Master's Thesis

Homotopy theory of simplicial sets

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1 Introduction

In this thesis, we will introduce the standard model structure on the category of simplicial sets in detail. This will equip us with the necessary prerequisites to prove the main results of the thesis that the category of simplicial sets forms a model category (Theorem 4.6.10) which is Quillen equivalent to the model category of topological spaces (Theorem 4.6.11). The modern simplicial homotopy theory has its origins in the mid-20th century thanks to the groundbreaking work by Daniel Kan in the 1950s and Daniel Quillen in the 1960s, who first stated and proved the theorems in [Qui67]. The two main references we will follow for the proofs are [GJ99] Chapter I and [Hov99] Chapter 3.

A simplicial set is a contravariant set valued functor from the simplex category, the category with objects of all finite non-empty ordered sets, with monotonous maps between them. These functors form a category whose morphisms are the natural transformations. In Chapter 2, we will introduce the category of simplicial sets in more detail and we will describe some important examples and basic properties. A simplicial set gives rise to a topological space formed by gluing topological simplices together along the abstract information given by the functor. This topological space is called the geometric realization of the simplicial set and gives rise to a functor between the two categories of simplicial sets and topological spaces, as defined in Section 2.2. This geometric realization functor leads to a fundamental adjunction, and later we will see that this is a Quillen equivalence between the model categories of simplicial sets and topological spaces.

In Chapter 3, we will give an overview of the categorical definitions as well as presenting the necessary properties for model structures and Quillen equivalences, needed later on in this thesis. A model structure (also called a Quillen model structure or a Quillen homotopy structure) on a given category consists of three classes of morphisms, the so called fibrations, cofibrations and weak equivalences, satisfying a certain list of properties (cf. Definition 3.1.1). A model category, first introduced by Quillen in 1967, is a complete and cocomplete category endowed with a model structure (cf. Definition 3.1.5). A Quillen functor is an adjoint functor between two model categories, preserving half of the model structure, either fibrations and trivial fibrations (in case of a right adjoint), or cofibrations and trivial cofibrations (in case of a left adjoint; cf. Definition 3.1.10). Trivial fibrations are defined to be fibrations which are also weak equivalences; analogously for trivial cofibrations. A Quillen equivalence is a pair of adjoint Quillen functors compatible with the weak equivalences in a suitable way (cf. Definition 3.1.12). The notion of an equivalence of categories is fundamental in modern mathematics. A Quillen equivalence is something weaker, however, since it is not an equivalence between the categories themselves. Rather, it induces an equivalence between the corresponding homotopy categories of the two respective model categories. The homotopy category of a model category is obtained by formally inverting the morphism class of weak equivalences, also called the localization at the weak equivalences. A main feature of model

categories is that the additional structure given by the classes of fibrations and cofibrations allows us to describe the localization at the weak equivalences in a better way (cf. [Hov99] Theorem 1.2.10).
In Section 3.2, we will briefly introduce the model structure on the category of topological spaces, and subsequently arrive at the result (without proof) that this structure indeed satisfies the required properties and forms a model category (cf. Theorem 3.2.2).

In the last Chapter, we will construct the model structure on the category of simplicial sets in a stepwise procedure, by presenting the necessary morphism classes and techniques. This will be the main focus of this thesis, starting with the class of fibrations, the so called Kan fibrations, in Section 4.1. This definition of fibrations turns out to be natural, as we will see that fibrations in the category of topological spaces map to Kan fibrations under the fundamental adjunction. Ending this section, we will give some examples of fibrant simplicial sets. An object in a model category is called fibrant if the unique morphism to the terminal object is a fibration. In Section 4.2, we will introduce the next class of morphisms in the category of simplicial sets, the so called anodyne extensions. These morphisms play an important role in simplicial homotopy theory, as it turns out that these are exactly the trivial cofibrations of the model structure.
In the following two sections, we will introduce homotopies between morphisms of simplicial sets and function complexes. Historically, the latter turned out to be an important discovery since the concept provided an indispensable tool on the way to prove the axioms of a model structure, as we will frequently see throughout this thesis.
Next, using the knowledge about simplicial homotopies, we will introduce the concept of simplicial homotopy groups for fibrant simplicial sets in Section 4.5. As we will see in the following section, there are isomorphisms between the homotopy groups of a simplicial set and the topological homotopy groups of its geometric realization. This will lead us to the definition of weak equivalences in the category of simplicial sets, since a morphism between simplicial sets is defined to be a weak equivalence, if and only if its realization is in the category of topological spaces. Within this category, the class of weak equivalences is just the class of (weak) homotopy equivalences (cf. Section 3.2).
Finally, in the last Section 4.6, we will bring all the previous results together and prove Theorem 4.6.10 (i.e. we really get a model structure on the category of simplicial sets) and Theorem 4.6.11 (i.e. the model categories of simplicial sets and topological spaces are Quillen equivalent via the geometric realization functor).

This thesis presents a concise summary of simplicial homotopy theory, which is an important subject in modern algebraic topology, as well as in general abstract homotopy theory. The thesis aims to give a readable and detailed summary of most concepts needed for working with the model structure on the category of simplicial sets. On our way, we state all the necessary definitions and properties concerning model categories; no prior knowledge is needed.

I want to thank Professor Jan Kohlhaase, for his support and mentorship throughout my studies at the University of Duisburg-Essen. Without his guidance and tireless supervision, this thesis would not have been possible. I also wish to thank Professor Marc Levine and his former group at the University of Duisburg-Essen for their extraordinary effort in teaching, especially Dr. Tariq Syed and Dr. Daniel Harrer. To give just one example, they held an

extra-curricular seminar on higher homotopy theory upon students' requests. This seminar gave me a much deeper understanding and appreciation of Algebraic Topology.

2 The Category of Simplicial Sets

2.1 Basic definitions

In this very first chapter we define the category of simplicial sets, give some basic examples and show some essential properties. For example, there exists a fundamental adjunction between this category and the category of topological spaces. Therefore, we start by introducing the simplicial category, i.e., the category of finite ordinal numbers, denoted by $\mathbf{\Delta}$. More precisely, the objects of $\mathbf{\Delta}$ are finite totally ordered sets $\underline{n} := \{0 < 1 < \ldots < n\}$ for $n \geq 0$, $n \in \mathbb{N}_0$, whose morphisms are all order-preserving maps $f : \underline{n} \to \underline{m}$, i.e. all maps f such that for $i, j \in \underline{n}$ with $i \leq j$ we also have $f(i) \leq f(j)$. These are also called monotonous.

Among all of the morphisms $\underline{m} \to \underline{n}$ appearing in $\mathbf{\Delta}$ there are the following special ones.

Definition 2.1.1. *(i) For each $n > 0$ and $0 \leq i \leq n$, the i-th coface map is the unique monotonous injection $d^i : \underline{n-1} \hookrightarrow \underline{n}$ satisfying $i \notin d^i(\underline{n-1})$, i.e.*

$$d^i : \underline{n-1} \to \underline{n}, \quad d^i(k) = \begin{cases} k, & \text{if } 0 \leq k < i, \\ k+1, & \text{if } i \leq k \leq n-1. \end{cases}$$

(ii) For each $n \geq 0$ and $0 \leq i \leq n$, the i-th codegeneracy map is the unique monotonous surjection $s^i : \underline{n+1} \twoheadrightarrow \underline{n}$ satisfying $s^i(i) = s^i(i+1)$, i.e.

$$s^i : \underline{n+1} \to \underline{n}, \quad s^i(k) = \begin{cases} k, & \text{if } 0 \leq k \leq i, \\ k-1, & \text{if } i < k \leq n+1. \end{cases}$$

As a matter of fact, every morphism $f : \underline{n} \to \underline{m}$ in $\mathbf{\Delta}$ can be decomposed uniquely as $f = ds$, where $d : \underline{k} \to \underline{m}$ is injective, and $s : \underline{n} \to \underline{k}$ is surjective. More precisely, f even can be factored into coface and codegeneracy maps, i.e. as

$$f = d^{i_p} \circ \ldots \circ d^{i_1} \circ s^{j_1} \circ \ldots \circ s^{j_q},$$

where the non-negative integers p and q satisfy $n + p - q = m$ and the superscripts i and j satisfy

$$0 \leq i_1 < \ldots < i_p \leq m \text{ and } 0 \leq j_1 < \ldots < j_q \leq n+1.$$

Here the empty composite is taken to be the identity map. The maps d^j and s^i satisfy the following list of relations which are called the *cosimplicial identities*:

(i) $d^j d^i = d^i d^{j-1}, \quad i < j$

(ii) $s^j d^i = \begin{cases} d^i s^{j-1}, & i < j, \\ id, & i = j \text{ or } i = j+1, \\ d^{i-1} s^j, & i > j+1. \end{cases}$

(iii) $s^j s^i = s^i s^{j+1}, \quad i \leq j.$

Hence, the maps d^i, s^j and these relations can be viewed as a set of generators and relations of $\mathbf{\Delta}$.

Example 2.1.2. By **Top**, we denote the category of topological spaces with continuous maps between them as morphisms. We recall some notations from algebraic topology first. The topological standard n-simplex $\Delta_n \subseteq \mathbb{R}^{n+1}$ is the space

$$\Delta_n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} t_i = 1, t_i \geq 0\},$$

endowed with the subspace topology. These standard n-simplices Δ_n are homeomorphic to the n-th unit disc $D^n \subseteq \mathbb{R}^n$. The i-th vertex of Δ_n is the vector

$$e_i := (0, \dots 0, 1, 0, \dots, 0) \in \Delta_n$$

where $t_i = 1$ and $t_j = 0$ for $j \neq i$. The i-th face map is the map

$$d^i_\Delta : \Delta_{n-1} \to \Delta_n, \quad (t_0, \dots, t_{n-1}) \mapsto (t_0, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n)$$

and the j-th degeneracy map is

$$s^j_\Delta : \Delta_{n+1} \to \Delta_n, \quad (t_0, \dots, t_{n+1}) \mapsto (t_0, \dots, t_{i-1}, t_i + t_{i+1}, \dots, t_{n+1}).$$

We indicate these maps with Δ to avoid misunderstandings regarding previous notations. The i-th face is the image under d^i which is homeomorphic to Δ_{n-1}.
There is a standard covariant functor

$$r : \mathbf{\Delta} \to \mathbf{Top},$$

which is given on objects by $\underline{n} \mapsto \Delta_n$. On morphisms it is defined as follows. A map $\theta : \underline{n} \to \underline{m}$ in $\mathbf{\Delta}$ induces a map $\theta_* : \Delta_n \to \Delta_m$ in **Top** which is defined by

$$\theta_*(t_0, \dots, t_n) = (s_0, \dots, s_m),$$

where

$$s_i = \sum_{j \in \theta^{-1}(i)} t_j.$$

This, indeed, defines a functor. We also note that $(d^i)_* = d^i_\Delta$ and $(s^j)_* = s^j_\Delta$.

Definition 2.1.3. *A simplicial set is a functor*

$$X : \mathbf{\Delta}^{op} \to \boldsymbol{Set}$$

between the categories $\mathbf{\Delta}$ *and* $\boldsymbol{Set}$*, the category of sets. We denote the category of simplicial sets by* $\boldsymbol{sSet} := [\mathbf{\Delta}^{op}, \boldsymbol{Set}]$*. Its morphisms are the natural transformations.*

Remark 2.1.4. For a simplicial set X, we will write $X_n := X(\underline{n})$ and $f^* = X(f)$ for a map $f \in \mathrm{Hom}_{\mathbf{\Delta}}(\underline{n}, \underline{m})$. The set X_n is called the set of n*-simplices* or simplices of dimension n. Specifically, an element in X_0 is called a *vertex* of X. By the above remark, any map in $\mathbf{\Delta}$ may be uniquely factorized as a composition of codegeneracies and cofaces. Thus, in order to define a simplicial set X, it suffices to write down a collection of sets X_n, $n \geq 0$ together with maps

$$\begin{aligned} d_i &:= X(d^i) : X_n \to X_{n-1}, \qquad 0 \leq i \leq n, \quad n > 0 \qquad (faces) \\ s_j &:= X(s^j) : X_n \to X_{n+1}, \qquad 0 \leq j \leq n \qquad (degeneracies) \end{aligned}$$

satisfying the analogous relations, the so called *simplicial identities*:

(i) $d_i d_j = d_{j-1} d_i, \qquad i < j$

(ii) $d_i s_j = \begin{cases} s_{j-1} d_i, & i < j, \\ id, & i = j \text{ or } i = j+1, \\ s_j d_{i-1}, & i > j+1. \end{cases}$

(iii) $s_i s_j = s_{j+1} s_i, \qquad i \leq j.$

This is the classical way of describing the combinatorial data for a simplicial set X.

Example 2.1.5. (i) For two given simplicial sets $X, Y \in \mathbf{sSet}$ one can construct new simplicial sets termwise, corresponding to the product and coproduct in **Set**. The product $X \times Y$ of X and Y, now to be defined in the category **sSet**, is the simplicial set with

$$(X \times Y)_n := X_n \times Y_n, \text{ for all } n \geq 0.$$

The face and degeneracy maps are defined to be

$$\begin{aligned} d_i^{X \times Y} &:= d_i^X \times d_i^Y : (X \times Y)_n \to (X \times Y)_{n-1} \text{ and} \\ s_j^{X \times Y} &:= s_j^X \times s_j^Y : (X \times Y)_n \to (X \times Y)_{n+1}, \text{ for all } i, j \in \mathbb{N}_0. \end{aligned}$$

One checks easily that these maps satisfy the simplicial identities, by using the properties of d_i and s_j on X_n and Y_n, respectively. A similar construction can be done for the disjoint union. We define the simplicial set $X \sqcup Y$ via

$$(X \sqcup Y)_n := X_n \sqcup Y_n, \text{ for all } n \geq 0$$

and

$$d_i^{X \sqcup Y} := d_i^X \sqcup d_i^Y, \quad s_j^{X \sqcup Y} := s_j^X \sqcup s_j^Y.$$

We note that these constructions yield a product and coproduct in **sSet**, respectively, since they are constructed component wise from those in the category **Set**.

(ii) A similar construction can be made for pullbacks and pushouts. If $X, Y, Z \in \mathbf{sSet}$ are simplicial sets with morphisms $X \xrightarrow{f} Z \xleftarrow{g} Y$, then we have a pullback diagram

$$\begin{tikzcd} X_m \underset{Z_m}{\times} Y_m \arrow[r, "pr_2^m"] \arrow[d, "pr_1^m"'] & Y_m \arrow[d, "g_m"] \\ X_m \arrow[r, "f_m"'] & Z_m \end{tikzcd}$$

for each degree $\underline{m} \in \mathbf{\Delta}$. We define the simplicial set $(X \underset{Z}{\times} Y)$ pointwise via

$$(X \underset{Z}{\times} Y)_m := (X_m \underset{Z_m}{\times} Y_m), \text{ for each } m \geq 0,$$

and for a map $f : \underline{m} \to \underline{n}$ in $\mathbf{\Delta}$ we define

$$(X \underset{Z}{\times} Y)(f) : (X \underset{Z}{\times} Y)_n \to (X \underset{Z}{\times} Y)_m$$

to be the product map $\bar{f} = X(f) \times Y(f)$ received by the universal property of the diagram

$$\begin{tikzcd} X_n \underset{Z_n}{\times} Y_n \arrow[rr, bend left, "pr_2^n"] \arrow[d, bend right, "pr_1^n"'] \arrow[drr, dashed, "\bar{f}"] & & Y_n \arrow[dr, bend left, "Y(f)"] & \\ X_n \arrow[drr, bend right, "X(f)"'] & & X_m \underset{Z_m}{\times} Y_m \arrow[r] \arrow[d] & Y_m \arrow[d] \\ & & X_m \arrow[r] & Z_m \end{tikzcd}$$

in **Set**. If we define the natural transformations $p_1 := (p_1^n)_{n \in \mathbf{\Delta}}$ and $p_2 := (p_2^n)_{n \in \mathbf{\Delta}}$, then the simplicial set $X \underset{Z}{\times} Y$ satisfies the universal property of a pushout for the diagram

$$\begin{tikzcd} X \underset{Z}{\times} Y \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

in the category **sSet**. The universal property of the pullback follows again from the universal property in **Set** by using the universal property in each component. With an analogous construction one defines the pushout $X \underset{Z}{\sqcup} Y$ in **sSet**. Similar constructions also give arbitrary small (co)products.

Remark 2.1.6. The categories **Set** and **Top** are both complete and cocomplete, which means that all small limits and small colimits exist, since all pullbacks and (small) products and, respectively, all pushouts and (small) coproducts exist. Hence, so does the category **sSet** of simplicial sets by working component wise, as seen above.

Example 2.1.7. A main example of simplicial sets are the contravariant representable functors $\mathrm{Hom}_{\mathbf{\Delta}}(-,\underline{n})$. For a given $n \in \mathbb{N}_0$ we call this simplicial set the standard (combinatorial) n-simplex and denote it Δ^n. In other words, $\Delta^n \in \mathbf{sSet}$ is the functor

$$\begin{aligned} \Delta^n : \mathbf{\Delta}^{op} \to \mathbf{Set}, \quad \underline{m} \mapsto (\Delta^n)_m := \mathrm{Hom}_{\mathbf{\Delta}}(\underline{m},\underline{n}) \quad \text{and} \\ f : \underline{m} \to \underline{k} \mapsto (- \circ f) : (\Delta^n)_k \to (\Delta^n)_m. \end{aligned}$$

Yoneda's Lemma (see the following result) implies that simplicial maps $\Delta^n \to Y$ classify n-simplices of a given simplicial set Y in the sense that there is a natural bijection

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, Y) \cong Y_n = Y(\underline{n})$$

between the set Y_n and the set $\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, Y)$ of simplicial maps from Δ^n to Y.

Since we will use it very often, we will state Yoneda's Lemma once.

Lemma 2.1.8. ***(Yoneda Lemma, contravariant version)***
Let $\mathcal{C}$ be a locally small category and $X \in \mathcal{C}$ an object of $\mathcal{C}$. Furthermore, let $F \in \hat{\mathcal{C}} := [\mathcal{C}^{op}, \mathbf{Set}]$, and $h_X := \mathrm{Hom}_{\mathcal{C}}(-, X)$ be a representable functor in $\hat{\mathcal{C}}$. Then:

(i) The map

$$\mathrm{Hom}_{\hat{\mathcal{C}}}(h_X, F) \to F(X), \quad \mu \mapsto \mu_X(id_X)$$

is bijective and functorial in X, with inverse map

$$F(X) \to \mathrm{Hom}_{\hat{\mathcal{C}}}(h_X, F), \quad \xi \mapsto \mu^\xi = (\mu^\xi_Y)_{Y \in \mathcal{C}}$$

where

$$\mu^\xi_Y : h_X(Y) \to F(Y), \quad f \mapsto F(f)(\xi).$$

(ii) The Yoneda embedding $\mathcal{C} \to \hat{\mathcal{C}}$ with

$$X \mapsto h_X \text{ and } f : X \to Y \mapsto \mu^f := (f \circ -) : h_X \to h_Y$$

is fully faithful, i.e. for all $X, Y \in \mathcal{C}$ the map

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\hat{\mathcal{C}}}(h_X, h_y), \quad f \mapsto \mu^f$$

is bijective.

(iii) If $F \in \hat{\mathcal{C}}$ is representable then the representing object $X \in \mathcal{C}$ is unique up to isomorphism.

Proof. See [KS06] Proposition 1.4.3 and Corollary 1.4.4 or [Mac00] Chapter III Section 2. □

Definition 2.1.9. *Let $X : \mathbf{\Delta}^{op} \to \mathbf{Set}$ be a simplicial set. A simplicial set $Y : \mathbf{\Delta}^{op} \to \mathbf{Set}$ is called a simplicial subset of X, denoted by $Y \subseteq X$, if it is a subfunctor. More precisely, if* *(i) for all* $\underline{n} \in \mathbf{\Delta}$*,* $Y(\underline{n}) = Y_n$ *is a subset of* $X_n = X(\underline{n})$ *and*

(ii) for all morphisms $f : \underline{n} \to \underline{m}$ *in* $\mathbf{\Delta}$ *we have*

$$X(f)(Y_m) \subseteq (Y_n) \text{ and } Y(f) = X(f)|_{Y_m}.$$

Remark 2.1.10. Of course, it suffices to prove the properties in (ii) for the faces and degeneracies.

Example 2.1.11. (i) For two simplicial subsets $X, Y \subseteq Z$ of a given $Z \in \mathbf{sSet}$ we again can define new simplicial sets. The union $X \cup Y$ is the simplicial subset of Z defined via

$$(X \cup Y)_n := X_n \cup Y_n \text{ for all } n \geq 0,$$

and the face and degeneracy maps are the induced ones from X and Y. We note that the d_i's and s_i's of X and Y agree on $X_n \cap Y_n$, since both, X and Y are simplicial subsets.

(ii) Another important simplicial subset is the boundary of the standard n-simplex Δ^n. To define this, we take a look at the natural transformations

$$d^i = (d^i \circ -)_{\underline{m} \in \mathbf{\Delta}} : \Delta^{n-1} \to \Delta^n,$$

induced by the injections $d^i : \underline{n-1} \hookrightarrow \underline{n}$, for $0 \leq i \leq n$. Its image

$$\partial^i \Delta^n = im(d^i : \Delta^{n-1} \to \Delta^n)$$

forms a simplicial subset of Δ^n, called the i-th $(n-1)$-face of Δ^n. More precisely,

$$(\partial^i \Delta^n)_m := \{d^i \circ f \mid f \in \mathrm{Hom}_{\mathbf{\Delta}}(\underline{m}, \underline{n-1})\} \subseteq \mathrm{Hom}_{\mathbf{\Delta}}(\underline{m}, \underline{n}).$$

Since all natural transformations $d^i : \Delta^{n-1} \to \Delta^n$ are pointwise monomorphisms, we have isomorphisms $\partial^i \Delta^n \cong \Delta^{n-1}$ in $\mathbf{sSet}$. The boundary of Δ^n then is defined as the union of all $(n-1)$-faces,

$$\partial \Delta^n := \bigcup_{i=0}^{n} \partial^i \Delta^n.$$

Therefore any element in $(\partial \Delta^n)_m$ is of the form $d^i \circ f$ for some $0 \leq i \leq n$ and $f \in \mathrm{Hom}_{\mathbf{\Delta}}(\underline{m}, \underline{n-1})$. Since any map $f : \underline{m} \to \underline{n}$ in $\mathbf{\Delta}$ can be factorized into a composition of coface and codegeneracy maps, we can give an explicit description of the m-simplices of $\partial \Delta^n$, namely

$$(\partial \Delta^n)_m = \{\alpha : \underline{m} \to \underline{n} \mid \alpha \text{ is not surjective}\}.$$

In particular we have that $\partial \Delta^n_k = \Delta^n_k$ for all $k < n$.

(iii) By $*$ we denote the simplicial set Δ^0. In each degree $(\Delta^0)_m = \mathrm{Hom}_{\mathbf{\Delta}}(\underline{m}, \underline{0})$ is the one-point set, since it consists of only one map. Hence, $*$ is the terminal object in the category $\mathbf{sSet}$ of simplicial sets. The boundary $\partial \Delta^0 \subseteq \Delta^0$ is $\partial \Delta^0 = \emptyset$, the initial object in $\mathbf{sSet}$.

Another example of a simplicial set is the nerve of a small category.

