LOCALLY ACYCLIC MORPHISMS AND BASE CHANGE

Recall that when studying the quasi-coherent cohomology of schemes, we have the following base change result.

Theorem 1. *Let $f: Y \to S$ be a qcqs morphism and $g: S' \to S$ be flat. Consider the cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

Then the base change morphism $g^* Rf_*(\mathscr{F}) \xrightarrow{\sim} Rf'_* g'^*(\mathscr{F})$ is an isomorphism as soon as $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module.

In étale cohomology the analogous base-change holds, but the quasi-coherent condition has to be replaced by torsion of invertible order, and flat morphisms have to be replaced by *locally acyclic morphisms*.

1 Locally acyclic morphisms

Let S be any scheme and $s \to S$ a geometric point. We assume, although later we will show that it doesn't matter, that all geometric points are of the form $s = \operatorname{Spec} \widetilde{k} \to S$, where $\widetilde{k}$ is the closure of the image of $s \to S$. (In other words no trancendental extensions allowed.)

We define denote for ease of notation, the strict henselization of S at s by

$$S_s = \operatorname{Spec} \mathscr{O}_{S,s}^{\mathrm{hs}}.$$

Given a morphism $X \to S$, we similarly denote by X_S the base change $X = X \times_S S_s \to S_s$.

Definition 1. Let S be a scheme, and $s \to S$ a geometric point. An *étale generalization* of s is a geometric point $t \to S_s$. Note that this has an underlying Zariski generalization given by the image via the canonical morphism $S_s \to \operatorname{Spec}\mathcal{O}_{S,s}$.

Let $X \to S$ be a morphism, $x \to X$ a geomtric point above a geometric point $s \to S$ and $t \to S_s$ an étale generalization of s. The *scheme (or variety) of vanishing cycles* of X/S at $x \to s \leftarrow t$ is defined to be the t-scheme

$$\widetilde{X}_t^x = X_x \times_{S_s} t \to t.$$

That is, the scheme of vanishing cycles is a scheme, over an algebraically closed field, parametrizing étale generalizations of $x \to X$ which lie over the specified generalization $t \to S_s$ of the base. This scheme is almost never of finite type, but it is noetherian as soon as X is locally noetherian.

We can now define locally acyclic sheaves and morphisms. To make things cleaner we restrict to the setting of Λ-modules wher $\Lambda = \mathbf{Z}/p^n$ for some n.

Definition 2. Let X/S be an S-scheme. We say that an étale sheaf $K \in D(X_{\text{ét}}, \Lambda)$ is *locally acyclic* if for all $x \to X$ over $s \to S$ and generalizations $t \to S_s$ we have that the canonical map

$$K_x \xrightarrow{\sim} R\Gamma(\widetilde{X}_t^x, K_{|\widetilde{X}_t^x})$$

is an isomorphism. A sheaf K is called *universally locally acyclic* if for any map $S' \to S$ the sheaf $K_{S'}$ is locally acyclic as a sheaf on $X' = X \times_S S'$.

A morphism $f : X \to S$ is said to be (universally) locally acyclic if Λ is (universally) locally acyclic.

We recall that if Y is a local strictly henselian scheme with closed point $y \to Y$ then for any étale sheaf F on Y we have that $F_y = R\Gamma(Y, F)$. In particular, the Leray isomorphism gives us

$$R\Gamma(\widetilde{X}_t^x, K_{|\widetilde{X}_t^x}) = R\Gamma(X_x, Rj_* K_{|\widetilde{X}_t^x})$$

where $j : \widetilde{X}_t^x \to X_x$. The canonical morphism above is just induced from the counit $K \to Rj_* j^* K$.

Example. Every étale morphism is (universally) locallly acyclic. Indeed, any étale morphism induces an isomorphism on strict henselizations.

Example. Locally acyclic morphisms lifts tautologically lifts specializations. In particular, any locally acyclic morphism which is locally of finite presentation is open (Chevalley's Theorem).

