ERRATA TO «RECIPROCITY LAWS FOR BALANCED DIAGONAL CLASSES»

by

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- After Equation (82) on page 113 of [BSV22a], it is implicitly stated that the pairing $\det_U^*$ induces a *perfect* duality of $\Lambda_U[p^{-1}]$-modules

$$\det_U^* : H^1(\Gamma, D_{U,m})^{\leqslant h} \otimes_{\Lambda_U[p^{-1}]} H^1_c(\Gamma, D'_{U,m})^{\leqslant h} \longrightarrow \Lambda_U[p^{-1}]$$

for each non-negative rational number h. In this generality, this is false. It is true in the following special cases.

  - $h = 0$.

    Proof. — With the notation of Section 4.3 of [BSV22a], there are Shapiro isomorphisms of $\Lambda_U[p^{-1}]$-modules (cf. Equation (87) of loco citato)

$$\mathrm{Sh}_\gamma : H^1_\gamma(\Gamma, D'_{U,m})^{\leqslant 0} \simeq e' \cdot \mathbf{T}_\gamma \otimes_\diamond \Lambda_U[p^{-1}],$$

    where $\gamma = \emptyset, c$ and $\mathbf{T}_c$ is defined by replacing $H^1_{\text{ét}}(Y_1(Np^r)_{\bar{\mathbf{Q}}}, \mathbf{Z}_p)$ with $H^1_{\text{ét},c}(Y_1(Np^r)_{\bar{\mathbf{Q}}}, \mathbf{Z}_p)$ in the definition of $\mathbf{T}$. According to Section 1.3 of [Oht03] the pairing (cf. Equation (114) of [BSV22a])

$$\det_U^* \circ \left(w_{Np}^{-1} \circ \mathrm{Sh}^{-1} \otimes \mathrm{Sh}_c^{-1}\right) : (e' \cdot \mathbf{T} \otimes_\diamond e' \cdot \mathbf{T}_c) \otimes_\diamond \Lambda_U[p^{-1}] \longrightarrow \Lambda_U[p^{-1}]$$

    is the $\Lambda_U[p^{-1}]$-linear extension of a perfect pairing $e' \cdot \mathbf{T} \otimes_\diamond e' \cdot \mathbf{T}_c \longrightarrow \diamond$ between finite free $\diamond$-modules. □

  - U is (sufficiently small and) centred at an integer $k \geqslant h + 2$.

    Proof. — Set $r = k - 2 \geqslant 0$, set $L_r = L_r(\mathbf{Z}_p)$, set $\mathscr{L}_r = L_r^{\text{ét}}$, and let

$$\det^*_{r,\mathbf{Z}_p} : H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{L}_r) \otimes_{\mathbf{Z}_p} H^1_{\text{ét},c}(Y_{\bar{\mathbf{Q}}}, \mathscr{L}_r) \longrightarrow \mathbf{Z}_p$$

    be the morphism arising from the pairing $\mathtt{det}_r^* : L_r \otimes_{\mathbf{Z}_p} L_r \longrightarrow \mathbf{Z}_p$ defined by $\mathtt{det}_r^*(\mu \otimes \nu) = \mu \otimes \nu\,((x_1y_2 - x_2y_1)^r)$ for all μ and ν in L_r (cf. page 96 of [BSV22a]). The $\mathbf{Q}_p$-linear extension of $\mathtt{det}_r^*$ is perfect, hence (by Poincaré duality) so is $\det^*_{r,\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} L$. The latter induces a perfect pairing

$$\det_r^* : H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{L}_r)_L^{\leqslant h} \otimes_L H^1_{\text{ét},c}(Y_{\bar{\mathbf{Q}}}, \mathscr{L}_r)_L^{\leqslant' h} \longrightarrow L,$$

where $M_L^{\leqslant^{\cdot} h} = (M \otimes_{\mathbf{Z}_p} L)^{\leqslant^{\cdot} h}$ (cf. Section 4.1.4 of [BSV22a]). Proposition 4.2 and Equation (73) of [BSV22a] give isomorphisms

$$\varrho^{\cdot}_{k,\gamma} : H^1_\gamma(\Gamma, D^{\cdot}_{U,m})^{\leqslant h} \otimes_k L \simeq H^1_\gamma(\Gamma, D^{\cdot}_{r,m})^{\leqslant h} \simeq H^1_{\text{ét},\gamma}(Y_{\bar{\mathbf{Q}}}, \mathscr{L}_r)^{\leqslant^{\cdot} h}_L,$$

where $\gamma = \emptyset, c$, $\cdot = \emptyset, \prime$ and $M \otimes_k L$ denotes the base change of M along evaluation at k on Λ_U. (The assumption $h < k-1$ is used here to guarantee that the comparison morphisms $H^1_\gamma(\Gamma, D^{\cdot}_{r,m}) \longrightarrow H^1_{\text{ét},\gamma}(Y_{\bar{\mathbf{Q}}}, \mathscr{L}_r)_L$ induce isomorphisms on the slope $\leqslant^{\cdot} h$ parts.) By construction

$$\det{}^*_U \otimes_k L = \det{}^*_r \circ \varrho_k \otimes \varrho'_{k,c},$$

hence the specialisation $\det^*_U \otimes_k L$ of $\det^*_U$ at weight k is a perfect pairing. Shrinking U if necessary, this implies that the finite $\Lambda_U[p^{-1}]$-modules $H^1_\gamma(\Gamma, D^{\cdot}_{U,m})^{\leqslant h}$ are free, and that $\det^*_U$ is perfect. □

In Section 4.2 of [BSV22a], the morphisms $\zeta_{U,m}, \zeta'_{U,m}, \mathbf{s}_{U,h}$ and $\mathbf{s}'_{U,h}$ are then defined assuming either $h = 0$ or U centred at an integer $k \geqslant h + 2$. The rest of the paper and its sequel [BSV22b] (where $h = 0$) are unaffected.

- In the line before Equation (92), *isomorphisms* should be *isomorphism*. The full stop after the isomorphism displayed in Equation (92) should be a comma.

- In the second line of Remark 4.4, $h < k_o - 2$ should be $h < k_o - 1$.

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References

[BSV22a] Massimo Bertolini, Marco Adamo Seveso, and Rodolfo Venerucci. Reciprocity laws for balanced diagonal classes. *Astérisque*, 434:77–174, 2022. 1, 2

[BSV22b] Massimo Bertolini, Marco Adamo Seveso, and Rodolfo Venerucci. Balanced diagonal classes and rational points on elliptic curves. *Astérisque*, 434:175–201, 2022. 2

[Oht03] Masami Ohta. Congruence modules related to Eisenstein series. *Ann. Sci. École Norm. Sup. (4)*, 36(2):225–269, 2003. 1

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