Example 2.1.12. Let $\mathcal{C}$ be a small category. One assigns a simplicial set $B\mathcal{C}$ to $\mathcal{C}$, the so called nerve (or classifying space), where the set of n-simplices $B\mathcal{C}_n$ is defined to be the set of all n-composable morphisms

$$c_0 \to c_1 \to ... \to c_n$$

in $\mathcal{C}$. By convention, the 0-simplices $B\mathcal{C}_0$ is the set of objects of C. Moreover, the 1-simplices are the morphisms in C. The face maps

$$d_i : B\mathcal{C}_n \to B\mathcal{C}_{n-1}$$

are defined by composing the morphisms at the i-th object and sending a string of length n

$$c_0 \to ... \to c_{i-1} \to c_i \to c_{i+1} \to ... \to c_n$$

to the string

$$c_0 \to ... \to c_{i-1} \to c_{i+1} \to ... \to c_n$$

of length $n-1$. The degeneracy maps

$$s_i : B\mathcal{C}_n \to B\mathcal{C}_{n+1}$$

are defined similarly by adding the identity map at the i-th object and sending

$$c_0 \to ... \to c_i \to c_{i+1} \to ... \to c_n$$

to

$$c_0 \to ... \to c_i \to c_i \to c_{i+1} \to ... \to c_n.$$

If we view $\underline{n}$ as a category with objects $0, 1, ..., n$ and consisting only of one morphism $i \to j$ if $i \leq j$, then one could also define the nerve as

$$B\mathcal{C}_n := [\underline{n}, \mathcal{C}],$$

where $[\underline{n}, \mathcal{C}]$ denotes the set of functors from $\underline{n}$ to $\mathcal{C}$.

2.2 The geometric realization

Our next step is to define a realization functor $|\ | : \mathbf{sSet} \to \mathbf{Top}$. This means that we want to assign to any simplicial set X a topological space which is given by a disjoint union of some discs D^n glued together according to the abstract information given by the X_n and $X(f)$. There is a quick way to construct this which uses the so called simplex category (also called the comma category often) $\mathbf{\Delta} \downarrow X$ of a given simplicial set X. The objects of $\mathbf{\Delta} \downarrow X$

are all maps $\sigma : \Delta^n \to X$, i.e., the simplices of X. A morphism of $\mathbf{\Delta} \downarrow X$ is a commutative diagram of simplicial sets

$$\begin{tikzcd} \Delta^n \arrow[rr, "\theta"] \arrow[dr, "\sigma"'] & & \Delta^m. \arrow[dl, "\tau"] \\ & X & \end{tikzcd}$$

Note that θ is induced by a unique map $\theta : \underline{n} \to \underline{m}$, by Yoneda's lemma, since it is an element of

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, \Delta^m) \cong \Delta^m_n = \mathrm{Hom}_{\mathbf{\Delta}}(\underline{n}, \underline{m}).$$

By Yoneda's lemma, every simplicial set X can be written as a colimit of representable functors as follows. We see that

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{sSet}}(\varinjlim_{\substack{\Delta^n \to X \\ \text{in } \mathbf{\Delta}\downarrow X}} \Delta^n, Y) &\cong \varprojlim_{\substack{\Delta^n \to X \\ \text{in } \mathbf{\Delta}\downarrow X}} \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, Y) \cong \varprojlim_{\substack{\Delta^n \to X \\ \text{in } \mathbf{\Delta}\downarrow X}} Y_n \\
&= \{y = (y_{x,n})_{x,n} \in \prod_{\substack{x \in X_n \\ n \in \mathbb{N}_0}} Y_n \mid Y(f)(y_{x,m}) = y_{X(f)(x),n} \text{ for all } x \in X_m, f \in \mathrm{Hom}_{\mathbf{\Delta}}(\underline{n}, \underline{m})\} \\
&= \{t = (t_n)_n \in \prod_{n \in \mathbb{N}_0} \mathrm{Hom}_{\mathbf{Set}}(X_n, Y_n) \mid f^* \circ t_j = t_i \circ f^* \text{ for all } f \in \mathrm{Hom}_{\mathbf{\Delta}}(\underline{n}, \underline{m})\} \\
&= \mathrm{Hom}_{\mathbf{sSet}}(X, Y),
\end{aligned}$$

using the properties of limits and colimits, Yoneda's lemma, and the definition of natuaral transformations. And hence, by the covariant version of Yoneda's lemma, we have

$$X \cong \varinjlim_{\substack{\Delta^n \to X \\ \text{in } \mathbf{\Delta}\downarrow X}} \Delta^n.$$

Definition 2.2.1. *The geometric realization $|X|$ of a simplicial set X is defined as the colimit*

$$|X| = \varinjlim_{\substack{\Delta^n \to X \\ \text{in } \mathbf{\Delta}\downarrow X}} \Delta_n$$

in the category of topological spaces, where the transition maps are the θ_* as in Example 2.1.2.

If $f : X \to Y$ is a morphism between simplicial sets, then any simplex $\Delta^n \to X$ gives rise to $\Delta^n \to X \xrightarrow{f} Y$ and therefore, by the universal property of colimits, it induces a continuous map $|f| : |X| \to |Y|$. The realization $|\Delta^n|$ of the representable functors $\Delta^n \in \mathbf{sSet}$ is the topological space Δ_n, since $\mathbf{\Delta} \downarrow \Delta^n$ has terminal object $1 : \Delta^n \to \Delta^n$. Thus we just use the notation $|\Delta^n|$ for the topological standard n-simplex Δ_n in the following. This yields a covariant functor $|\ | : \mathbf{sSet} \to \mathbf{Top}$, which extends the functor r from Example 2.1.2 in the way that we have a commutative triangle

$$\begin{tikzcd} \mathbf{\Delta} \arrow[rr, "\Delta"] \arrow[dr, "r"'] & & \mathbf{sSet}, \arrow[dl, "|-|"] \\ & \mathbf{Top} & \end{tikzcd}$$

where Δ denotes the Yoneda embedding $\mathrm{Hom}_{\mathbf{\Delta}}(-,-)$ viewed as a functor in the second variable.

Remark 2.2.2. The geometric realization of a simplicial set X is the topological space

$$|X| = \coprod_{n\geq 0} (X_n \times |\Delta^n|) \Big/ \sim,$$

where $\sim$ is the equivalence relation defined by the rule that $(X(f)(x), t) \sim (x, f_*(t))$ with $x \in X_n$, $t \in |\Delta^m|$ and $f \in \mathrm{Hom}_{\Delta}(\underline{m}, \underline{n})$. The X_n are endowed with the discrete topology. Product, coproduct and the quotient are endowed with the corresponding topologies. In other words, $|X|$ is endowed with the final topology with respect to the maps $(X_n \times |\Delta^n| \to |X|)_{n\geq 0}$.

Definition 2.2.3. *Let T be a topological space. The singular set $S(T)$ is the simplicial set given by*

$$\underline{n} \mapsto \mathrm{Hom}_{\boldsymbol{Top}}(|\Delta^n|, T)$$

and

$$(\theta : \underline{n} \to \underline{m}) \mapsto (- \circ \theta_* : \mathrm{Hom}_{\boldsymbol{Top}}(|\Delta^m|, T) \to \mathrm{Hom}_{\boldsymbol{Top}}(|\Delta^n|, T)),$$

where $\theta_* : |\Delta^n| \to |\Delta^m|$ is the induced map of Example 2.1.2. In other words, the face maps $d_i : S(T)_n \to S(T)_{n-1}$ and degeneracy maps $s_j : S(T)_n \to S(T)_{n+1}$ are induced by the corresponding standard maps between topological simplices d^i_Δ and s^j_Δ, respectively. From this, we get a functor

$$S : \boldsymbol{Top} \to \boldsymbol{sSet}, \quad T \mapsto S(T),$$
$$f : T \to T' \mapsto \eta^f := (f \circ -)_{\underline{n} \in \mathbf{\Delta}},$$

called the singular functor.

Proposition 2.2.4. The realization functor is left adjoint to the singular functor in the sense that there is a bijection of sets

$$\mathrm{Hom}_{\mathbf{Top}}(|X|, Y) \cong \mathrm{Hom}_{\mathbf{sSet}}(X, S(Y)),$$

which is natural in simplicial sets X and topological spaces Y.

Proof. (Cf. [GJ99] Chapter I Proposition 2.2) Let $X \in \mathbf{sSet}$ be a simplicial set and $Y \in \mathbf{Top}$ a topological space. Then, using the properties of colimits, we see that

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Top}}(|X|, Y) &= \mathrm{Hom}_{\mathbf{Top}}(\varinjlim_{\Delta^n \to X} |\Delta^n|, Y) \cong \varprojlim_{\Delta^n \to X} \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, Y) \\ &\cong \varprojlim_{\Delta^n \to X} \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, S(Y)) \cong \mathrm{Hom}_{\mathbf{sSet}}(\varinjlim_{\Delta^n \to X} \Delta^n, S(Y)) \\ &\cong \mathrm{Hom}_{\mathbf{sSet}}(X, S(Y)).\end{aligned}$$

Note that the isomorphism $X \cong \varinjlim_{\Delta^n \to X} \Delta^n$ was constructed above. Moreover, we use that

$$\mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, Y) = S(Y)_n = \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, S(Y)).$$

□

Since left adjoints preserve colimits we obtain the following result.

Corollary 2.2.5. The realization functor $|\ | : \mathbf{sSet} \to \mathbf{Top}$ preserves colimits.

Later in this chapter, we are going to show that the realization of a simplicial set is a CW-complex. First we need to discuss some fundamental properties to understand the structure of simplicial sets.

Definition 2.2.6. *Let X be a simplicial set. An n-simplex $x \in X_n$ is called degenerate if there is a surjection $\eta : \underline{n} \to \underline{m}$ with $m < n$ and an m-simplex $y \in X_m$ such that $X(\eta)(y) = \eta^* y = x$. We write*

$$e(X)_n := \{x \in X_n |\ x \textit{ is not of the form } \eta^* y \textit{ for any } m < n,\ y \in X_m \textit{ and } \eta : \underline{n} \twoheadrightarrow \underline{m}\}$$

for the set of nondegenerate n-simplices.

Example 2.2.7. An m-simplex f of Δ^n is degenerate if it is of the form

$$\underline{m} \xrightarrow{f} \underline{n} = \underline{m} \xrightarrow{\eta} \underline{k} \xrightarrow{g} \underline{n},$$

where $k < m$ and $\eta : \underline{m} \twoheadrightarrow \underline{k}$ is a surjection in $\mathbf{\Delta}$. Using the factorization in coface and codegeneracy maps, we can determine the number of nondegenerate simplices of Δ^n in dimension n and $n+1$. The only nondegenerate n-simplex is the identity as the empty product. Every other factorization starts with a codegeneracy map s^i. Similarly, Δ^n has precisely $n+1$ nondegenerate $n-1$-simplices, corresponding to the coface maps $d^i : \underline{n-1} \to \underline{n}$, for $0 \leq i \leq n$ (cf. [HP14] Observe 3.17).

Proposition 2.2.8. ***(The Eilenberg-Zilber Lemma)***
For each $x \in X_n$ there exists a unique surjection $\eta : \underline{n} \to \underline{m}$ and a unique non-degenerate $y \in Y_m$, such that $x = \eta^* y$.

Proof. (Cf. [JT] Proposition 1.2.2 and [Rue17] Lemma 2.19) We start proving the existence. For every $x \in X_n$ we consider the set of pairs $y \in X_m$ and $\eta : \underline{n} \twoheadrightarrow \underline{m}$ with $\eta^* y = x$. This set is non-empty, as it contains $(\eta, y) = (id_{\underline{n}}, x)$. So if x is non-degenerate, we are done.
The set has a pair (η, y) which is minimal in m, since the appearing dimensions m, 0 is a lower bound. If x is degenerate we choose this minimal pair (η, y) with $y \in X_m$, $m < n$. Then y must be non-degenerate, because otherwise we would find a pair of lower dimension.
To show the uniqueness, we suppose (η, y) and (η', y') to be two such pairs. We look at the pushout diagram

$$\begin{tikzcd} \underline{n} \arrow[r, two heads, "\eta"] \arrow[d, two heads, "\eta'"'] & \underline{m} \arrow[d, "\eta_2"] \\ \underline{m'} \arrow[r, "\eta_1"'] & \underline{p} \end{tikzcd}$$

in the category $\mathbf{\Delta}$ as a subcategory of **Set**. Since the covariant Hom functor preserves pushouts, we get the pushout diagram

$$\begin{tikzcd} \Delta^n \arrow[r, "\eta"] \arrow[d, "\eta'"'] & \Delta^m \arrow[d, "\eta_2"] \\ \Delta^{m'} \arrow[r, "\eta_1"'] & \Delta^p \end{tikzcd}$$

of simplicial sets in **sSet**. By assumption, we have $\eta^* y = x = \eta'^* y'$. Using the universal property of the pushout we get

$$\begin{tikzcd} \Delta^n \arrow[r, "\eta"] \arrow[d, "\eta'"'] & \Delta^m \arrow[d, "\eta_2"] \arrow[ddr, bend left, "y"] & \\ \Delta^{m'} \arrow[r, "\eta_1"] \arrow[drr, bend right, "y'"'] & \Delta^p \arrow[dr, dotted, "\exists! z"] & \\ & & X, \end{tikzcd}$$

i.e. there is $z : \Delta^p \to X$ such that $y' = \eta_1^* z$ and $y = \eta_2^* z$. Since y and y' are non-degenerate, $\eta_1 = \eta_2 = id$, and thus $y = y'$ and $\eta = \eta'$. $\square$

2.3 The skeleton of a simplicial set

Our goal now is to show that the realization of a simplicial set X is a CW-complex. To this aim, we construct a filtration for X by so-called skeletons. These are subcomplexes of X which are generated by the simplices of X of lower degree. Then by applying the realization functor to the filtration, we obtain a CW-structure on the topological space $|X|$.

Definition 2.3.1. *(i) Let $\mathbf{\Delta_n}$ denote the full subcategory of $\mathbf{\Delta}$ whose objects are the totally ordered sets $\underline{m}$ for $m \leq n$. A functor $X : \mathbf{\Delta_n}^{op} \to Set$ is called an n-truncated* *simplicial set. By*

$$\boldsymbol{sSet_n} := [\mathbf{\Delta_n}^{op}, \boldsymbol{Set}]$$

we denote the category of n-truncated simplicial sets.

(ii) Let $X \in \boldsymbol{sSet}$. The restriction functor

$$tr^n : \boldsymbol{sSet} \to \boldsymbol{sSet_n}$$

induced by the inclusion $\mathbf{\Delta_n} \to \mathbf{\Delta}$ is called the n-th truncation functor. It truncates a simplicial set X at n. For $p \leq n$ the representable functor $\mathrm{Hom}_{\mathbf{\Delta}_n}(-, \underline{p})$ in $\boldsymbol{sSet_n}$ on $\underline{p}$ is denoted by Δ^p_n. It is the truncation of $\Delta^p \in \boldsymbol{sSet}$ at n.

(iii) Let $X \in \boldsymbol{sSet_n}$ be an n-truncated simplicial set. We define the simplicial set $sk^n X \in \boldsymbol{sSet}$ as

$$sk^n X := \varinjlim_{\Delta^p_n \to X} \Delta^p.$$

This defines a functor $sk^n : \boldsymbol{sSet_n} \to \boldsymbol{sSet}$.

Proposition 2.3.2. The functor sk^n is left-adjoint to tr^n.

Proof. (Cf. [HP14] Proposition 3.11) Similar to the proof of Proposition 2.2.4, one has

$$\begin{aligned}\mathrm{Hom}_{\mathbf{sSet}}(sk^n X, Y) = \mathrm{Hom}_{\mathbf{sSet}}(\varinjlim_{\Delta^p_n \to X} \Delta^p, Y) &\cong \varprojlim_{\Delta^p_n \to X} Y_p \\ &\cong \varprojlim_{\Delta^p_n \to X} \mathrm{Hom}_{\mathbf{sSet_n}}(\Delta^p_n, tr^n Y) \cong \mathrm{Hom}_{\mathbf{sSet_n}}(X, tr^n Y).\end{aligned}$$

□

Remark 2.3.3. (i) Since $\mathbf{\Delta_n}$ is a full subcategory of $\mathbf{\Delta}$, the inclusion $\mathbf{\Delta_n} \to \mathbf{\Delta}$ is fully faithful. Hence, $(sk^n X)_m = X_m$ for $m \leq n$. It follows that the unit $X \to tr^n sk^n X$ of the adjunction is an isomorphism, and sk^n is fully faithful, too.

(ii) As $sk^n X$ is defined as such a colimit, it is a quotient of a sum of some Δ^p for $p \leq n$, and the m-simplices of Δ^p are degenerate for $m > n$. It follows that $(sk^n X)_m$ consists only of degenerate simplices for $m > n$ (Cf. [JT] pages 7/8).

Proposition 2.3.4. The counit $sk^n tr^n X \to X$ of the adjunction is a monomorphism.

Proof. (Cf. [JT] Proposition 1.2.3) By the above remark, the map $(sk^n tr^n X)_m \to X_m$ is a bijection for $m \leq n$. Therefore, it suffices to prove that if $f : Y \to X$ is a natural transformation of simplicial sets such that $f_m : Y_m \to X_m$ is injective for $m \leq n$ and the m-simplices of Y are degenerate for $m > n$ then f_m is injective for all m.
Let $y, y' \in Y_m$ for $m > n$. By the Eilenberg-Zilber-Lemma, there are surjections $\eta : \underline{m} \twoheadrightarrow \underline{p}$ and $\eta' : \underline{m} \twoheadrightarrow \underline{p'}$ and non-degenerate simplices $z \in Y_p$ and $z' \in Y_{p'}$ such that $\eta^* z = y$ and $\eta'^* z' = y'$. Since $p, p' \leq m$ and $f_p, f_{p'}$ are injective, it follows that $f_p(z)$ and $f_{p'}(z)$ are

non-degenerate. Indeed, if for example $f_p(z) = \alpha^* x$ with $\alpha : \underline{p} \twoheadrightarrow \underline{q}$ is surjective, $q < p$ and $x \in X_q$, then α has a section $\varepsilon : \underline{q} \to \underline{p}$ such that $\alpha\varepsilon = id$. It follows that

$$\varepsilon^* f_p(z) = \varepsilon^* \alpha^* x = x \quad \text{and so} \quad \alpha^*(\varepsilon^* f_p(z)) = \alpha^* x = f_p(z).$$

Since f is a natural transformation we have $\alpha^*\varepsilon^* f_p(z) = f_p(\alpha^*\varepsilon^* z))$. As seen above, the left hand side is equal to $f_p(z)$. The injectivity of f_p therefore gives $\alpha^*\varepsilon^* z = z$ and thus z is degenerate which yields a contradiction. Since $f_m(y) = \eta^* f_p(z)$ and $f_m(y') = \eta'^* f_{p'}(z')$, if $f_m(y) = f_m(y')$ we have $\eta = \eta'$ and $f_p(z) = f_{p'}(z')$ by the Eilenberg-Zilber-Lemma. Thus, we have $p = p'$ and since f_p is injective $z = z'$, and $y = y'$. □

Definition 2.3.5. *We write $Sk^n X$ for the image of the monomorphism $sk^n tr^n X \to X$ and call it the n-skeleton of X. We say X is of dimension n if $Sk^n X = X$.*

Remark 2.3.6. (i) We have $(Sk^n X)_m = X_m$ for $m \leq n$, and for $m > n$ $(Sk^n X)_m$ consists of those m-simplices $x \in X_m$ for which there is a surjection $\eta : \underline{m} \to \underline{p}$ with $p \leq n$ and a $y \in X_p$ such that $x = \eta^* y$. This follows from the Eilenberg-Zilber lemma.

(ii) Any $X \in \mathbf{sSet}$ is the union of its skeleta

$$\bigcup_{n \leq 0} Sk^n X = X$$

with

$$Sk^0 X \subseteq Sk^1 X \subseteq ... \subseteq Sk^{n-1} X \subseteq Sk^n X \subseteq ...$$

(cf. [JT] page 8).

Proposition 2.3.7. For each simplicial set $X \in \mathbf{sSet}$, the geometric realization $|X|$ is a CW-complex.

Proof. (Cf. [GJ99] Chapter I Proposition 2.3 and [JT] pages 9/10) We want to construct a filtration for the topological space $|X|$, such that it is a CW-complex. Therefore we start with the filtration of the simplicial set X in its skeletons $Sk^n X$ and claim that each $Sk^n X$ can be obtained from the previous skeleton $Sk^{n-1} X$ in the following way:

Let $e(X)_n$, $n \geq 0$, be the set of non-degenerate n-simplices of X. For each $x \in e(X)_n$, $n \geq 1$, we have the diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \Delta^n \arrow[d, "x"] \\ Sk_{n-1} X \arrow[r] & Sk_n X, \end{tikzcd}$$

where the vertical maps send f to $f^* x$ and the horizontal maps are the injections. By summing over all $x \in e(X)_n$ we get the diagram

$$\begin{tikzcd} \coprod\limits_{x \in e(X)} \partial\Delta^n \arrow[r] \arrow[d] & \coprod\limits_{x \in e(X)} \Delta^n \arrow[d] \\ Sk^{n-1} X \arrow[r] & Sk^n X. \end{tikzcd}$$

This square is a pushout. Indeed, since all the simplicial sets are of dimension $\leq n$, it suffices to show that this diagram is a pushout after applying the functor tr^n, i.e. that

$$\begin{tikzcd} (\coprod_{x\in e(X)} \partial\Delta^n)_m \arrow[r] \arrow[d] & (\coprod_{x\in e(X)} \Delta^n)_m \arrow[d] \\ (Sk^{n-1}X)_m \arrow[r] & (Sk^n X)_m \end{tikzcd}$$

is a pushout of sets for all $m \leq n$. For $m \leq n-1$ this is clear since the two horizontal maps are isomorphisms. For $m = n$ the complement of $(\partial\Delta^n)_n$ in $(\Delta^n)_n$ consists only of one element, $id_{\underline{n}}$. Thus, the complement of $\coprod_{x\in e(X)}(\partial\Delta^n)_n$ in $\coprod_{x\in e(X)}(\Delta^n)_n$ is isomorphic to $e(X)_n$. But $(Sk^nX)_n = (Sk^{n-1}X)_n \cup e(X)_n$ so that the diagram is indeed a pushout. Next we look at the diagram

$$\begin{tikzcd} \Delta^{n-2} \arrow[r, "d^{j-1}"] \arrow[d, "in_{i<j}"'] & \Delta^{n-1} \arrow[d, "in_i"] \arrow[dr, "d^i"] & \\ \coprod_{0\leq i<j\leq n} \Delta^{n-2} \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\psi"'] & \coprod_{i=0}^{n} \Delta^{n-1} \arrow[r] & \partial\Delta^n \\ \Delta^{n-2} \arrow[r, "d^i"'] \arrow[u, "in_{i<j}"] & \Delta^{n-1} \arrow[u, "in_j"'] \arrow[ur, "d^j"'] & \end{tikzcd}$$

where in denote the various injections. I.e., the maps ϕ and ψ are determined by $\phi \circ in_{i<j} = in_i \circ d^{j-1}$ and $\psi \circ in_{i<j} = in_j \circ d^i$. The row in the middle is a coequalizer diagram in **sSet**. To prove this, one constructs a coequalizer diagram

$$\coprod_{0\leq i<j\leq n} \Delta^{n-1} \underset{\partial\Delta^n}{\times} \Delta^{n-1} \overset{p_1}{\underset{p_2}{\rightrightarrows}} \coprod_{0\leq i\leq n} \Delta^{n-1} \longrightarrow \partial\Delta^n,$$

over the fibre product $\Delta^{n-1} \underset{\partial\Delta^n}{\times} \Delta^{n-1}$. By using the isomorphisms

$$\Delta^{n-1} \underset{\partial\Delta^n}{\times} \Delta^{n-1} \cong \Delta^{n-1} \underset{\Delta^n}{\times} \Delta^{n-1} \cong \Delta^{n-2},$$

the statement follows. We will not give the details here, since later on we will check the coequalizer property in a very similar situation (cf. Lemma 4.1.1). Since $|-|$ preserves colimits it follows that there is a coequalizer diagram of spaces

$$\coprod_{0\leq i<j\leq n} |\Delta^{n-2}| \rightrightarrows \coprod_{i=0}^{n} |\Delta^{n-1}| \longrightarrow |\partial\Delta^n|.$$

Applying this to the diagram

$$\begin{tikzcd} \coprod_{i<j} \Delta^{n-2} \arrow[r, shift left] \arrow[r, shift right] & \coprod_i \Delta^{n-1} \arrow[r] \arrow[dr, "\xi"'] & \partial\Delta^n \arrow[d] \\ & & \Delta^n, \end{tikzcd}$$

with $\xi \circ in_i = d^i$, it follows that $|\partial\Delta^n| \cong \partial|\Delta^n|$ and the induced map $|\partial\Delta^n| \to |\Delta^n|$ maps $|\partial\Delta^n|$ onto the $(n-1)$-sphere bounding $|\Delta^n|$. Thus, $|X|$ has a filtration $|Sk^0 X| \subseteq |Sk^1 X| \subseteq ... \subseteq |Sk^n X| \subseteq ...$ where $|X|_n := |Sk^n X|$ is obtained from $|X|_{n-1} := |Sk^{n-1} X|$ by attaching n-cells according to the pushout diagram

$$\begin{tikzcd} \coprod\limits_{x \in e(X)} |\partial\Delta^n| \arrow[r] \arrow[d, hook] & |Sk^{n-1} X| \arrow[d, hook] \\ \coprod\limits_{x \in e(X)} |\Delta^n| \arrow[r] & |Sk^n X|. \end{tikzcd}$$

$\square$

By **KTop** we denote the full subcategory of **Top** consisting of k-spaces. A topological space T is called a *k-space* (or *Kelly-space*) if every compactly open subset is open (a subset U of T is called compactly open if for every continuous map $f : K \to X$ where K is compact Hausdorff, $f^{-1}(U)$ is open in K). Since the realization $|X|$ of a simplicial set X is a CW-complex (in particular a compactly generated Hausdorff space), we can view the realization functor as a functor $|-| : \mathbf{sSet} \to \mathbf{KTop}$ (cf. [Hov99] Definition 2.4.21 and page 77). One can use this to prove the following important property of the realization functor.

