > 0$ so that A and A^{-1} both have entries in $\tilde{\mathcal{R}}^r$. For $d \in \mathbb{Q}_{>0}$ and $n \geq 1$ to be specified later, we define the following.

- Define the splitting function f_d^+, f_d^- as follows: Given $x = \sum_{i \in \mathbb{Q}} x_i u^i \in \tilde{\mathcal{R}}$, set

$$f_d^+(x) = \sum_{i \geq d} x_i u^i \quad \text{and} \quad f_d^-(x) = \sum_{i < d} x_i u^i = x - f_d^+(x).$$

  We extend the definition to vectors as usual. Once d is fixed, we write $\mathbf{w}^\pm$ for $f_d^\pm(\mathbf{w})$.

- Let $g : M \longrightarrow M$ be the map

$$g(\mathbf{w}) = \pi^{-n} A \varphi(\mathbf{w}^+) + \varphi^{-1}(\pi^n A^{-1} \mathbf{w}^-).$$

We inspect $|\varphi(f_d^+(x))|_r$ and $|\varphi^{-1}(f_d^-(x))|_r$ for x as above:

$$|\varphi(f_d^+(x))|_r = \sup_{i \geq d}\{|x_i| e^{-riq}\} \leq e^{-rd(q-1)} \sup_{i \in \mathbb{Q}}\{|x_i| e^{-ri}\} = e^{-rd(q-1)} |x|_r;$$
$$|\varphi^{-1}(f_d^-(x))|_r = \sup_{i < d}\{|x_i| e^{-ri/q}\} \leq e^{-rd(q^{-1}-1)} \sup_{i \in \mathbb{Q}}\{|x_i| e^{-ri}\} = e^{-rd(q^{-1}-1)} |x|_r$$

Of course, this generalizes to vectors so if we are given d, n, we have

$$|g(\mathbf{w})|_r \leq \max\{|\pi|^{-n} |A|_r e^{-rd(q-1)}, |\pi|^n |\varphi^{-1}(A^{-1})|_r e^{-rd(q^{-1}-1)}\} |\mathbf{w}|_r$$

by the triangle inequality. If we can choose d, n such that the quantities in the maximum are both strictly less than 1, then g is contractive towards zero. One checks directly that this happens if

$$d \in \left(\frac{n(-\log|\pi|) + \log|A|_r}{r(q-1)}, \frac{qn(-\log|\pi|) - q \log|A^{-1}|_{qr}}{r(q-1)} \right).$$

The interval is nonempty if we take n sufficiently large because $n(-\log|\pi|) < qn(-\log|\pi|)$ and all other quantities appearing are constant. In particular, once we have one $n > 0$ for which the interval is nonempty, then any $m \geq n$ will work as well. Fix such an n and fix $d \in \mathbb{Q}_{>0}$ in the interval and define the map g as above.

We first show that $H^1(M(-n)) = 0$, that is, the cokernel of $\pi^{-n} A \varphi$ is trivial. Given $\mathbf{w}$ with entries in $\tilde{\mathcal{R}}^r$, define the sequence $\{\mathbf{w}_l\}_{l \geq 0}$ by $\mathbf{w}_0 = \mathbf{w}$ and $\mathbf{w}_{l+1} = g(\mathbf{w}_l)$ for $l \geq 0$. The series

$$\mathbf{v} = \sum_{l=0}^{\infty} (\mathbf{w}_l^+ - \varphi^{-1}(\pi^n A^{-1} \mathbf{w}_l^-)) \tag{4.4.1}$$

converges under $|\cdot|_r$ since $\mathbf{w}_l^+$ and $\varphi^{-1}(\pi^n A^{-1}\mathbf{w}_l^-)$ both converge to 0 under $|\cdot|_r$ by choice of n, d. We have $|\mathbf{v}|_r \le |\mathbf{w}_0^+ - \varphi^{-1}(\pi^n A^{-1}\mathbf{w}_0^-)|_r \le |\mathbf{w}|_r$ and

$$\begin{aligned}\mathbf{v} - \pi^{-n}A\varphi(\mathbf{v}) &= \sum_{l=0}^{\infty}(\mathbf{w}_l^+ - \varphi^{-1}(\pi^n A^{-1}\mathbf{w}_l^-) - \pi^{-n}A\varphi(\mathbf{w}_l^+) + \mathbf{w}_l^-)\\ &= \sum_{l=0}^{\infty}(\mathbf{w}_l - g(\mathbf{w}_l)) = \sum_{l=0}^{\infty}(\mathbf{w}_l - \mathbf{w}_{l+1}) = \mathbf{w}_0 = \mathbf{w}\end{aligned}$$

so that $\mathbf{v}$ is a potential preimage of $\mathbf{w}$. A priori, we only know that the sum defining $\mathbf{v}$ converges under $|\cdot|_r$, but the equation $\mathbf{v} = \pi^{-n}A\varphi(\mathbf{v}) + \mathbf{w}$ implies that it also converges under $|\cdot|_{r/q}$ since $\pi^{-n}A\varphi(\mathbf{v}), \mathbf{w} \in \mathcal{R}^{r/q}$. Inductively, we see that the series defining $\mathbf{v}$ converges under $|\cdot|_{r/q^m}$ for any $m \ge 0$ (cf. the proof of Proposition 1.20) so $\mathbf{v}$ has entries in $\tilde{\mathcal{R}}^r$. Hence $H^1(M(-n)) = 0$.

We modify the construction slightly to show that $H^0(M(-n)) \neq 0$, that is, the kernel of $\pi^{-n}A\varphi - 1$ is nontrivial. Let $\mathbf{w} = (u^d, 0, \dots, 0)$ and construct $\mathbf{v}$ as in (4.4.1). Then define the sequence $\{\mathbf{w}'_l\}_{l\ge 0}$ by $\mathbf{w}'_0 = \mathbf{w}$, $\mathbf{w}'_1 = \varphi^{-1}(\pi^n A^{-1}\mathbf{w}'_0)$ and $\mathbf{w}'_{l+1} = g(\mathbf{w}'_l)$ for $l \ge 1$. Since $\mathbf{w} = \mathbf{w}^+$, this means we just swap $\mathbf{w}^+$ and $\mathbf{w}^-$ in the definition of g in the first step. Define the series

$$\mathbf{v}' = -\varphi^{-1}(\pi^n A^{-1}\mathbf{w}) + \sum_{l=1}^{\infty}((\mathbf{w}'_l)^+ - \varphi^{-1}(\pi^n A^{-1}(\mathbf{w}'_l)^-))).$$

One checks that we have $\mathbf{v}' - \pi^{-n}A\varphi(\mathbf{v}') = \mathbf{w}$ as before. However, $|\mathbf{v}|_r = |\mathbf{w}_0^+| = |u^d|_r$ whereas $|\mathbf{v}'|_r = |-\varphi^{-1}(\pi^n A^{-1}\mathbf{w}_0)|_r < |\mathbf{w}_0|_r = |u^d|_r$. Hence $\mathbf{v} - \mathbf{v}'$ is a nonzero element of $H^0(M(-n))$. □

4.5 Twisted polynomials and their Newton polygons

We adapt the tool of Newton polygons to the setting of twisted polynomials. Some additional care is needed to treat the case of polynomials over K and over $\kappa(\!(u^{\mathbb{Q}})\!)$ simultaneously. We follow [16, §2.4], but some details are inspired by [15, §6].

Notation. *Throughout this subsection, fix a real number $s \ge 1$, and let (F, v_F) be a field equipped with an automorphism $\varphi = \varphi_F$ and a valuation v_F for which it is complete and which satisfies $v_F(\varphi_F(x)) = s v_F(x)$ for all $x \in F$.*

Definition 4.31. *Let $i \in \mathbb{Z}$. If $s = 1$, we set $[i] = i$. If $s > 1$, we set $[i] = \frac{s^i - 1}{s-1}$. One checks that in either case $[0] = 0$, $[1] = 1$ and $[i + j] = [i] + s^i[j]$ for all $i, j \in \mathbb{Z}$. Let $P(T) = \sum_{i\in\mathbb{Z}} a_i T^i \in F\{T^{\pm}\}$. Define the homogeneous Newton polygon* Newt_P *of P as the lower convex hull of the set*

$$\{(-[i], v_F(a_i)) \mid i \in \mathbb{Z}\}.$$

In other words, Newt_P *is the convex polygon which is largest in terms of value such that all points $(-[i], v_F(a_i))$ lie on or above it. We refer to the slopes of this polygon as the homogeneous (Newton) slopes of P and to the points where the slope changes as the breakpoints of* Newt_P. *Clearly,* Newt_P *is finite only on a segment of finite length and when we talk about the Newton polygon of P, we are really only interested in this part.*

As in the usual theory of Newton polygons, the following definition will be crucial.

Definition 4.32. *For $r \in \mathbb{R}$ and $P(T) = \sum_{i\in\mathbb{Z}} a_i T^i \in F\{T^{\pm}\}$, set*

$$v_r(P) = \min_{i\in\mathbb{Z}}\{v_F(a_i) + r[i]\}$$

That is, $v_r(P)$ is the value at zero of the supporting line of slope r of Newt_P. *Note that v_r satisfies the strict triangle inequality, but it is not quite a valuation as it is only submultiplicative in general (see below).*

Remark 4.33. *The data of the Newton polygon of P is now equivalent to giving $v_r(P)$ for all $r \in \mathbb{R}$. This is because the graph of a convex continuous function is determined by the positions of its supporting lines of all slopes (see [17, Remark 2.1.7]). Intuitively, the breakpoints of* Newt_P *are exactly the points where several supporting lines of* Newt_P *intersect.*

Remark 4.34. *Clearly, multiplying P by a nonzero element $a \in F$ just shifts the Newton polygon vertically by $v_F(a)$. For $m \in \mathbb{Z}$, we have*

$$\begin{aligned} v_r(T^m P) &= \min_{i\in\mathbb{Z}}\{v_F(\varphi_F^m(a_i)) + r[i+m]\} \\ &= \min_{i\in\mathbb{Z}}\{s^m v_F(a_i) + r[m] + s^m r[i]\} \\ &= s^m \min_{i\in\mathbb{Z}}\{v_F(a_i) + r[i]\} + r[m] \\ &= s^m v_r(P) + r[m]. \end{aligned}$$

So multiplying by a power of T results in a horizontal shift which moreover preserves slopes, but the segments of a given slope may get longer or shorter (if $s \neq 1$).

The point of introducing Newton polygons is to find factorizations. This is what we will do now. We start by analyzing the behavior of slopes in products of a special kind.

Lemma 4.35. *For $P(T) \in F\{T\}$ and $Q(T) \in F\{T^-\}$ such that $v_r(Q) \geq 0$, we have $v_r(PQ) \geq v_r(P) + v_r(Q)$.*

Proof. Write $P(T) = \sum_{i\geq 0} a_i T^i$ and $Q(T) = \sum_{j\leq 0} b_j T^j$. By definition of multiplication in $F\{T^\pm\}$, we have

$$(PQ)(T) = \sum_{k\in\mathbb{Z}}\sum_{i+j=k} a_i\varphi^i(b_j)T^k = \sum_{k\in\mathbb{Z}} c_k T^k.$$

Hence for any $k \in \mathbb{Z}$,

$$v_F(c_k) \geq \min_{i+j=k}\{v_F(a_i\varphi^i(b_j))\} = \min_{i+j=k}\{v_F(a_i) + s^i v_F(b_j)\}.$$

Choose i_k, j_k attaining the minimum on the right. Note that for any i, j,

$$v_F(a_i\varphi^i(b_j)) + r[i+j] = v_F(a_i) + r[i] + s^i(v_F(b_j) + r[j]). \tag{4.5.1}$$

Choose k with $v_r(PQ) = v_F(c_k) + r[k]$. Then the above inequalities give

$$v_F(c_k) + r[k] \geq v_F(a_{i_k}) + r[i_k] + s^{i_k}(v_F(b_{j_k}) + r[j_k]) \geq v_r(P) + v_r(Q),$$

using that $s \geq 1$, $i_k \geq 0$ and $v_r(Q) \geq 0$. This finishes the proof. □

Proposition 4.36. *Let $r_0 \in \mathbb{R}$. Suppose that $P(T) \in F\{T\}$ and $Q(T) \in F\{T^{-1}\}$ are such that P has constant coefficient 1 and all slopes $\leq r_0$, and Q has constant coefficient 1 and all slopes $\geq r_0$. Then the set of slopes of PQ is obtained by taking the union (with multiplicities) of the sets of slopes of P and Q.*

Proof. P has all slopes $\leq r_0$ and constant coefficient 1. In particular, $v_r(P) \leq 0$ for all r and $v_r(P) = 0$ whenever $r \geq r_0$ because in this range the lowest possible supporting line of slope r through a vertex of Newt_P must have right endpoint $(0,0)$. We have a similar relation for Q so altogether,

$$\begin{aligned} r \geq r_0 &\implies v_r(P) = 0, v_r(Q) \leq 0 \\ r \leq r_0 &\implies v_r(P) \leq 0, v_r(Q) = 0. \end{aligned}$$

The claim of the proposition can only be true if the slopes of PQ arise by taking first the slopes of P and then the ones of Q because the slopes have to increase from left to right by convexity. Since Newt_{PQ} is determined by the data of $v_r(PQ)$ for all $r \in \mathbb{R}$, it suffices to show that

$$v_r(PQ) = \begin{cases} v_r(P) & r < r_0 \\ 0 & r = r_0 \\ v_r(Q) & r > r_0. \end{cases}$$

We use the notation from the proof of Lemma 4.35. If $r \leq r_0$, take the smallest i with $v_F(a_i) + r[i] = v_r(P)$. Note that $v_r(Q) = 0$ in this range, so $v_r(PQ) \geq v_r(P) + v_r(Q) = v_r(P)$ by Lemma 4.35. On the other hand, (4.5.1) equals $v_r(P)$ for $j = 0$ but is strictly bigger for any other pair k, l with sum i. Hence $v_r(PQ) \leq v_r(P)$ and we must have equality. This proves the cases $r < r_0$ and $r = r_0$. The case $r \geq r_0$ is similar by choosing instead the smallest j with $v_F(b_j)+r[j] = v_r(Q)$. □

Remark 4.37. *Conceptually, this means that to obtain the Newton polygon of PQ, one patches together the Newton polygon of P and that of Q at the common vertex* $(0, 0)$.

We now prove a factorization result and examine afterwards when it can be applied.

Proposition 4.38. *Let $r \in \mathbf{R}$ and suppose that $R \in F\{T^{\pm}\}$ satisfies $v_r(R - 1) > 0$. Then there exist $c \in F$ and $P(T) \in F\{T\}$, $Q \in F\{T^{-1}\}$ such that $v_F(c - 1) > 0$, P has constant term 1 and all slopes $< r$, Q has constant term 1 and all slopes $> r$, and $cPQ = R$.*

Proof. We inductively construct $c_i \in F$, $P_i(T) \in F\{T\}$ and $Q_i(T) \in F\{T^{-}\}$ for $i \geq 0$ with the following properties:

1. $\min\{v_F(c_i - 1), v_r(P_i - 1), v_r(Q_i - 1)\} \geq v_r(R - 1)$;
2. P_i has constant term 1 and all slopes $< r$, and Q_i has constant term 1 and all slopes $> r$;
3. $v_r(R - c_iP_iQ_i) \geq (i + 1)v_r(R - 1)$.