Example. If $S = \operatorname{Spec} k$, then every $K \in D(X_{\text{ét}})$ is locally acyclic. Indeed, there are no specializations on the base, so the local acyclicity is equivalent to asking if

$$K_x \xrightarrow{\sim} R\Gamma(X_x, K)$$

is an isomorphism, which is a basic property of étale cohomology.

It is a deep theorem (I believe due to Gabber) that if X is a variety then it is furthremore universally locally acyclic. The problem here is that we lose control of the strict henselizations when we pass to the product $X \times Y \to Y$.

Example. Let X be a nodal curve and consider the degree 2 branched covering $X \to \mathbf{P}^1$ (lets assume we are in characteristic not 2 for simplicity). Then this map is not locally acyclic. Indeed, taking $x \to X$ to be the nodal point and the non-trivial generalization in the base, we see that the variety of vanishing cycles is disconnected[1] and hence $R\Gamma(\widetilde{X}_t^x, \Lambda)$ has bigger rank than expected.

We see that singularities seem to break local acyclicity. One of the two main theorems of local acyclicity is that smooth morphisms are (universally) locally acyclic when Λ has invertible order in the base. This takes some effort, and we will come back to this later.

Proposition 1. Let $S' \to S$ be a quasi-finite morphism. If X/S is locally acyclic then so is $X' = X \times_S S' \to S'$. If X'/S' is locally acyclic, and $S' \to S$ is also surjective, then so is X/S.

Proof. It follows from the fact that we can identify the scheme of vanishing cycles of the two via the following lemma. □

Lemma 1. Let $S' \to S$ be a quasi-finite morphism and $T \to S$ be an S-scheme. Then if $T' = T \times_S S'$ and $t' \to T'$ is a point mapping to s', t, s respectively, then

$$\operatorname{Spec} \mathcal{O}_{T,t}^{\mathrm{hs}} \otimes^{\mathrm{hs}}_{\operatorname{Spec} \mathcal{O}_{S,s}} \operatorname{Spec} \mathcal{O}_{S',s'}^{\mathrm{hs}} \xrightarrow{\sim} \operatorname{Spec} \mathcal{O}_{T',t'}^{\mathrm{hs}}$$

is an isomorphism.

[1]Crucially here we are using that the étale topology can "separate" the two branches of the node. In the Zariski topology such example would fail because the nodal curve is still integral.

Observe such result must be false without strict henselization: just take two finite extensions of fields.

Proof. We can assume S, S' and T are strictly local, the points to be the corresponding closed points and the morphisms to be local. Then $S' \to S$ finite by properties of henselian local rings. By the same reasoning T' is finite over S' and hence a product of henselian local rings. We must then show that there is a unique closed point. But the closed fiber is the fibered product $s' \times_s t$ which must inject into the point s' since $t \to s$ is radiciel. Now the morphism in question is a local morphism of henselian rings which is an isomorphism at the closed point, hence an isomorphism. □

2 Vanishing cycles

Let S be a henselian trait, that is, the spectrum of a henselian DVR. Let s, η be the closed and generic points respectively. If X/S is an S-scheme we can consider the fibers

$$X(s) = X \times_S s, \quad X(\eta) = X \times_S \eta.$$

We have a diagram in which we label the inclusions i and j.

$$\begin{tikzcd} X(s) \arrow[r, "i"] \arrow[d] & X \arrow[d] & X(\eta) \arrow[l, "j"'] \arrow[d] \\ s \arrow[r] & S & \eta \arrow[l] \end{tikzcd}$$

The same definition also applies for S henselian local, $s \to S$ the closed point and $\eta = t \to S_s$ an étale generalization.

Definition 3. Let X/S with S a henselian trait (or more generally henselian local as above). Let K be an étale sheaf in $D(X_{\text{ét}}, \Lambda)$. Then the *nearby cycles* of K is defined to be

$$R\Psi(K) = i^* R j_* j^* K \in D(X(s)_{\text{ét}}, \Lambda).$$

It comes equipped with the (restriction of the) counit morphism $K \to R\Psi(K)$. We also define the *vanishing cycles* of K to be the cone $R\Phi(K)$ of this morphism, that is, we have an exact triangle

$$K \to R\Psi(K) \to R\Phi(K)$$

in $D(X(s)_{\text{ét}}, \Lambda)$.