Proposition 2.3.8. The realization functor $|-| : \mathbf{sSet} \to \mathbf{KTop}$ preserves finite limits.

Proof. See [Hov99] Lemma 3.2.4. $\square$

3 Model categories

3.1 Model structure and Quillen equivalence

After introducing the category of simplicial sets, we start with the basic theory of model categories. In particular, this section briefly gives the main definitions and properties of Quillen homotopy theory also including Quillen equivalences.

Definition 3.1.1. *Let $\mathcal{C}$ be a category. A model structure on $\mathcal{C}$ (also called a Quillen model structure or a Quillen homotopy structure) consists of three classes of morphisms of $\mathcal{C}$, the so called fibrations, cofibrations and weak equivalences satisfying the following properties:*

Q1: (2-out-of-3) If $f : X \to Y$ and $g : Y \to Z$ are morphisms of $\mathcal{C}$ such that two of f, g or gf are weak equivalences, then so is the third.

Q2: (Retracts) Let $f : X \to Y$ and $f' : X' \to Y'$ be morphisms of $\mathcal{C}$ such that f is a retract of f', i.e. there is a commutative diagram of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y \arrow[r] & Y' \arrow[r] & Y, \end{tikzcd}$$

where the horizontal composites are identities. If f' is a fibration, cofibration or weak equivalence, then so is f.

Q3: (Lifting) Let

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed] & Y, \end{tikzcd}$$

be a commutative diagram in $\mathcal{C}$ where i is a cofibration and f is a fibration. If either i or f is also a weak equivalence, then the dotted arrow exists, making both triangles commute.

Q4: (Factorization) Any morphism $f : X \to Y$ can be factored as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & E \arrow[ur, "p"'] & \end{tikzcd}$$

in two ways. One in which p is a fibration and i is a cofibration and a weak equivalence. And one way in which i is a cofibration and p is a fibration and a weak equivalence.

Definition 3.1.2. *Let $\mathcal{C}$ be a category with a model structure on it. A morphism which is both a cofibration and a weak equivalence is called a trivial cofibration. And, analogously, a morphism is called a trivial fibration, if it is a fibration and a weak equivalence.*

Definition 3.1.3. *Let $i : A \to B$ and $p : X \to Y$ be two morphisms in a category $\mathcal{C}$. We say i has the left lifting property (LLP) with respect to p and p has the right lifting property (RLP) with respect to i if, for every commutative diagram*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "g"'] & Y, \end{tikzcd}$$

there is a lift $h : B \to X$ such that $h \circ i = f$ and $p \circ h = g$.

Remark 3.1.4. With these two definitions the Lifting and Factorization properties can be reformulated in an easier way as follows:

Q3: Trivial cofibrations have the left lifting property with respect to fibrations, and trivial fibrations have the right lifting property with respect to cofibrations.

Q4: Any morphism $f : X \to Y$ can be factored as:

(a) $f = p \circ i$ where p is a fibration and i is a trivial cofibration, and

(b) $f = q \circ j$ where q is a trivial fibration and j is a cofibration.

Definition 3.1.5. *A model category is a category $\mathcal{C}$ with all small limits and colimits together with a model structure on $\mathcal{C}$.*

We point out that the definition of a model category can vary rather significantly, depending on the references. For example, the reader should compare with the definition given in [Hov99] Section 1.1.

Example 3.1.6. (i) Let $\mathcal{C}$ be a category with all small colimits and limits. We obtain a trivial model structure on $\mathcal{C}$ by choosing one of the classes of fibrations, cofibrations and weak equivalences to be the isomorphisms and the other two classes to be all morphisms in $\mathcal{C}$.

(ii) Let $\mathcal{C}$ and $\mathcal{D}$ be two model categories. Then the product category $\mathcal{C} \times \mathcal{D}$ is the category whose objects are pairs (X, Y) with $X \in \mathcal{C}$, $Y \in \mathcal{D}$ and the morphisms from (X_1, Y_1) to (X_2, Y_2) are pairs (f, g), where $f : X_1 \to X_2$ is a morphism of $\mathcal{C}$ and $g : Y_1 \to Y_2$ is a morphism of $\mathcal{D}$. We obtain a model structure on $\mathcal{C} \times \mathcal{D}$ from those of $\mathcal{C}$ and $\mathcal{D}$ in the following way: a morphism (f, g) in $\mathcal{C} \times \mathcal{D}$ is a fibration (cofibration, weak equivalence) if and only if so are both f and g in $\mathcal{C}$ and $\mathcal{D}$, respectively.

(iii) Let $\mathcal{C}$ be a model category with classes of fibration, cofibration and weak equivalence denoted by $(\mathcal{F}, \mathcal{G}, \mathcal{W})$. Then the dual category $\mathcal{C}^{op}$ becomes a model category with morphism classes $(\mathcal{F}^{op}, \mathcal{G}^{op}, \mathcal{W}^{op})$ by choosing $\mathcal{F}^{op} := \mathcal{G}$, $\mathcal{G}^{op} := \mathcal{F}$ and $\mathcal{W}^{op} := \mathcal{W}$. All axioms follow immediately from $\mathcal{C}$ being a model category. For the factorization axiom Q4, for any $f \in \mathrm{Hom}_{\mathcal{C}^{op}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ the first factorization in $\mathcal{C}$ becomes the second in $\mathcal{C}^{op}$ and, similarly, the other way around. The lifting diagram in axiom Q3 just flips.

Since a model category $\mathcal{C}$ has all finite limits and colimits, it has an initial object, the colimit of the empty diagram, and a terminal object, the limit of the empty diagram. This leads us to the following definition.

Definition 3.1.7. *Let $\mathcal{C}$ be a model category with initial object I and terminal object T. An object of $X \in \mathcal{C}$ is called cofibrant if the map $I \to X$ is a cofibration, and fibrant if the map $X \to T$ is a fibration.*

If $\mathcal{C}$ is a model category and $X \in \mathcal{C}$ we can apply the second factorization in axiom Q4 to the map $I \to X$ from the initial object to X. This gives us a factorization

$$I \to QX \to X,$$

where the left morphism is a cofibration and right one is a trivial fibration. Hence, we obtain a cofibrant object QX together with a trivial fibration $QX \xrightarrow{q_X} X$. This is called the *cofibrant replacement* of X. Depending on the axioms of a model category or depending on the situation, this construction can be made functorial and $X \mapsto QX$ is called the *cofibrant replacement functor*. Similarly, by applying first factorization to the map from X to the terminal object we obtain a fibrant object RX, called the *fibrant replacement*, together with a trivial cofibration $X \xrightarrow{r_X} RX$. Accordingly, if the construction is made functorial, we get a functor $X \mapsto RX$, called the *fibrant replacement functor*.

A useful result in order to work with model categories, is the following.

Lemma 3.1.8. Let $\mathcal{C}$ be a model category. Then a map is a cofibration (a trivial cofibration) if and only if it has the left lifting property with respect to all trivial fibrations (fibrations). Dually, a map is a fibrations (a trivial fibrations) if and only if it has the right lifting property with respect to all trivial cofibrations (cofibrations).

Proof. See [Hov99] Lemma 1.1.10. □

Remark 3.1.9. In particular, the classes of fibrations and cofibrations are both closed under composition, as can be seen by iteratively applying the lifting property twice. Moreover, every isomorphism in a model category is a trivial cofibration and a trivial fibration.

Definition 3.1.10. *Let $\mathcal{C}$ and $\mathcal{D}$ be model categories.*

(i) We call a functor $F : \mathcal{C} \to \mathcal{D}$ a left Quillen functor if F is a left adjoint and preserves cofibrations and trivial cofibrations.

(ii) *We call a functor $U : \mathcal{D} \to \mathcal{C}$ a right Quillen functor if U is a right adjoint and preserves fibrations and trivial fibrations.*

(iii) *Suppose (F, U, φ) is an adjunction from $\mathcal{C}$ to $\mathcal{D}$. That is, F is a functor $\mathcal{C} \to \mathcal{D}$, U is a functor $\mathcal{D} \to \mathcal{C}$, and φ is a natural isomorphism $\mathrm{Hom}_{\mathcal{D}}(FA, B) \to \mathrm{Hom}_{\mathcal{C}}(A, UB)$ expressing U as a right adjoint of F. We call (F, U, φ) a Quillen adjunction if F is a left Quillen functor.*

Remark 3.1.11. A triple (F, U, φ) as in part (iii) of the definition is a Quillen adjunction if and only if U is a right Quillen functor (cf. [Hov99] Lemma 1.3.4). Hence, the definition is in fact symmetric in F and U.

Definition 3.1.12. *A Quillen adjunction $(F, U, \varphi) : \mathcal{C} \to \mathcal{D}$ is called a Quillen equivalence if and only if, for all cofibrant X in $\mathcal{C}$ and fibrant Y in $\mathcal{D}$, a map $f : FX \to Y$ is a weak equivalence in $\mathcal{D}$ if and only if $\varphi(f) : X \to UY$ is a weak equivalence in $\mathcal{C}$.*

An important property of Quillen equivalences is that the functors induce an inverse equivalence between the corresponding homotopy categories (cf. Hovey section 1.3.2 and proposition 1.3.13). We will not introduce the homotopy category of a model structure. However, here is a list of criteria to check whether a Quillen adjunction is a Quillen equivalence.

Proposition 3.1.13. Let $(F, U, \varphi) : \mathcal{C} \to \mathcal{D}$ be a Quillen adjunction. Then the following are equivalent:

(i) (F, U, φ) is a Quillen equivalence.

(ii) For all cofibrant objects $X \in \mathcal{C}$ the composite

$$X \to U(F(X)) \xrightarrow{U(r_{F(X)})} U(RF(X))$$

is a weak equivalence and for all fibrant $Y \in \mathcal{D}$ and the composite

$$F(QU(Y)) \xrightarrow{F(q_{U(Y)})} F(U(Y)) \to Y$$

is a weak equivalence as well.

(iii) F reflects weak equivalences between cofibrant objects and, for every fibrant $Y \in \mathcal{D}$, the map $F(QU(Y)) \to Y$ is a weak equivalence.

(iv) U reflects weak equivalences between fibrant objects and, for every cofibrant $X \in \mathcal{C}$, the map $X \to U(RF(X))$ is a weak equivalence.

Proof. See [Hov99] Proposition 1.3.13 and Corollary 1.3.16. □

Remark 3.1.14. A functor is said to reflect some property of morphisms if any morphism f has this property whenever $F(f)$ does. All maps in this proposition without explicit description are induced by the isomorphisms coming from the adjunction $\mathrm{Hom}_{\mathcal{D}}(F(C), D) \cong \mathrm{Hom}_{\mathcal{C}}(C, U(D))$.

3.2 The model structure on Top

Now we turn to **Top**, the category of topological spaces. We want to construct a non-trivial model structure on **Top**. First, we give a categorical definition, which helps us define the classes of fibrations and cofibrations, concisely.

Definition 3.2.1. *Let $\mathcal{I}$ be a class of morphisms in a category $\mathcal{C}$.*

(i) A morphism in $\mathcal{C}$ is called $\mathcal{I}$-injective if it has the right lifting property with respect to every map in $\mathcal{I}$. The class of $\mathcal{I}$-injective maps is denoted by $\mathcal{I}$-inj.

(ii) A morphism in $\mathcal{C}$ is called $\mathcal{I}$-projective if it has the left lifting property with respect to every map in $\mathcal{I}$. The class of $\mathcal{I}$-projective maps is denoted by $\mathcal{I}$-proj.

(iii) A morphism in $\mathcal{C}$ is called an $\mathcal{I}$-cofibration if it has the left lifting property with respect to every $\mathcal{I}$-injective map. The class of $\mathcal{I}$-cofibrations is the class ($\mathcal{I}$-inj)-proj and is denoted by $\mathcal{I}$-cof.

(iv) A morphism in $\mathcal{C}$ is called an $\mathcal{I}$-fibration if it has the right lifting property with respect to every $\mathcal{I}$-projective map. The class of $\mathcal{I}$-fibrations is the class ($\mathcal{I}$-proj)-inj and is denoted by $\mathcal{I}$-fib.

To define the required classes on **Top**, we fix some notations. By $D^n \subseteq \mathbb{R}^n$ we denote the n-th unit disc and by $S^{n-1} \subseteq \mathbb{R}^n$ the unit sphere, so that we have the boundary inclusion $S^{n-1} \hookrightarrow D^n$, for $n \geq 0$. For $n = 0$ we set $D^0 = \{0\}$ and $S^{-1} = \emptyset$. By $I := [0,1]$ we denote the unit interval, as usual. For all $n \geq 0$, define the set of maps $\mathcal{I}$ to consist of the inclusions $D^n \to D^n \times I$ and the set $\mathcal{J}$ to consist of all boundary inclusions $S^{n-1} \hookrightarrow D^n$. Note that the notation of the maps in $\mathcal{I}$ is an abbreviation for $D^n \times \{0\} \to D^n \times I$. For our model structure, we start with constructing the fibrations in **Top**, the so called *Serre fibrations*. A continuous map $f : E \to X$ of topological spaces is defined to be a fibration if it is in $\mathcal{I}$-inj, i.e. if $h : D^n \times I \to X$ is a homotopy and $p : D^n \to E$ such that $f \circ p = h_0$ then there is a homotopy $\bar{h} : D^n \times I \to E$ such that $\bar{h}_0 = p$ and $f \circ \bar{h} = h$. Schematically:

$$\begin{tikzcd} D^n \arrow[r, "p"] \arrow[d] & E \arrow[d, "f"] \\ D^n \times I \arrow[r, "h"'] \arrow[ur, dashed, "\exists \bar{h}"] & X. \end{tikzcd}$$

A map $i : A \to B$ in **Top** is defined to be a cofibration if it is in $\mathcal{J}$-cof, i.e. if we have the following commutative diagram of solid arrows,

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & E \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed, "\exists"] & X, \end{tikzcd}$$

with $f : E \to X$ having the RLP with respect to all boundary inclusions $S^{n-1} \hookrightarrow D^n$, then the diagonal map exists, making both triangles commute. And last, the class of weak equivalences in **Top** are the (weak) homotopy equivalences $f : X \to Y$, i.e. all maps f for which the induced map

$$f^* : \pi_0(X) \to \pi_0(Y)$$

is a bijection, and the induced group homomorphisms

$$\pi_n(f, x) : \pi_n(X, x) \to \pi_n(Y, f(x)), \quad [\alpha] \mapsto [f \circ \alpha]$$

are isomorphisms for all $n \geq 1$ and all $x \in X$, where $\pi_n(X, x)$ denotes the n-th homotopy group of X at x.

Theorem 3.2.2. With the above definitions of fibrations, cofibrations and weak equivalences, the category **Top** admits a model structure.

Proof. See [Hov99] Theorem 2.4.19. □

Since we will use it later for our model structure on **sSet**, we prove property Q1 in Definition 3.1.1 for the category **Top**.

Proposition 3.2.3. The weak equivalences in **Top** as defined above satisfy the two out of three axiom.

Proof. (Cf. [Hov99] Lemma 2.4.4) Suppose we are given the commutative diagram

$$\begin{tikzcd} X \arrow[rr, "g\circ f"] \arrow[dr, "f"'] & & Z \\ & Y \arrow[ur, "g"'] & \end{tikzcd}$$

of topological spaces. The only case which is not clear is the one where f and $g \circ f$ are weak equivalences. We need to show that

$$g_* : \pi_n(Y, y) \to \pi_n(Z, g(y))$$

is an isomorphism for all $n \geq 0$ and all $y \in Y$. But y may not be in the image of f. Since $f_* : \pi_0(X) \to \pi_0(Y)$ is a bijection between the sets of path components, there exists a point $x \in X$ and a path $\alpha : I \to Y$ from $f(x)$ to y in Y. Therefore we get a commutative diagram

$$\begin{tikzcd} \pi_n(Y, y) \arrow[r, "g_*"] \arrow[d] & \pi_n(Z, g(y)) \arrow[d] \\ \pi_n(Y, f(x)) \arrow[r] & \pi_n(Z, f(g(f(x))), \end{tikzcd}$$

where the vertical maps are the composition with the paths α and $g \circ \alpha$, respectively and their inverse paths. As known, these maps are isomorphism for different choices of the basepoint in the same path-component (cf. [Hat01] page 341). The bottom horizontal map is an isomorphism by applying the assumption. Therefore, the upper horizontal map is an isomorphism as well, as required. □

Example 3.2.4. (i) An easy example for a Serre-fibration in **Top** we will need later on, are projection maps $X \times Y \xrightarrow{pr_X} X$ onto the first component. If we have a diagram of the form

$$\begin{tikzcd}
D^n \arrow[r, "p"] \arrow[d] & X \times Y \arrow[d, "pr_X"] \\
D^n \times I \arrow[r, "h"'] & X,
\end{tikzcd}$$

we can lift the homotopy by defining $\bar{h} : D^n \times I \to X \times Y$ as $\bar{h}(r,t) := (h(r,t), pr_Y(p(r)))$. Similarly, the projections onto the second component $X \times Y \xrightarrow{pr_X} Y$ are Serre-fibrations.

(ii) A better known class of morphisms in the category **Top** are the Hurewicz fibrations, playing an important role in higher homotopy theory. This is a continuous map $f : E \to X$ of topological spaces with the homotopy lifting property, i.e. for any diagram

$$\begin{tikzcd}
Y \arrow[r, "h_0"] \arrow[d] & E \arrow[d, "f"] \\
Y \times I \arrow[r, "g"'] \arrow[ur, dashed] & X,
\end{tikzcd}$$

the lifted homotopy $h : Y \times I \to E$ exists. Clearly, every Hurewicz fibration is a Serre fibration. This gives us a bunch of more examples of Serre fibrations like covering maps (cf. [Ark11] Sections 3.3 and 3.4).

Remark 3.2.5. A Hurewicz cofibration is a continuous map $i : A \to X$ such that the homotopy extension in the diagram

$$\begin{tikzcd}
A \arrow[rr] \arrow[dd, "i"'] & & A \times I \arrow[dl, "g"'] \arrow[dd, "i \times id_I"] \\
& Y & \\
X \arrow[rr] \arrow[ur, "h_0"] & & X \times I \arrow[ul, dashed, "h"']
\end{tikzcd}$$

exists for every space Y, map $h_0 : X \to Y$ and homotopy $g : A \times I \to Y$ (cf. [Ark11] Definition 3.2.1). These Hurewicz fibrations and cofibrations build another model structure on **Top**, where the weak equivalences are the strong homotopy equivalences, i.e. all morphisms $f : X \to Y$ with a homotopy inverse $g : Y \to X$ such that $g \circ f \sim id_X$ and $f \circ g \sim id_Y$. This model structure is called the *Strøm model structure* (cf. [Str72] Theorem 3). The identity is a Quillen equivalence $\mathbf{Top}_{\text{Strom}} \xrightarrow{id} \mathbf{Top}_{\text{Quillen}}$, since any strong homotopy equivalence is a (weak) homotopy equivalence (cf. [Hat01] page 342) and any Hurewicz fibration is a Serre fibration.

Remark 3.2.6. (i) Every map in $\mathcal{J}$-inj is a trivial fibration and every trivial fibration is in $\mathcal{J}$-inj (cf. [Hov99] Proposition 2.4.10 and Theorem 2.4.12). Hence the class of cofibrations can be defined as maps that have the left lifting property with respect to all trivial fibrations.

(ii) Every map in $\mathcal{I}$-cof is a trivial cofibration (cf. [Hov99] Proposition 2.4.9).

(iii) Every topological space is fibrant (cf. [Hov99] Corollary 2.4.14).