Start by setting $c_0 = P_0 = Q_0 = 1$. These clearly satisfy all conditions. Given c_i, P_i and Q_i, write $R - c_iP_iQ_i = \sum_{j\in\mathbb{Z}} r_jT^j$. For convenience, write $R^+ = \sum_{j>0} r_jT^j$ and $R^- = \sum_{j<0} r_jT^j$. Now set

$$c_{i+1} = c_i + r_0, \quad P_{i+1} = P_i + R^+, \quad Q_{i+1} = Q_i + R^-.$$

Clearly, P_{i+1} and Q_{i+1} have constant term 1. Condition (3.) ensures that $v_r(R^+), v_r(R^-), v_F(r_0) \geq v_r(\sum_{j\in\mathbb{Z}} r_jT^j) \geq ((i + 1)v_r(R - 1)$. Hence, using (1.),

$$v_r(P_{i+1} - 1) = v_r(P_i - 1 + R^+) \geq \min\{v_r(P_i - 1), v_r(R^+)\} \geq v_r(R - 1)$$

and similarly for c_{i+1} and Q_{i+1}. Next, we show that P_{i+1} has all slopes $< r$. Note that the condition on the slopes of P_i is equivalent to saying that $v_s(P_i - 1) > 0$ for all $s \geq r$. We have, for $s \geq r$,

$$\begin{aligned} v_s(P_{i+1} - 1) &\geq \min\{v_s(P_i - 1), v_s(R^+)\} \\ &\geq \min\{v_s(P_i - 1), v_r(R^+)\} \\ &> \min\{0, v_r(R - 1)\} = 0 \end{aligned}$$

using in the second line that only positive powers of T appear in R^+. Hence P_{i+1} has all slopes $< r$. Analogously, one shows that Q_{i+1} has all slopes $> r$. To check that condition (3.) is satisfied for $c_{i+1}, P_{i+1}, Q_{i+1}$, write

$$R - c_{i+1}P_{i+1}Q_{i+1} = \underbrace{r_0(1 - P_iQ_i)}_{=A} + \underbrace{(1 - c_iP_i)R^-}_{=B} + \underbrace{R^+ - c_iR^+Q_i}_{=C} + \underbrace{c_{i+1}R^+R^-}_{=D}$$

(NB: $R - c_iP_iQ_i = r_0 + R^+ + R^-$). Note that $1 - P_iQ_i = (1 - P_i) + P_i(Q_i - 1)$ whence $v_r(1 - P_iQ_i) \geq v_r(R - 1) \geq 0$ using the triangle inequality and Lemma 4.35. It follows from Lemma 4.35 and the induction hypothesis that

$$v_r(A) = v_r(r_0(1 - P_iQ_i)) \geq v_r(r_0) + v_r(1 - P_iQ_i) \geq (i + 1)v_r(R - 1) + v_r(R - 1) = (i + 2)v_r(R - 1).$$

Similarly, one shows that $v_r(B), v_r(C) \geq (i+2)v_r(R-1)$ (NB: $C = (1 - c_i)R_i^+ - c_iR_i^+(Q_i - 1)$). Since $v_F(c_{i+1}) = 0$, we also have

$$v_r(D) = v_r(R_i^+R_i^-) \geq 2(i + 1)v_r(R - 1),$$

using Lemma 4.35 and $v_r(R_i^+), v_r(R_i^-) \geq (i+1)v_r(R-1)$ by induction hypothesis. Altogether, we have

$$v_r(R - c_{i+1}P_{i+1}Q_{i+1}) \geq \min\{(i + 2)v_r(R - 1), 2(i + 1)v_r(R - 1)\} = (i + 2)v_r(R - 1),$$

as desired. This finishes the inductive construction of c_i, P_i, Q_i. Condition (3.) implies that the coefficients of $R - c_iP_iQ_i$ tend to zero as $i \longrightarrow \infty$. Hence c_i, P_i, Q_i converge to limits c, P, Q with the desired properties. Here we use that the induction procedure never introduces monomials in T which don't already appear in R. Therefore, the limits really are polynomials (and not formal power series) in T, T^{-1}. ☐

Remark 4.39. *The proof given above is inspired by that of [15, Lemma 6.3.2] which is a bit easier to follow than that of [16, Proposition 2.4.5].*

Lemma 4.40. *Write $R(T) = \sum_{i\in\mathbb{Z}} a_iT^i$. The condition $v_r(R - 1) > 0$ is equivalent to the following two conditions:*

1. *$v_F(a_0 - 1) > 0$. In particular, $a_0 \in \mathfrak{o}_F^\times$;*
2. *The supporting line of slope r touches the Newton polygon of R only in $(0, 0)$ and nowhere else. In particular, r is not a slope of Newt_R.*

Proof. If $v_r(R - 1) > 0$, then $v_F(a_i) + r[i] > 0$ for all $i \neq 0$ and $v_F(a_0 - 1) > 0$. This implies that $v_F(a_0) = 0$ whence $v_r(R) = 0$. Hence $(0, 0)$ lies on the Newton polygon of R and it is the only point in which the supporting line of slope r touches Newt_R. Conversely, conditions (1.) and (2.) imply that $v_F(a_0) = 0$ and that $v_F(a_i) + r[i] > 0$ for all $i \neq 0$ since otherwise there would be at least two points of Newt_R on the supporting line of slope r. Hence $v_r(R - 1) > 0$. ☐

Remark 4.41. *If R satisfies condition (2.), then we can apply the proposition to $a_0^{-1}R$, i.e. the first condition can essentially be neglected.*

If R has at least two slopes $r_1 < r_2$, then we can get into a situation where we can apply the proposition as follows. Let $(-[k], a_k)$ be a breakpoint of Newt_R, i.e. a point where the slope changes. We may and do assume that it changes from r_1 to r_2. In particular, for any $r \in (r_1, r_2)$,

$$v_r(R) = \min_{i\in\mathbb{Z}}\{v_F(a_i) + r[i]\} = v_F(a_k) + r[k]$$

and k is the only index realizing this minimum. In other words, the supporting line of slope r touches Newt_R only in $(-[k], v_F(a_k))$ and nowhere else for any r in this range. If we can shift this point to $(0, 0)$ without changing the Newton polygon too much, then we can apply the proposition. This is done by multiplying with a_k^{-1} and then with T^{-k} from the left (the order is important if $s > 1$!) using Remark 4.34. It follows that we get a factorization whenever we have at least two slopes. This implies the following corollary.

Corollary 4.42. *If $R(T) \in F\{T^\pm\}$ is irreducible, then it has only one slope.* ☐

4.6 Classification of pure φ-modules

We show that $\kappa(\!(u^{\mathbb{Q}})\!)$ is strongly difference-closed. From this we can then deduce that the categories of pure φ-modules over $\kappa(\!(u^{\mathbb{Q}})\!)$ and over $\tilde{\mathcal{R}}$ of a given slope s are equivalent.

Notation. *With the notation of the previous subsection, we let $F = \kappa(\!(u^{\mathbb{Q}})\!)$, $s = q$, v_F the u-adic valuation and φ_F an automorphism of the form $\sum_{i\in\mathbb{Q}} a_i u^i \mapsto \sum_{i\in\mathbb{Q}} \varphi_\kappa(a_i)u^{qi}$ for some automorphism φ_κ of κ.*

Lemma 4.43. *Let $P(T) \in F\{T\}$ be a polynomial over F with all Newton slopes equal to 0. Then there exists $x \in \mathfrak{o}_F^\times$ such that $P(\varphi_F)(x) = 0$.*

Proof. Write $P(T) = a_n T^n + a_{n-1}T^{n-1} + \cdots + a_0$. Since all Newton slopes of P are zero, we have $v_F(a_j) \geq v_F(a_0) = v_F(a_n)$ for all j. Dividing by a_0, we can thus assume that P has all coefficients in $\mathfrak{o}_F$, constant coefficient 1 and leading coefficient in $\mathfrak{o}_F^\times$. This ensures that $M = \mathfrak{o}_F\{T\}/\mathfrak{o}_F\{T\}P$ is free of rank n over $\mathfrak{o}_F$. We equip it with multiplication by T from the left which defines a φ-semilinear map $\varphi_M : M \longrightarrow M$. Its representing matrix has determinant $\pm a_0 = \pm 1 \in \mathfrak{o}_F^\times$ so that M is a φ-module over $\mathfrak{o}_F$. Its reduction $\bar{M} = M/\mathfrak{m}_F M$ is then a φ-module over $\kappa = \mathfrak{o}_F/\mathfrak{m}_F$ and so must be trivial by Hypothesis 4.3. In particular, there is a nonzero $\mathbf{v} \in \bar{M}$ which is fixed by scalar multiplication with T. Denote the reduction of P modulo $\mathfrak{m}_F$ by Q. Then

$$\begin{aligned} Q(T)\cdot \mathbf{v} = Q(\varphi)(\mathbf{v}) &= \bar{a}_n\varphi^n(\mathbf{v}) + \cdots + \bar{a}_1\varphi(\mathbf{v}) + \bar{a}_0\mathbf{v} \\ &= \bar{a}_n\mathbf{v} + \cdots + \bar{a}_1\mathbf{v} + \bar{a}_0\mathbf{v} \\ &= Q(\varphi)(1)\cdot\mathbf{v} \end{aligned}$$

where $\bar{(\cdot)}$ denotes the reduction modulo $\mathfrak{m}_F$. However, multiplication with $Q(T)$ is the reduction of multiplication with $P(T)$ which is zero on M. It follows that $Q(\varphi)(1)\cdot\mathbf{v} = 0$ which implies that $Q(\varphi)(1) = 0$ in κ since $\mathbf{v} \neq 0$. Hence $P(\varphi)(1) \in \mathfrak{m}_F$. Now note that for any $x \in \mathfrak{m}_F$,

$$\begin{aligned} v_F((P-1)(\varphi)(x)) &= v_F(a_n\varphi^n(x) + \cdots + a_1\varphi(x)) \\ &\geq \min\{v_F(a_n) + q^n v_F(x), \dots, v_F(a_1) + q v_F(x)\} \\ &> v_F(x) \end{aligned}$$

using in the last line that $q > 1$, $v_F(x) \geq 1$ and $v_F(a_j) \geq 0$ for all j. In other words, $(P-1)(\varphi)$ is norm decreasing on $\mathfrak{m}_F$ whence $P(\varphi) = 1 + (P-1)(\varphi)$ is bijective on $\mathfrak{m}_F$ with inverse

$$\sum_{k=0}^{\infty}(-1)^k(P-1)(\varphi)^k$$

by the geometric series. This means that we can find $y \in \mathfrak{m}_F$ with $P(\varphi)(y) = P(\varphi)(1)$. It follows that $x = 1 - y \in \mathfrak{o}_F^\times$ satisfies $P(\varphi)(x) = 0$, as desired. □

Lemma 4.44. *Let $P(T) \in F\{T\}$ be a monic polynomial of degree n over F with all Newton slopes equal to zero. Then there exist $a_1, \dots, a_n \in \mathfrak{o}_F^\times$ such that $P(T) = \prod_{j=1}^n (T - a_j)$.*

Proof. By the previous lemma, there is $x \in \mathfrak{o}_F^\times$ such that $P(\varphi)(x) = 0$. Let $a = \varphi(x)/x$. By the Euclidean algorithm (cf. [18, §I.2]), P can be divided by $Q(T) = (T - a)$ from the right, that is, there are $P', R \in F\{T\}$ such that $P = P'Q + R$ with R of degree zero. Since $P(\varphi)(x) = Q(\varphi)(x) = 0$, we deduce that $R = 0$. By inspection of the equality $P = P'Q$, we see that $P' \in \mathfrak{o}_F\{T\}$ is monic with all Newton slopes equal to zero because the constant and leading coefficient have valuation zero. The lemma now follows by induction. □

Lemma 4.45. *Every irreducible φ-module over F is trivial.*

Proof. Let M be an irreducible φ-module over F. Write $M = F\{T^{\pm}\}/F\{T^{\pm}\}P$ for some monic irreducible polynomial $P(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_0 \in F\{T\}$. By Corollary 4.42, P has only one slope. Since the value group of F is $\mathbb{Q}$, we can find $b \in F^\times$ with $v_F(b) = -(q-1)/(q^n-1)v_F(a_0)$. The polynomial $Q(T) = b^{-1} \cdots \varphi^{n-1}(b^{-1})P(bT)$ is monic and irreducible because P is irreducible. It follows that $Q(T)$ also has only one Newton slope and by construction its constant and leading coefficient have valuation 0. Hence the only slope of Q must be 0. Since $R(T) \mapsto R(bT)$ is an automorphism of $F\{T^{\pm}\}$ it follows that we can replace P by Q and assume that the only slope of P was 0 to begin with. In this case P can be factored into linear polynomials by Lemma 4.44. Since P is irreducible, this forces $P(T) = T - a$ for some $a \in \mathfrak{o}_F^\times$. Now the equation $\varphi(x) = ax$ has a solution $x \in \mathfrak{o}_F^\times$ by Lemma 4.43 so we conclude that $M = F\{T^{\pm}\}/F\{T^{\pm}\}(T-a)$ is isomorphic to F as a φ-module via $Q + F\{T^{\pm}\}P \mapsto Q(\varphi)(x)$. □

Proposition 4.46. *Every φ-module over F is trivial, that is, $\kappa(\!(u^{\mathbb{Q}})\!)$ is strongly difference-closed.*

Proof. Any φ-module over F can be written as a successive extension of irreducible ones using a composition series. Now irreducible φ-modules over F are trivial by the lemma above so that it suffices to show that any extension between trivial φ-modules over $\kappa(\!(u^{\mathbb{Q}})\!)$ splits. This follows was already remarked after Corollary 4.10. □

To solve inhomogeneous φ-equations over F, we introduce the following variation of the homogeneous Newton polygon.

Definition 4.47. *Let $P(T) = \sum_{i\in\mathbb{Z}} a_i T^i \in F\{T^{\pm}\}$ be nonzero and $z \in F$. Define the inhomogeneous Newton polygon of the pair (P, z) as the lower convex hull of the set*

$$\{(-q^i, v_F(a_i)) \mid i \in \mathbb{Z}\} \cup \{(0, v_F(z))\}.$$

The slopes of this convex polygon will be called the inhomogeneous (Newton) slopes of P.

Remark 4.48. *Any slope of the inhomogeneous Newton polygon not involving the point $(0, v_F(z))$ is just a multiple of a slope of the homogeneous Newton polygon. More precisely, any such slope is $q-1$ times a slope of the homogeneous Newton polygon. This is easily seen by comparing the distances between $-[i]$ and $-[j]$ and between $-q^i$ and $-q^j$ for $i, j \in \mathbb{Z}$.*

Proposition 4.49. *Let $P \in F\{T^{\pm}\}$ nonzero and $z \in F$. If $r \in \mathbb{R}$ occurs as a slope of the inhomogeneous Newton polygon of (P, z), then there exists $x \in F$ with $v_F(x) = r$ such that $P(\varphi_F)(x) = z$.*

Proof. The proof is done by induction on the number of homogeneous slopes, ultimately reducing the claim to Corollary 4.10 and its proof.

Since P is nonzero, the homogeneous Newton polygon of P consists of at least one vertex. If it is only one vertex, then $P(T) = aT^m$ for some $m \in \mathbb{Z}$ and some nonzero $a \in F$. The only inhomogeneous slope is then the slope of the line connecting $(-q^m, v_F(a))$ and $(0, v_F(z))$ which is $q^{-m}(v_F(z) - v_F(a))$. Let $x = \varphi_F^{-m}(a^{-1}z) \in F$. Then x satisfies $P(\varphi_F)(x) = a\varphi^m(x) = z$ and has u-adic valuation $v_F(x) = q^{-m}(-v_F(a) + v_F(z))$, as desired.

Next, we treat the case that P has exactly one homogeneous slope. By the same calculation as above, we see that multiplying with some power of T doesn't affect the problem so we may assume that $P(T) = \sum_{i=0}^m a_i T^i \in F\{T\}$ with nonzero constant term. After multiplying by a_0^{-1}, if necessary, we may then assume that $a_0 = 1$. Choose $b \in F$ with $v_F(b) = -v_F(a_m)/(q^m - 1)$ and consider $Q(T) = b^{-1}P(T)b = \sum_{i=0}^m a_i b^{-1}\varphi_F^i(b)T^i \in F\{T\}$. Since P has only one homogeneous slope, the same is true for Q. Moreover, the constant coefficient of Q is still 1 and its leading coefficient has u-adic valuation zero by choice of b. Hence the only homogeneous slope of Q is zero. Assume we were able to find $x' \in F$ with $Q(\varphi_F)(x') = b^{-1}z$. Let $x = bx'$, then $P(\varphi_F)(bx') = bQ(\varphi_F)(x') = z$. Now note that x has exactly the right u-adic valuation since any inhomogeneous slopes of (P, z) is $v_F(b)$ plus an inhomogeneous slope of $(Q, b^{-1}z)$. Hence, after replacing P by Q if necessary, we may assume that the only homogeneous slope of P is zero. Then P can be factored into linear polynomials by Lemma 4.44. Assume that $P = P_1P_2$ where $P_i(T) = T - a_i$ with $a_i \in \mathfrak{o}_F^\times$. If we can solve $P_1(\varphi_F)(y) = z$ and $P_2(\varphi_F)(x) = y$, then $P(\varphi_F)(x) = P_1(\varphi_F)(P_2(\varphi_F)(x)) = P_1(\varphi_F)(y) = z$ (NB: of course, one needs to check that x has the desired u-adic valuation, but we will see a similar argument later so we omit it here). By induction on the number of factors, we may thus reduce to the case that $P(T) = T - a$ for some $a \in \mathfrak{o}_F^\times$. In this situation, the inhomogeneous Newton polygon of (P, z) has either one or two slopes. If $v_F(z) \geq 0$, then it consists of a segment of slope 0 from $(-q, 0)$ to $(-1, 0)$, followed by a segment of slope $v_F(z)$ from $(-1, 0)$ to $(0, v_F(z))$. If $v_F(z) < 0$, then it consists only of a segment of slope $v_F(z)/q$ from $(-q, 0)$ to $(0, v_F(z))$. By Lemma 4.43, there is $x_0 \in \mathfrak{o}_F^\times$ such that $\varphi_F(x_0) = ax_0$. Assume we could find $x_1 \in F$ with $v_F(x_1) = r$ and $\varphi_F(x_1) - x_1 = (ax_0)^{-1}z$. Then $x = x_0x_1$ satisfies $v_F(x) = v_F(x_1) = r$ and

$$\varphi_F(x_0x_1) - ax_0x_1 = ax_0\varphi_F(x_1) - ax_0x_1 = ax_0(ax_0)^{-1}z = z.$$

Since $(T - 1, (ax_0)^{-1}z)$ and $(T - a, z)$ have the same inhomogeneous Newton polygon, we may therefore assume that $a = 1$. By Corollary 4.10, the map $\varphi_F - 1$ is surjective on F so we can find $x \in F$ with $P(\varphi_F)(x) = \varphi_F(x) - x = z$. If $v_F(z) < 0$, then $v_F(z) = v_F(\varphi_F(x) - x) = qv_F(x)$. Hence $v_F(x) = v_F(z)/q$. If $v_F(z) \geq 0$, then the construction in the proof of Corollary 4.10 actually shows that $v_F(x) = v_F(\varphi_F(x) - x) = v_F(z)$. If $v_F(z) > 0$, then $r = 0$ is also an inhomogeneous slope of P. In this case, choose $x' \in F$ with $v_F(x') = v_F(z) > 0$ and $\varphi_F(x') - x' = z$. The element $x = 1 + x' \in F$ has u-adic valuation 0 and satisfies $\varphi_F(x) - x = \varphi_F(x') - x' = z$.

We are left with the induction step. That is, we assume that the proposition holds for any Laurent polynomial with at most $n \geq 1$ homogeneous slopes. Assume that $P(T) = \sum_{i\in\mathbb{Z}} a_iT^i \in F\{T^\pm\}$ has $n + 1$ homogeneous slopes $r_1 < r_2 < \cdots < r_{n+1}$. Let $(-[m], v_F(a_m))$ be the breakpoint between the segment of slope r_n and the segment of slope r_{n+1}. As seen at the end of the previous subsection, $v_r(T^{-m}a_m^{-1}P - 1) > 0$ for any $r \in (r_n, r_{n+1})$. It follows from Proposition 4.38 that there exists $c \in F$ with $v_F(c - 1) > 0$, $P' \in F\{T\}$ with all slopes $< r$, and $Q \in F\{T^-\}$ with all slopes $> r$ such that $T^{-m}a_m^{-1}P = cP'Q$. We may rewrite this as $P = a_mT^mcP'Q$. By the case treated in the beginning, we can ignore the factor a_mT^mc and hence assume that $P = P'Q$. By Proposition 4.36, the homogeneous slopes of P' are $r_1, \dots, r_n$ and Q has only one homogeneous slope, namely r_{n+1}. If $r \in \mathbb{R}$ is an inhomogeneous slope of (P, z), then there are two possible cases to consider. Note that $(0, 0)$ is a breakpoint of the homogeneous Newton polygon of P so $(-1, 0)$ is potentially a point of the inhomogeneous Newton polygon of (P, z).

<u>Case 1:</u> If $r \leq (q - 1)r_n$, then r is also a slope of the inhomogeneous Newton polygon of (P', z) because it must either be $(q-1)r_i$ for some $1 \leq i \leq n$ or the slope from $(-1, 0)$ to $(0, v_F(z))$. By the induction hypothesis, we can find $x' \in F$ with $v_F(x') = r$ and $P'(\varphi_F)(x') = z$. Since $r < (q - 1)r_{n+1}$, the inhomogeneous Newton polygon of (Q, x') consists of a single line of slope r from $(-1, 0)$ to $(0, v_F(x'))$. Hence we may find $x \in F$ with $Q(\varphi_F)(x) = x'$. It follows that

$$P(\varphi_F)(x) = P'(\varphi_F)(Q(\varphi_F)(x)) = P'(\varphi_F)(x') = z.$$

<u>Case 2:</u> If $r > (q - 1)r_n$, then $(-1, 0)$ lies on the inhomogeneous Newton polygon of (P', z) because r could only possibly be the slope on the segment from $(-1, 0)$ to $(0, v_F(z))$. Hence the inhomogeneous Newton polygon of (P', z) certainly has a segment of slope $v_F(z)$. By the

induction hypothesis, there is $x' \in F$ with $v_F(x') = v_F(z)$ and $P(\varphi_F)(x') = z$. Note that the inhomogeneous Newton polygon of (Q, x') is the same as that of (Q, z) so that both must have a segment of slope r. Hence we can find $x \in F$ with $v_F(x) = r$ and $Q(\varphi_F)(x) = z$. Then $P(\varphi_F)(x) = z$ as before. □

Remark 4.50. *Note that there is seemingly a contradiction here. For example, if $v_F(z) = 0$ and φ_κ is the identity on κ, then the equation $\varphi_F(x) - x = z$ cannot be solved since the left side of the equation has constant term 0 whereas the right side does not. This can not occur under the assumption of Hypothesis 4.3 because κ equipped with the identity is not a strongly difference-closed φ-field (cf. [17, Remark 14.3.2]).*

Proposition 4.51. *Let A be an $n \times n$ matrix with entries in $\tilde{\mathcal{R}}^{\mathrm{int}}$. If $\mathbf{v} \in \tilde{\mathcal{E}}^n$ satisfies $A\mathbf{v} = \varphi(\mathbf{v})$, then $\mathbf{v} \in (\tilde{\mathcal{R}}^{\mathrm{b}})^n$.*

Proof. We note that as in Lemma 1.9, $\tilde{\mathcal{E}}^{\mathrm{int}}$ is the ring of series $\sum_{i\in\mathbb{Q}} a_i u^i$ with coefficients in $\mathfrak{o}_K$ such that for any $c > 0$, the set of $i \in \mathbb{Q}$ with $|a_i| \geq c$ is well-ordered and $\lim_{i\longrightarrow-\infty} v_K(a_i) = \infty$.

Rescaling by a factor of u as in the proof of Proposition 4.21, we may assume that the entries of A are bounded by 1 under $|\cdot|_r$. Moreover, we may assume that $\mathbf{v}$ has entries in $\tilde{\mathcal{E}}^{\mathrm{int}}$ by replacing $\mathbf{v}, A$ by $\pi^n\mathbf{v}, A\varphi(\pi^n)/\pi^n$, if necessary. To show that $\mathbf{v}$ in fact has entries in $\tilde{\mathcal{R}}^{\mathrm{int}}$, we only need to see that $\mathbf{v}$ has entries in $\tilde{\mathcal{R}}^s$ for some $s > 0$ because then $\mathbf{v}$ has entries in $\tilde{\mathcal{R}} \cap \tilde{\mathcal{E}}^{\mathrm{int}} = \tilde{\mathcal{R}}^{\mathrm{int}}$. Write $\mathbf{v} = \sum_{j=1}^n \sum_{i\in\mathbb{Q}} c_{ij} u^i \mathbf{e}_j$ and $A\mathbf{v} = \sum j = 1^n \sum_{i\in\mathbb{Q}} d_{ij} u^i \mathbf{e}_j$ where $\mathbf{e}_1, \dots, \mathbf{e}_n$ denote the standard basis vectors. We first prove the following.

Claim. $|c_{ij} u^i|_r \leq 1$ *for all i, j, that is,* $|\mathbf{v}|_r \leq 1$.

Proof of Claim. Assume that $|\mathbf{v}|_r > 1$. Then also $|\varphi^{-1}(A\mathbf{v})|_r = |\mathbf{v}|_r > 1$. Since $\mathbf{v} = \varphi^{-1}(A\mathbf{v})$ has coefficients in $\mathfrak{o}_K$, any pair (i, j) with $|\varphi^{-1}(d_{ij} u^i)|_r > 1$ satisfies $i < 0$ and this quantity becomes bigger as i decreases. By well-ordering, the supremum $|\varphi^{-1}(A\mathbf{v})|_r$ is attained, i.e. we may choose i, j with $|\varphi^{-1}(d_{ij} u^i)|_r = |\varphi^{-1}(A\mathbf{v})|_r > 1$. Since $i < 0$, this gives the contradiction $|\mathbf{v}|_r = |\varphi^{-1}(A\mathbf{v})|_r = |\varphi^{-1}(d_{ij} u^i)|_r = |c_{ij} u^{i/q}|_r < |c_{ij} u^i|_r \leq |\mathbf{v}|_r$. □

To prove that $\mathbf{v}$ has entries in $\tilde{\mathcal{R}}$, we need to show that $|c_{ij}| e^{-ri} \longrightarrow 0$ as $i \longrightarrow -\infty$ as the same will then be true for any $s \in (0, r]$. From the equation $A\mathbf{v} = \varphi(\mathbf{v})$, we deduce that

$$|\mathbf{v}|_{rq} = |\varphi(\mathbf{v})|_r = |A\mathbf{v}|_r \leq |\mathbf{v}|_r$$

because $|A|_r \leq 1$. It therefore follows from the claim that $|c_{ij} u^i|_{rq} \leq 1$ for all i, j. By induction, we get $|c_{ij} u^i|_{rq^n} \leq 1$ for any $n \geq 0$ and all i, j. But this is only possible if $|c_{ij}| e^{-ri} \longrightarrow 0$ since $|u^i|_{rq^n} = e^{-riq^n} \longrightarrow \infty$ as $i \longrightarrow -\infty$ for any given $n \geq 0$. This finishes the proof. □

Remark 4.52. *For invertible matrices A over $\tilde{\mathcal{R}}^{\mathrm{int}}$, we can also deduce from an equation of the form $A\varphi(\mathbf{v}) = \mathbf{v}$ that $\mathbf{v}$ has entries in $\tilde{\mathcal{R}}^{\mathrm{b}}$ by applying the proposition to the equation $\varphi(\mathbf{v}) = A^{-1}\mathbf{v}$. This implies that the base change functor from étale φ-modules over $\tilde{\mathcal{R}}^{\mathrm{b}}$ to étale φ-modules over $\tilde{\mathcal{E}}$ is fully faithful (cf. the proof of Proposition 2.48).*

Corollary 4.53. *Let φ be any relative Frobenius lift on $\mathcal{R}$, and let A be an $n \times n$ matrix over $\mathcal{R}^{\mathrm{int}}$. If $\mathbf{v} \in \mathcal{E}^n$ satisfies $A\mathbf{v} = \varphi(\mathbf{v})$, then $\mathbf{v} \in (\mathcal{R}^{\mathrm{b}})^n$.*

Proof. Follows immediately from the proposition since all inclusions are well-behaved. □

Remark 4.54. *As above, this implies that the base change functor from étale φ-modules over $\mathcal{R}^{\mathrm{b}}$ to étale φ-modules over $\mathcal{E}$ is fully faithful.*

We can now prove Proposition 4.26: The categories of pure φ-modules over K and over $\tilde{\mathcal{R}}$ of a given slope $s \in \mathbb{Q}$ are equivalent.

Proof of Proposition 4.26. We check full faithfulness first. Recall that by Lemma 2.55, a φ-module M over $\tilde{\mathcal{R}}$ or K is pure of slope s if and only if $[a]_* M$ is. So by twisting and using the fact that $[a]_*$ commutes with tensor products, we may reduce to the case that $s = 0$. As in the proof of Proposition 2.48, we can then formulate the claim in terms of H^0 so we must show the following: Given an étale φ-module M_0 over K, we have $H^0(M_0) = H^0(M_0 \otimes_K \tilde{\mathcal{R}})$. But M_0 is trivial by Hypothesis 4.3 so that $M_0 \otimes_K \tilde{\mathcal{R}}$ is also trivial. Hence what we need to show is that a vector with entries in $\tilde{\mathcal{R}}$ whose entries are fixed by φ must actually have entries in K. This follows from Lemma 4.20.

To prove essential surjectivity, we again reduce to the case that $s = 0$. Since étale φ-modules over K are trivial by Hypothesis 4.3, we need to show that étale φ-modules over $\tilde{\mathcal{R}}$ are trivial. Let M be an étale φ-module over $\tilde{\mathcal{R}}$. It is obtained by base change from an étale φ-module over $\tilde{\mathcal{R}}^{\mathrm{b}}$ (Proposition 2.48). We denote this φ-module by M again. The base extension $\tilde{M} = \tilde{\mathcal{E}} \otimes_{\tilde{\mathcal{R}}^{\mathrm{b}}} M$ of M to $\tilde{\mathcal{E}}$ is an étale φ-module over $\tilde{\mathcal{E}}$. Choose some basis of M such that φ_M acts on M by an invertible matrix A over $\tilde{\mathcal{R}}^{\mathrm{int}}$. Note that the residue field of $\tilde{\mathcal{E}}$ is $\kappa(\!(u^{\mathbb{Q}})\!)$ which is strongly difference-closed by Proposition 4.46. Now $\tilde{\mathcal{E}}$ is $\mathfrak{m}_K$-adically complete so by the same argument as in the proof of Lemma 4.5, we can find a basis invariant under $\varphi_{\tilde{M}}$. This basis consists of vectors $\mathbf{v}$ with entries in $\tilde{\mathcal{E}}$ satisfying $A\varphi(\mathbf{v}) = \mathbf{v}$. By Proposition 4.51, applied to the equation $A^{-1}\mathbf{v} = \varphi(\mathbf{v})$, these vectors must have entries in $\tilde{\mathcal{R}}^{\mathrm{b}}$ and hence correspond to elements of M. It follows that M is trivial, as desired. □

4.7 The local calculation

We make the explicit calculation proving Proposition 4.28. To do this, we first translate the proposition into certain φ-equations.

Definition 4.55. *Denote by $\tilde{\mathcal{R}}^{\mathrm{tr}}$ the set of elements of $\tilde{\mathcal{R}}$ whose support has a least element. This forms a subring of $\tilde{\mathcal{R}}$ which carries a u-adic valuation v. By an argument as for the ring of formal Laurent series over a field, we see that $\tilde{\mathcal{R}}^{\mathrm{tr}}$ is u-adically complete.*

Remark 4.56. *Note that an element $x = \sum_{i\in\mathbb{Q}} a_i u^i \in \tilde{\mathcal{R}}$ whose support has a least element $j \in \mathbb{Q}$ such that $|a_i| \leq |a_j|$ for all $i \in \mathbb{Q}$ is a unit in $\tilde{\mathcal{R}}^{\mathrm{tr}}$. This is clear because any such element must lie in $\tilde{\mathcal{R}}^{\mathrm{b}}$, but another way of seeing this is to reduce to $j = 0$ and $a_0 = 1$ by multiplying with $a_j^{-1}u^{-j}$. In this case we can construct x^{-1} by the usual geometric series argument. It follows that any element which is invertible over $\tilde{\mathcal{R}}^{\mathrm{tr}}$ must be of this form. The upshot is that $(\tilde{\mathcal{R}}^{\mathrm{tr}})^\times \subseteq \tilde{\mathcal{R}}^{\mathrm{b}}$ so we can apply the valuation w to such elements.*

Lemma 4.57. *Let P be a φ-module over K of rank 1 and degree $n > 0$, and fix a generator $\mathbf{v}$ of P. We have the following:*

1. *For any $x \in \tilde{\mathcal{R}}^{\mathrm{tr}}$ with support in $[0, +\infty)$, the class of $x\mathbf{v}$ in $H^1(P \otimes_K \tilde{\mathcal{R}})$ vanishes;*
2. *Each class in $H^1(P \otimes_K \tilde{\mathcal{R}})$ has a representative of the form $\sum_{j=0}^{n-1} u_j\mathbf{v}$, where for each j, either $u_j = 0$, or $u_j \in (\tilde{\mathcal{R}}^{\mathrm{tr}})^\times$, $w(u_j) = j$, and $v(u_j) < 0$.*