4 The model structure on simplicial sets

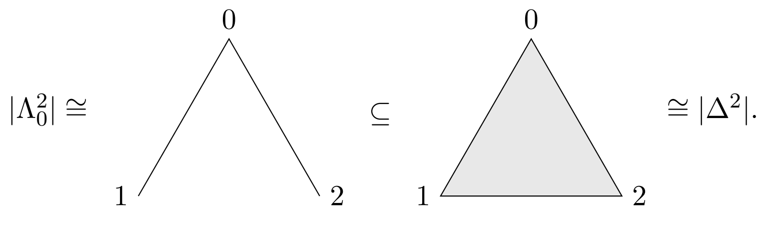
4.1 Kan Fibrations

In the subsequent sections, our goal is to construct a model structure on the category **sSet**, the category of simplicial sets. Therefore, we are going to define five classes of morphisms. Later on, we will prove that these five classes overlap and actually constitute only three classes, which satisfy the required properties of a model structure. In this section, we will be starting with the fibrations in **sSet**, the so called *Kan fibrations*. For this, we introduce the concept of the *k-th horn* in addition to the boundary $\partial\Delta^n$. Recall that the i-th face of Δ^n is the simplicial set $\partial^i\Delta^n := im(d^i : \Delta^{n-1} \to \Delta^n)$. For $n \geq 1$ and $0 \leq k \leq n$ the k-th horn of Δ^n is defined to be the simplicial subset

$$\Lambda^n_k := \bigcup_{i \neq k} \partial^i \Delta^n.$$

Thus, for $m \geq 0$, the set $(\Lambda^n_k)_m \subseteq \mathrm{Hom}_{\mathbf{\Delta}}(\underline{m}, \underline{n})$ consists of all maps $f : \underline{m} \to \underline{n}$ which can be factored as $f = d^i \circ f'$ with $f' \in \mathrm{Hom}_{\mathbf{\Delta}}(\underline{m}, \underline{n-1})$ and $i \neq k$. Since any map $f \in \mathrm{Hom}_{\mathbf{\Delta}}(\underline{m}, \underline{n})$ has a unique representation in cofaces and codegeneracies as in the beginning of Chapter 2, one can make this explicit for the different cases of m. For $m < n-1$, we have $(\Lambda^n_k)_m = (\Delta^n)_m$ and for $m = n-1$, $(\Lambda^n_k)_{n-1}$ consists of all maps in $(\Delta^n)_{n-1}$ whose unique representation does not contain d^k. In the cases $m \geq n$, any map in $(\Lambda^n_k)_m$ contains codegeneracies in their representation, i.e. all m-simplices are degenerate (cf. [Sin18] Remark 2.2.3).

As we will see later, the geometric realization of Λ^n_k is the union of all faces of $|\Delta^n|$ that contain the k-th vertex. For example, one could represent the topological space $|\Lambda^2_0|$ by the picture



In the last chapter we had the coequalizer diagram

$$\coprod_{0 \leq i < j \leq n} \Delta^{n-2} \rightrightarrows \coprod_{i=0}^{n} \Delta^{n-1} \longrightarrow \partial\Delta^n$$

for the boundary $\partial\Delta^n$ of Δ^n. There is a similar diagram for the horn Λ^n_k as follows:

$$\begin{tikzcd}
\Delta^{n-2} \arrow[r, "d^{j-1}"] \arrow[d, "in_{i<j}"'] & \Delta^{n-1} \arrow[d, "in_i"] \arrow[dr, "d^i"] & \\
\coprod\limits_{0\le i<j\le n} \Delta^{n-2} \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\psi"'] & \coprod\limits_{i\neq k} \Delta^{n-1} \arrow[r] & \Lambda^n_k \\
\Delta^{n-2} \arrow[r, "d^i"'] \arrow[u, "in_{i<j}"] & \Delta^{n-1} \arrow[u, "in_j"'] \arrow[ur, "d^j"'] &
\end{tikzcd}$$

where in denote the injections. I.e., the maps ϕ and ψ are determined by $\phi\circ in_{i<j} = in_i \circ d^{j-1}$ and $\psi \circ in_{i<j} = in_j \circ d^i$.

Lemma 4.1.1. The diagram

$$\coprod_{0\le i<j\le n} \Delta^{n-2} \rightrightarrows \coprod_{\substack{0\le i\le n\\ i\neq k}} \Delta^{n-1} \longrightarrow \Lambda^n_k$$

is a coequalizer in **sSet**.

Proof. (Cf. [GJ99] Chapter I Lemma 3.1) The diagram

$$\coprod_{0\le i<j\le n} \Delta^{n-1} \underset{\Lambda^n_k}{\times} \Delta^{n-1} \overset{p_1}{\underset{p_2}{\rightrightarrows}} \coprod_{\substack{0\le i\le n\\ i\neq k}} \Delta^{n-1} \longrightarrow \Lambda^n_k,$$

where the fibre product $\Delta^{n-1} \underset{\Lambda^n_k}{\times} \Delta^{n-1}$ comes from the square

$$\begin{tikzcd}
\Delta^{n-1} \underset{\Lambda^n_k}{\times} \Delta^{n-1} \arrow[r, "p_2"] \arrow[d, "p_1"'] & \Delta^{n-1} \arrow[d, "d^i"] \\
\Delta^{n-1} \arrow[r, "d^j"'] & \Lambda^n_k
\end{tikzcd}$$

is a coequalizer. We check this pointwise. Let $\underline{m}$ be an object in $\mathbf{\Delta}$, then we have

$$(\Delta^{n-1} \underset{\Lambda^n_k}{\times} \Delta^{n-1})_m = \{(f_1, f_2) \in \Delta^{n-1}_m \times \Delta^{n-1}_m \mid d^i \circ f_1 = d^j \circ f_2\}.$$

The above coequalizer in degree m is the set

$$\coprod_{\substack{0\le i\le n\\ i\neq k}} \Delta^{n-1}_m \Big/ \sim,$$

where the equivalence relation is induced by p_1 and p_2, but this means by definition for $f, g \in \coprod_{i \neq k} \Delta^{n-1}_m$, that $f \sim g$ if and only if $d^i \circ f = d^j \circ g$. And therefore, it is in bijection with the union of the images under the maps d^i, which is by definition the horn in degree m

$$(\Lambda^n_k)_m = \bigcup_{i \neq k} im(d^i : \Delta^{n-1}_m \to \Delta^n_m).$$

But the fibre product $\Delta^{n-1} \underset{\Lambda^n_k}{\times} \Delta^{n-1}$ is isomorphic to $\Delta^{n-1} \underset{\Delta^n}{\times} \Delta^{n-1}$, because $\Lambda^n_k \subseteq \Delta^n$ is a simplicial subset. Next, for $i < j$ we look at the diagram

$$\begin{tikzcd} \underline{n-2} \arrow[r, "d^{j-1}"] \arrow[d, "d^i"'] & \underline{n-1} \arrow[d, "d^i"] \\ \underline{n-1} \arrow[r, "d^j"'] & \underline{n} \end{tikzcd}$$

in $\mathbf{\Delta}$, which is commutative by the simplicial identities. Hence, we get a pushout square

$$\begin{tikzcd} \Delta^{n-2} \arrow[r, "d^{j-1}"] \arrow[d, "d^i"'] & \Delta^{n-1} \arrow[d, "d^i"] \\ \Delta^{n-1} \arrow[r, "d^j"'] & \Delta^n \end{tikzcd}$$

and therefore,

$$\Delta^{n-1} \underset{\Delta^n}{\times} \Delta^{n-1} \cong \Delta^{n-2}.$$

□

To proceed, it is useful to characterize the set of morphisms $\mathrm{Hom}_{\mathbf{sSet}}(\Lambda^n_k, X)$ for the horn Λ^n_k and a simplicial set $X \in \mathbf{sSet}$ as it is in Yoneda's Lemma for the representing functors Δ^n. One could show this directly but we use our previous lemma to prove the following result.

Corollary 4.1.2. The set $\mathrm{Hom}_{\mathbf{sSet}}(\Lambda^n_k, X)$ of simplicial set maps from Λ^n_k to X is in bijective correspondence with the set of n-tuples $(y_0, \dots, \hat{y_k}, \dots, y_n)$ of (n-1)-simplices y_i of X such that $d_i y_j = d_{j-1} y_i$ if $i < j$, and $i, j \neq k$, i.e.,

$$\mathrm{Hom}_{\mathbf{sSet}}(\Lambda^n_k, X) = \{(y_0, \dots, \hat{y_k}, \dots, y_n) \in (X_{n-1})^n \mid d_i y_j = d_{j-1} y_i \ \forall\ i < j,\ i \neq k \text{ and } j \neq k\}.$$

Proof. By applying the functor $\mathrm{Hom}_{\mathbf{sSet}}(-, X)$ to the above coequalizer diagram, it turns into the equalizer

$$\coprod_{0 \leq i < j \leq n} \mathrm{Hom}_{\mathbf{sSet}}(\Delta^{n-2}, X) \leftleftarrows \coprod_{\substack{0 \leq i \leq n \\ i \neq k}} \mathrm{Hom}_{\mathbf{sSet}}(\Delta^{n-1}, X) \longleftarrow \mathrm{Hom}_{\mathbf{sSet}}(\Lambda^n_k, X)$$

since it is contravariantly representable. By using the two isomorphisms

$$\mathrm{Hom}_{\mathbf{sSet}}(\coprod_{\substack{0 \leq i \leq n \\ i \neq k}} \Delta^{n-1}, X) \cong \prod_{\substack{0 \leq i \leq n \\ i \neq k}} \mathrm{Hom}_{\mathbf{sSet}}(\Delta^{n-1}, X)$$

and

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^{n-1}, X) \cong X_{n-1},$$

as well as the functoriality of the latter, the statement follows. □

Remark 4.1.3. A similar consideration can be done for the boundary $\partial\Delta^n$. Using the coequalizer diagram in the proof of Proposition 2.3.7, one gets that

$$\mathrm{Hom}_{\mathbf{sSet}}(\partial\Delta^n, X) = \{(y_0, \dots, y_n) \in (X_{n-1})^n \mid d_i y_j = d_{j-1} y_i \ \forall\ i < j\}.$$

Now, we can define when a morphism in **sSet** is a fibration.

Definition 4.1.4. *For $n > 0$ and $0 \leq k \leq n$ let $\iota : \Lambda^n_k \hookrightarrow \Delta^n$ be the inclusion, i.e. the natural transformation $\iota = (\iota_{\underline{m}})_{\underline{m} \in \mathbf{\Delta}}$, where*

$$\iota_{\underline{m}} : (\Lambda^n_k)_m \xrightarrow{\subseteq} (\Delta^n)_m$$

*is the inclusion of sets. A map $f : X \to Y$ of simplicial sets is a (Kan) fibration in **sSet**, if it has the RLP with respect to all inclusions $\iota : \Lambda^n_k \hookrightarrow \Delta^n$. That is if for every commutative diagram*

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, "\iota"'] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y, \end{tikzcd}$$

there is a dotted arrow, making the diagram commute.

Remark 4.1.5. By Yoneda's Lemma and our previous Corollary 4.1.2, the definition of being a Kan fibration for $f : X \to Y$ amounts to the condition that if $(x_0 ... \hat{x_k} ... x_n)$ is an n-tuple of $n-1$-simplices of X such that $d_i x_j = d_{j-1} x_i$ for $i < j$, $i, j \neq k$, and if there is an n-simplex y of Y such that $d_i y = f_{n-1}(x_i)$, then there is an n-simplex x of X such that $d_i x = x_i$, $i \neq k$, and such that $f_n(x) = y$.

One can show that there are homeomorphisms $|\Delta^n| \cong D^n \cong D^{n-1} \times I$ and $|\Lambda^n_k| \cong D^{n-1}$ such that the inclusion $\Lambda^n_k \to \Delta^n$ becomes the inclusion $D^{n-1} \to D^{n-1} \times I$. Thus a continuous map of spaces $f : T \to U$ is a Serre fibration if it has the right lifting property for all diagrams

$$\begin{tikzcd} {|\Lambda^n_k|} \arrow[r] \arrow[d] & T \arrow[d, "f"] \\ {|\Delta^n|} \arrow[r] \arrow[ur, dashed] & U. \end{tikzcd}$$

And by the adjointness of the functors $|-| \dashv S(-)$ (cf. Proposition 2.2.4), all such diagrams can be identified with their corresponding diagrams

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & S(T) \arrow[d, "S(f)"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & S(Y), \end{tikzcd}$$

so that $f : T \to U$ is a Serre fibration if and only if $S(f) : S(T) \to S(U)$ is a Kan fibration. This is a main part of the motivation, to define Kan fibrations in such this way. The other statement that the realization of a Kan fibration is again a Serre fibration does not follow immediately and the proof is a long technical result.

Lemma 4.1.6. For every space $T \in$ **Top**, the map $S(T) \to * = \Delta^0$ is a fibration.

Proof. (Cf. [GJ99] Chapter I Lemma 3.3) Since $|\Lambda^n_k| \subseteq |\Delta^n|$ is a retract, the inclusion $i : |\Lambda^n_k| \hookrightarrow |\Delta^n|$ has a section $r : |\Delta^n| \to |\Lambda^n_k|$. Now we consider a commutative diagram of solid arrows in **Top** of the form

$$\begin{tikzcd} {|\Lambda^n_k|} \arrow[r, "g"] \arrow[d, "i"'] & T \arrow[d] \\ {|\Delta^n|} \arrow[r] \arrow[ur, dashed, "\exists h"] & *, \end{tikzcd}$$

where $*$ is the one-point space. Then the dotted arrow exists by defining $h = g \circ r$, showing that $T \to *$ is a Serre fibration for any topological space T. And therefore, $S(T) \to S(*) = \Delta^0 = *$ is a Kan fibration. $\square$

A simplicial set $X \in$ **sSet** is called a *Kan complex* if the the canonical morphism $X \to *$ is a fibration, i.e. if it is fibrant (cf. Definition 3.1.7). Alternatively, by Remark 4.1.5 above, a simplicial set X is a Kan complex if and only if one of the following equivalent conditions holds:

K1: Every morphism $\alpha : \Lambda^n_k \to X$ can be extended to a map defined on Δ^n in the sense that there is a commutative diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\alpha"] \arrow[d] & Y \\ \Delta^n \arrow[ur] & \end{tikzcd}$$

K2: For each n-tuple of $(n-1)$-simplices $(x_0 ... \hat{x_k} ... x_n)$ of X such that $d_i x_j = d_{j-1} x_i$ if $i < j$, $i, j \neq k$, there is an n-simplex x such that $d_i x = x_i$ for $i \neq k$.

As we have seen in Lemma 4.1.6, any singular complex $S(T)$ for $T \in \mathbf{Top}$ is a Kan complex. Another standard example of Kan complexes is that of simplicial groups. A simplicial group is a simplicial set $G \in \mathbf{sSet}$ such that for any $\underline{n} \in \mathbf{\Delta}$, the set G_n carries a group structure and for any map $\theta : \underline{m} \to \underline{n}$ in $\mathbf{\Delta}$, the map $\theta^* : G_n \to G_m$ is a group homomorphism. In other words, a simplicial group is a group functor

$$G : \mathbf{\Delta}^{op} \to \mathbf{Grp},$$

where $\mathbf{Grp}$ denotes the category of groups. We denote the unit element of the group G_n by e_n. That these simplicial groups are fibrant, will be shown in the following proposition which was a result by Robert Lee Moore in 1954. The proof needs some work but uses no other techniques than easy group calculations and the simplicial identities. There is a more "geometric" version of the proof in [JT] Theorem 3.1.3, using the theory of *anodyne extensions*, which we will introduce in the next section.

Proposition 4.1.7. ***(Moore)***
The underlying simplicial set of a simplicial group

$$G : \mathbf{\Delta}^{op} \to \mathbf{Grp}$$

is a Kan complex.

Proof. (Cf. [May67] Theorem 17.1 and [HP14] Proposition 4.9) Let

$$x_0, ..., x_{k-1}, x_{k+1}, ..., x_n \in G_{n-1}$$

be a family of $(n-1)$-simplices of G which satisfy the condition $d_i x_j = d_{j-1} x_i$ for all $i < j$ and $i, j \neq k$. We want to inductively construct a family $y_0, ..., y_{k-1} \in G_n$ such that for any $i = 0, ..., k-1$ we have $d_j(y_i) = x_j$, $\forall j < k$. Note that we may assume $k > 0$ in this step. Having finished that we recursively define elements $y_{k+1}, ..., y_n \in G_n$, starting with the construction of y_n and inductively going downward to $y_n, \dots y_i$ for $n \geq i > k$, such that such that $d_j(y_i) = x_j$ for all $j < k$ and all $i < j \leq n$. Then $y_{k+1} \in G_n$ is our required element with $d_j(y_{k+1}) = x_j$ for all $j \in \{0, ..., k-1, k+1, ...n\}$ and by the above condition K2, G is fibrant. We start defining our family with $y_0 := s_0(x_0)$ and

$$y_i := s_i(x_i \cdot d_i(y_{i-1}^{-1})) \cdot y_{i-1}, \text{ for } 0 < i < k.$$

Then we have

$$d_i(y_i) = d_i(s_i(x_i)) \cdot d_i(s_i(d_i(y_{i-1}^{-1}))) \cdot d_i(y_{i-1}) = x_i \cdot d_i(y_{i-1}^{-1}) \cdot d_i(y_{i-1}) = x_i,$$

because $d_i s_i = id$. We need to show that the same equation holds for $j < i$. Using the assumption $d_j x_i = d_{i-1} x_j$ and the defining properties of the previous element y_{i-1}, we get

$$d_j(x_i) = d_{i-1}(x_j) = d_{i-1}(d_j(y_{i-1})) = d_j(d_i(y_{i-1})),$$

and

$$\begin{aligned} d_j(x_i \cdot d_i(y_{i-1}^{-1})) \cdot d_j(x_i) =& d_j(x_i \cdot d_i(y_{i-1}^{-1})) \cdot d_j(d_i(y_{i-1})) \\ =& d_j(x_i \cdot d_i(y_{i-1}^{-1}) \cdot d_i(y_{i-1})) = d_j(x_i) \end{aligned}$$

and therefore $d_j(x_i \cdot d_i(y_{i-1}^{-1})) = e_{n-2}$. Hence, we get for $j < i$

$$\begin{aligned} d_j(y_i) =& d_j(s_i(x_i \cdot d_i(y_{i-1}^{-1})) \cdot y_{i-1}) = d_j(s_i(x_i \cdot d_i(y_{i-1}^{-1}))) \cdot d_j(y_{i-1}) \\ =& s_{i-1}(d_j(x_i \cdot d_i(y_{i-1}^{-1}))) \cdot x_j = e_{n-1} \cdot x_j = x_j, \end{aligned}$$

as required. Now we want to construct the rest of the family $y_{k+1}, ..., y_n$, starting with the last one and defining the rest inductively backwards. To do so, we set

$$y_n := s_{n-1}(x_n \cdot d_n(y_{k-1}^{-1})) \cdot y_{k-1}$$

(if $k = 0$ one can take the unit element e_n instead of y_{k-1}) and we define y_i from y_{i+1} as

$$y_i := s_{i-1}(x_i \cdot d_i(y_{i+1}^{-1})) \cdot y_{i+1},$$

for $k + 1 \leq i \leq n - 1$. Then again, we have

$$d_i(y_i) = d_i(s_{i-1}(x_i \cdot d_i(y_{i+1}^{-1})) \cdot d_i(y_{i+1}) = x_i \cdot d_i(y_{i+1}^{-1}) \cdot d_i(y_{i+1}) = x_i.$$

It remains to show that $d_j(y_i) = x_j$ for $j > i$ and $j < k$. We have

$$d_j(y_i) = d_j(s_{i-1}(x_i \cdot d_i(y_{i+1}^{-1}))) \cdot d_j(y_{i+1}) = s_{i-1}(d_{j-1}(x_i \cdot d_i(y_{i+1}^{-1}))) \cdot x_j \text{ for } j > i$$

and

$$d_j(y_i) = d_j(s_{i-1}(x_i \cdot d_i(y_{i+1}^{-1}))) \cdot d_j(y_{i+1}) = s_{i-2}(d_j(x_i \cdot d_i(y_{i+1}^{-1}))) \cdot x_j \text{ for } j < k,$$

so we just need to show that $d_{j-1}(x_i \cdot d_i(y_{i+1}^{-1})) = e_{n-2} = d_j(x_i \cdot d_i(y_{i+1}^{-1}))$ for both cases. In the second case $j < k < i + 1$ we have

$$d_j(x_i) = d_{i-1}(x_j) = d_{i-1}(d_j(y_{i+1})) = d_j(d_i(y_{i+1}))$$

and therefore

$$d_j(x_i \cdot d_i(y_{i+1}^{-1})) \cdot d_j(x_i) = d_j(x_i \cdot d_i(y_{i+1}^{-1})) \cdot d_j(d_i(y_{i+1})) = d_j(x_i).$$

For the first case $j > i$, we calculate similarly

$$d_{j-1}(x_i) = d_i(x_j) = d_i(d_j(y_{i+1})) = d_{j-1}(d_i(y_{i+1}))$$

and therefore

$$d_{j-1}(x_i \cdot d_i(y_{i+1}^{-1})) \cdot d_{j-1}(x_i) = d_{j-1}(x_i \cdot d_i(y_{i+1}^{-1})) \cdot d_{j-1}(d_i(y_{i+1})) = d_{j-1}(x_i).$$

Hence, we have $d_{j-1}(x_i \cdot d_i(y_{i+1}^{-1})) = e_{n-2}$ for $j > i$ and $d_j(x_i \cdot d_i(y_{i+1}^{-1})) = e_{n-2}$ for $j < k$. $\square$

Our last example of a fibrant simplicial set is that of a nerve of a group (cf. Example 2.1.12). Any group G gives rise to a category also denoted by G, with one single object $*$ and morphisms $g : * \to *$ for each element $g \in G$. Composition is defined by multiplication in G and the neutral element forms the identity morphism. In particular, every morphism in the category is an isomorphism, i.e. the category G is a so-called groupoid. Then we can talk about the nerve BG of G and we have the following statement, which is in fact true for groupoids in general (cf. [GJ99] Chapter I Lemma 3.5).

Lemma 4.1.8. For every group G, the nerve BG is a Kan complex.

Proof. See [GJ99] Chapter I Lemma 3.5. □

Remark 4.1.9. The realization $|BG|$ for the nerve of a group G is the Eilenberg-Mac Lane space $K(G,1)$. In other words all homotopy groups $\pi_n(|BG|)$ of $|BG|$ vanish except the first one, for which $\pi_1(|BG|) \cong G$. We will see this later in the last chapter, by defining simplicial homotopy groups for fibrant simplicial sets and giving isomorphisms between both the simplicial and the topological homotopy groups of a Kan complex X and $|X|$, respectively.

Example 4.1.10. For the rest of the work we fix some notations, often used in the following sections. In the 1-simplex Δ^1 there are the two faces $\partial^1\Delta^1 = im(d^1 : \Delta^0 \to \Delta^1)$ and $\partial^0\Delta^1 = im(d^0 : \Delta^0 \to \Delta^1)$, building the boundary $\partial\Delta^1 \subseteq \Delta^1$. Since $d^1 \circ f = 0 : \underline{m} \to \underline{1},\ m \mapsto 0$ and $d^0 \circ f = 1 : \underline{m} \to \underline{1},\ m \mapsto 1$ are the constant maps mapping to 0 or 1, respectively for all $f \in \Delta^0(\underline{m})$, the faces are one-pointed sets in each degree. Thus, they are disjoint, building the two vertices in Δ^1. We denote the faces by

$$\{0\} := \partial^1\Delta^1 \text{ and } \{1\} := \partial^0\Delta^1,$$

with $\{0\} \mathbin{\dot\cup} \{1\} = \partial\Delta^1 \subseteq \Delta^1$.

Example 4.1.11. The standard n-simplex Δ^n is no Kan-complex for $n > 0$ (cf. [GJ99] page 15). For $n = 1$ we consider the lifting diagram

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "\alpha"] \arrow[d] & \Delta^1 \\ \Delta^2 \arrow[ur] & \end{tikzcd}$$

over the horn inclusion $\Lambda^2_0 \subseteq \Delta^2$ as depicted above. Here $\alpha \in \mathrm{Hom}_{\mathbf{sSet}}(\Lambda^2_0, \Delta^1)$ is the natural transformation corresponding $(-,1,0) \in (\Delta^1_1)^2$ (cf. Corollary 4.1.2). But the lifting property fails since there is no map $f \in \mathrm{Hom}_{\mathbf{sSet}}(\Delta^2, \Delta^1) = \Delta^1_2 = \mathrm{Hom}_{\mathbf{\Delta}}(\underline{2}, \underline{1})$ such that $f(1) = 1$ and $f(2) = 0$, since it would not be order-preserving.