Proof. (1.) It is equivalent to show that the class of any element $x \in \tilde{\mathcal{R}}^{\mathrm{tr}}$ with support in $[0, +\infty)$ vanishes in $H^1(\tilde{\mathcal{R}}(n))$, that is, $x = y - \pi^n\varphi(y)$ for some $y \in \tilde{\mathcal{R}}$. Here $\pi \in K$ is some uniformizer, as usual. Define the twisted powers $\pi^{\{i\}}$, $i \geq 0$ by the recurrence

$$\pi^{\{0\}} = 1, \quad \pi^{\{i+1\}} = \varphi_K(\pi^{\{i\}})\pi^n.$$

Write $x = \sum_{i\geq 0} x_i u^i \in \tilde{\mathcal{R}}^{\mathrm{tr}}$. Since the map $1 - \pi^n\varphi_K$ is surjective on K by Hypothesis 4.3, the class of x is equivalent to the class of $x_+ = \sum_{i>0} x_i u^i$. Since x_+ has no constant term, the series $y = \sum_{i=0}^{\infty} \pi^{\{i\}}\varphi^i(x_+)$ converges in $\tilde{\mathcal{R}}$ and it is easily seen that it satisfies $y - \pi^n\varphi(y) = x_+$. Hence the class of x_+ in $H^1(\tilde{\mathcal{R}}(n))$ vanishes, as claimed.
(2.) As in (1.), we can reformulate the assertion for classes in $H^1(\tilde{\mathcal{R}}(n))$. By (1.), we can always find a representative of the form $x = \sum_{i<0} x_i u^i$. We only sketch how one can find a representative of the desired form in a special case. The full calculation can be found in [14, Lemma 4.3.2]. Let $l \in \mathbb{Z}$ and assume that $w(x_i) = l$ for all $i < 0$ with $x_i \neq 0$. The conditions on elements of $\tilde{\mathcal{R}}$ then imply that the support of x has a least element. As all coefficients of x have the same valuation this implies that x is a unit in $\tilde{\mathcal{R}}^{\mathrm{tr}}$ by the previous remark. Let $m \in \mathbb{Z}$ be the unique integer such that $l + mn \in \{0, \dots, n-1\}$. Then x and $x' = \pi^{mn}\varphi^n(x)$ represent the same class in $H^1(\tilde{\mathcal{R}}(n))$ and x' is of the desired form (where we set $u_{l+mn} = x'$ and $u_j = 0$ for all other j). The general case uses a clever iteration of this idea (NB: in places where minima are chosen, use the well-orderedness condition instead). $\square$

The sequence in Proposition 4.28 is isomorphic to one of the form

$$0 \longrightarrow [n]^*N' \longrightarrow E_\alpha \longrightarrow P \longrightarrow 0$$

where $E_\alpha = P \oplus [n]^*N'$ with the φ-action given by $\varphi_{E_\alpha}(n,p) = (\varphi_{[n]^*N'}(n) + \alpha\circ\varphi_P(p), \varphi_P(p))$ for some class $\alpha \in \mathrm{Ext}(P, [n]^*N') = H^1(P^\vee \otimes [n]^*N')$. The snake lemma yields an exact sequence

$$H^0(E_h) \longrightarrow H^0(P) \xrightarrow{\delta} H^1([n]^*N')$$

To prove the proposition, it suffices to show that the first map has nonzero image. By exactness, this is equivalent to showing that δ has nonzero kernel. Going through the construction of δ, we see that in our situation it is given by pairing with the class α, that is,

$$H^0(P) \ni p \longmapsto \delta(p) = \alpha(p) + \mathrm{im}(\varphi_{[n]^*N'} - 1) \in H^1([n]^*N').$$

Under the identifications $H^1(P^\vee \otimes [n]^*N') \simeq H^1([n]^*([n]_*P^\vee \otimes N')) = H^1([n]_*P^\vee \otimes N')$ and $H^1([n]^*N') = H^1(N')$ (cf. Lemma 2.20 and 2.21) this can also be viewed as the composition of $H^0(P) \hookrightarrow H^0([n]_*P)$ with the map $H^0([n]_*P) \longrightarrow H^1(N')$ given by pairing with $\alpha \in H^1([n]_*P^\vee \otimes N')$. If the class of α vanishes, then we are done because $H^0(P) \neq 0$ by Proposition 4.22 and 4.26. We assume hereafter that α does not vanish.

Notation. *By Proposition 4.26, P and N' are obtained by base change from a φ-module P_0 and a φ^n-module N'_0 over K, respectively. These have rank 1 so we may choose generators $\mathbf{v}$ and $\mathbf{w}$ and define $\lambda, \mu \in K^\times$ by $\varphi_P(\mathbf{v}) = \lambda\mathbf{v}$ and $\varphi_{N'}(\mathbf{w}) = \mu\mathbf{w}$. We view P as $\tilde{\mathcal{R}}$ equipped with the φ-semilinear map $\lambda\varphi$ and N' as $\tilde{\mathcal{R}}$ equipped with the φ^n-semilinear map $\mu\varphi^n$. Set $Q_0 = [n]_*P_0^\vee \otimes N'_0$ and $Q = [n]_*P^\vee \otimes N' \simeq Q_0 \otimes_K \tilde{\mathcal{R}}$. We write $\mathbf{v}^\vee$ for the generator of $P^\vee$ dual to $\mathbf{v}$ and let $\mathbf{x}$ be the generator $\mathbf{v}^\vee \otimes \mathbf{w}$ of Q_0. Define the twisted powers $\lambda^{\{m\}}$ and $\mu^{\{m\}}$ of λ and μ by the two-way recurrences*

$$\lambda^{\{0\}} = 1, \lambda^{\{m+1\}} = \varphi(\lambda^{\{m\}})\lambda \quad \textit{and} \quad \mu^{\{0\}} = 1, \mu^{\{m+1\}} = \varphi^n(\mu^{\{m\}})\mu.$$

We then have $\varphi_P^m = \lambda^{\{m\}}\varphi^m$ and $\varphi_{N'}^m = \mu^{\{m\}}\varphi^{nm}$ for all $m \in \mathbb{Z}$.

By Lemma 4.57, we can represent the class $\alpha \in H^1(Q)$ by a nonzero element of Q of the form $\sum_{j=0}^{n} u_j\mathbf{x}$ where each u_j is either zero or a unit in $\tilde{\mathcal{R}}^{\mathrm{tr}}$ with $w(u_j) = j$ and $v(u_j) < 0$. The map $\delta : H^0(P) \longrightarrow H^1(N')$ is then given by $p\mathbf{v} \mapsto \sum_{j=0}^n u_j p\mathbf{x}$. Viewing everything in $\tilde{\mathcal{R}}$, the proof of Proposition 4.28 therefore amounts to finding $x, y \in \tilde{\mathcal{R}}$ solving the two equations

$$\lambda\varphi(x) - x = 0, \quad \sum_{j=0}^{n} u_j x = \mu\varphi^n(y) - y.$$

(The first equation translates to $x \in H^0(P)$ and the second translates to $\delta(x) \in H^1(N')$)

Proposition 4.58. *The equations above can be solved over* $\tilde{\mathcal{R}}$.

Proof. We follow the proof of [13, Lemma 4.12]. For $j \in \{0, \dots, n\}$ such that $u_j \neq 0$, $l \in \mathbb{Z}$ and $m \in (0, +\infty)$, define

$$e(j, l, m) = (v(u_j) + mq^{-l})q^{-n(j+l)}.$$

If we fix j, m, then we make the following observations:

- $e(j, l, m)$ approaches 0 from below as $l \longrightarrow +\infty$;
- $e(j, l, m)$ tends to $+\infty$ as $l \longrightarrow -\infty$.

In particular, the minimum $h(m) = \min_{j,l}\{e(j, l, m)\}$ is well-defined. The function $h : (0, +\infty) \longrightarrow \mathbb{R}$ is clearly continuous. In fact, it is piecewise linear since it is the minimum of linear functions in m. Moreover, it is strictly increasing because $e(j, l, m)$ is strictly increasing in m. Observe that $e(j, l + 1, qm) = q^{-n}e(j, l, m)$ whence $h(qm) = q^{-n}h(m)$. Since $v(u_j) < 0$, we have $h(m) < 0$ for m sufficiently small. It follows that $h(q^j m) = q^{-nj}h(m) < 0$ for all $j \in \mathbb{Z}$ so that h only takes on negative values. Altogether, h is a continuous increasing bijection $(0, +\infty) \longrightarrow (-\infty, 0)$. Another way of interpreting this is as follows. Let H be the lower convex hull of the set of points

$$\{(-q^{-nj-(n+1)l}, q^{-n(j+l)}v(u_j)) \mid j = 0, \dots, n \text{ and } l \in \mathbb{Z}\}.$$

The point corresponding to (j, l) lies on H if and only if there is some $r \in \mathbb{R}$ such that the value of the supporting line of slope r of H is $q^{-n(j+l)}v(u_j) + rq^{-nj-(n+1)l}$ which is nothing other than $e(j, l, r)$. In fact, this is the case if and only if $e(j, l, r) = h(r)$. In other words, if $r \in \mathbb{R}$ is a slope of H, then r is a breakpoint of h and vice versa.

Now choose $r \in (0, +\infty)$ at which h changes its slope, i.e. r is a slope of H. Let T be the set of all ordered pairs (j, l) such that $e(j, l, r) < 0$. This is an infinite set because $e(j, l, r) < 0$ for infinitely many $l \geq 0$. However, for fixed j, the possible values for l such that $(j, l) \in T$ are bounded below because $e(j, l, r)$ tends to $+\infty$ as $l \longrightarrow -\infty$. We denote by S the subset of T consisting of those pairs (j, l) with $e(j, l, r) < q^{-n}h(r)$. For fixed j, the possible values for l such that $(j, l) \in S$ are again bounded below, but they are also bounded above because $e(j, l, r) < q^{-n}h(r) < 0$ only for finitely many $l \geq 0$ since $e(j, l, r)$ tends to 0 as $l \longrightarrow +\infty$. That is, S is a finite set. For any pair (j, l), set $s(j, l) = \lfloor \log_{q^n}(h(r)/e(j, l, r)) \rfloor$.

Claim. *The following assertions are true:*

1. *For* $(j, l) \in T$, *we have* $s(j, l) \geq 0$.
2. *For* $(j, l) \in T$, *we have* $e(j, l, r)q^{ns(j,l)} \in [h(r), q^{-n}h(r))$.
3. *We have* $(j, l) \in S$ *if and only if* $(j, l) \in T$ *and* $s(j, l) = 0$.
4. *For any* $c > 0$, *there are only finitely many pairs* $(j, l) \in T$ *with* $s(j, l) \leq c$.

Proof of Claim. (1.) For $(j,l) \in T$, we have $h(r) \le e(j,l,r) < 0$ so that $h(r)/e(j,l,r) \ge 1$.
(2.) We have $e(j,l,r)q^{ns(j,l)} \ge e(j,l,r) \ge h(r)$ by (1.). This gives the lower bound. Moreover, $e(j,l,r)q^{ns(j,l)} \le e(j,l,r)h(r)/e(j,l,r) = h(r) < q^{-n}h(r)$ since $h(r) < 0$ and $q^{-n} < 1$. This gives the upper bound.
(3.) If $(j,l) \in S \subseteq T$, then $e(j,l,r) < q^{-n}h(r)$. It follows that $h(r)/e(j,l,r) > q^n$ so that $s(j,l) = \lfloor \log_{q^n} h(r)/e(j,l,r) \rfloor < 1$. Since $s(j,l) \ge 0$ by (1.), we deduce that $s(j,l) = 0$. Conversely, if $s(j,l) = 0$, then $\log_{q^n} h(r)/e(j,l,r)$ must be at least 0, but strictly smaller than 1. It follows that $h(r)/e(j,l,r) > q^n$, as desired.
(4.) For fixed j, the possible values for l such that $(j,l) \in T$ are bounded below so it suffices to show that $s(j,l) \le c$ only for finitely many $l > 0$. Since $h(r)$ is independent of l and $e(j,l,r)$ tends to zero from below as $l \longrightarrow +\infty$, we have $s(j,l) = \lfloor \log_{q^n} h(r)/e(j,l,r) \rfloor \longrightarrow +\infty$ for $l \longrightarrow +\infty$. This gives the desired result. □

For $c \in \mathbb{R}$, let U_c be the set of $z \in \tilde{\mathcal{R}}^{\mathrm{tr}} \cap \tilde{\mathcal{R}}^{\mathrm{int}}$ with $v(z) \ge c$. For any c, the set U_c is π-adically complete because the support of elements in U_c is bounded below. That is, we got rid of the sequences that show that $\tilde{\mathcal{R}}^{\mathrm{int}}$ is not π-adically complete. For $z \in U_r$, define the function

$$R(z) = \sum_{(j,l)\in T} \mu^{\{-j-l+s(j,l)\}} \varphi^{-n(j+l-s(j,l))}(u_j \lambda^{\{-l\}} \varphi^{-l}(z)).$$

To see that this series converges π-adically, we compute the π-adic valuation of each summand.

$$\begin{aligned} w(\mu^{\{-j-l+s(j,l)\}} &\varphi^{-n(j+l-s(j,l)}(u_j\lambda^{\{-l\}}\varphi^{-l}(z))) \\ &= w(\mu^{\{-j-l+s(j,l)\}}) + w(u_j) + w(\lambda^{\{-l\}} + w(z) \\ &= (-j-l+s(j,l))w(\mu) + j + (-l)w(\lambda) \\ &= -j-l+s(j,l)+j+l = s(j,l) \end{aligned}$$

Here we use multiple times that w is φ-invariant and the definitions of u_j, λ and μ. Hence the series converges π-adically by (4.) and because the term in brackets lies in $\tilde{\mathcal{R}}^{\mathrm{tr}}$. In fact, we can show that it lies in $U_{h(r)}$. To see this, we check that each summand has u-adic valuation at least $h(r)$. We ignore the twisted powers of μ and λ since these don't change the u-adic valuation.

$$\begin{aligned} v(\varphi^{-n(j+l-s(j,l))}(u_j\varphi^{-l}(z))) &= q^{-n(j+l-s(j,l))}(v(u_j) + q^{-l}v(z)) \\ &= q^{ns(j,l)}e(j,l,v(z)) \\ &\ge q^{ns(j,l)}e(j,l,r) \ge h(r) \end{aligned}$$

where the final estimate holds by (2.). If we reduce $R(z)$ modulo π, we get

$$\begin{aligned} R(z) &\equiv \sum_{(j,l)\in S} \mu^{\{-j-l\}}\varphi^{-n(j+l)}(u_j\lambda^{\{-l\}}\varphi^{-l}(z)) \\ &= \sum_{(j,l)\in S} \mu^{\{-j-l\}}\varphi^{-n(j+l)}(u_j\lambda^{\{-l\}})\varphi^{-nj-(n+1)l}(z) \mod \pi. \end{aligned}$$

This is because $s(j,l) = 0$ for all $(j,l) \in S$ by (3.) and $s(j,l) \ge 1$ for all $(j,l) \in T \setminus S$ by (1.) and the definition of $s(j,l)$. Note that the values $-nj-(n+1)l$ are all distinct because j runs only through $\{0,\dots,n\}$. Hence the summands in the bottom line are all distinct monomials in $\varphi(z)$. Since S is finite, we may therefore write this sum as $Q(\varphi)(z)$ for some Laurent polynomial $Q(T) \in \kappa(\!(u^{\mathbb{Q}})\!)\{T^{\pm}\}$. Let $w \in U_{h(r)}$ be arbitrary. By construction, the inhomogeneous Newton polygon of (Q,w) is the lower convex hull of the set of points

$$\{(-q^{-nj-(n+1)l}, q^{-n(j+l)}v(u_j)) \mid (j,l) \in S\} \cup \{(0, v(w))\}.$$

We claim that it has a segment of slope r. To see this, consider the line $y = rx + h(r)$. Since $e(j,l,r) \geq h(r)$, the point corresponding to (j,l) either lies on or above this line, depending on whether $e(j,l,r) = h(r)$ or $e(j,l,r) > h(r)$. Moreover, since $v(w) \geq h(r)$, the point $(0, v(w))$ also lies either on or above this line. Since H has a segment of slope r, there must be at least two pairs (j,l) which lie on the line. Now by definition any such pair must lie in T. In fact, these must be precisely the pairs in S by (2.) and (3.). Hence the inhomogeneous Newton polygon of (Q, w) indeed has a segment of slope r. By Proposition 4.49, there exists $z \in U_r$ with $v(z) = r$ and $Q(\varphi)(z) = w$, that is, $R(z) \equiv w \mod \pi$. Note that if $R(z) = w + \pi w'$, then the support of w' must also lie in $[h(r), +\infty)$ so $w' \in U_{h(r)}$. By iteration, we conclude that the image of U_r under R is dense in $U_{h(r)}$ with respect to the π-adic topology. Since U_r is complete, it follows that U_r surjects onto $U_{h(r)}$. Choosing $w = 0 \in U_{h(r)}$, there is z_0 with $v(z_0) = r$ and $R(z_0) \equiv 0 \mod \pi$. In particular, z_0 is nonzero modulo π. We may then choose $z_1 \in U_r$ with $R(z_1) = R(z_0)/\pi$ and set $z = z_0 - \pi z_1$. Then $z \equiv z_0 \not\equiv 0 \mod \pi$ and hence is nonzero, but $R(z) = 0$.