4.2 Anodyne extensions

A leading role in homotopy theory of simplicial sets is played by the class of anodyne extensions, which we will discuss in this section. As we will see later on, this class forms the trivial cofibrations on **sSet**.

Definition 4.2.1. *A class $\mathcal{M}$ of monomorphisms of **sSet** is said to be saturated if the following conditions are satisfied:*

*(i) $\mathcal{M}$ contains all isomorphisms of **sSet***

(ii) *$\mathcal{M}$ is closed under pushouts. That is, if*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & C \arrow[d, "i'"] \\ B \arrow[r] & B \sqcup_A C \end{tikzcd}$$

is a pushout diagram and if i lies in $\mathcal{M}$ then so does i'.

(iii) *$\mathcal{M}$ is closed under retracts. That is, if*

$$\begin{tikzcd} A' \arrow[r, "j"] \arrow[d, "i'"'] & A \arrow[r, "u"] \arrow[d, "i"'] & A' \arrow[d, "i'"] \\ B' \arrow[r, "k"'] & B \arrow[r, "v"'] & B' \end{tikzcd}$$

is a commutative diagram with $uj = id_{A'}$, $vk = id_{B'}$ and if $i \in \mathcal{M}$, then $i' \in \mathcal{M}$.

(iv) *$\mathcal{M}$ is closed under coproducts. That is, if $(i_j : A_j \to B_j)_{j \in J}$ s a family of monomorphisms with $i_j \in \mathcal{M}$ for each $j \in J$, then the induced morphism*

$$\coprod_{j \in J} i_j : \coprod_{j \in J} A_j \to \coprod_{j \in J} B_j$$

is in $\mathcal{M}$.

(v) *$\mathcal{M}$ is closed under ω-composites. That is, if*

$$A_1 \xrightarrow{i_1} A_2 \xrightarrow{i_2} A_3 \xrightarrow{i_3} \ldots$$

is a countable family of morphisms of $\mathcal{M}$, then the morphism

$$A_1 \to \varinjlim_{n \geq 1} A_n$$

is in $\mathcal{M}$.

Remark 4.2.2. Note that in **Set**, the category of sets, if we have an injective map $i : X \to X_1$ and a pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i"'] & X_2 \arrow[d, "i'"] \\ X_1 \arrow[r] & X_1 \sqcup_X X_2 \end{tikzcd}$$

then the map i' is injective, as well, by commutativity of the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i"'] & X_2 \arrow[d, "i'"] \arrow[ddr, bend left, "id"] & \\
X_1 \arrow[r] \arrow[drr, bend right, "\bar{i}"'] & X_1 \coprod_X X_2 \arrow[dr, "\phi"] & \\
& & X_2,
\end{tikzcd}$$

where $\bar{i}$ is defined as follows. For $x \in i(X)$ there is a unique $a \in X$ such that $x = i(a)$ and we set $\bar{i}(x) := f(a)$. For $x \notin i(X)$ we set $\bar{i}(a) := \hat{z}$, for fixed $\hat{z} \in X_2 \neq \emptyset$. Alternatively, one could work with the explicit definition of a pushout in **Set**. Hence, if we have a pushout diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & C \arrow[d, "i'"] \\
B \arrow[r] & B \coprod_A C
\end{tikzcd}$$

in **sSet**, where i is a pointwise monomorphism then i' is a pointwise monomorphism, as it holds for any required morphism in the above definition.

Lemma 4.2.3. Let $p : X \to Y$ be a fixed morphism of simplicial sets. Then the class M_p of all monomorphisms which have the LLP with respect to p is saturated.

Proof. (Cf. [GJ99] Chapter I Lemma 4.1 and [HP14] Lemma 5.3) Clearly, all isomorphisms have the LLP. The other conditions follow from construction and the universal properties. We want to show at least some of them, starting with the property of being closed under pushouts. Let $i : A \to B$ be a morphism in M_p and consider a commutative diagram of the form

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"] & C \arrow[r] \arrow[d, "j"] & X \arrow[d, "p"] \\
B \arrow[r] & B \coprod_A C \arrow[r] & Y,
\end{tikzcd}$$

where the left square is a pushout. Since $i \in M_p$, there is a map $\theta : B \to X$ such that

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & X \arrow[d] \\
B \arrow[r] \arrow[urr, "\theta"] & B \coprod_A C \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd}$$

the diagram of solid arrows commutes. By the universal property of the pushout, the dotted arrow $B \coprod_A C \to X$ exists. Hence, $j \in M_p$ using that j is a monomorphism by our discussion above. The closedness under countable composition also follows from the universal property of the colimit in the following way. Consider the diagram

$$
\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "i_1"'] & X \arrow[dddd, "p"] \\
\vdots \arrow[d] & \\
A_n \arrow[uur] \arrow[d, "i_n"'] & \\
\vdots \arrow[d] & \\
\varinjlim A_n \arrow[r] \arrow[uuuur, dotted] & Y,
\end{tikzcd}
$$

where the maps $A_n \to X$ are constructed inductively using that $i_n : A_n \to A_{n+1}$ is in M_p for all n. The commutativity of all solid diagrams then gives the dotted arrow $\varinjlim A_n \to X$ making everything commutative. Again, it is clear from the construction of colimits in sSet that together with all i_n also the map $A_1 \to \varinjlim A_n$ is a monomorphism. The statements that M_p is closed under coproducts follows in a similar way and for retracts it follows immediately from the definition. $\square$

Definition 4.2.4. *Let B be a class of monomorphisms of **sSet**. The intersection of all saturated classes containing B is called the saturated class generated by B and denoted by M_B.*

Example 4.2.5. To get used to this new terminology, we want to look at a first example. Therefore, consider the family of inclusions $B := \{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geq 0\}$. We claim that the saturated class M_B generated by this family consists of all monomorphisms in **sSet**. To see this, let $i : A \to X$ be an arbitrary monomorphism and let

$$e(X \setminus A)_n := \{x \in X_n \setminus A_n \mid x \text{ is not of the form } s_i y \text{ for any } 0 \leq i \leq n-1 \text{ and } y \in X_{n-1}\}$$

be the set of non-degenerate n-simplices of X which are not in A, where we view $A \subseteq X$ as a subcomplex. By condition (iv) in Definition 4.2.1, together with $\partial\Delta^n \hookrightarrow \Delta^n$ also

$$\coprod_{e(X\setminus A)_n} \partial\Delta^n \to \coprod_{e(X\setminus A)_n} \Delta^n$$

has to be in M_B. Similarly to the proof of Proposition 2.3.7, one can show that the diagram

$$
\begin{tikzcd}
\coprod_{e(X\setminus A)_n} \partial\Delta^n \arrow[r] \arrow[d] & \coprod_{e(X\setminus A)_n} \Delta^n \arrow[d] \\
Sk^{n-1}X \cup A \arrow[r] & Sk^n X \cup A
\end{tikzcd}
$$

is a pushout for all $n \geq 0$ and by axioms (ii) and (v) also the map

$$Sk^{-1}X \cup A \to \varinjlim_{n \geq -1} Sk^n X \cup A$$

is in M_B, where we introduce by convention $Sk^{-1}X = \emptyset$. But since $Sk^{n-1}X \subseteq Sk^n X \subseteq \ldots \bigcup_n Sk^n X = X$ we have

$$Sk^{-1}X \cup A = A \text{ and } \varinjlim_{n \geq -1} Sk^n X \cup A = X,$$

and therefore, the above map is precisely $i : A \to X$. Hence, the saturated class generated by B contains all monomorphisms in **sSet** (cf. [JT] pages 38/39).

Definition 4.2.6. *The saturated class generated by the family of inclusions*

$$\{\Lambda^n_k \hookrightarrow \Delta^n |\, 0 \leq k \leq n,\ n > 0\}.$$

is called the class of anodyne extensions.

Proposition 4.2.7. A morphism $p : X \to Y$ is a Kan fibration if and only if it has the RLP with respect to all anodyne extensions.

Proof. Clearly, by definition, if a morphism has the RLP with respect to all anodyne extensions it is a fibration. For the other direction let $p : X \to Y$ be a fibration in **sSet**. By Lemma 4.2.3, M_p, the class of monomorphisms with LLP with respect to p, is saturated. But the anodyne extensions are defined to be the saturated class generated by all $\Lambda^n_k \hookrightarrow \Delta^n$ which all have the LLP with respect to p. Hence, by minimality, the anodyne extensions are contained in M_p. □

Later we will interprete this result in the sense that the anodyne extensions are exactly the cofibrations of a model structure which are also weak equivalences (cf. Proposition 4.6.8).

Proposition 4.2.8. Consider the three classes of monomorphisms

$$\mathbf{B1} := \text{the set of all inclusions } \Lambda^n_k \hookrightarrow \Delta^n, 0 \leq k \leq n, n > 0,$$

$$\mathbf{B2} := \text{the set of all inclusions } (\Delta^1 \times \partial\Delta^n) \cup (\{e\} \times \Delta^n) \hookrightarrow (\Delta^1 \times \Delta^n), \quad e = 0, 1,$$

where $0, 1 \subseteq \Delta^1$ are the two simplicial subsets only containing the constant maps in each degree, and

$$\mathbf{B3} := \text{the set of all inclusions } (\Delta^1 \times Y) \cup (\{e\} \times X) \hookrightarrow (\Delta^1 \times X),$$

where $Y \subseteq X$ is an inclusion of simplicial sets, and $e = 0, 1$. Then the families **B1**, **B2** and **B3** all generate the same saturated class.

Proof. See [GJ99] Chapter I Proposition 4.2. □

Corollary 4.2.9. Let $j : K \hookrightarrow L$ be an anodyne extension let $I : Y \hookrightarrow X$ be an arbitrary inclusion. Then the induced map

$$(K \times X) \cup (L \times Y) \to (L \times X)$$

and the induced map

$$(K \times X) \underset{(K \times Y)}{\sqcup} (L \times Y) \to (L \times X)$$

from the pushout are anodyne extensions.

Proof. See [GJ99] Chapter I Corollary 4.6 for the first statemant and [Hov99] Theorem 3.3.2 for the second statemant. $\square$

Next, we define a new class of morphisms in **sSet**, the class of trivial fibrations. This is an explicit definition. Of course, once we have our model structure on **sSet** there will be a second definition of a trivial fibration. Namely, this will be a morphism which is both a fibration and a weak equivalence. However, we will see later, that these two definitions coincide.

Definition 4.2.10. *A morphism $p : X \to Y$ is called a trivial fibration if it has the RLP with respect to the family $\{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geq 0\}$.*

Remark 4.2.11. Let $p : X \to Y$ be a trivial fibration, and let M_p be the class of all monomorphisms with the LLP with respect to p. This, again, is saturated and contains all $\partial\Delta^n \hookrightarrow \Delta^n$ by definition. But as we have discussed in Example 4.2.5, the saturated class generated by $\{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geq 0\}$, contains all monomorphisms. Hence, p has the RLP with respect to all $\Lambda^n_k \hookrightarrow \Delta^n$. So at least we can say, that any trivial fibration is also a fibration, as expected.

An important step for our model structure on **sSet** is to show the factorization axiom in the definition. To this end, we will need the following theorems.

Theorem 4.2.12. Any morphism $f : X \to Y$ in **sSet** can be factored as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & E \arrow[ur, "p"'] & \end{tikzcd}$$

where i is an anodyne extension and p is a fibration.

Proof. (Cf. [JT] Theorem 3.1.1) Let L be the set of all commutative diagrams of the form

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] & Y, \end{tikzcd}$$

with $n \geq 1$ and $0 \leq k \leq n$. Taking the coproduct over L this yields to the diagram

$$\begin{tikzcd} \coprod_L \Lambda^n_k \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\ \coprod_L \Delta^n \arrow[r] & Y, \end{tikzcd}$$

where i is anodyne, since the inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ are anodyne. Now we take the pushout

$$\begin{tikzcd} \coprod_L \Lambda^n_k \arrow[r] \arrow[d, "i"] & X \arrow[d, "i^0"] \\ \coprod_L \Delta^n \arrow[r] & X^1, \end{tikzcd}$$

of i and the upper horizontal map, denoted by X^1, where again, by definition, i^0 is anodyne. By the universal property, we get a map $f^1 : X^1 \to Y$ such that the diagram

$$\begin{tikzcd} \coprod_L \Lambda^n_k \arrow[r] \arrow[d, "i"] & X \arrow[d, "i^0"] \arrow[ddr, bend left, "f"] & \\ \coprod_L \Delta^n \arrow[r] \arrow[drr, bend right] & X^1 \arrow[dr, "f^1"] & \\ & & Y, \end{tikzcd}$$

commutes. Thus, we have a factorization

$$\begin{tikzcd} X \arrow[rr, "i^0"] \arrow[dr, "f"'] & & X^1 \arrow[dl, "f^1"] \\ & Y & \end{tikzcd}$$

with i^0 anodyne. We now repeat this whole process with f^1 instead of f and obtain the diagram

$$\begin{tikzcd} X \arrow[r, "i^0"] \arrow[dr, "f"'] & X^1 \arrow[r, "i^1"] \arrow[d, "f^1"] & X^2 \arrow[dl, "f^2"] \\ & Y & \end{tikzcd}$$

where, again, i^1 is anodyne and we keep repeating it with any obtained f^n. We set $X^0 = X$ and $f^0 = f$ and $E = \varinjlim_{n \geq 0} X^n$ for the colimit. By the universal property we get an induced map $p : E \to Y$ by all f^n and we denote the map $X \to E$ by i. This gives us a factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & E \arrow[ur, "p"'] & \end{tikzcd}$$

with i anodyne. It remains to show that p is a fibration. Consider any diagram of the form

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "h"] \arrow[d] & E \arrow[d, "p"] \\ \Delta^n \arrow[r] & Y, \end{tikzcd}$$

with $n \geq 1$ and $0 \leq k \leq n$. Since Λ^n_k has only finitely many non-degenerate simplices, h factors through some X^j for $j \geq 0$ as in

$$\begin{tikzcd} \Lambda^n_k \arrow[rr, "h"] \arrow[dr] \arrow[dd] & & E \arrow[dd, "p"] \\ & X^j \arrow[ur] \arrow[dr, "f^j"] & \\ \Delta^n \arrow[rr] & & Y. \end{tikzcd}$$

To see why, we assume given a map of simplicial sets $\Lambda^n_k \to E$. For any $m \geq 0$ the set E_m is the union of the $(X^i)_m$. Therefore, there is an index i such that $(\Lambda^n_k)_m$ maps to X^i_m for all $0 \leq m \leq n$, since the set of simplices of Λ^n_k is finite. It remains to see that $(\Lambda^n_k)_m$ maps to $(X^i)_m$ also if $m > n$. To see this, let $f : \underline{m} \to \underline{n}$ be an m-simplex of Λ^n_k. It can be factorized as $\underline{m} \xrightarrow{g} \underline{j} \xrightarrow{h} \underline{n}$, where g is surjective and h is injective. Then we have $j \leq n$ and there is a commutative diagram

$$\begin{tikzcd} (\Lambda^n_k)_j \arrow[r] \arrow[d, "g^*"'] & E_j \arrow[d, "g^*=E(g)"] \\ (\Lambda^n_k)_m \arrow[r] & E_m. \end{tikzcd}$$

Since $f = h \circ g \in (\Lambda^n_k)_m$ is the image of $h \in (\Lambda^n_k)_j$ under g^* and since h maps into $(X^i)_j$, using that $j \leq n$, we get that f maps into $(X^i)_m$.
Using this factorization of h means we have reduced our lifting problem to the diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "h"] \arrow[d] & X^j \arrow[r, "i^j"] \arrow[d, "f^j"'] & X^{j+1} \arrow[dl, "f^{j+1}"] \\ \Delta^n \arrow[r] \arrow[urr, dashed] & Y & \end{tikzcd}$$

But here the dotted map exists, because X^{j+1} and f^{j+1} were constructed from such a pushout as above, satisfying the required condition. Hence, we obtain a lift from Δ^n to E, making the diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "h"] \arrow[d] & E \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

commute and therefore p is a fibration. $\square$

Corollary 4.2.13. Any morphism $f : X \to Y$ that has the LLP with respect to the class of all Kan fibrations is anodyne.

Proof. (Cf. [JT] Corollary 3.1.1) By our previous Theorem 4.2.12, we can factorize f as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & E \arrow[ur, "p"'] & \end{tikzcd}$$

where i is an anodyne extension and p is a fibration. From this factorization we get the commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & E \arrow[d, "p"] \\ Y \arrow[r, "id_Y"'] \arrow[ur, dashed, "k"] & Y, \end{tikzcd}$$

where the dotted arrow exists, since f has the LLP with respect to p. But then f is a retract of i via

$$\begin{tikzcd} X \arrow[r, "id_X"] \arrow[d, "f"'] & X \arrow[r, "id_X"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ Y \arrow[r, "k"'] & E \arrow[r, "p"'] & Y \end{tikzcd}$$

and therefore, f is anodyne. $\square$

Theorem 4.2.14. Any morphism $f : X \to Y$ in **sSet** can be factored as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\ & E \arrow[ur, "p"'] & \end{tikzcd}$$

where i is a monomorphism, and p is a trivial fibration.

Proof. The proof is analogous to the one of Theorem 4.2.12. Replacing the family of inclusions $\{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geq 0\}$ with $\{\Lambda^n_k \hookrightarrow \Delta^n | \; 0 \leq k \leq n, \; n > 0\}$ will yield to the proof of the theorem at hand. $\square$

4.3 Function complexes

Given two simplicial sets $X, Y \in \mathbf{sSet}$ we construct a new simplicial set $\mathrm{Hom}(X, Y)$. It will help us later to work with simplicial homotopies. As we will see, this construction behaves nicely under the property of being fibrant in the sense that if Y is fibrant then $\mathrm{Hom}(X, Y)$ is fibrant for all $X \in \mathbf{sSet}$.

Definition 4.3.1. *Let X and Y be simplicial sets. The function complex (sometimes called mapping complex or mapping space)* $\mathrm{Hom}(X, Y)$ *is the simplicial set defined by*

$$\mathrm{Hom}(X, Y)(\underline{n}) = \mathrm{Hom}(X, Y)_n := \mathrm{Hom}_{\boldsymbol{sSet}}(X \times \Delta^n, Y)$$

on objects. And on morphisms $\theta : \underline{m} \to \underline{n}$ the induced function

$$\mathrm{Hom}(X, Y)(\theta) = \theta^* : \mathrm{Hom}(X, Y)_n \to \mathrm{Hom}(X, Y)_m$$

is defined by

$$(X \times \Delta^n \xrightarrow{f} Y) \mapsto (X \times \Delta^m \xrightarrow{id \times \theta} X \times \Delta^n \xrightarrow{f} Y).$$

Remark 4.3.2. The simplicial set $\mathrm{Hom}(X, Y)$ depends contravariantly on X and covariantly on Y, hence, we have a functor

$$\mathrm{Hom}(-, -) : \mathbf{sSet}^{op} \times \mathbf{sSet} \to \mathbf{sSet}.$$

There is an important natural transformation

$$ev : X \times \mathrm{Hom}(X, Y) \to Y,$$

the so called evaluation map, defined as follows. Given $x \in X_n$ and $f : X \times \Delta^n \to Y \in \mathrm{Hom}(X, Y)_n$ we set

$$ev_n(x, f) := f_n(x, 1_n) \in Y_n,$$

where $1_n \in \Delta^n(\underline{n}) = \mathrm{Hom}_{\boldsymbol{\Delta}}(\underline{n}, \underline{n})$ is the identity. To see that the evaluation map is indeed a natural transformation of simplicial sets, it suffices to show the commutativity with the face and degeneracy maps. For a coface map $d^j : \underline{n-1} \to \underline{n}$ we consider the diagram

$$\begin{tikzcd}
X_{n-1} \times \mathrm{Hom}(X, Y)_{n-1} \arrow[r, "ev_{n-1}"] & Y_{n-1} \\
X_n \times \mathrm{Hom}(X, Y)_n \arrow[r, "ev_n"'] \arrow[u, "d_j^X \times (d^j)^*"] & Y_n, \arrow[u, "d_j^Y"']
\end{tikzcd}$$

where we denote $d_j^X := X(d^j)$ and $d_j^Y := Y(d^j)$ the face maps on X and Y, respectively. Let $x \in X_n$ and $f \in \mathrm{Hom}(X, Y)_n$. Then we have

$$\begin{aligned}
ev_{n-1}(d_j^X x, (d^j)^*(f)) &= f_{n-1} \circ (id \times d^j)(d_j^X x, 1_{n-1}) = f_{n-1}(d_j^X x, d^j \circ 1_{n-1}) \\
&= f_{n-1}(d_j^X x, 1_n \circ d^j) = d_j^Y(f_n(x, 1_n)) \\
&= d_j^Y(ev_n(x, f)),
\end{aligned}$$

since $f : X \times \Delta^n \to Y$ is a natural transformation itself. The similar calculation holds for degeneracies and therefore ev commutes with all θ^* for $\theta : \underline{m} \to \underline{n}$ and hence, is a morphism of simplicial sets.

Proposition 4.3.3. ***(The Exponential Law)***
Let K, X, Y be simplicial sets. The map

$$ev_* : \mathrm{Hom}_{\mathbf{sSet}}(K, \mathrm{Hom}(X,Y)) \to \mathrm{Hom}_{\mathbf{sSet}}(X \times K, Y)$$

which is defined by sending the simplicial map $g : K \to \mathrm{Hom}(X,Y)$ to the composition

$$X \times K \xrightarrow{id \times g} X \times \mathrm{Hom}(X,Y) \xrightarrow{ev} Y,$$

is a bijection which is contravariantly natural in K and X, and covariantly natural in Y.

Proof. (Cf. [GJ99] Chapter I Proposition 5.1) We construct an explicit inverse

$$ev_*^{-1} : \mathrm{Hom}_{\mathbf{sSet}}(X \times K, Y) \to \mathrm{Hom}_{\mathbf{sSet}}(K, \mathrm{Hom}(X,Y))$$

of the map ev_* by sending a morphism $g : X \times K \to Y$ to $ev_*^{-1}(g) = g_* : K \to \mathrm{Hom}(X,Y)$, where g_* is defined as follows. Let $x \in K_n$ be an n-simplex of K which is, by Yoneda's Lemma, associated to a unique simplicial map $\iota_x \in \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, K) \cong K_n$. Then $(g_*)_n(x) \in \mathrm{Hom}(X,Y)_n$ is defined to be the composite

$$X \times \Delta^n \xrightarrow{id \times \iota_x} X \times K \xrightarrow{ev} Y.$$

This gives an inverse to ev_*, since ev evaluates at $1_n \in \mathrm{Hom}_{\mathbf{\Delta}}(\underline{n}, \underline{n})$ and ι_x is characterized by $(\iota_x)_n(1_n) = x$. $\square$

Proposition 4.3.4. Let $i : K \hookrightarrow L$ be an inclusion of simplicial sets and $p : X \to Y$ be a Kan fibration. Then the map

$$\mathrm{Hom}(L,X) \xrightarrow{(i^*, p_*)} \mathrm{Hom}(K,X) \underset{\mathrm{Hom}(K,Y)}{\times} \mathrm{Hom}(L,Y),$$

which is induced by the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(L,X) \arrow[dr, dashed] \arrow[drr, bend left, "p_*"] \arrow[ddr, bend right, "i^*"'] & & \\
& \mathrm{Hom}(K,X) \underset{\mathrm{Hom}(K,Y)}{\times} \mathrm{Hom}(L,Y) \arrow[r] \arrow[d] & \mathrm{Hom}(L,Y) \arrow[d, "i^*"] \\
& \mathrm{Hom}(K,X) \arrow[r, "p_*"'] & \mathrm{Hom}(K,Y),
\end{tikzcd}$$

is a Kan fibration, where $i^* = (- \circ (i \times id))_{\underline{m} \in \mathbf{\Delta}}$ and $p_* = (p \circ -)_{\underline{m} \in \mathbf{\Delta}}$.