Claim. *The element*

$$\sum_{l\in\mathbb{Z}} \varphi_P^{-l}(z\mathbf{v}) = \sum_{l\in\mathbb{Z}} \lambda^{\{-l\}} \varphi^{-l}(z)\mathbf{v} \in H^0(P)$$

pairs to zero with the class of α, that is, $\sum_{j=0}^n \sum_{l\in\mathbb{Z}} u_j \lambda^{\{-l\}} \varphi^{-l}(z) \in \operatorname{im}(\mu\varphi^n - 1)$.

It can be shown that the series $\sum_{l\in\mathbb{Z}} \lambda^{\{-l\}} \varphi^{-l}(z)$ converges in $\tilde{\mathcal{R}}$ using that $\varphi^{-l}(z)$ has support bounded below (NB: $v(\varphi^{-l}(z)) = q^{-l} v(z) \geq q^{-l} r$) and π-adic completeness of $\tilde{\mathcal{R}}^{\mathrm{tr}} \cap \tilde{\mathcal{R}}^{\mathrm{int}}$. We have

$$\lambda\varphi(\sum_{l\in\mathbb{Z}} \lambda^{\{-l\}} \varphi^{-l}(z)) = \sum_{l\in\mathbb{Z}} \lambda\varphi(\lambda^{\{-l\}})\varphi^{-l+1}(z) = \sum_{l\in\mathbb{Z}} \lambda^{\{-l+1\}} \varphi^{-l+1}(z) = \sum_{l\in\mathbb{Z}} \lambda^{\{-l\}} \varphi^{-l}(z)).$$

This proves that $\sum_{l\in\mathbb{Z}} \lambda^{\{-l\}} \varphi^{-l}(z)\mathbf{v} \in H^0(P)$. Now write

$$\sum_{j=0}^{n} \sum_{l\in\mathbb{Z}} u_j \lambda^{\{-l\}} \varphi^{-l}(z) = \sum_{(j,l)\in T} u_j \lambda^{\{-l\}} \varphi^{-l}(z) + \sum_{(j,l)\notin T} u_j \lambda^{\{-l\}} \varphi^{-l}(z)$$

and denote the two series on the right by A and B, respectively. Recall that z was chosen such that $v(z) \geq r$. It follows that $v(\varphi^{-l}(z)) \geq rq^{-l}$. For $(j,l) \notin T$, we have $e(j,l,r) \geq 0$ which is equivalent to $v(u_j) \geq -rq^{-l}$. Hence $v(u_j\varphi^{-l}(z)) \geq 0$ for all $(j,l) \notin T$ so that the class of B is zero in $H^1(N')$ by Lemma 4.57 (1.). It remains to show that the class of A is also zero. Set

$$a = \sum_{(j,l)\in T} \sum_{k=0}^{j+l-s(j,l)-1} \mu^{\{k-j-l+s(j,l)\}} \varphi^{n(k-j-l+s(j,l))}(u_i \lambda^{\{-l\}} \varphi^{-l}(z)).$$

It can be shown that this series converges as well (cf. the end of the proof of [13, Lemma 4.12]).

Claim. *We have $a - \mu\varphi^n(a) = R(z) - A$.*

We are done if we can prove this claim since $R(z) = 0$ by choice of z and the claim then implies that the class of A is zero in $H^1(N')$. Note that applying $\mu\varphi^n$ to a has the same effect as letting the sum over k, for given $(j,l) \in T$, run from $k = 1$ to $k = j + l - s(j,l)$ instead. In particular, for given $(j,l) \in T$, the sum over k in $a - \mu\varphi^n(a)$ telescopes, leaving the summand corresponding to $k = 0$ minus the summand corresponding to $k = j + l - s(j,l)$, that is,

$$\mu^{\{-j-l+s(j,l)\}} \varphi^{-nj-nl+ns(j,l)}(u_j \lambda^{\{-l\}} \varphi^{-l}(z)) - \lambda^{\{-l\}} u_j \varphi^{-l}(z).$$

This is exactly the summand corresponding to (j,l) in $R(z) - A$. □

5 Faithfully flat descent

In this section, we finish the proof of Theorem 3.1 by constructing the field extension of K demanded in Hypothesis 4.3 and then descending the result of Theorem 4.29 back down to $\mathcal{R}$.

5.1 Construction of a coefficient field

We go back and construct the field demanded in Hypothesis 4.3. To simplify some statements, we make the following definition. It will only be used in this subsection.

Definition 5.1. *Suppose that K is inversive. An admissible extension of K is a field extension L of K, complete for a nonarchimedean absolute value extending the one on K with the same value group, and equipped with an isometric field automorphism φ_L extending φ_K.*

Remark 5.2. *If we are given K as in Section 1 which is not inversive, then we can embed it into its φ-perfection. This is the completed direct limit of the direct system $K \longrightarrow K \longrightarrow \dots$ where the transition maps are all φ_K. It is an inversive φ-field which shares all the properties of K we need (cf. [17, Hypothesis 14.4.1]). In other words, the assumption that K is inversive is not a problem because we can simply replace K by its φ-perfection, if necessary.*

For K as in the above definition, we now construct an extension with strongly difference-closed residue field. We need two preliminary results.

Lemma 5.3. *For any $z \in K^\times$, there exists an admissible extension L of K such that the equation $\varphi_L(x) - x = z$ has a solution $x \in L$.*

Proof. For $\rho > 0$, consider on the polynomial ring $K[x]$ in the variable x over K the ρ-norms. That is, $|\sum_{n\geq 0} a_n x^n|_\rho = \max_{n\geq 0}\{|a_n|\rho^n\}$ which extends to the rational function field $K(x)$ by multiplicativity. Choose the normalization $\rho = |z| \in |K^\times|$ and complete $K(x)$ with respect to $|\cdot|_\rho$. We obtain a complete nonarchimedean valued field L whose absolute value extends the one on K. Since $K(x)$ is dense in L, it suffices to define the value of φ_L on the variable x. Set $\varphi_L(x) = x + z$. Then φ_L is an isometry since $|\varphi_L(x)|_\rho = |x + z|_\rho = |x|_\rho$ by construction. Hence L is an admissible extension of K and $x \in L$ is a solution of the equation. $\square$

Remark 5.4. *This means in particular that if we are given an extension of trivial φ-modules over K, then the extension splits if we extend scalars to a suitably large admissible extension of K (Lemma 2.13).*

Lemma 5.5. *Let $P(T) = T^n + a_{n-1}T^{n-1} + \dots + a_0$ be a twisted polynomial over $\mathfrak{o}_K$ with $|a_0| = 1$. Then there exists an admissible extension L of K such that the equation $P(\varphi_L)(x) = 0$ has a solution $x \in \mathfrak{o}_L^\times$.*

Proof. We repeat the previous construction with more variables. Let L be the completion of the rational function field $K(y_0, \dots, y_{n-1})$ under the Gauss norm normalized such that the variables $y_0, \dots, y_{n-1}$ all have norm 1. Extend φ_K to an isometric automorphism φ_L of L by setting $\varphi_L(y_i) = y_{i+1}$ for $i = 0, \dots, n-2$ and $\varphi_L(y_{n-1}) = -a_{n-1}y_{n-1} - \dots - a_0 y_0$. Then

$$\begin{aligned} P(\varphi_L(y_0)) &= \varphi_L^n(y_0) + a_{n-1}\varphi_L^{n-1}(y_0) + \dots + a_0 y_0 \\ &= -a_{n-1}y_{n-1} - \dots - a_0 y_0 + a_{n-1}y_{n-1} + \dots + a_0 y_0 \\ &= 0 \end{aligned}$$

so $x = y_0 \in \mathfrak{o}_L^\times$ is a solution of the equation. $\square$

Proposition 5.6. *There exists a complete extension L of K with the same value group, equipped with an extension of φ_K to an automorphism φ_L, such that any étale φ-module over L is trivial. In other words, κ_L is strongly difference-closed.*

Proof. If we can find for an arbitrary étale φ-module M over K an admissible extension L' such that $M_{L'} = L' \otimes_K M$ is trivial, then the field L demanded by the proposition exists by a standard argument using Zorn's lemma. So let M be an étale φ-module over K. We can write M as a successive extension of irreducible φ-modules using the composition series. The first step, M_1, is an irreducible φ-submodule of M. Let M_0 be an étale lattice of M. Then $N_0 = M_0 \cap M_1$ is a φ-module over $\mathfrak{o}_K$ (cf. Lemma 2.51) and is an $\mathfrak{o}_K$-lattice of M_1. Therefore, M_1 is étale. By a similar argument, we see that all the quotients in the decomposition series are irreducible étale φ-modules. If we can show that these are all trivial, then we can split the filtration by passing to a suitably large admissible extension (cf. Lemma 5.3 and the remark thereafter).

It remains to show that if M is an irreducible étale φ-module over K, then M becomes trivial over some extension of K. Write $M = K\{T^{\pm}\}/K\{T^{\pm}\}P$ for some irreducible $P(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_0 \in K\{T\}$. Since M is étale, we have $0 = \deg(M) = v_K(a_0)$ so P has constant and leading term of valuation 0. Since P has only one homogeneous slope (Corollary 4.42), we must have $v_K(a_j) \geq 0$ for all j so P actually has coefficients in $\mathfrak{o}_K$. By Lemma 5.5, there exists an admissible extension L of K over which the equation $P(\varphi_L)(x) = 0$ has a solution $x \in \mathfrak{o}_L^{\times}$. This solution gives rise to a factorization of $P(T)$ over $\mathfrak{o}_L\{T\}$ and we can repeat the construction to obtain an admissible extension, still denoted L, over which P splits into linear factors. This is nothing but a filtration of $L \otimes_K M$ by φ-modules such that the successive quotients all have rank 1. In fact, these quotients are all étale because every linear factor has coefficients in $\mathfrak{o}_L^{\times}$. Note that any étale φ-module of rank 1 is trivial so we have found a filtration of $L \otimes_K M$ by trivial φ-modules. We now enlarge L using Lemma 5.3 to split this filtration, finishing the proof. □

5.2 The descent argument

We set up the descent argument and prove the remaining statements to finish the proof of Theorem 3.1.

Definition 5.7. *Let $R \longrightarrow S$ be a faithfully flat morphism of φ-rings and let M be a φ-module over R. Write $M_S = M \otimes_R S$ and let N_S be a φ-submodule of M_S. We say that N_S descends to R if there exists a φ-submodule N of M such that the image of $N \otimes_R S$ in M_S coincides with N_S.*

Proposition 5.8. *Let $R \longrightarrow S$ be a faithfully flat morphism of φ-rings which are integral domains. Write $S' = S \otimes_R S$ and define $i_1, i_2 : S \longrightarrow S'$ by $i_1(s) = s \otimes 1$ and $i_2(s) = 1 \otimes s$. Let M be an R-module equipped with an isomorphism $\varphi^* M \longrightarrow M$ and write $M_S = M \otimes_R S$ as in the above definition. A φ-submodule N_S of M_S descends to R if and only if $N \otimes_{i_1,R} S' = N \otimes_{i_2,R} S'$ inside $M \otimes_R S'$. Moreover, if this occurs, then N is the unique R-submodule of M with $N_S = N \otimes_R S$ which is finite locally free and comes equipped with an isomorphism $\varphi^* N \longrightarrow N$.*

Proof. See [11, Exposé VIII, Corollaire 1.3] for the descent of N_S as a module, op. cit. [Corollaire 1.2] for the action of φ and op. cit. [Proposition 1.10] for the finite local freeness. □

Remark 5.9. *Note that if R is Bézout, then the R-submodule N will in fact be a φ-module over R because finite locally free modules over a Bézout domain are free.*

We introduce another triplet of rings. Let L be the field constructed in Proposition 5.6.

Definition 5.10. *Define*

$$\mathcal{S}^{\mathrm{int}} = \tilde{\mathcal{R}}_L^{\mathrm{int}} \otimes_{\mathcal{R}^{\mathrm{int}}} \tilde{\mathcal{R}}_L^{\mathrm{int}}, \quad \mathcal{S}^{\mathrm{b}} = \tilde{\mathcal{R}}_L^{\mathrm{b}} \otimes_{\mathcal{R}^{\mathrm{b}}} \tilde{\mathcal{R}}_L^{\mathrm{b}}, \quad \mathcal{S} = \tilde{\mathcal{R}}_L \otimes_{\mathcal{R}} \tilde{\mathcal{R}}_L$$

Remark 5.11. *S^{b} carries an $\mathfrak{m}_K$-adic valuation extending the $\mathfrak{m}_K$-adic valuation w on $\tilde{\mathcal{R}}_L^{\mathrm{b}}$. On simple tensors it satisfies $w(x \otimes y) = w(x) + w(y)$ (cf. [2, Lemma 7]).*

Notation. *We write $[0,1)_{\mathbb{Q}} = [0,1) \cap \mathbb{Q}$ to make the following results look slightly nicer.*

Proposition 5.12. *There exists a continuous, $\mathcal{R}$-linear map $f : \tilde{\mathcal{R}}_L \longrightarrow \mathcal{R}$ which sections the inclusion $\mathcal{R} \longrightarrow \tilde{\mathcal{R}}_L$. If $r_0 > 0$ is as in Remark 1.18, then we have, for any $r \in (0, r_0)$ and any $x \in \tilde{\mathcal{R}}_L^r$,*

$$|x|_r = \sup_{\alpha \in [0,1)_{\mathbb{Q}}, a \in L^\times} \{|a|^{-1} e^{-r\alpha} |f(au^{-\alpha}x)|_r\}. \tag{5.2.1}$$

Proof. The idea is more or less the same as in the proof of Lemma 4.18 in that we define compatible maps f_n for any $n \geq 1$ which induce a map f with the desired properties. Fix a basis $\overline{B}$ of κ_L over κ_K containing 1 and lift it to a subset B of $\mathfrak{o}_L$ containing 1. We also fix a uniformizer π of K. Note that for any $n \geq 1$, the ring $\tilde{\mathcal{R}}_L^{\mathrm{int}}/\pi^n\tilde{\mathcal{R}}_L^{\mathrm{int}}$ is naturally isomorphic to $(\mathfrak{o}_L/\pi^n\mathfrak{o}_L)(\!(u^{\mathbb{Q}})\!)$. Indeed, if we truncate elements of $\tilde{\mathcal{R}}_L^{\mathrm{int}}$ modulo π^n, then we just get Hahn series with the coefficients reduced modulo π^n. In particular, we can write $x \in \tilde{\mathcal{R}}_L^{\mathrm{int}}/\pi^n\tilde{\mathcal{R}}_L^{\mathrm{int}}$ as $x = \sum_{i \in \mathbb{Q}} a_i u^i$ with $a_i \in \mathfrak{o}_L/\pi^n\mathfrak{o}_L$. Since any $i \in \mathbb{Q}$ can be written uniquely as $i = \alpha + n$ for some $n \in \mathbb{Z}$, we may then write

$$x = \sum_{i \in \mathbb{Q}} a_i u^i = \sum_{\alpha \in [0,1)_{\mathbb{Q}}} \Big(\sum_{n \in \mathbb{Z}} a_{\alpha+n} u^n\Big) u^\alpha = \sum_\alpha x_\alpha u^\alpha.$$

We use the morphism ψ from Lemma 4.18 to embed $\mathcal{R}^{\mathrm{int}}/\pi^n\mathcal{R}^{\mathrm{int}}$ into $\tilde{\mathcal{R}}_L^{\mathrm{int}}/\pi^n\tilde{\mathcal{R}}_L^{\mathrm{int}}$ (i.e. we identify the variable t with $\psi(t)$). In particular, the formula for the r-norms in the proposition holds at best for $r \in (0, r_0)$ since this is the only range where we know that ψ preserves r-norms. It is easily seen that B is a basis of $(\mathfrak{o}_L/\pi^n\mathfrak{o}_L)(\!(t)\!)$ over $(\mathfrak{o}_K/\pi^n\mathfrak{o}_K)(\!(t)\!)$. If we write each x_α in this basis, then we see that x can be written uniquely as

$$x = \sum_{\alpha \in [0,1)_{\mathbb{Q}}} \sum_{b \in B} x_{\alpha,b} b u^\alpha \tag{5.2.2}$$

where each $x_{\alpha,b} \in \mathcal{R}^{\mathrm{int}}/\pi^n\mathcal{R}^{\mathrm{int}}$. By construction, this presentation has the following two properties:

- For each $\alpha \in [0,1)_{\mathbb{Q}}$, we have $x_{\alpha,b} \neq 0$ only for finitely many $b \in B$;
- If S_c denotes the set of $\alpha \in [0,1)_{\mathbb{Q}}$ for which there is at least one $b \in B$ such that the t-adic valuation of $x_{\alpha,b}$ is less than c (the t-adic valuation of $x_{\alpha,b}$ is well-defined because it is truncated modulo π^n), then S_c is well-ordered for all c and empty for c sufficiently small.

If x is presented as above, we let $f_n(x) = x_{0,1}$. This is well-defined and compatible with the transition maps by uniqueness of the presentation. Moreover, $f_n(b^{-1}u^{-\alpha}x) = x_{\alpha,b}$ for any α, b and $f_n(\lambda x) = \lambda f_n(x)$ for any $\lambda \in \mathcal{R}^{\mathrm{int}}/\pi^n\mathcal{R}^{\mathrm{int}}$. The latter equality uses implicitly that $f_n(\lambda) = f_n(\psi(\lambda)) = \lambda$. It follows that the map $f : \tilde{\mathcal{E}}_L^{\mathrm{int}} \longrightarrow \mathcal{E}^{\mathrm{int}}$ induced by the f_n sections the map ψ and is $\mathcal{E}^{\mathrm{int}}$-linear. By localizing, we obtain an $\mathcal{E}$-linear map $\tilde{\mathcal{E}}_L \longrightarrow \mathcal{E}$ sectioning ψ. If we can show that it is compatible with the r-norms in the sense of the proposition, then it restricts to an $\mathcal{R}^{\mathrm{b}}$-linear map $\tilde{\mathcal{R}}_L^{\mathrm{b}} \longrightarrow \mathcal{R}^{\mathrm{b}}$. This map then extends by continuity to an $\mathcal{R}$-linear map $f : \tilde{\mathcal{R}}_L \longrightarrow \mathcal{R}$ with the desired properties. By construction, the map $f : \tilde{\mathcal{E}}^{\mathrm{int}} \longrightarrow \mathcal{E}^{\mathrm{int}}$ satisfies $x = \sum_{\alpha,b} f(b^{-1}u^{-\alpha}x) b u^\alpha$ for all $x \in \tilde{\mathcal{E}}^{\mathrm{int}}$. As in [14, Lemma 2.2.19], one shows that if $x \in \tilde{\mathcal{R}}^{\mathrm{int}} \cap \tilde{\mathcal{R}}^r$ then also $f(b^{-1}u^{-\alpha}x) \in \mathcal{R}^{\mathrm{int}} \cap \mathcal{R}^r$ for all α, b. Note that $|\sum_{b \in B} r_b b| = \sup_b |r_b|$ for all $r_b \in \mathfrak{o}_K$ whence $|\sum_b f(b^{-1}u^{-\alpha}x) b|_r = \sup_b |f(b^{-1}u^{-\alpha}x)|_r$ for all α. Moreover, if $\alpha \neq \beta$, then the supports of $\sum_b f(b^{-1}u^{-\alpha}x) b u^\alpha$ and $\sum_b f(b^{-1}u^{-\beta}x) b u^\beta$ are disjoint. This gives $|x|_r = \sup_{b,\alpha}\{e^{-r\alpha}|f(b^{-1}u^{-\alpha}x)|_r\}$ and proves (5.2.1) (where $L^\times$ could be replaced by a smaller set). $\square$

Remark 5.13. • *Note that f preserves bounded elements, that is, f maps $\tilde{\mathcal{R}}_L^{\mathrm{b}}$ into $\mathcal{R}^{\mathrm{b}}$.*

- *Let $x \in \tilde{\mathcal{R}}^r$. The formula for $|x|_r$ together with Lemma 1.13 implies that $x \in \tilde{\mathcal{R}}_L^{\mathrm{b}}$ if and only if the quantity*
$$|a|^{-1} e^{-\alpha r} \lfloor f(au^{-\alpha} x)|_r$$
is uniformly bounded for all $a \in L^\times$, all $\alpha \in [0,1)_{\mathbb{Q}}$ and all $s \in (0,r]$.

Proposition 5.14. *The multiplication map $\mu : \tilde{\mathcal{R}}_L^{\mathrm{b}} \otimes_{\mathcal{R}^{\mathrm{b}}} \mathcal{R} \longrightarrow \tilde{\mathcal{R}}_L$, $x \otimes y \mapsto xy$, is injective.*

Proof. Assume that $x \neq 0$ lies in the kernel of μ. Write $x = \sum_{i=1}^n y_i \otimes z_i$ with $y_i \in \tilde{\mathcal{R}}_L^{\mathrm{b}}$, $z_i \in \mathcal{R}$ and n minimal. By Corollary 1.26, the z_i are linearly independent over $\mathcal{R}^{\mathrm{b}}$. Since $x \neq 0$, we may assume without loss of generality that $y_1 \neq 0$. Then $|y_1|_r \neq 0$ and as a consequence of (5.2.1) there must be $a \in L^\times$, $\alpha \in [0,1)_{\mathbb{Q}}$ with $f(au^{-\alpha} y_1) \neq 0$. But then $0 = \lfloor f(\mu(x))|_r = |\sum_{i=1}^n f(au^{-\alpha} y_1) z_i|_r$ using that f is $\mathcal{R}$-linear. In particular, $0 = \sum_{i=1}^n f(au^{-\alpha} y_i) z_i$ which is a non-trivial dependence relation over $\mathcal{R}^{\mathrm{b}}$ because f maps $\tilde{\mathcal{R}}_L^{\mathrm{b}}$ into $\mathcal{R}^{\mathrm{b}}$. This contradicts the linear independence of the z_i so μ must be injective. □

We now explain which faithfully flat ring homomorphisms we will be considering and why they are faithfully flat.

Proposition 5.15. *The morphisms $\mathcal{R}^{\mathrm{b}} \longrightarrow \mathcal{R}$, $\mathcal{R} \longrightarrow \tilde{\mathcal{R}}_L$, $\tilde{\mathcal{R}}_L^{\mathrm{b}} \longrightarrow \tilde{\mathcal{R}}_L$, $\mathcal{R}^{\mathrm{b}} \longrightarrow \tilde{\mathcal{R}}_L^{\mathrm{b}}$ and $\mathcal{R}^{\mathrm{int}} \longrightarrow \tilde{\mathcal{R}}_L^{\mathrm{int}}$ are all faithfully flat. Moreover,*
$$\tilde{\mathcal{R}}_L^{\mathrm{b}} \otimes_{\mathcal{R}^{\mathrm{b}}} \tilde{\mathcal{R}}_L^{\mathrm{b}} \hookrightarrow \tilde{\mathcal{R}}_L^{\mathrm{b}} \otimes_{\mathcal{R}^{\mathrm{b}}} \tilde{\mathcal{R}}_L \simeq (\tilde{\mathcal{R}}_L^{\mathrm{b}} \otimes_{\mathcal{R}^{\mathrm{b}}} \mathcal{R}) \otimes_{\mathcal{R}} \tilde{\mathcal{R}}_L \hookrightarrow \tilde{\mathcal{R}}_L \otimes_{\mathcal{R}} \tilde{\mathcal{R}}_L,$$
that is, $\mathcal{S}^{\mathrm{b}} \longrightarrow \mathcal{S}$ is injective.

Proof. $\mathcal{R}^{\mathrm{b}} \longrightarrow \mathcal{R}$ is faithfully flat by Proposition 1.23 because $\mathcal{R}^\times = (\mathcal{R}^{\mathrm{b}})^\times$. The morphism $\tilde{\mathcal{R}}_L^{\mathrm{b}} \longrightarrow \tilde{\mathcal{R}}_L$ is faithfully flat by the same argument. The inclusion $\mathcal{R} \longrightarrow \tilde{\mathcal{R}}_L$ is faithfully flat because $\tilde{\mathcal{R}}_L^\times = (\tilde{\mathcal{R}}_L^{\mathrm{b}})^\times$ and $(\tilde{\mathcal{R}}_L^{\mathrm{b}})^\times \cap \mathcal{R} = (\mathcal{R}^{\mathrm{b}})^\times = \mathcal{R}^\times$ by construction (cf. Lemma 4.57). Since this inclusion respects bounded and integral elements, the embeddings $\mathcal{R}^{\mathrm{b}} \longrightarrow \tilde{\mathcal{R}}_L^{\mathrm{b}}$ and $\mathcal{R}^{\mathrm{int}} \longrightarrow \tilde{\mathcal{R}}_L^{\mathrm{int}}$ are also faithfully flat. The final remark follows from Proposition 5.14 and the faithful flatness of $\tilde{\mathcal{R}}_L^{\mathrm{b}} \longrightarrow \tilde{\mathcal{R}}_L$ we just proved. □

To calculate on $\mathcal{S}$, we will need the following two-variable analog of the previous remark.

Lemma 5.16. *For $x \in \mathcal{S}$, we have $x \in \mathcal{S}^b$ if and only if for some $r > 0$, the quantity*
$$|ab|^{-1} e^{-\alpha s - \beta s} |(f \otimes f)((au^{-\alpha} \otimes bu^{-\beta})x)|_s \tag{5.2.3}$$
is uniformly bounded for all $s \in (0,r]$, all $a, b \in L^\times$, and all $\alpha, \beta \in [0,1)_{\mathbb{Q}}$.

Proof. If $x \in \mathcal{S}^b$, choose a presentation $x = \sum_{i=1}^n y_i \otimes z_i$ with $y_i, z_i \in \tilde{\mathcal{R}}_L^{\mathrm{b}}$. Using (5.2.1), we may bound $|a|^{-1} e^{-\alpha r} \lfloor f(au^{-\alpha} y_i)|_s$ for all $\alpha \in (0,1]_{\mathbb{Q}}$, all $a \in L^\times$ and all $s \in (0,r]$ for some $r > 0$ and we may do the same for the z_i. Since there are only finitely many i, we may choose these bounds, C and C', respectively, to work for all i simultaneously. Since f is $\mathcal{R}$-linear, we have
$$(f \otimes f)((au^{-\alpha} \otimes bu^{-\beta}) \sum_{i=1}^n y_i \otimes z_i) = \sum_{i=1}^n f(au^{-\alpha} y_i) f(bu^{-\beta} z_i).$$
We can thus bound (5.2.3) by CC' for all $\alpha, \beta \in [0,1)_{\mathbb{Q}}$, all $a, b \in L^\times$ and all $s \in (0,r]$.

Conversely, assume that $x \in \mathcal{S}$ and that (5.2.3) is bounded. We may also assume that $x \neq 0$. Choose a presentation $x = \sum_{i=1}^n y_i \otimes z_i$ with $y_i, z_i \in \tilde{\mathcal{R}}_L$ and n minimal. In particular, the z_i are linearly independent over $\mathcal{R}$ by Corollary 1.26. We proceed by induction on $n \geq 1$. Since $y_1 \neq 0$ by minimality of n, we can choose a, α with $f(au^{-\alpha}y_1) \neq 0$. The case $n = 1$ follows from (5.2.1) by inspection of (5.2.3) after first varying β, b for fixed α, a and then varying α, a for fixed β, b.

Now let $b = 1$ and $\beta = 0$ in (5.2.3). Then we see that

$$|a|^{-1}e^{-\alpha r}|(f \otimes f)(au^{-\alpha} \otimes 1)x)|_s = |a|^{-1}e^{-r\alpha}|\sum_{i=1}^{n} f(au^{-\alpha}y_i)f(z_i)|_s$$

is bounded for all $\alpha \in [0,1)_{\mathbb{Q}}$, all $a \in L^\times$ and all $s \in (0, r]$. Note that the quantity on the right is exactly what we get if we plug $x' = \sum_{i=1}^{n} f(au^{-\alpha}y_i)z_i$ into (5.2.1). It follows that $x' \in \tilde{\mathcal{R}}_L^{\mathrm{b}}$. Since $x' \neq 0$, this implies that x' is a unit in $\tilde{\mathcal{R}}_L$. Hence the ideal generated by the $f(au^{-\alpha}y_i)$ in $\mathcal{R}$ extends to the unit ideal in $\tilde{\mathcal{R}}_L$. Now both $\mathcal{R}$ and $\tilde{\mathcal{R}}_L$ are Bézout so the ideals generated by these elements are principal. By the previous observation, the generator in $\tilde{\mathcal{R}}_L$ is a unit, hence so must be the generator in $\mathcal{R}$ by Proposition 1.23. To summarize, the $f(au^{-\alpha}y_i)$ generate the unit ideal in $\mathcal{R}$ so Lemma 1.25 yields another presentation $x = \sum_{i=1}^{n} y_i' \otimes z_i'$ where $z_1' = \sum_{i=1}^{n} f(au^{-\alpha}y_i)z_i \in \tilde{\mathcal{R}}_L^{\mathrm{b}}$. As noted before, this element is nonzero.

Pick b, β such that $f(bu^{-\beta}z_1')$ is nonzero (use (5.2.1)). Then $f(bu^{-\beta}z_1')$ is a unit in $\mathcal{R}$ because it is an element of $\mathcal{R}^{\mathrm{b}}$ by Remark 5.13. Write $c_i = f(bu^{-\beta}z_i')/f(bu^{-\beta}z_1')$ for $i = 2, \dots, n$, and set

$$y_i'' = \begin{cases} y_1' + c_2y_2' + \cdots + c_ny_n' & i = 1 \\ y_i' & i > 1 \end{cases} \quad \text{and} \quad z_i'' = \begin{cases} z_i' & i = 1 \\ z_i' - c_iz_1' & i > 1 \end{cases}$$

We then have $x = \sum_{i=1}^{n} y_i'' \otimes z_i''$ by definition. Since f is $\mathcal{R}$-linear, we have $f(bu^{-\beta}z_i'') = 0$ for $i = 2, \dots, n$ so that $y_1''f(bu^{-\beta}z_1'') = \sum_{i=1}^{n} y_i''f(bu^{-\beta}z_i'') \in \tilde{\mathcal{R}}_L^{\mathrm{b}}$ (cf. Remark 5.13). Since $f(bu^{-\beta}z_1'') \in \mathcal{R}^{\mathrm{b}}$, we deduce that $y_1'' \in \tilde{\mathcal{R}}_L^{\mathrm{b}}$. The induction hypothesis applied to $x - y_1'' \otimes z_1'' = \sum_{i=2}^{n} y_i'' \otimes z_i''$ now yields the claim. □

Proposition 5.17. *Let A be an $n \times n$ matrix with entries in $\mathcal{S}^{\mathrm{int}}$. If $\mathbf{v} \in \mathcal{S}^n$ satisfies $\mathbf{v} = A\varphi(\mathbf{v})$, then $\mathbf{v}$ has entries in $\mathcal{S}^{\mathrm{b}}$.*

Proof. For each entry $\mathbf{v}_i$ of $\mathbf{v}$, choose a presentation $\mathbf{v}_i \sum_j y_{ij} \otimes z_{ij}$ with $y_{ij}, z_{ij} \in \tilde{\mathcal{R}}_L$. As in the proof of Proposition 1.20 (or 4.21), after rescaling by a power of u, we may choose $r \in (0, r_0)$ such that each term in a presentation of the entries of A has entries in $\tilde{\mathcal{R}}_L^r$ and is bounded by 1 on the annulus $e^{-r} \leq |u| < 1$. After shrinking r if necessary, we can also assume that $y_{ij}, z_{ij} \in \tilde{\mathcal{R}}_L^r$ for all i, j. Choose $c > 0$ such that $|y_{ij}|_s, |z_{ij}|_s \leq c$ for all i, j and any $s \in [r/q, r]$. This is possible because we pick s from a closed interval (cf. Lemma 4.14). As usual, we have $|\varphi^m(y_{ij})|_{s/q^m} = |y_{ij}|_s \leq c$ and $|\varphi^m(z_{ij})|_{s/q^m} = |z_{ij}|_s \leq c$ for all i, j and any integer $m \geq 0$.