Proof. (Cf. [GJ99] Chapter I Proposition 5.2) Every diagram of the form

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d] & \mathrm{Hom}(L,X) \arrow[d, "(i^*, p_*)"] \\
\Delta^n \arrow[r] & \mathrm{Hom}(K,X) \underset{\mathrm{Hom}(K,Y)}{\times} \mathrm{Hom}(L,Y)
\end{tikzcd} \tag{4.1}$$

can be identified with a diagram

$$\begin{tikzcd}
(\Lambda^n_k \times L) \sqcup_{(\Lambda^n_k \times K)} (\Delta^n \times K) \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\Delta^n \times L \arrow[r] & Y,
\end{tikzcd} \tag{4.2}$$

using the exponential law and the universal properties of pushouts and pullbacks. To see this, we start by constructing the lower diagram from a given upper one. The map $\Lambda^n_k \to \mathrm{Hom}(L, X)$ gives a map $\Lambda^n_k \times L \to X$ by the exponential law. Similarly, the map

$$\Lambda^n_k \to \mathrm{Hom}(L, X) \xrightarrow{i^*} \mathrm{Hom}(K, X)$$

gives a map $\Lambda^n_k \times K \to X$. This map equals the composition

$$\Lambda^n_k \times K \xrightarrow{id \times i} \Lambda^n_k \times L \to X,$$

by the functoriality of the exponential law. Next, the map

$$\Lambda^n_k \hookrightarrow \Delta^n \to \mathrm{Hom}(K, X) \times_{\mathrm{Hom}(K,Y)} \mathrm{Hom}(L, Y) \xrightarrow{pr_1} \mathrm{Hom}(K, X)$$

gives a map $\Lambda^n_k \times K \hookrightarrow \Delta^n \times K \to X$. By the commutativity of our given diagram, these two maps from Λ^n_k to X coincide. Using the universal property in the pushout diagram

$$\begin{tikzcd}
\Lambda^n_k \times K \arrow[r, "id \times i"] \arrow[d] & \Lambda^n_k \times L \arrow[d] \arrow[ddr, bend left] & \\
\Delta^n \times K \arrow[r] \arrow[drr, bend right] & (\Lambda^n_k \times L) \sqcup_{(\Lambda^n_k \times K)} (\Delta^n \times K) \arrow[dr, dashed] & \\
& & X
\end{tikzcd}$$

gives us the upper horizontal map in (4.2). Finally, applying the exponential law to the map

$$\Delta^n \to \mathrm{Hom}(K, X) \times_{\mathrm{Hom}(K,Y)} \mathrm{Hom}(L, Y) \xrightarrow{pr_2} \mathrm{Hom}(L, Y)$$

gives us the remaining map in (4.2). The commutativity of the upper square (4.1) and the functoriality of the exponential law then imply that (4.2) commutes. The inverse construction is done in a similar manner. One starts by constructing the lower horizontal map in (4.1), again, using the exponential law and the universal property of the pullback. Once we have shown that each of these two diagrams (4.1) and (4.2) are equivalent, the rest of the proof is mostly a collection of previous results. The left vertical map in (4.2) is anodyne by Corollary 4.2.9. Since p is a fibration it has the RLP with respect to all anodyne extensions by Proposition 4.2.7, hence there is a lift $\Delta^n \times L \to X$ in (4.2). Applying the exponential law to this lift, we get a lift $\Delta^n \to \mathrm{Hom}(L, X)$ in (4.1). And therefore (i^*, p_*) is a fibration, as required. □

Corollary 4.3.5. (i) If $p : X \to Y$ is a fibration, then so is $p_* : \mathrm{Hom}(K, X) \to \mathrm{Hom}(K, Y)$ for any $K \in \mathbf{sSet}$.

(ii) If $i : K \hookrightarrow L$ is an inclusion and X is fibrant, then $i^* : \mathrm{Hom}(L, X) \to \mathrm{Hom}(K, X)$ is a fibration. In particular, if X is fibrant then so is $\mathrm{Hom}(L, X)$ for any $L \in \mathbf{sSet}$.

Proof. (Cf. [GJ99] Chapter I Corollary 5.3)

(i) The diagram

$$\begin{tikzcd}
\mathrm{Hom}(K,Y) \arrow[r, "id"] \arrow[d] & \mathrm{Hom}(K,Y) \arrow[d] \\
* \arrow[r] & *
\end{tikzcd}$$

is a pullback and we consider the inclusion $i : \emptyset \hookrightarrow K$, where $\emptyset$ is the simplicial set with the empty set in each degree. Then we have $\mathrm{Hom}(\emptyset, X) = *$ for any $X \in \mathbf{sSet}$ and the diagram

$$\begin{tikzcd}
\mathrm{Hom}(K,X) \arrow[dr] \arrow[drr, bend left, "p_*"] \arrow[ddr, bend right] & & \\
& \mathrm{Hom}(K,Y) \arrow[r, "id"] \arrow[d] & \mathrm{Hom}(K,Y) \arrow[d] \\
& \mathrm{Hom}(\emptyset,X) \arrow[r, "p_*"'] \arrow[d, equal] & \mathrm{Hom}(\emptyset,Y) \arrow[d, equal] \\
& * & *
\end{tikzcd}$$

commutes, where the morphism $\mathrm{Hom}(K, X) \to \mathrm{Hom}(\emptyset, X)$ is uniquely determined. Hence, by the above Proposition 4.3.4, the map

$$(i^*, p_*) = p_* : \mathrm{Hom}(K, X) \to \mathrm{Hom}(K, Y)$$

is a fibration.

(ii) If X is fibrant, than the map $p : X \to *$ is a fibration and we have $\mathrm{Hom}(K, *) = *$ for any $K \in \mathbf{sSet}$. Again, applying this to the pullback

$$\begin{tikzcd}
\mathrm{Hom}(K,X) \arrow[r] \arrow[d, "id"'] & * \arrow[d] \\
\mathrm{Hom}(K,X) \arrow[r] & *
\end{tikzcd}$$

we get the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(L,X) \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right, "i^*"'] & & \\
& \mathrm{Hom}(K,X) \arrow[r] \arrow[d, "id"] & \mathrm{Hom}(L,*) = * \arrow[d] \\
& \mathrm{Hom}(K,X) \arrow[r] & \mathrm{Hom}(K,*) = *
\end{tikzcd}$$

and hence, the map $i^* : \mathrm{Hom}(L,X) \to \mathrm{Hom}(K,X)$ is a fibration. The special case follows immediately by taking the fibration $X \to *$ and $K = \emptyset$. Then the map

$$\mathrm{Hom}(L,X) \to * \underset{*}{\times} * = *$$

is a fibration and therefore, $\mathrm{Hom}(L,X)$ is fibrant.

$\square$

Remark 4.3.6. If in the setting of Proposition 4.3.4 the inclusion $i : K \hookrightarrow L$ is anodyne, then the map

$$\mathrm{Hom}(L,X) \xrightarrow{(i^*,p_*)} \mathrm{Hom}(K,X) \underset{\mathrm{Hom}(K,Y)}{\times} \mathrm{Hom}(L,Y),$$

is even a trivial fibration. This follows from the fact that as in the above proof every diagram of the form

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & \mathrm{Hom}(L,X) \arrow[d, "(i^*,p_*)"] \\
\Delta^n \arrow[r] & \mathrm{Hom}(K,X) \underset{\mathrm{Hom}(K,Y)}{\times} \mathrm{Hom}(L,Y)
\end{tikzcd}$$

can be identified with a diagram

$$\begin{tikzcd}
(\partial\Delta^n \times L) \underset{(\partial\Delta^n \times K)}{\sqcup} (\Delta^n \times K) \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\Delta^n \times L \arrow[r] & Y.
\end{tikzcd}$$

Note that in the lower diagram there exists a lift of p since the inclusion is anodyne (cf. Corollary 4.2.9 using that $i : K \hookrightarrow L$ is anodyne). Applying the inverse construction yields the claim.

4.4 Simplicial homotopy

For two given morphisms of simplicial sets, we need to introduce a concept of a homotopy between them. This turns out to be an equivalence relation in certain situations and we use it later on to define homotopy groups for fibrant simplicial sets.

Definition 4.4.1. *Let $f, g : X \to Y$ be two maps of simplicial sets. A (simplicial) homotopy from f to g is a map*

$$h : X \times \Delta^1 \to Y,$$

such that the diagram

$$\begin{tikzcd}
X \times \Delta^0 \arrow[r, equal] \arrow[d, "id\times d^1"'] & X \arrow[dr, "f"] & \\
X \times \Delta^1 \arrow[rr, "h"] & & Y \\
X \times \Delta^0 \arrow[r, equal] \arrow[u, "id\times d^0"] & X \arrow[ur, "g"'] &
\end{tikzcd}$$

commutes. We will say that f is homotopic to g if there exists a (simplicial) homotopy from f to g, and write $f \simeq g$.

Remark 4.4.2. By our above notation for the two vertices in $\partial\Delta^1 \subseteq \Delta^1$, we could also write

$$h(x, 0) = f(x) \text{ and } h(x, 1) = g(x) \text{ for all } x \in X,$$

as a shorthand notation. One also often writes I instead of the simplicial set Δ^1, related to homotopies of continuous maps and the fact that $|\Delta^1| = I$.

Definition 4.4.3. *Let $i : A \hookrightarrow X$ be an inclusion of simplicial sets and let $f, g : X \to Y$ be simplicial maps such that that the restrictions $f|_A = g|_A$ to A coincide. We say there is a homotopy from f to g relative to A and write it $f \simeq g$ (rel A) if there exists an h as in the previous definition and, additionally, h is stationary on A in the sense that the diagram*

$$\begin{tikzcd}
A \times \Delta^1 \arrow[r, "pr_1"] \arrow[d, "i\times id"'] & A \arrow[d, "\alpha"] \\
X \times \Delta^1 \arrow[r, "h"'] & Y
\end{tikzcd}$$

commutes, where $\alpha = f|_A = g|_A$ and pr_1 denotes the projection onto the first factor.

Lemma 4.4.4. If X is a fibrant simplicial set then simplicial homotopy is an equivalence relation on the set of its vertices $X_0 \cong \mathrm{Hom}_{\mathbf{sSet}}(\Delta^0, X)$.

Proof. (Cf. [GJ99] Chapter I Lemma 6.1) For the proof we introduce the notation of the boundary $\partial(\sigma)$ of an n-simplex σ which is the collection of faces $\partial(\sigma) = (d_0\sigma, ..., d_n\sigma)$. By definition, two vertices x and y are homotopic if and only if there is a 1-simplex v of X such that $d_1 v = x$ and $d_0 v = y$, or expressed in our new notation $\partial v = (y, x)$. For the reflexivity of the homotopy relation we choose $v = s_0 x$, for $x \in X_0$ and since $\partial(s_0 x) = (d_0 s_0 x, d_1 s_0 x) = (x, x)$, we have $x \simeq x$. Now suppose that $x \simeq y$ and $y \simeq z$, i.e. we have $v_0, v_2 \in X_1$ such that $\partial(v_2) = (y, x)$ and $\partial(v_0) = (z, y)$. Then $d_0 v_2 = y = d_1 v_0$ and so v_0 and v_2 induce a well-defined natural transformation $(v_0, \ , v_2) : \Lambda^2_1 \to X$ defined as

$$(\Lambda^2_1)_m \ni d^i \circ g \mapsto g^*(v_i) \in X_m \text{ for } i = 0, 2.$$

Alternatively, we could use our characterizing Corollary 4.1.2 for $\mathrm{Hom}_{\mathbf{sSet}}(\Lambda^2_1, X)$. Since X is fibrant, we can lift the morphism to

$$\begin{tikzcd} \Lambda^2_1 \arrow[r, "{(v_0,\ ,v_2)}"] \arrow[d] & X \\ \Delta^2 \arrow[ur, "\theta"'] & \end{tikzcd}$$

with $\theta \in \mathrm{Hom}_{\mathbf{sSet}}(\Delta^2, X) \cong X_2$. By taking $d_1\theta \in X_1$, we get

$$\partial(d_1\theta) = (d_0 d_1 \theta, d_1 d_1 \theta) = (d_0 d_0 \theta, d_1 d_2 \theta) = (d_0 v_0, d_1 v_2) = (z, x).$$

Hence, $d_1\theta$ is the required homotopy from x to z and so the relation is transitive. It remains to show the symmetry. For this, we proceed similarly. For a given homotopy $v_2 \in X_1$ from x to y with $\partial(v_2) = (y, x)$, we set $v_1 := s_0 x \in X_1$ with $d_1 v_1 = x = d_1 v_2$. As above, this induces a $(\ , v_1, v_2) : \Lambda^2_1 \to X$ and from the Kan condition, we choose a lift

$$\begin{tikzcd} \Lambda^2_1 \arrow[r, "{(\ ,v1,v2)}"] \arrow[d] & X \\ \Delta^2 \arrow[ur, "\theta'"'] & \end{tikzcd}$$

Then for $d_0\theta' \in X_1$ we have

$$\partial(d_0\theta') = (d_0 d_0 \theta', d_1 d_0 \theta') = (d_0 d_1 \theta', d_0 d_2 \theta') = (d_0 v_1, d_0 v_2) = (x, y)$$

and therefore, $y \simeq x$. $\square$

Corollary 4.4.5. Let $A \hookrightarrow X$ be an inclusion and Y fibrant. Then

(i) homotopy of maps $X \to Y$ is an equivalence relation.

(ii) homotopy of maps $X \to Y$ (rel A) is an equivalence relation.

Proof. (Cf. [GJ99] Chapter I Corollary 6.2) Part (i) follows from part (ii) by choosing $A = \emptyset$. By Corollary 4.3.5 (ii) the map

$$i^* : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(A, Y)$$

is a fibration with $i : A \hookrightarrow X$. Let $f, g : X \to Y$ be two simplicial maps matching on A, then $v = f|_A = g|_A \in \mathrm{Hom}(A, Y)_0$ is a vertex of the function complex. We take the fiber

$$\begin{tikzcd} F_v \arrow[r] \arrow[d] & \mathrm{Hom}(X,Y) \arrow[d, "i^*"] \\ \Delta^0 \arrow[r, "v"'] & \mathrm{Hom}(A,Y) \end{tikzcd}$$

over the vertex v. The fibre over a fibration is a Kan complex, as will be discussed in detail in the next section. Then homotopy between maps $f, g : X \to Y$ (rel A) corresponds to homotopy of vertices in the fibre $F_v = (i^*)^{(-1)}(v)$. But this is an equivalence relation by the previous Lemma 4.4.4. □

4.5 Simplicial homotopy groups

In this section, we use our knowledge about simplicial homotopies to construct the homotopy groups of a fibrant simplicial set. And, in the course of this, we define our last class of morphisms in **sSet**, the weak equivalences, first for maps between Kan complexes, later on generalized to arbitrary simplicial sets. For many results we will use explicit calculations to construct equivalence classes or simplicial homotopies in this section.

Definition 4.5.1. *Let X be a fibrant simplicial set. $\pi_0(X)$ is defined to be the set of homotopy classes of vertices of X and it is called the set of path components of X. A simplicial set X is said to be connected if $\pi_0(X)$ is a one-point set.*

Lemma 4.5.2. Let X be a fibrant simplicial set. Then the map $\pi_0(X) \to \pi_0(|X|)$, mapping an equivalence class $[v]$ for $v \in X_0$ to the path component of $|X|$ containing $|v|$, is a bijection of sets. Here we view $v : \Delta^0 \to X$ as a morphism of simplicial sets and let $|v|$ be the image of the associated map $|\Delta^0| \to |X|$. Note that $|\Delta^0| = *$ is a singleton.

Proof. (Cf. [Hov99] Lemma 3.4.3) The realization of X is the topological space

$$|X| = \coprod_{n \geq 0} X_n \times |\Delta^n| \Big/ \sim$$

with $(x, d^i t) \sim (d_i x, t)$ and $(x, s^j t) \sim (s_j x, t)$ for all $x \in X_n$, $t \in |\Delta^n|$ and $0 \leq i, j \leq n$. Since $|\Delta^n|$ is path connected for all $n \geq 0$, every point of $|X|$ is contained in a path component of a vertex. On the other hand, for $\alpha = [v] \in \pi_0(X)$ we define the simplicial subset X_α of X containing all simplices x of X with a vertex in α, i.e. $(X_\alpha)_n$ consists of those $x \in X_n$ with $d_0 \ldots d_0 x \simeq v$. Then X is the disjoint union of its path components

$$X = \coprod_{\alpha \in \pi_0(X)} X_\alpha$$

and since the realization functor preserves colimits, the above map is injective. This also uses that (obviously) $\pi_0(X_\alpha)$ is a singleton. Therefore, we get a bijection $\pi_0(X) \cong \pi_0(|X|)$, as required. □

Definition 4.5.3. *Let X be a fibrant simplicial set, $x \in X_0$ vertex and $n \geq 1$. We define $\pi_n(X, v)$ to be the set of homotopy classes of n-simplices $\alpha : \Delta^n \to X$ (rel $\partial\Delta^n$) for all $\alpha : \Delta^n \to X$ for which there exists a commutative diagram of the form*

$$\begin{tikzcd} \Delta^n \arrow[r, "\alpha"] & X \\ \partial\Delta^n \arrow[u] \arrow[r] & \Delta^0. \arrow[u, "v"'] \end{tikzcd}$$

The set $\pi_n(X, v)$ is called the n-th homotopy group of X at v.

Remark 4.5.4. To fix some short hand notation, we will write $v : \partial\Delta^n \to X$ for the composition

$$\partial\Delta^n \to \Delta^0 \xrightarrow{v} X.$$

Further, we denote the homotopy class of α by $[\alpha]$, in all contexts.

To get used to the terminology of simplicial homotopy, we spell it out one more time: An element $[\alpha] \in \pi_n(X, v)$ for an n-simplex $\alpha : \Delta^n \to X$ that sends $\partial\Delta^n$ to v, is the equivalence class defined by the relation $\alpha \sim \beta$ if there is a simplicial homotopy $h : \Delta^n \times \Delta^1 \to X$ such that h equals α on $\Delta^n \times \{0\}$, β on $\Delta^n \times \{1\}$, and is the constant map v on $\partial\Delta^n \times \Delta^1$.

As for topological spaces in Algebraic Topology, we want to show that these homotopy groups $\pi_n(X, v)$ are indeed groups and even abelian for $n \geq 2$. We will construct the group structure first and show the commutativity later on. To do this, let $n \geq 1$ and let $[\alpha], [\beta]$ be two elements of $\pi_n(X, v)$ represented by $\alpha, \beta : \Delta^n \to X$. We fix the family

$$v_i = \begin{cases} v, & 0 \leq i \leq n-2, \\ \alpha, & i = n-1, \\ \beta, & i = n+1 \end{cases}$$

of n-simplices in X, where v is the n-simplex $\Delta^n \to \Delta^0 \to X$. In other words, we view v as the projection $v = s_0 ... s_0 v = X(\underline{n} \to \underline{0})(v)$ to X_n. These simplices satisfy $d_i v_j = d_{j-1} v_i$ for $i < j$ and $i, j \neq n$, since all faces of all simplices v_i map through the vertex v by definition of α and β on $\partial\Delta^n$. Thus, the family v_i determines a morphism of simplicial sets $(v_0, ..., v_{n-1}, \ , v_{n+1}) : \Lambda^{n+1}_n \to X$, and since X is fibrant there is an extension ω in the diagram

$$\begin{tikzcd} \Lambda^{n+1}_n \arrow[r, "{(v_0, ..., v_{n-1}, \ , v_{n+1})}"] \arrow[d] & X. \\ \Delta^{n+1} \arrow[ur, "\omega"'] & \end{tikzcd}$$

For $d_n\omega \in X_n$ the boundary is

$$\begin{aligned} \partial(d_n\omega) =& (d_0 d_n \omega, ..., d_{n-1} d_n \omega, d_n d_n \omega) \\ =& (d_{n-1} d_0 \omega, ..., d_{n-1} d_{n-1} \omega, d_n d_{n+1} \omega) \\ =& (d_{n-1} v, ..., d_{n-1} \alpha, d_n \beta) \\ =& (v, ..., v) \end{aligned}$$

and therefore $d_n\omega$ is equal to v on $\partial\Delta^n$. Hence, $d_n\omega$ represents an element in $\pi_n(X, v)$. This new homotopy class is meant to be the multiplication of $[\alpha]$ and $[\beta]$. But first we need to prove that this class is well-defined. To see this, we are going to construct an explicit homotopy, by using the lifting property for fibrant simplicial sets. Since we have to use this technique often in this chapter, we will do this first case in detail and abbreviate it in later proofs.

Lemma 4.5.5. The homotopy class of $d_n\omega$ (rel $\partial\Delta^n$) is independent of the choices of representatives of $[\alpha]$ and $[\beta]$ and of the choice of ω.

Proof. (Cf. [GJ99] Chapter I Lemma 7.1) Let $\alpha \simeq \alpha'$ and $\beta \simeq \beta'$ for $\alpha, \alpha', \beta, \beta' : \Delta^n \to X$. We choose homotopies h_{n-1} between α and α' (rel $\partial\Delta^n$) and h_{n+1} between β and β' (rel $\partial\Delta^n$). Let ω and ω' be lifts as above, such that

$$\partial\omega = (v, ..., v, \alpha, d_n\omega, \beta)$$

and

$$\partial\omega' = (v, ..., v, \alpha', d_n\omega', \beta').$$

We need to show that $d_n\omega \simeq d_n\omega'$ (rel $\partial\Delta^n$), so that they represent the same homotopy class in $\pi_n(X, v)$. To see this, consider the map

$$(\Delta^{n+1} \times \partial\Delta^1) \cup (\Lambda^{n+1}_n \times \Delta^1) \xrightarrow{((\omega',\omega),(v,...,v,h_{n-1},\ ,h_{n+1}))} X,$$

where $(\omega', \omega) : \Delta^{n+1} \times \partial\Delta^1 \to X$ denotes the map which is $\omega' \times id$ on the vertex $\Delta^{n+1} \times \{0\}$ and $\omega \times id$ on $\Delta^{n+1} \times \{1\}$. The order of ω' and ω is crucial here, otherwise we would get a homotopy the other way round. The map $f := (v, ..., v, h_{n-1},\ , h_{n+1}) : \Lambda^{n+1}_n \times \Delta^1 \to X$ is defined as

$$f \circ (d^i \times id) = v \text{ for } i < n-1 \text{ and } f \circ (d^{n\pm1} \times id) = h_{n\pm1}.$$

We note that this map is well-defined, since h_{n-1} and h_{n+1} are constant with value v on $\partial\Delta^n \times \Delta^1$. Next, we choose an extension in the diagram

$$\begin{tikzcd}
(\Delta^{n+1} \times \partial\Delta^1) \cup (\Lambda^{n+1}_n \times \Delta^1) \arrow[r, "{((\omega',\omega),(v,...,v,h_{n-1},\ ,h_{n+1}))}"] \arrow[d] & X, \\
\Delta^{n+1} \times \Delta^1 \arrow[ur, "\bar{\omega}"'] &
\end{tikzcd}$$

which exists since X is fibrant and the inclusion is anodyne by Corollary 4.2.9, since the horn inclusion is anodyne. Then the composite

$$\Delta^n \times \Delta^1 \xrightarrow{d^n \times id} \Delta^{n+1} \times \Delta^1 \xrightarrow{\bar{\omega}} X$$

is a homotopy $d_n\omega \simeq d_n\omega'$ (rel $\partial\Delta^n$) because the diagram

$$\begin{tikzcd}
\Delta^n \times \Delta^0 \arrow[d, "id\times d^1"'] \arrow[rrrd, "d_n\omega"] & & & \\
\Delta^n \times \Delta^1 \arrow[r, "d^n\times id"] & \Delta^{n+1}\times\Delta^1 \arrow[rr, "\bar{\omega}"] & & X \\
\Delta^n \times \Delta^0 \arrow[u, "id\times d^0"] \arrow[rrru, "d_n\omega'"'] & & &
\end{tikzcd}$$

commutes. $\square$

Now we can define our multiplication

$$m : \pi_n(X,v) \times \pi_n(X,v) \to \pi_n(X,v)$$

on the n-th homotopy group $\pi_n(X,v)$ as

$$([\alpha],[\beta]) \mapsto [\alpha]\cdot[\beta] := [d_n\omega],$$

where ω is a lift as above, such that $\partial\omega = (v,...,v,\alpha,d_n\omega,\beta)$. This operation is well-defined by the previous lemma. By e we denote the homotopy class $[v] \in \pi_n(X,v)$ which is represented by the constant map

$$\Delta^n \to \Delta^0 \xrightarrow{v} X.$$

Theorem 4.5.6. With multiplication m and unit element e, we obtain a group structure on the set $\pi_n(X,v)$ for $n \geq 1$.