Claim. *Fix any $m \geq 0$. Then for all i, all $\alpha, \beta \in [0,1)_{\mathbb{Q}}$, all $a, b \in L^\times$ and all $s \in \left[r/q^{m+1}, r/q^m\right]$,*

$$|ab|^{-1}e^{-\alpha s - \beta s}|(f \otimes f)((au^{-\alpha} \otimes bu^{-\beta})\mathbf{v}_i)|_s \leq c^2.$$

If we can prove the claim, then we get the above inequality for all $s \in (0, r]$ by varying $m \geq 0$. This implies that $\mathbf{v}_i \in \mathcal{S}^{\mathrm{b}}$ by Lemma 5.16, proving that $\mathbf{v}$ has entries in $\mathcal{S}^{\mathrm{b}}$.

We prove the claim by induction on $m \geq 0$. For $m = 0$ the inequality holds by choice of c and Lemma 5.12. Explicitly, for any i, any α, β, any a, b, and all $s \in [r/q, r]$, we have

$$\begin{aligned} |ab|^{-1}e^{-\alpha s - \beta s}|(f \otimes f)((au^{-\alpha} \otimes bu^{-\beta})\mathbf{v}_i|_s &= |ab|^{-1}e^{-\alpha s - \beta s}|f(au^{-\alpha}y_i)f(bu^{-\beta}z_i)|_s \\ &= |a|^{-1}e^{-\alpha s}|f(au^{-\alpha}y_i)|_s|b|^{-1}e^{-\beta s}|f(bu^{-\beta}z_i)|_s \\ &\leq |y_i|_s|z_i|_s \leq c^2. \end{aligned}$$

Here we use in the last line the formula for the s-norm from Proposition 5.12 and the choice of c. For the induction step, we use that $A\mathbf{v} = \varphi(\mathbf{v})$. For any $s \in \left[r/q^{m+1}, q^m\right]$, all α, β and all a, b,

$$\begin{aligned}
|ab|^{-1} e^{-\alpha s - \beta s} |(f \otimes f)((au^{-\alpha} \otimes bu^{-\beta})\mathbf{v}_i)|_s &= |ab|^{-1} e^{-\alpha s - \beta s} |(f \otimes f)((au^{-\alpha} \otimes bu^{-\beta}) \sum_j A_{ij} \varphi(\mathbf{v}_i))|_s \\
&\leq |ab|^{-1} e^{-\alpha s - \beta s} \max_j |(f \otimes f)((au^{-\alpha} \otimes bu^{-\beta}) \varphi(\mathbf{v}_j))|_s \\
&= |ab|^{-1} e^{-\alpha s - \beta s} \max_j |(f \otimes f)((au^{-\alpha} \otimes bu^{-\beta}) \mathbf{v}_j|_{sq} \\
&\leq c^2.
\end{aligned}$$

Here we use in the second line that the entries of A are bounded by 1 and in the last line the induction hypothesis. This finishes the induction and proves the claim. □

Remark 5.18.
- *One can do without the induction and just use that* $\mathbf{v} = A\varphi(\mathbf{v})$ *implies* $\mathbf{v} = A\varphi(A) \cdots \varphi^{m-1}(A)\varphi^m(\mathbf{v})$ *for any* $m \geq 1$*, but this makes the calculation a bit less apparent.*
- *The proposition implies that* $\mathrm{Hom}_{\mathcal{S}^{\mathrm{b}},\varphi}(M, N) \longrightarrow \mathrm{Hom}_{\mathcal{S},\varphi}(M \otimes_{\mathcal{S}^{\mathrm{b}}} \mathcal{S}, N \otimes_{\mathcal{S}^{\mathrm{b}}} \mathcal{S})$ *is bijective. Indeed, we can argue as in the proof of Proposition 2.48.*

Now we can finally finish the proof of the slope filtration theorem.

Theorem 5.19. *Let* M *be a semistable* φ*-module over* $\mathcal{R}$*. Then* $M_L = M \otimes_{\mathcal{R}} \tilde{\mathcal{R}}_L$ *is semistable.*

Proof. Assume that M_L is not semistable and let $0 = M_{L,0} \subsetneq \cdots \subset M_{L,l} = M_L$ be its HN filtration. We wish to show that $M_{L,1}$ descends to $\mathcal{R}$ along the faithfully flat inclusion $\mathcal{R} \longrightarrow \tilde{\mathcal{R}}_L$. To do so, we prove that $M_{L,1} \otimes_{i_2} \mathcal{S} \subseteq M_{L,j} \otimes_{i_1} \mathcal{S}$ for $j = l, l-1, \ldots, 1$ by descending induction. The case $j = l$ is clear because $M_{L,l} = M$. Now assume that $M_{L,1} \otimes_{i_2} \mathcal{S} \subseteq M_{L,j} \otimes_{i_1} \mathcal{S}$ for some $j > 1$. We then have a homomorphism of φ-modules over $\mathcal{S}$,

$$M_{L,1} \otimes_{i_2} \mathcal{S} \longrightarrow M_{L,j} \otimes_{i_1} \mathcal{S} \longrightarrow (M_{L,j}/M_{L,j-1}) \otimes_{i_1} \mathcal{S}. \tag{5.2.4}$$

By definition of the HN filtration, both $M_{L,1}$ and $M_{L,j}/M_{L,j-1}$ are semistable and hence pure (cf. Theorem 4.29). Moreover, we have $\mu(M_{L,1}) < \mu(M_{L,j}/M_{L,j-1})$. By Theorem 2.58, $M_{L,1}$ and $M_{L,j}/M_{L,j-1}$ descend to $\tilde{\mathcal{R}}_L^{\mathrm{b}}$. Write $M_{L,1} = M_{L,1}^{\mathrm{b}} \otimes_{\tilde{\mathcal{R}}_L^{\mathrm{b}}} \tilde{\mathcal{R}}_L$ and $M_{L,j}/M_{L,j-1} = (M_{L,j}/M_{L,j-1})^{\mathrm{b}} \otimes_{\tilde{\mathcal{R}}_L^{\mathrm{b}}} \tilde{\mathcal{R}}_L$. Now Proposition 5.17 implies that also the homomorphism $M_{L,1} \otimes_{i_2} \mathcal{S} \longrightarrow (M_{L,j}/M_{L,j-1}) \otimes_{i_1} \mathcal{S}$ descends (i.e. arises via base change along $(\,\cdot\,) \otimes_{\mathcal{S}^{\mathrm{b}}} \mathcal{S}$). Therefore, it suffices to show that the homomorphism $M_{L,1}^{\mathrm{b}} \otimes_{i_2^{\mathrm{b}}} \mathcal{S}^{\mathrm{b}} \longrightarrow (M_{L,j}/M_{L,j-1})^{\mathrm{b}} \otimes_{i_1^{\mathrm{b}}} \mathcal{S}^{\mathrm{b}}$ is trivial. Since $(M_{L,1}^{\mathrm{b}})^\vee \otimes_{\tilde{\mathcal{R}}_L^{\mathrm{b}}} (M_{L,j}/M_{L,j-1})^{\mathrm{b}}$ is pure of positive slope and $\mathcal{S}^{\mathrm{b}}$ carries an $\mathfrak{m}_K$-adic valuation extending that of $\tilde{\mathcal{R}}_L^{\mathrm{b}}$ (cf. Remark 5.11), one can argue as in Lemma 2.60 and Corollary 2.61. It follows that $M_{L,1} \otimes_{i_2} \mathcal{S} \subseteq M_{L,j-1} \otimes_{i_1} \mathcal{S}$, completing the induction. In particular, $M_{L,1} \otimes_{i_2} \mathcal{S} \subseteq M_{L,1} \otimes_{i_1} \mathcal{S}$ and we have equality by a symmetric argument. Thus, $M_{L,1}$ descends to a φ-submodule M_1 of M (cf. Proposition 5.8). But then $\mu(M_1) = \mu(M_{L,1}) < \mu(M_L) = \mu(M)$ so M is not semistable either. □

Remark 5.20. *Although we do not need it, the* φ*-submodule* M_1 *of* M *obtained by descending* $M_{L,1}$ *to* $\mathcal{R}$ *is the first step of the HN filtration of* M*. See also the proof of [14, Theorem 6.4.1].*

Theorem 5.21. *Let* M *be a* φ*-module over* $\mathcal{R}$*. If* $M_L = M \otimes_{\mathcal{R}} \tilde{\mathcal{R}}_L$ *is pure, then so is* M*.*

Proof. We have $\mu(M) = \mu(M_L)$. Applying $[\mathrm{rk}(M)]_*$ and invoking Lemma 2.55, we may assume that $\mu(M) \in \mathbb{Z}$. By twisting, we can then assure that $0 = \mu(M) = \mu(M_L)$. In other words, it suffices to show that if M_L is étale then so is M. By Proposition 4.26, M_L is obtained by base change from an étale φ-module over L which must be trivial by choice of L (cf. Proposition 5.6). Hence we can find a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of M_L on which φ acts by the $n \times n$ identity matrix E_n.

Claim. *M descends to $\mathcal{R}^{\mathrm{b}}$.*

Proof of Claim. It seems that we cannot quite use Proposition 5.8, but we can argue as follows. Let N' be the $\tilde{\mathcal{R}}_L^{\mathrm{b}}$-submodule of M_L generated by $\mathbf{v}_1, \dots, \mathbf{v}_n$, that is, N' is the φ-module over $\tilde{\mathcal{R}}_L^{\mathrm{b}}$ obtained by descending M_L to $\tilde{\mathcal{R}}_L^{\mathrm{b}}$ as in Proposition 2.48. We have bases $\mathbf{v}_1 \otimes_{i_1} 1, \dots, \mathbf{v}_n \otimes_{i_1} 1$ and $\mathbf{v}_1 \otimes_{i_2} 1, \dots, \mathbf{v}_n \otimes_{i_2} 1$ of $M \otimes_{i_1} \mathcal{S}$ and $M \otimes_{i_2} \mathcal{S}$ (using the notation of Proposition 5.8). Let U be the change-of-basis matrix (with entries in $\mathcal{S}$) from the basis $\mathbf{v}_1 \otimes_{i_1} 1, \dots, \mathbf{v}_n \otimes_{i_1} 1$ to the basis $\mathbf{v}_1 \otimes_{i_2} 1, \dots, \mathbf{v}_n \otimes_{i_2} 1$, that is,

$$\mathbf{v}_j \otimes_{i_1} 1 = \sum_{i=1}^{n} U_{ij}(\mathbf{v}_i \otimes_{i_2} 1)$$

for all $j = 1, \dots, n$. This may be rewritten as $E_n \otimes_{i_1} 1 = U^{-1}(E_n \otimes_{i_2} 1)\varphi(U)$ which implies that $U = \varphi(U)$. By Proposition 5.17 applied to each column of U separately, U must have entries in $\mathcal{S}^{\mathrm{b}}$. Now note that $M_L^\vee = M^\vee \otimes_{\mathcal{R}} \tilde{\mathcal{R}}_L$ is also étale so that the same argument with M replaced by $M^\vee$ and $\mathbf{v}_1, \dots, \mathbf{v}_n$ replaced by the dual basis shows that U^{-1} also has entries in $\mathcal{S}^{\mathrm{b}}$.

By the above calculation, the images of N' under the two maps $M_L \longrightarrow M \otimes_{i_j} \mathcal{S}$, $j = 1, 2$, generate the same $\mathcal{S}^{\mathrm{b}}$-submodule. We obtain an induced descent datum on N' with respect to the faithfully flat morphism $\mathcal{R}^{\mathrm{b}} \longrightarrow \tilde{\mathcal{R}}_L^{\mathrm{b}}$. Thus, by [20, Proposition 35.3.9], $N' = N \otimes_{\mathcal{R}^{\mathrm{b}}} \tilde{\mathcal{R}}_L^{\mathrm{b}}$ where N is the difference kernel of the two maps. By the canonical descent of $M \otimes_{\mathcal{R}} \tilde{\mathcal{R}}_L$, the $\mathcal{R}^{\mathrm{b}}$-module N is an $\mathcal{R}^{\mathrm{b}}$-submodule of M. By construction, we now have

$$M \otimes_{\mathcal{R}} \tilde{\mathcal{R}}_L = M_L = N' \otimes_{\tilde{\mathcal{R}}_L^{\mathrm{b}}} \tilde{\mathcal{R}}_L = (N \otimes_{\mathcal{R}^{\mathrm{b}}} \tilde{\mathcal{R}}_L^{\mathrm{b}}) \otimes_{\tilde{\mathcal{R}}_L^{\mathrm{b}}} \tilde{\mathcal{R}}_L = (N \otimes_{\mathcal{R}^{\mathrm{b}}} \mathcal{R}) \otimes_{\mathcal{R}} \tilde{\mathcal{R}}_L.$$

Since $\mathcal{R} \longrightarrow \tilde{\mathcal{R}}_L$ is faithfully flat, we deduce that $N \otimes_{\mathcal{R}^{\mathrm{b}}} \mathcal{R} \longrightarrow M$ is an isomorphism. $\square$

Let N be the φ-module obtained by descending M to $\mathcal{R}^{\mathrm{b}}$. We now have access to Lemma 2.51 to construct an étale lattice of N (hence of M) as follows. Choose any $\mathcal{R}^{\mathrm{b}}$-basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ of N and let P be the $\mathcal{R}^{\mathrm{int}}$-span of the images of the basis elements under powers of φ_M. Write $\mathbf{e}_1, \dots, \mathbf{e}_n$ in terms of the $\mathbf{v}_i$ (with coefficients in $\mathcal{R}^{\mathrm{b}}$). Since φ acts by the identity matrix on the basis $\mathbf{v}_1, \dots, \mathbf{v}_n$, we see that P is contained in the $\mathcal{R}^{\mathrm{int}}$-submodule of N generated by the basis $\mathbf{e}_1, \dots, \mathbf{e}_n$. It follows that P is a φ_M-stable $\mathcal{R}^{\mathrm{int}}$-lattice in M. Hence $P \otimes_{\mathcal{R}^{\mathrm{int}}} \tilde{\mathcal{R}}_L^{\mathrm{int}}$ is a φ_{M_L}-stable $\tilde{\mathcal{R}}_L^{\mathrm{int}}$-submodule of the étale φ-module M_L and so is a φ-module over $\tilde{\mathcal{R}}_L^{\mathrm{int}}$ by Lemma 2.51. Now the inclusion $\mathcal{R}^{\mathrm{int}} \longrightarrow \tilde{\mathcal{R}}_L^{\mathrm{int}}$ is faithfully flat so that P must itself be a φ-module over $\mathcal{R}^{\mathrm{int}}$ (if $\varphi^*(P \otimes_{\mathcal{R}^{\mathrm{int}}} \tilde{\mathcal{R}}^{\mathrm{int}}) \longrightarrow P \otimes_{\mathcal{R}^{\mathrm{int}}} \tilde{\mathcal{R}}^{\mathrm{int}}$ is an isomorphism, then so must be $\varphi^* P \longrightarrow P$). Hence P is an étale lattice for M. $\square$

6 Triangguline representations

The original goal of this section was to define trianguline representations and talk about the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. However, to do this in some detail one has to introduce a lot of terminology and theory having very little to do with the slope filtration theorem. It therefore seems more reasonable to instead focus on the impact of the theorem on p-adic Hodge theory and especially the works of Berger on (φ, Γ)-modules. This is what we will do in the first subsection. What was originally meant to be a section on trianguline representations has been condensed into a very brief second subsection culminating in the statement of the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

Notation. *Fix a prime number p . Throughout this section, K denotes a finite extension of $\mathbb{Q}_p$ and v_p denotes the p-adic valuation. We write $\mathcal{R} = \mathcal{R}_K$ for the Robba ring with coefficients in K and denote by φ the Frobenius lift $t \mapsto (1+t)^p - 1$. $\mathcal{R}$ is endowed with an action of the group $\Gamma = \mathbb{Z}_p^\times$ via $([\gamma]f)(t) = f((1+t)^\gamma - 1)$ for $\gamma \in \Gamma$.*

Remark 6.1. *In some references the assumption $p \neq 2$ is made, but it seems that all statements also hold for $p = 2$. It should therefore be of no harm to include this case.*

6.1 (φ, Γ)-modules and p-adic representations

One of the original goals of p-adic Hodge theory is to distinguish those p-adic Galois representations which arise as étale cohomology of varieties over K. For this purpose, Fontaine introduced the rings of periods $\mathbb{B}_{\mathrm{cris}}$, $\mathbb{B}_{\mathrm{st}}$ and $\mathbb{B}_{\mathrm{dR}}$, to define and study semistable, crystalline and de Rham representations. We will explain this briefly and outline the progress made in this theory using the slope filtration theorem.

Definition 6.2. *A (φ, Γ)-module over $\mathcal{R}$ is a φ-module M over $\mathcal{R}$, endowed with a semilinear continuous action of the group Γ which is compatible with φ_M. We say that M is étale (resp. pure of slope $s \in \mathbb{Q}$) if the underlying φ-module is étale (resp. pure of slope s).*

An important example is the following. It is similar to the twists $\mathcal{R}(n)$ we considered earlier.

Example 6.3. *Let $\delta : \mathbb{Q}_p^\times \longrightarrow K^\times$ be a character (i.e a continuous group homomorphism). We define $\mathcal{R}(\delta)$, called the twist of $\mathcal{R}$ by δ, to be the φ-module of rank 1 with generator $\mathbf{e}_\delta$ satisfying $\varphi(\mathbf{e}_\delta) = \delta(p)\mathbf{e}_\delta$. It becomes a (φ, Γ)-module via $[a](\mathbf{e}_\delta) = \delta(a)\mathbf{e}_\delta$ for $a \in \Gamma$. By definition, the slope of $\mathcal{R}(\delta)$ is $\mu(\mathcal{R}(\delta)) = v_p(\delta(p))$. It follows that $\mathcal{R}(\delta)$ is étale if and only if $\delta(p) \in \mathfrak{o}_E^\times$. One may similarly define the twist of any (φ, Γ)-module by δ.*

Notation. *We write $u(\delta) = \mu(\mathcal{R}(\delta)) = v_p(\delta(p))$.*

We now describe what we mean by a p-adic Galois representation.

Definition 6.4. *A p-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p|K)$ is a finite dimensional $\mathbb{Q}_p$-vector space V, equipped with a continuous $\mathbb{Q}_p$-linear action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p|K)$ where V is given the topology induced from $\mathbb{Q}_p$ and $\mathrm{Gal}(\overline{\mathbb{Q}}_p|K)$ carries the profinite topology. Equivalently, a p-adic representation is a finite dimensional $\mathbb{Q}_p$-vector space V together with a group homomorphism $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}_p|K) \longrightarrow \mathrm{GL}_{\mathbb{Q}_p}(V)$ where $\mathrm{GL}_{\mathbb{Q}_p}(V)$ denotes the group of $\mathbb{Q}_p$-linear automorphisms of V.*

Fontaine constructed the rings of periods $\mathbb{B}_{\mathrm{cris}}, \mathbb{B}_{\mathrm{st}}$ and $\mathbb{B}_{\mathrm{dR}}$ to sort through p-adic representations. All three rings are $\mathbb{Q}_p$-algebras equipped with an action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$ and each have certain additonal structures. A very good overview and some historical background of this is given in [3]. One now defines crystalline, semistable and de Rham representations. We follow [5, §3.1]. Let $K_0 = W(\kappa_K)[1/p]$ be the maximal unramified subextension of $K|\mathbb{Q}_p$.

Definition 6.5. *If V is a p-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p|K)$ and $* \in \{\mathrm{cris}, \mathrm{st}, \mathrm{dR}\}$, set $D_*(V) = (\mathbb{B}_* \otimes_{\mathbb{Q}_p} V)^{\mathrm{Gal}(\overline{\mathbb{Q}}_p|K)}$ where $(\,\cdot\,)^{\mathrm{Gal}(\overline{\mathbb{Q}}_p|K)}$ denotes the elements invariant under the group action. Then $D_{\mathrm{cris}}(V)$ (resp. $D_{\mathrm{st}}(V)$, resp. $D_{\mathrm{dR}}(V)$) is a vector space over K_0 (resp. K_0, resp. K) of dimension at most $\dim_{\mathbb{Q}_p}(V)$. We say that V is crystalline (resp. semistable; resp. de Rham) if we have equality of dimensions for $* = \mathrm{cris}$ (resp. $* = \mathrm{st}$; resp. $* = \mathrm{dR}$).*

Remark 6.6.

- *We say that a p-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p|K)$ is potentially semistable if its restriction to $\mathrm{Gal}(\overline{\mathbb{Q}}_p|L)$ is semistable for some finite extension L of K. Potentially semistable representations are always de Rham.*
- *If X is a proper smooth variety over K, then the étale cohomology groups $H^i_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p)$ are p-adic representations. One can shown that these are de Rham (cf. [3, §IV.5]).*

- *Our definition of a semistable φ-module and the above definition of a semistable p-adic representation are not directly related!*

Berger later constructed another ring of periods, denoted $\tilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}}$ (roughly comparable to $\tilde{\mathcal{R}}$), which is endowed with an endomorphism φ and a commuting action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$. The inclusion of $\mathcal{R}$ into $\tilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}}$ is φ-equivariant and compatible with the action of Γ in an explicit way (cf. the discussion after [5, Definition 2.3]).

Definition 6.7. *If M is a (φ,Γ)-module over $\mathcal{R}$, then we let $V(M) = (\tilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\mathcal{R}} M)^{\varphi=1}$. Here $(\,\cdot\,)^{\varphi=1}$ denotes the elements invariant under the φ-action. This is a (finite or infinite-dimensional) $\mathbb{Q}_p$-vector space, endowed with an action of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p|K)$ *(loc. cit.).*

One has the following theorem which is a variant of results of Fontaine and Cherbonnier-Colmez (see [17, §24.2]) complemented by the slope filtration theorem.

Theorem 6.8 (Berger). *If M is an étale (φ,Γ)-module over $\mathcal{R}$, then $V(M) = (\tilde{\mathbb{B}}^{\dagger}_{\mathrm{rig}} \otimes_{\mathcal{R}} M)^{\varphi=1}$ is a p-adic representation of of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p|K)$. *The resulting functor induces an equivalence of categories between the categories of étale (φ,Γ)-modules over $\mathcal{R}$ and the category of p-adic representations of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p|K)$. *We denote the inverse functor by $V \mapsto D(V)$.*

Proof. See [17, Theorem 24.2.8] or [5, Theorem 1.3]. □

The essential influence of the slope filtration theorem to p-adic Hodge theory is that it allows one to study p-adic representations via their associated (φ,Γ)-modules (note that the functor D also has an explicit description). Berger has advanced this theory tremendously by extending the definitions of crystalline and semistable representations to (φ,Γ)-modules and proving that these definitions are compatible with those of Fontaine (cf. [5, Theorem 3.1]). One obtains as a corollary an affirmative answer to a former conjecture of Fontaine, the p-adic monodromy conjecture.

Corollary 6.9. *Every de Rham representation is potentially semistable.*

Proof. See [17, Corollary 24.4.5]. □

6.2 2-dimensional trianguline representations

We have already seen that the slope filtration theorem has lead to many advancements in the theory of (φ,Γ)-modules. We will now look at what one can do with (φ,Γ)-modules. Essentially all results in this section are due to Colmez. We follow Berger's survey [5] and explain how one can construct a parameter space for all (irreducible) 2-dimensional trianguline representations and conclude with a few words on the the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$.

Definition 6.10. *A p-adic representation V of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p|K)$ *is called trianguline if $D(V)$ is a successive extension of (φ,Γ)-modules of rank 1 over $\mathcal{R}$.*

This means that there should exist a basis of $D(V)$ such that the representing matrices of $\varphi_{D(V)}$ and of the elements of Γ are all upper triangular. One can classify all (φ,Γ)-modules of rank 1 over $\mathcal{R}$ and the possible extensions between them explicitly as follows.

Theorem 6.11. *We denote by x the character $\mathbb{Q}_p^\times \longrightarrow K^\times$ induced by the inclusion $\mathbb{Q}_p \subseteq K$, and by $|\cdot|_p$ the character which sends $z \in \mathbb{Q}_p^\times$ to $p^{-v_p(z)}$.*

1. *If M is a (φ, Γ)-module of rank 1 over $\mathcal{R}$, then there exists a unique character $\delta : \mathbb{Q}_p^\times \longrightarrow K^\times$ such that $M \simeq \mathcal{R}(\delta)$.*

2. *If $\delta_1, \delta_2 : \mathbb{Q}_p^\times \longrightarrow K^\times$ are two characters, then $\mathrm{Ext}(\mathcal{R}(\delta_2), \mathcal{R}(\delta_1))$ is a 1-dimensional K-vector space, unless $\delta_1\delta_2^{-1}$ is either of the form x^{-i} for some integer $i \geq 0$ or of the form $|x|_p x^i$ for some integer $i \geq 1$. In the latter case, $\mathrm{Ext}(\mathcal{R}(\delta_2), \mathcal{R}(\delta_1))$ is 2-dimensional.*

Proof. The theorem is stated as a forward reference in [7, Théorème 0.2]. The proofs can be found in op. cit. [Proposition 3.1] and [Théorème 2.9]. □

Remark 6.12. *Part (2.) of the theorem implies that there is either only one nonsplit extension of $\mathcal{R}(\delta_1)$ by $\mathcal{R}(\delta_2)$ or the set of such extensions is parametrized by the projective line $\mathbb{P}^1(K)$. The parameter of such an extension is called the $\mathcal{L}$-invariant.*

In particular, if V is a 2-dimensional trianguline representation, then there exists a short exact sequence of (φ, Γ)-modules

$$0 \longrightarrow \mathcal{R}(\delta_1) \longrightarrow D(V) \longrightarrow \mathcal{R}(\delta_2) \longrightarrow 0$$

That is, $D(V)$ is determined by two characters $\delta_1, \delta_2 : \mathbb{Q}_p^\times \longrightarrow K^\times$ and an $\mathcal{L}$-invariant. Since $D(V)$ comes from the p-adic representation V, we also know that $D(V)$ is étale (cf. Theorem 6.8). It follows from Lemma 2.30 that $u(\delta_1) + u(\delta_2) = 0$. Moreover, $u(\delta_1) \geq 0$ by Theorem 2.62. Given $s = (\delta_1, \delta_2, \mathcal{L})$, we write $D(s)$ for the étale (φ, Γ)-module associated to s in this way.

Remark 6.13. *If $u(\delta_1) = 0$, then $u(\delta_2) = 0$. In this case $\mathcal{R}(\delta_1)$ and $\mathcal{R}(\delta_2)$ are both étale so that the extension above corresponds to an extension of p-adic representations, that is, V is then itself an extension of two representations.*

Colmez proceeds to construct a parameter space for all 2-dimensional trianguline representations and determines, using the invariant $u(\delta)$ of a character $\delta : \mathbb{Q}_p^\times \longrightarrow E^\times$ and another invariant $w(\delta)$ (the weight of δ), when they are semistable, crystalline or non-geometric and when they are irreducible (cf. the introduction of [7]). Now one of the reasons one might be interested in trianguline representations is their appearance in the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. This is a correspondence between certain 2-dimensional p-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$ and certain (not necessarily finite dimensional) representations of $\mathrm{GL}_2(\mathbb{Q}_p)$. The first examples of such a correspondence were given by Breuil for special representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$ (see [4, §2.3]). To reproduce these examples in a functorial way, Colmez realized that the correct condition on the associated (φ, Γ)-modules was to be an extension of two (φ, Γ)-modules of rank 1 which lead him to study trianguline representations. To conclude, we present the aforementioned correspondence.

Theorem 6.14. *There is a functor $\Pi \mapsto V(\Pi)$, called the Montreal functor, from $\mathrm{Rep}_K(\mathrm{GL}_2(\mathbb{Q}_p))$ to $\mathrm{Rep}_K(\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p))$ which induces a bijection between the isomorphism classes of*

1. *absolutely irreducible non-ordinary $\Pi \in \mathrm{Ban}^{\mathrm{adm}}_{\mathrm{GL}_2(\mathbb{Q}_p)}(K)$;*

2. *2-dimensional absolutely irreducible continuous K-linear representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$.*

The rough strategy of the proof is the following (where the terminology and notation is explained in [10, §1]). Given an irreducible trianguline representation one can attach to it in a functorial way a so-called p-adic unitary Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ ([8, §V.1]).

One then attaches as in [9, §6] to any 2-dimensional p-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$ a representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ coinciding with the previous construction for trianguline representations. Since there are "enough" trianguline representations, it remains to show that the functor is suitably well-behaved and has the desired properties for trianguline representations (cf. [5, Theorem 4.1] and the discussion thereafter).

Remark 6.15. *The general idea is that certain n-dimensional representations of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$ *should correspond to certain representations of* $\mathrm{GL}_n(\mathbb{Q}_p)$*. Moreover, this correspondence should satisfy a list of nice properties to make it canonical/unique. For the p-adic local Langlands correspondence such a list does not yet seem to exist. That the correspondence of the theorem above satisfies a number of desirable properties is the content of [10].*

References

[1] Y. André. Slope Filtrations. *Confluentes Mathematici*, 01(01):1–85, 2009.

[2] I. Ben Yaacov. Tensor products of valued fields. *Bulletin of the London Mathematical Society*, 47(1):42–46, 2015.

[3] L. Berger. An introduction to the theory of p-adic representations. In W. de Gruyter, editor, *Geometric Aspects of Dwork Theory*, pages 255–292, 2001.

[4] L. Berger. La correspondance de Langlands locale p-adique pour $\mathrm{GL}_2(\mathbb{Q}_p)$. In *Séminaire Bourbaki*, number 339 in Astérisque. Société mathématique de France, 2011.

[5] L. Berger. Trianguline representations. *Bulletin of the London Mathematical Society*, 43(4):619–635, 2011.

[6] P. L. Clark. Commutative Algebra. http://math.uga.edu/~pete/integral2015.pdf, 2008.

[7] P. Colmez. Représentations triangulines de dimension 2. In *Représentations p-adiques de groupes p-adiques I : représentations galoisiennes et* (φ, Γ)*-modules*, number 319 in Astérisque, pages 213–258. Société mathématique de France, 2008.

[8] P. Colmez. La série principale unitaire de $\mathrm{GL}_2(\mathbb{Q}_p)$. In *Représentations p-adiques de groupes p-adiques II : Représentations de* $\mathrm{GL}_2(\mathbb{Q}_p)$ *et* (φ, Γ)*-modules*, number 330 in Astérisque, pages 213–262. Société mathématique de France, 2010.

[9] P. Colmez. Représentations de $\mathrm{GL}_2(\mathbb{Q}_p)$ et (φ, Γ)-modules. *Astérisque*, 330:283–511, 2010.

[10] P. Colmez, G. Dospinescu, and V. Paskunas. The p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. https://arxiv.org/pdf/1310.2235.pdf, 2013. Preprint.

[11] A. Grothendieck. *Revêtements étales et groupe fondamental (SGA 1)*, volume 224 of *Lecture notes in mathematics*. Springer-Verlag, 1971.

[12] U. Hartl and R. Pink. Vector bundles with a Frobenius structure on the punctured unit disc. *Compositio Mathematica*, 140:689–716, 2004.

[13] K. S. Kedlaya. A p-adic local monodromy theorem. *Annals of Mathematics*, 160:93–184, 2004.

[14] K. S. Kedlaya. Slope filtrations revisited. *Doc. Math.*, 10:447–525, 2005.

[15] K. S. Kedlaya. Finite automata and algebraic extensions of function fields. *Journal de Théorie des Nombres de Bordeaux*, 18:379–420, 2006.

[16] K. S. Kedlaya. Slope filtrations for relative Frobenius. In *Représentations p-adiques de groupes p-adiques I : représentations galoisiennes et* (φ, Γ)*-modules*, number 319 in Astérisque, pages 259–301. Société mathématique de France, 2008.

[17] K. S. Kedlaya. *p-adic Differential Equations*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2010.

[18] O. Ore. Theory of Non-Commutative Polynomials. *Annals of Mathematics*, 34(3):480–508, 1933.

[19] P. Schneider. Die Theorie des Anstiegs. https://ivv5hpp.uni-muenster.de/u/pschnei/publ/lectnotes/Theorie-des-Anstiegs.pdf, 2006/07. Lecture notes WWU Münster.

[20] The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu, 2018.

Versicherung an Eides statt

Ich versichere an Eides statt durch meine untenstehende Unterschrift,

- dass ich die vorliegende Arbeit - mit Ausnahme der Anleitung durch die Betreuer - selbstständig ohne fremde Hilfe angefertigt habe und
- dass ich alle Stellen, die wörtlich oder annähernd wörtlich aus fremden Quellen entnommen sind, entsprechend als Zitate gekennzeichnet habe und
- dass ich ausschließlich die angegebenen Quellen (Literatur, Internetseiten, sonstige Hilfsmittel) verwendet habe und
- dass ich alle entsprechenden Angaben nach bestem Wissen und Gewissen vorgenommen habe, dass sie der Wahrheit entsprechen und dass ich nichts verschwiegen habe.

Mir ist bekannt, dass eine falsche Versicherung an Eides Statt nach §156 und nach §163 Abs. 1 des Strafgesetzbuches mit Freiheitsstrafe oder Geldstrafe bestraft wird.

Ort, Datum

Unterschrift