Proof. (Cf. [GJ99] Chapter I Theorem 7.2) We start to prove the associativity of the multiplication. To show this we take three n-simplices $\alpha,\beta,\gamma : \Delta^n \to X$, representing elements in $\pi_n(X,v)$. Let $\omega_{n-1}, \omega_{n+1}$ and ω_{n+2} be $(n+1)$-simplices such that $[\alpha]\cdot[\beta] = [d_n\omega_{n-1}]$, $([\alpha]\cdot[\beta])\cdot[\gamma] = [d_n\omega_{n+1}]$ and $[\beta]\cdot[\gamma] = [d_n\omega_{n+2}]$. This means, these are lifts with the property

$$\begin{aligned}
\partial(d_n\omega_{n-1}) =& (v,...,v,\alpha,d_n\omega_{n-1},\beta),\\
\partial(d_n\omega_{n+1}) =& (v,...,v,d_n\omega_{n-1},d_n\omega_{n+1},\gamma), \text{ and}\\
\partial(d_n\omega_{n+2}) =& (v,...,v,\beta,d_n\omega_{n+2},\gamma).
\end{aligned}$$

Then we get a map $\Lambda^{n+2}_n \xrightarrow{(v,...,v,\omega_{n-1},\ ,\omega_{n+1},\omega_{n+2})} X$, since this family satisfies the required property for $\mathrm{Hom}_{\mathbf{sSet}}(\Lambda^{n+2}_n, X)$ (cf. Corollary 4.1.2). Since X is fibrant the map can be extended to

$$\begin{tikzcd}
\Lambda^{n+2}_n \arrow[r, "{(v,...,v,\omega_{n-1},\ ,\omega_{n+1},\omega_{n+2})}"] \arrow[d] & X, \\
\Delta^{n+2} \arrow[ru, "\omega"'] &
\end{tikzcd}$$

where the n-th face $d_n\omega$ of the extension has the boundary

$$\begin{aligned}\partial(d_n\omega) =&(d_0d_n\omega, ..., d_{n-2}d_n\omega, d_{n-1}d_n\omega, d_nd_n\omega, d_{n+1}d_n\omega)\\ =&(d_{n-1}d_0\omega, ..., d_{n-1}d_{n-2}\omega, d_{n-1}d_{n-1}\omega, d_nd_{n+1}\omega, d_nd_{n+2}\omega)\\ =&(v, ..., v, \alpha, d_n\omega_{n+1}, d_n\omega_{n+2}).\end{aligned}$$

And hence, we get

$$\begin{aligned}([\alpha]\cdot[\beta])\cdot[\gamma] &= [d_n\omega_{n-1}]\cdot[\gamma]\\ &= [d_n\omega_{n+1}]\\ &= [d_nd_n\omega]\\ &= [\alpha]\cdot[d_n\omega_{n+2}]\\ &= [\alpha]\cdot([\beta]\cdot[\gamma]).\end{aligned}$$

Now, let $[\alpha]\in\pi_n(X,v)$. For the product $e\cdot[\alpha]$, the $n+1$-simplex $s_n\alpha$ satisfies

$$\partial(s_n\alpha) = (v, ..., v, v, \alpha, \alpha)$$

and so we have $e\cdot[\alpha] = [\alpha]$. The same holds for $[\alpha]\cdot e = [\alpha]$ using the lift $s_{n-1}\alpha$ and hence, e is the unit element. For the existence of inverse elements it remains to prove that for any $[\alpha]\in\pi_n(X,v)$, the map $\pi_n(X,v)\to\pi_n(X,v)$ induced by left multiplication by $[\alpha]$ is bijective. Injectivity follows from composing a homotopy between $[d_n\omega]$ and $[d_n\omega']$ by degeneracy and face maps, receiving a homotopy between the two sources. To show the surjectivity, one can lift the map $(v,\dots,v,\alpha,\gamma,\ ):\Lambda^{n+1}_{n+1}\to X$. □

If $f: X\to Y$ is a morphism of fibrant simplicial sets, it induces a map between the n-th simplicial homotopy groups of X and Y. This induced map is defined as

$$f_*:\pi_n(X,v)\to\pi_n(Y,f_0(v)),\quad [\alpha]\mapsto[f_n(\alpha)].$$

This is well-defined, because for two n-simplices $\alpha,\beta:\Delta^n\to X$ that represent the same class in $\pi_n(X,v)$, the homotopy between them extends to a homotopy between $f\circ\alpha, f\circ\beta:\Delta^n\to Y$ by composing with f. For $n\geq 1$, this is also a group homomorphism, since the defined lifting for the multiplication also extends through f.

The following lemma is a useful criterion as to when a homotopy class in $\pi_n(X,v)$ equals the unit element. We will need this later on.

Lemma 4.5.7. Let $\alpha:\Delta^n\to X$ represent an element of $\pi_n(X,v)$. Then $[\alpha]=e$ if and only if there is an $(n+1)$-simplex ω of X such that $\partial\omega = (v, ..., v, \alpha)$.

Proof. (Cf. [Hov99] Lemma 3.4.5) We first suppose that $[\alpha]=[v]$. Then there is a homotopy $h:\Delta^n\times\Delta^1\to X$ between α and v, which is equal to v on $\partial\Delta^n\times\Delta^1$ and $\Delta^n\times\{1\}$. We consider the lifting diagram

$$\begin{tikzcd}(\Delta^{n+1}\times\{1\})\cup(\partial\Delta^{n+1}\times\Delta^1) \arrow[r, "{(v,(v,...,v,h))}"] \arrow[d] & X, \\ \Delta^{n+1}\times\Delta^1 \arrow[ur, "\bar{\omega}"'] & \end{tikzcd}$$

where $\bar{\omega} : \Delta^{n+1} \times \Delta^1 \to X$ exists since X is fibrant and the inclusion is anodyne. Then the restriction $\omega := \bar{\omega}|_{\Delta^{n+1}\times\{0\}} : \Delta^{n+1} \to X$ is the desired $(n+1)$-simplex such that $d_i\omega = v$ for $i \leq n$ and $d_{n+1}\omega = \alpha$. Conversely, we consider an $n+1$-simplex ω such that $d_{n+1}\omega = \alpha$ and $d_i\omega = v$ for $i \leq n$. Again, we look at a lifting diagram

$$\begin{tikzcd}
(\Delta^{n+1} \times \partial\Delta^1) \cup (\Lambda^{n+1}_{n+1} \times \Delta^1) \arrow[r, "{((\alpha,v),(v,\dots,v,))}"] \arrow[d] & X. \\
\Delta^{n+1} \times \Delta^1 \arrow[ur, "\theta"'] &
\end{tikzcd}$$

Then the composite

$$\Delta^n \times \Delta^1 \xrightarrow{d^{n+1}\times id} \Delta^{n+1} \times \Delta^1 \xrightarrow{\theta} X$$

is the desired homotopy between α and v (rel $\partial\Delta^n$). Hence, $[\alpha] = e$ in $\pi_n(X, v)$. $\square$

Our next goal is to show that the groups $\pi_n(X, v)$ are abelian for $n \geq 2$. To see this, we are going to define the so called loop-space ΩX for a simplicial set such that $\pi_n(X, v) \cong \pi_{n-1}(\Omega X, v)$ and show that $\pi_{n-1}(\Omega X, v)$ is abelian for $i \geq 1$. In order to do so, we need to construct the long exact sequence of a fibration, similarly to Algebraic Topology. We start with the homotopy groups for fibrations. Let X and Y be fibrant simplicial sets and $p : X \to Y$ be a Kan fibration. Suppose $v \in X_0$ is a vertex and let F denote the fibre of p over $* := p_0(v) \in Y_0$ in the sense that the diagram

$$\begin{tikzcd}
F \arrow[r, "i"] \arrow[d] & X \arrow[d, "p"] \\
\Delta^0 \arrow[r, "*"'] & Y
\end{tikzcd}$$

is a pullback diagram. In other words F is the simplicial set $p^{-1}(*)$. We note that F is fibrant, as well. Indeed, for a given map

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
\Delta^n \arrow[r] & \Delta^0 \arrow[r] & Y
\end{tikzcd}$$

there is a lift $\Delta^n \to X$, since p is a fibration. By the universal property of the pullback, we get a map $\Delta^n \to F$, such that the left square in the diagram commutes.

We want to construct a map ∂ between the n-th homotopy group of Y and the $(n-1)$-th homotopy group of F with vertex $v \in F_0$. To do this, we look at a commutative diagram of the form

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "{(,v\dots,v)}"] \arrow[d] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "\alpha"'] \arrow[ur, "\theta"] & Y,
\end{tikzcd}$$

where α represents an element of $\pi_n(Y,*)$. Here θ exists since p is a fibration. By the commutativity of the diagram, $d_0\theta$ lies in F and $d_i d_0\theta = v$ for all $0 \leq i \leq n-1$. Hence, $[d_0\theta]$ is an element in $\pi_{n-1}(F,v)$ and it is independent of the choice of the lift θ and representative of $[\alpha]$. To see this, we take another representative $\alpha' : \Delta^n \to Y$ for the class in $\pi_n(Y,*)$ and a lift in the diagram

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "{(,v\dots,v)}"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "\alpha'"'] \arrow[ur, "\theta'"] & Y. \end{tikzcd}$$

Since $[\alpha] = [\alpha']$, there is a homotopy $h : \Delta^n \times \Delta^1 \to Y$ between these two simplices, which is constant with value $*$ on $\partial\Delta^n \times \Delta^1$. Choose an extension $\bar{h}$ in the diagram

$$\begin{tikzcd} (\Delta^n \times \partial\Delta^1) \cup (\Lambda^n_0 \times \Delta^1) \arrow[rr, "{((\theta',\theta),(\ ,v,\dots,v))}"] \arrow[d] & & X \arrow[d, "p"] \\ \Delta^n \times \Delta^1 \arrow[rr, "h"'] \arrow[urr, "\bar{h}"] & & Y, \end{tikzcd}$$

as in the proof of Lemma 4.5.5. Then the composite

$$\Delta^{n-1} \times \Delta^1 \xrightarrow{d^0 \times id} \Delta^n \times \Delta^1 \xrightarrow{\bar{h}} X$$

is the desired homotopy $d_0\theta \simeq d_0\theta'$ which takes values in the fiber F by the properties of h and the commutativity of the diagram. For a more detailed proof I refer to [Hov99] Lemma 3.4.8. Altogether, this leads to the well-defined map

$$\partial : \pi_n(Y,*) \to \pi_{n-1}(F,v), [\alpha] \mapsto [d_0\omega],$$

the so called *boundary map*. As in higher homotopy theory for topological spaces, it forms part of a long exact sequence.

Lemma 4.5.8. (i) The boundary map $\partial : \pi_n(Y,*) \to \pi_{n-1}(F,v)$ is a group homomorphism if $n > 1$.

(ii) The sequence

$$\begin{aligned} \dots \to \pi_n(F,v) \xrightarrow{i_*} \pi_n(X,v) \xrightarrow{p_*} \pi_n(Y,*) \xrightarrow{\partial} \pi_{n-1}(F,v) \to \dots \\ \dots \xrightarrow{p_*} \pi_1(Y,*) \xrightarrow{\partial} \pi_0(F) \xrightarrow{i_*} \pi_0(X) \xrightarrow{p_*} \pi_0(Y) \end{aligned}$$

is exact in the sense that kernel equals image everywhere (where we view the zeroth homotopy groups as pointed sets with base point the corresponding vertex, and the kernels are defined as the preimage of the basepoint and i is the projection from the above pullback diagram).

Proof. (Cf. [GJ99] Chapter I Lemma 7.3 and [Hov99] Lemma 3.4.9)

(i) We start by proving that $\partial : \pi_n(Y,*) \to \pi_{n-1}(F,v)$ is a group homomorphism for $n \geq 2$. To do this, we take three representatives $\alpha_{n-1}, \alpha_n, \alpha_{n+1} : \Delta^n \to Y$ of elements in $\pi_n(Y,*)$. Consider an $(n+1)$-simplex ω such that

$$\partial\omega = (*, \dots, *, \alpha_{n-1}, \alpha_n, \alpha_{n+1}),$$

i.e. $[\alpha_{n-1}][\alpha_{n+1}] = [d_n\omega] = [\alpha_n]$. For each three of these simplices we consider the diagrams

$$\begin{tikzcd} \Lambda_0^n \arrow[r, "{(,v\dots,v)}"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "\alpha_i"'] \arrow[ur, "\theta_i"] & Y, \end{tikzcd}$$

$i = n-1, n, n+1$. These lifts induce a diagram

$$\begin{tikzcd} \Lambda_0^{n+1} \arrow[r, "{(,v,\dots,v,\theta_{n-1},\theta_n,\theta_{n+1})}"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^{n+1} \arrow[r, "\alpha_i"'] \arrow[ur, "\gamma"] & Y, \end{tikzcd}$$

with an extension $\gamma : \Delta^{n+1} \to X$ such that

$$\begin{aligned} \partial(d_0\gamma) =& (d_0d_0\gamma, ..., d_{n-1}d_0\gamma, d_nd_0\gamma) \\ =& (d_0d_1\gamma, d_0d_2\gamma, \dots, d_0d_{n-1}\gamma, d_0d_n\gamma, d_0d_{n+1}\gamma) \\ =& (v, \dots, v, d_0\theta_{n-1}, d_0\theta_n, d_0\theta_{n+1}). \end{aligned}$$

Thus we have calculated that $[d_0\theta_n] = [d_0\theta_{n-1}][d_0\theta_{n+1}]$ and therefore

$$\partial([\alpha_{n-1}][\alpha_{n+1}]) = [d_0\theta_n] = [d_0\theta_{n-1}][d_0\theta_{n+1}] = \partial([\alpha_{n-1}])\partial([\alpha_{n+1}]),$$

as desired.

(ii) We prove the exactness of the sequence at

$$\pi_n(X,v) \xrightarrow{p_*} \pi_n(Y,*) \xrightarrow{\partial} \pi_{n-1}(F,v).$$

The rest of the proof is mostly straightforward using Lemma 4.5.7. We start by showing that $ker(\partial) \subseteq im(p_*)$. To see this, we choose an n-simplex $\alpha : \Delta^n \to Y$ with $d_i\alpha = * = p(v)$ for all i, representing an element in $\pi_n(Y,*)$, such that $\partial[\alpha] = [v]$. We choose an extension in the diagram

$$\begin{tikzcd} \Lambda_0^n \arrow[r, "{(,v\dots,v)}"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "\alpha"'] \arrow[ur, "\theta"] & Y, \end{tikzcd}$$

with $\partial[\alpha] = [d_0\theta] = [v]$. Hence, there is a homotopy $h : \Delta^{n-1} \times \Delta^1 \to F$ (rel $\partial\Delta^{n-1}$) between $d_0\theta$ and v. Consider the lifting diagram

$$\begin{tikzcd}
(\Delta^n \times \{0\}) \cup (\partial\Delta^n \times \Delta^1) \arrow[r, "{(\theta,(h,v,\ldots,v))}"] \arrow[d] & X \arrow[d, "p"] \\
\Delta^n \times \Delta^1 \arrow[r, "\alpha\circ pr_1"'] \arrow[ur, "\bar{h}"] & Y.
\end{tikzcd}$$

Then the restriction $\beta := \bar{h}|_{\Delta^n\times\{1\}} : \Delta^n \to X$ defines a class in $\pi_n(X, v)$ such that $\alpha \simeq p \circ \beta$ by the homotopy

$$\Delta^n \times \Delta^1 \xrightarrow{\bar{h}} X \xrightarrow{p} Y.$$

Hence, $[\alpha] = p_*[\beta]$ and $[\alpha] \in im(p_*)$, as desired. It remains to show that the image of p_* is contained in the kernel of the boundary map. If $p_*([\alpha]) \in im(p_*)$ for a given representative $\alpha : \Delta^n \to X$, then the diagram

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "{(\,,v\ldots,v)}"] \arrow[d] & X \arrow[d, "p"] \\
\Delta^n \arrow[r, "p\circ\alpha"'] \arrow[ur, "\alpha"] & Y
\end{tikzcd}$$

commutes. Therefore, $\partial(p_*([\alpha])) = [d_0\alpha] = [v]$.

$\square$

Before we can define the loop space, we need to introduce the concept of the so called path space.

Definition 4.5.9. *Let X be a fibrant simplicial set and $v \in X_0$ a vertex. The path space PX is defined to be the pullback in the diagram*

$$\begin{tikzcd}
PX \arrow[r, "pr"] \arrow[d] & \mathrm{Hom}(\Delta^1, X) \arrow[d, "(d^0)^*"] \\
\Delta^0 \arrow[r, "v"'] & \mathrm{Hom}(\Delta^0, X) \cong X.
\end{tikzcd}$$

Further, the map $\pi : PX \to X$ is defined to be the composite

$$PX \xrightarrow{pr} \mathrm{Hom}(\Delta^1, X) \xrightarrow{(d^1)^*} \mathrm{Hom}(\Delta^0, X) \cong X.$$

Remark 4.5.10. Since $(d^\epsilon)^*$, $\epsilon = 0, 1$ are fibrations by Corollary 4.3.5 (ii), the path space PX is fibrant, as discussed before.

Lemma 4.5.11. The group (respectively set) $\pi_i(PX, w)$ is trivial for $i \geq 0$ and for all vertices $w \in PX_0$, and π is a fibration.

Proof. (Cf. [GJ99] Chapter I Lemma 7.5) We will mostly sketch this proof, since all methods used have been demonstrated before. The maps $d^\epsilon : \Delta^0 \to \Delta^1$ are anodyne, since they are contained in the second set of inclusions in Proposition 4.2.8, for $n = 0$. We want to show that $(d^0)^*$ has the right lifting property with respect to all inclusions $\partial\Delta^n \hookrightarrow \Delta^n$. To see this, one uses the same technique as in the proof of Proposition 4.3.4 applied to $p : X \to \Delta^0$ and $i : \Delta^0 \to \Delta^1$. Then the diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & \mathrm{Hom}(\Delta^1, X) \arrow[d] & \\
\Delta^n \arrow[r] & \mathrm{Hom}(\Delta^0, X) \underset{\mathrm{Hom}(\Delta^0,\Delta^0)}{\times} \mathrm{Hom}(\Delta^1, \Delta^0) \arrow[r, equal] & \mathrm{Hom}(\Delta^0, X)
\end{tikzcd}$$

can be identified with a lifting diagram over the fibration p (cf. Remark 4.3.6). Note that the map obtained on the left is anodyne, as mentioned above. By pullback, $PX \to \Delta^0$ has the RLP with respect to $\partial\Delta^n \hookrightarrow \Delta^n$ as well. Now let $\alpha : \Delta^n \to PX$ represent an element in $\pi_n(PX, w)$. Then we can find an extension in the diagram

$$\begin{tikzcd}
\partial\Delta^{n+1} \arrow[r, "{(w,\dots,w,\alpha)}"] \arrow[d] & PX. \\
\Delta^{n+1} \arrow[ur] &
\end{tikzcd}$$

Therefore, $[\alpha] = e$ in $\pi_n(PX, w)$ by the homotopy $\Delta^{n+1} \cong \Delta^n \times \Delta^1 \to PX$. Any two vertices $w, w' \in PX_0$ are homotopic by choosing a lift in the diagram

$$\begin{tikzcd}
\partial\Delta^1 \arrow[r, "{(w',w)}"] \arrow[d] & PX. \\
\Delta^1 \arrow[ur] &
\end{tikzcd}$$

Altogether, we have that all groups $\pi_n(PX, w)$ are trivial for all vertices $w \in PX_0$. It remains to show that $\pi : PX \to X$ is a fibration. The map sits inside the pullback diagram

$$\begin{tikzcd}
PX \arrow[r] \arrow[dd, "\pi"'] & \mathrm{Hom}(\Delta^1, X) \arrow[d, "i^*"] \\
 & \mathrm{Hom}(\partial\Delta^1, X) \arrow[d, "\cong"] \\
X \arrow[r, "{(v, id_X)}"] \arrow[d] & X \times X \arrow[d, "pr_1"] \\
\Delta^0 \arrow[r] & X,
\end{tikzcd}$$

where i^* is a fibration by Corollary 4.3.5 with $i : \partial\Delta^1 \hookrightarrow \Delta^1$. $\square$

Now, we finally reach the definition of the loop space.

Definition 4.5.12. *Let X be a fibrant simplicial set with vertex $* \in X_0$. The loop space ΩX is the fiber of $\pi : PX \to X$ over the base point $*$.*

Remark 4.5.13. By the definition of π and PX, an n-simplex of ΩX is a simplicial map $f : \Delta^n \times \Delta^1 \to X$ such that the restriction of f to $\Delta^n \times \partial\Delta^1$ maps into $*$.

Lemma 4.5.14. Let X be a fibrant simplicial set and $* \in X_0$ a vertex. Then the homotopy groups $\pi_n(\Omega X, *)$ are abelian for $n \geq 1$. Here we view $* : \Delta^0 \to X$ as a 1-simplex $\Delta^1 \to \Delta^0 \xrightarrow{*} X$ of X (mapping to $*$ under both d_0 and d_1, hence it is an element of $(\Omega X)_0$).

Proof. (Cf. [GJ99] Chapter I Lemma 7.6) The set $\pi_n(\Omega X, *)$ consists of homotopy classes of maps of the form

$$\begin{tikzcd}
\Delta^n \times \Delta^1 \arrow[r, "\alpha"] & X \\
(\Delta^n \times \partial\Delta^1) \cup (\partial\Delta^n \times \Delta^1) \arrow[u] \arrow[r] & \Delta^0 \arrow[u, "*"']
\end{tikzcd}$$

relative $(\Delta^n \times \partial\Delta^1) \cup (\partial\Delta^n \times \Delta^1)$. One can define two multiplications on $\pi_n(\Omega X, *)$ as

$$\begin{tikzcd}
\Lambda^{n+1}_n \times \Delta^1 \arrow[r] \arrow[d] & X \\
\Delta^{n+1} \times \Delta^1 \arrow[ur, "\omega"']
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
\Delta^n \times \Lambda^2_1 \arrow[r] \arrow[d] & X, \\
\Delta^n \times \Delta^2 \arrow[ur, "\theta"']
\end{tikzcd}$$

where the first one is our known multiplication $[\alpha] \cdot [\beta] = [\omega \circ (d^n \times id)]$ on the simplicial homotopy groups. We denote the second one by $[\alpha] \star [\beta] = [\theta \circ (id \times d^1)]$. Similarly as above, one can show that this multiplication is well-defined with identity element $[*]$. Moreover, one can show that

$$([\alpha_1] \star [\beta_1]) \cdot ([\alpha_2] \star [\beta_2]) = ([\alpha_1] \cdot [\alpha_2]) \star ([\beta_1] \star [\beta_2])$$

for all $[\alpha_{1,2}], [\beta_{1,2}] \in \pi_n(\Omega X, *)$ and by the Eckmann–Hilton argument both multiplications are equal and abelian. $\square$

Corollary 4.5.15. Let X be a fibrant simplicial set and $* \in X_0$ a vertex. Then $\pi_i(X, *)$ is abelian if $i \geq 2$.

Proof. By the long exact sequence applied to $\pi : PX \to X$, we get the exact sequence

$$\ldots \to \pi_n(\Omega X, *) \to \pi_n(PX, *) \to \pi_n(X, *) \to \pi_{n-1}(\Omega X, *) \to \ldots$$

Since $\pi_n(PX, *) = 0$, it follows that $\pi_n(X, *) \cong \pi_{n-1}(\Omega X, *)$. Hence, $\pi_n(X, *)$ is abelian for $n \geq 2$. $\square$

Example 4.5.16. Let G be a group. For the simplicial homotopy groups of the nerve BG there are isomorphisms

$$\pi_i(BG, *) = \begin{cases} G, & \text{if } i = 1, \\ 0, & \text{if } i \neq 1, \end{cases}$$

for all vertices in BG (cf. [GJ99] Chapter I Proposition 7.8).

For maps of fibrant simplicial sets, we can define our next class of morphisms in the model structure on **sSet**. We will generalize this definition to arbitrary morphisms in our category later on.

Definition 4.5.17. *Let $f : X \to Y$ be a morphism of fibrant simplicial sets. Then f is called a weak equivalence if for each vertex x_0 of X the induced map*

$$f_* : \pi_i(X, x_0) \to \pi_i(Y, f(x_0))$$

is an isomorphism for all $i > 0$, and the map

$$f_* : \pi_0(X) \to \pi_0(Y)$$

is a bijection.

Remark 4.5.18. We have already seen, that a trivial fibration (cf. Definition 4.2.10) is indeed a fibration. If $p : X \to Y$ is a morphism of fibrant simplicial sets with the RLP with respect to all inclusions $\partial\Delta^n \hookrightarrow \Delta^n$, $n \geq 0$, then it is a weak equivalence as well. First of all, the induced map $p_* : \pi_0(X) \to \pi_0(Y)$ is bijective. Indeed, if w is a vertex in Y, then the lifting diagram

$$\begin{tikzcd} \partial\Delta^0 \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^0 \arrow[r, "w"'] \arrow[ur] & Y \end{tikzcd}$$

shows the surjectivity, where the upper horizontal map is the unique map from the initial object. If we have $v, v' \in X_0$ with $[p(v)] = [p(v')]$ in $\pi_0(Y)$, then the homotopy in Y lifts to a homotopy $v \simeq v'$ in X via the diagram

$$\begin{tikzcd} \partial\Delta^1 \arrow[r, "{(v', v)}"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^1 \arrow[r, "h"'] \arrow[ur] & Y. \end{tikzcd}$$

Hence, p_* is injective. To prove that $p_* : \pi_i(X, x) \to \pi_i(Y, p(x))$ is an isomorphism for $i > 0$, one takes the fibre F_x of p over $p(x)$. As we have seen above, F_x is fibrant and $F_x \to *$ has the RLP with respect to all $\partial\Delta^n \hookrightarrow \Delta^n$, $n \geq 0$, as well. With the same argument as in the proof of Lemma 4.5.11, one can show that $\pi_0(F_x) = *$ and $\pi_i(F_x, x) = 0$ for all $i > 0$. Applying this to the long exact sequence for fibrations, $p_* : \pi_i(X, x) \to \pi_i(Y, p(x))$ is an

isomorphism for all $i \geq 1$.
Conversely, it is also true that a map of fibrant simplicial sets which is a fibration and a weak equivalence, is a trivial fibration as defined in Definition 4.2.10. The proof is long and technical and is omitted here. However, we record this result for later reference:

Theorem 4.5.19. A map between fibrant simplicial sets has the right lifting property with respect to all inclusions $\partial\Delta^n \hookrightarrow \Delta^n$, $n \geq 0$, if and only if it is a fibration and a weak equivalence.

Proof. See [GJ99] Chapter I Theorem 7.10. $\square$

4.6 The model structure on sSet

In this last section, we define the remaining classes of morphisms for the model structure on the category of simplicial sets. We will discuss the necessary properties of these morphisms to show that the category of simplicial sets has the required properties Q1 to Q4 of a model category (cf. Definition 3.1.1 and Definition 3.1.5). This will then prove Theorem 4.6.10. We start with the definition of cofibrations in **sSet**.

Definition 4.6.1. *A morphism $i : X \to Y$ of simplicial sets is called a cofibration if it is an inclusion.*

Proposition 4.6.2. Let $f : X \to Y$ be a trivial fibration in **sSet**. Then $|f| : |X| \to |Y|$ is a Serre fibration.

Proof. (Cf. [Hov99] Lemma 3.2.5) By definition, f has the RLP with respect to all $\partial\Delta^n \to \Delta^n$, $n \geq 0$, and therefore also the RLP with respect to all inclusions in **sSet**, as we have seen before. In particular, we can find an extension $\bar{f}$ in the commutative diagram

$$\begin{tikzcd} X \arrow[r, "id"] \arrow[d, "{(id,f)}"'] & X \arrow[d, "f"] \\ X \times Y \arrow[r, "pr_2"'] \arrow[ur, "\bar{f}"] & Y. \end{tikzcd}$$

This extension turns f into a retract of pr_2 via the diagram

$$\begin{tikzcd} X \arrow[r, "{(id,f)}"] \arrow[d, "f"'] & X \times Y \arrow[r, "\bar{f}"] \arrow[d, "pr_2"'] & X \arrow[d, "f"'] \\ Y \arrow[r] & Y \arrow[r] & Y. \end{tikzcd}$$

Therefore, $|f|$ is a retract of $|pr_2| : |X| \times |Y| \to |Y|$, which is a fibration since the geometric realization preserves products (cf. Proposition 2.3.8) and by Example 3.2.4 (i). Thus, $|f|$ is a fibration. $\square$

An important result is the following theorem of Quillen, which states that the realization functor preserves fibrations.

Theorem 4.6.3. ***(Quillen)***
Let $p : X \to Y \in \mathbf{sSet}$ be a Kan fibration. Then its realization $|p| : |X| \to |Y|$ is a Serre fibration.

Proof. See [GJ99] Chapter I Theorem 10.10 or [Hov99] Corollary 3.6.2. □

The detailed proof can be found in the literature as cited above. Here, I will just give a rough outline of the argument. To start with, one defines when a fibration is called minimal (cf. [Hov99] Definition 3.5.5). Then one proves that any fibration $p : X \to Y$ can be factorized as $p = p'r$, where p' is a minimal fibration and r is a trivial fibration. The realization of a minimal fibration is a Serre fibration (cf. [GJ99] Chapter I Theorem 10.9 or [Hov99] Corollary 3.5.7 and Proposition 3.6.1). This result is also called the Gabriel-Zisman Theorem. We have already seen that the realization of a trivial fibration is a Serre fibration in Proposition 4.6.2. Thus, the theorem of Quillen follows since fibrations are closed under composition. The theory of minimal fibrations is slightly technical and would be worth a whole section of its own. For the details, I refer to [GJ99] Chapter I Section 10 or [Hov99] Section 3.5.

Proposition 4.6.4. Let X be a Kan complex and let $\eta_X : X \to S(|X|)$ be the unit of the adjunction $|\ | \dashv S$. Then η_X induces an isomorphism

$$(\eta_X)_* : \pi_i(X, v) \xrightarrow{\cong} \pi_i(S(|X|), \eta_X(v))$$

on the simplicial homotopy groups for all $i \geq 0$ and all $v \in X_0$.

Proof. (Cf. [GJ99] Chapter I Proposition 11.1) First, we note that $S(|X|)$ is fibrant again by Lemma 4.1.6. Therefore, the homotopy groups for $S(|X|)$ are defined. The proof is done by induction on i. The base case for $i = 0$ is similar to the proof of Lemma 4.5.2 (details can be found in the reference). For the induction step we assume that η_X induces an isomorphism

$$(\eta_X)_* : \pi_i(X, v) \xrightarrow{\cong} \pi_i(S(|X|), \eta_X(v))$$

for for all fibrant simplicial sets X and $0 \leq i \leq n$. Then we obtain a commutative diagram

$$\begin{tikzcd}
\pi_{n+1}(X, v) \arrow[r, "(\eta_X)_*"] \arrow[d, "\cong"'] & \pi_{n+1}(S(|X|), \eta_X(v)) \arrow[d] \\
\pi_n(\Omega X, v) \arrow[r, "\cong", "(\eta_{\Omega X})_*"'] & \pi_n(S(|\Omega X|), \eta_{\Omega X}(v)),
\end{tikzcd}$$

where we already know that the lower horizontal and left vertical arrows are isomorphisms (cf. the proof of Corollary 4.5.15). Here,

$$\begin{tikzcd}
S(|\Omega X|) \arrow[r] \arrow[d] & S(|PX|) \arrow[d] \\
\Delta^0 \arrow[r] & S(|X|),
\end{tikzcd}$$

is again a pullback diagram, since right adjoints always preserve limits. As in the proof of Corollary 4.5.15 it now remains to show that $S(|PX|)$ has trivial homotopy groups. To see this, we use the lifting diagram

$$\begin{tikzcd}
(PX \times \partial\Delta^1) \cup (\Delta^0 \times \Delta^1) \arrow[rr, "((id_{PX},v),v)"] \arrow[d] & & PX \arrow[d] \\
PX \times \Delta^1 \arrow[rru, "h"] \arrow[rr] & & \Delta^0,
\end{tikzcd}$$

where the extension h exists (cf. Remark 2.4.11), since $PX \to \Delta^0$ has the RLP with respect to all inclusions $\partial\Delta^n \subseteq \Delta^n$, for $n \geq 0$. Having this property, PX is also called contractible. Since the realization functor preserves products, this gives us a homotopy $|h|$ between the identity map and a constant map, so $|PX|$ is contractible and has trivial homotopy groups. As discussed in the direct follow-up to this proof (applied to the space $|PX|$), also $S(|PX|)$ has trivial simplicial homotopy groups. This completes the proof. $\square$

We suppose Y to be an arbitrary topological space and $v \in Y$ a point. If we recall the definition of simplicial homotopy groups, then $\pi_n(S(Y), v)$ consists of homotopy classes of maps $\alpha : \Delta^n \to S(Y)$ relative $\partial\Delta^n$ such that α is constant with value v on $\partial\Delta^n$ in the sense that the diagram

$$\begin{tikzcd}
\Delta^n \arrow[r, "\alpha"] & S(Y) \\
\partial\Delta^n \arrow[u] \arrow[r] & \Delta^0 \arrow[u, "v"']
\end{tikzcd}$$

commutes. The homotopy group $\pi_n(Y, v)$ consists of homotopy classes of maps $(I^n, \partial I^n) \to (Y, v)$ relative ∂I^n. Using the adjunction

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, S(Y)) \cong \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, Y)$$

we get an isomorphism between those homotopy groups, because any homotopy $h : \Delta^n \times \Delta^1 \to S(Y)$ between two n-simplices $\alpha, \beta : \Delta^n \to S(Y)$ induces a homotopy

$$h : |\Delta^n| \times |\Delta^1| \cong I^n \times I \to S(Y)$$

between their adjoints. One can also run this adjunction backwards in a similar way. Therfore, there are isomorphims between $\pi_n(Y, v)$ and $\pi_n(S(Y), v)$. Applying this to the topological space $|X|$ and using our previous Propostion 4.6.4, we get the following result.

Corollary 4.6.5. Let X be a Kan complex. Then there are isomorphims

$$\pi_i(X, v) \cong \pi_i(S(|X|), \eta_X(v)) \cong \pi_i(|X|, |v|)$$

for all $i \geq 0$ and all vertices $v \in X_0$ between the simplicial and topological homotopy groups.

Hence, a morphism between Kan complexes is a weak equivalence if and only if its realization is a weak equivalence in the category of topological spaces. This leads us to the definition of our next class of morphisms for our model structure on **sSet**, the weak equivalences, generalized for maps between arbitrary simplicial sets.

Definition 4.6.6. *A map $f : X \to Y$ is defined to be a weak equivalence in* **sSet***, if the map $|f| : |X| \to |Y|$ is a weak equivalence of spaces in* ***Top***.

With this definition, the following proposition follows directly from the corresponding result in the category **Top** Proposition 3.2.3.

Proposition 4.6.7. Let

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\ & E \arrow[ur, "h"'] & \end{tikzcd}$$

be a commutative diagram in **sSet**. If any two of f, g and h are weak equivalences, then so is the third.

Proposition 4.6.8. Let $p : X \to Y$ be a map between simplicial sets. Then p is a Kan fibration and a weak equivalence if and only if p has the RLP with respect to all inclusions $\partial\Delta^n \hookrightarrow \Delta^n$, $n \geq 0$.

Proof. (Cf. [GJ99] Chapter I Theorem 11.2) Suppose that $p : X \to Y$ has the RLP with respect to $\mathcal{J}$. Then p is a Kan fibration as we have mentioned earlier (cf. Remark 4.2.11). We want to prove that

$$S(|p|) : S(|X|) \to S(|Y|)$$

is a weak equivalence, since by the natural isomorphism $\pi_i(S(|X|), \eta_X(v)) \cong \pi_i(|X|, |v|)$, then $|p|$ is a weak equivalence in **Top**, as well. One starts by defining the set $\pi_0(Y)$ to be the set of equivalence classes of vertices of Y for the relation generated by the vertex homotopy relation for an arbitrary simplicial set Y. Formally, for two vertices $y, z \in Y_0$ we have $y \simeq z$ if and only if there are vertices

$$y = y_0, y_1, \ldots, y_n = z \in Y_0$$

and 1-simplices

$$v_1, \ldots v_n \in Y_1$$

such that $\partial v_i = (y_{i-1}, y_i)$ or $\partial v_i = (y_i, y_{i-1})$ for $i = 1, \ldots, n$. If Y is a Kan complex, then this definition coincides with the definition of $\pi_0(Y)$ as the zeroth homotopy group. The canonical map $\eta_Y : Y \to S(|Y|)$ induces a bijection $\pi_0(Y) \to \pi_0(S(|Y|))$ for all simplicial sets Y (similar to the base case in the proof of Proposition 4.6.4). By the required lifting property of p the map $p_* : \pi_0(X) \to \pi_0(Y)$ is a bijection as well, similarly to the discussion in Remark 4.5.18. Thus, also the induced map $S(|X|) \to S(|Y|)$ is a bijection. It remains to show that

$$\pi_i(S(|X|), x) \to \pi_i(S(|Y|), p(x))$$

is an isomorphism of simplicial homotopy groups for all vertices x of X and all $i \geq 1$. To see this, one proceeds as in the proof of Proposition 4.6.4 as follows. Let F_x be the fibre

$$\begin{tikzcd} F_x \arrow[r, "j"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ \Delta^0 \arrow[r, "p(x)"'] & Y, \end{tikzcd}$$

and thus $S(|F_x|)$ is the fibre of $S(|X|) \to S(|Y|)$ over $S(|p|(|x|))$. One proves analogously that F_x and hence $S(|F_x|)$ both have trivial homotopy groups by looking at lifting diagrams of the form

$$\begin{tikzcd} (F_x \times \partial\Delta^1) \cup (\Delta^0 \times \Delta^1) \arrow[r, "((id_{F_x},w),w)"] \arrow[d] & F_x \arrow[d] \\ F_x \times \Delta^1 \arrow[r] \arrow[ur, dashed] & \Delta^0. \end{tikzcd}$$

We note that together with $p : X \to Y$ also $F_x \to \Delta^0$ has the RLP with respect to all boundary inclusions. And finally, by using the long exact sequence, the statement follows. For the inverse implication one uses minimal fibrations to show that any Kan fibration which is additionally a weak equivalence has the required lifting property. For a detailed proof I refer to [GJ99] Chapter I Theorem 11.2. $\square$

Proposition 4.6.9. Let $i : X \to Y$ be a map between simplicial sets. Then i is anodyne if and only if it is a cofibration and a weak equivalence.

Proof. (Cf. [GJ99] Chapter I Theorem 11.3 and [JT] Proposition 3.4.2) By definition, any anodyne extension is a cofibration. We consider the class of morphisms in **sSet** consisting of all inclusions $i : X \hookrightarrow Y$ such that the realization $|i| : |X| \to |Y|$ is a trivial cofibration in **Top**. One can easily check that this class is saturated by using the colimit-preserving property of the geometric realization functor. It also contains all horn inclusions $\Lambda^n_k \to \Delta^n$, since their realizations have the LLP with respect to all fibrations, by definition, and all homotopy groups of $|\Lambda^n_k| \cong D^{n-1}$ and $|\Delta^n| \cong D^n$ vanish, since these spaces are contractible. Thus all anodyne extensions are contained in this class and are weak equivalences, as well. For the reverse implication we assume that $i : X \hookrightarrow Y$ is a weak equivalence. By Theorem 4.2.12 i can be factorized as

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "j"'] & & Y \\ & E \arrow[ur, "p"'] & \end{tikzcd}$$

where j is anodyne and p is a fibration. Since j is a weak equivalence, by the first implication, the 2-out-of-3 axiom gives that p is a weak equivalence as well. Hence, it is a trivial fibration in the sense of Definition 3.1.2. Applying this fact to the right lifting diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ Y \arrow[r, "id_Y"'] \arrow[ur, dashed, "s"] & Y, \end{tikzcd}$$

over the monomorphism i we get an extension $s : Y \to E$. This extension turns i into a retract of j via the diagram

$$\begin{tikzcd} X \arrow[r, "id_X"] \arrow[d, "i"'] & X \arrow[r, "id_X"] \arrow[d, "j"'] & X \arrow[d, "i"] \\ Y \arrow[r, "s"'] & Y \arrow[r, "p"'] & Y \end{tikzcd}$$

and therefore i is anodyne by the definition of saturated classes. □

Now we have all morphism classes for our model structure on the category of simplicial sets. Summarizing our previous definitions and properties, the fibrations are the class of $\mathcal{I} - inj$ maps where $\mathcal{I}$ is the set of horn inclusions

$$\mathcal{I} = \{\Lambda_k^n \hookrightarrow \Delta^n \mid n \geq 1,\ 0 \leq k \leq n\}.$$

The cofibrations are all inclusions of simplicial sets and a morphism in **sSet** is a weak equivalence if and only if its realization is so in the category of topological spaces. Let

$$\mathcal{J} = \{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geq 0\}$$

denote the set of all boundary inclusions. As seen in the previous two propositions (cf. Proposition 4.6.8 and Proposition 4.6.9), the trivial fibrations are precisely the class $\mathcal{J}$-inj and the trivial cofibrations are the anodyne extensions, i.e, the saturated class generated by $\mathcal{I}$. Taken all the results together, we can now state and prove the main result of this work.

Theorem 4.6.10. With the above fibrations, cofibrations and weak equivalences the category **sSet** of simplicial sets is a model category.

Proof. (Cf. [GJ99] Chapter I Theorem 11.3 and [Hov99] Theorem 3.6.5) The proof is mainly a summary of the previous results. The "2-out-of-3" axiom Q1 has been discussed in Proposition 4.6.7. Axiom Q2, stating that the structure is closed under retracts, is trivially true in all three cases. For example, the lifting property for Kan fibrations holds immediately for the retract morphism. For weak equivalences, the property follows from the corresponding axiom in **Top**, where it is trivial as well. The lifting axiom Q3 has been discussed in Proposition 4.2.7 for fibrations and trivial cofibrations and in Lemma 4.2.3, Example 4.2.5 and Remark 4.2.11 for the reverse case. The two possible factorizations in axiom Q4 are proven in Theorem 4.2.12 and 4.2.14. □

And finally, we prove the theorem, that the fundamental adjunction indeed is a Quillen equivalence.

Theorem 4.6.11. The realization and singular functors $|-| : \mathbf{sSet} \rightleftarrows \mathbf{Top} : S$ form a Quillen equivalence and hence induce inverse equivalences on the corresponding homotopy categories.

Proof. (Cf. [GJ99] Chapter I Theorem 11.4 and [Hov99] Theorem 3.6.7) First, we note that these two functors are a Quillen adjunction, since it is an adjunction (cf. Proposition 2.2.4) and the singular functor preserves fibrations and trivial fibrations (cf. the discussion after Remark 4.1.5 and Corollary 4.6.5). Clearly, any simplicial set $X \in \mathbf{sSet}$ is cofibrant and any topological space $Y \in \mathbf{Top}$ is fibrant. Therefore, it remains to show that the unit and counit map

$$\eta_X : X \to S(|X|) \text{ and } \varepsilon_Y : |S(Y)| \to Y$$

of the adjunction $|\,| \dashv S$ are both weak equivalences for all simplicial sets X and spaces Y in **sSet** and **Top**, respectively. We start to prove this property for ε. Let Y be a topological space. It suffices to show that the map $\pi_i(|S(Y)|, v) \to \pi_i(Y, v)$ is an isomorphism for every point v of Y. Since $S(Y)$ is fibrant by Lemma 4.1.6, we have seen in Corollary 4.6.5 that there are natural isomorphisms $\pi_i(S(Y), v) \cong \pi_i(|S(Y)|, v)$. It remains to see that there are isomorphisms $\pi_i(S(Y), v) \cong \pi_i(Y, v)$. But this has been discussed in general above, after the proof of Proposition 4.6.4. Next we look at the map η. Let $X \in \mathbf{sSet}$ be an arbitrary simplicial set. By axiom Q4, we get a factorization diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "j"'] & & * \\ & E \arrow[ur] & \end{tikzcd}$$

with j a trivial cofibration and E a Kan complex, since $E \to *$ is a fibration. Thus, X is weakly equivalent to a fibrant simplicial set. For $\eta_E : E \to S(|E|)$ we have already seen that this is a weak equivalence (cf. Proposition 4.6.4). And therefore, η_X has to be weak equivalence as seen in the diagram

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "j"'] & S(|X|) \arrow[d, "S(|j|)"] \\ E \arrow[r, "\eta_E"'] & S(|E|), \end{tikzcd}$$

where $S(|j|)$ is a weak equivalence as well, since the composite functor $S|\,|$ preserves them by definition and by above isomorphisms of homotopy groups.

$\square$

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