Remark. The nomencalture is somewhat unfortunate. Initially, on the SGA, Grothendieck called the nearby cycles vanishing cycles, and since then things still have not become quite normalized. For example our scheme of vanishing cycles should be called scheme of nearby cycles.

Proposition 2. Let $X \to S$ be a morphism. For every point $x \to X$ with image $s \to S$ and étale generalization $t \to S_s$, we have a canonical identification

$$R\Psi(K_s)_x = R\Gamma(\widetilde{X}_t^x, K_{|\widetilde{X}_t^x}).$$

In particular a morphism is locally acyclic if and only if for all $s \to S$ and generalizations $t \to S_s$ the map $K_{|X(s)} = K(s) \to R\Psi(K(s))$ is an isomrphism if and only if $R\Psi(K(s)) = 0$.

Proof. We can increase the diagram above to

$$\begin{tikzcd} x \arrow[r, "i"] \arrow[d] & X_x \arrow[d] & \widetilde{X}_t^x \arrow[l, "j"'] \arrow[d] \\ X(s) \arrow[r, "i"] \arrow[d] & X \arrow[d] & X(t) \arrow[l, "j"'] \arrow[d] \\ s \arrow[r] & S & t \arrow[l] \end{tikzcd}$$

and we note that, since pullbacks commute with pullbacks, the stalk of $R\Psi(K)$ at $x \to X(s)$ can be computed as the stalk of Rj_*j^*K at $x \to X$. Again, a basic computation of stalks of étale pushfoward yields

$$(Rj_*j^*K)_x = R\Gamma(X(t) \times_X X_x, K_{X_x}) = R\Gamma(\widetilde{X}_t^x, K_t)$$

since $X(t) \to X$ is qcqs (it is affine, because $t \to S$ is). $\square$

Recall the functoriality of étale cohomology: if $f : Y \to X$ is any morphism and K is a sheaf in $D(X_{\text{ét}})$ there is a canonical pullback map

$$R\Gamma(X, K) \to R\Gamma(Y, f^*K).$$

Applying this to our situation, and also using the Leray isomorphism, we have a natural map

$$R\Gamma(X(t), K(t)) = R\Gamma(X, Rj_*K(t)) \to R\Gamma(X(s), R\Psi(K)).$$

When K is locally acyclic, by the above theorem, we can define the following:

Definition 4. Let K be locally acyclic for X/S. The *cospecialization map* is the morphism

$$R\Gamma(X(t),K(t)) \to R\Gamma(X(s),K(s))$$

defined above.

So in some sense being locally acyclic means that you relate the cohomology of the fibers which are infinitesimally close to each other. In some sense this is analogous to the flatness of a connection, which we will make precise when X/S is furthermore proper (so that the cohomology of the fibers will be identified with the fibers of a sheaf on S).

3 The locally acyclic base change

Our goal in this section is to prove the analogous of the flat base change for the étale topology. We start with the statement. From now on, we assume that the order ℓ of $\Lambda = \mathbf{Z}/\ell^n$ is invertible in the base. (In other words we are only considering $\mathbf{Z}[1/\ell]$-schemes.)

Theorem 2 (Locally acyclic base change)**.** *Let $f: X \to S$ be a qcqs morphism and $g: S' \to S$ be locally acyclic. Let*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

be a cartesian diagram. Then the base change morphism $g^*Rf_*(K) \xrightarrow{\sim} Rf'_*g'^*(K)$ for $K \in D(X_{\acute{e}t}, \Lambda)$.

Lets start with some reductions. Since X/S is qcqs, we can compute the stalk of the pushfoward $Rf_*(K)$ at a geometric point $s \to S$ via the global sections of K pulled back to X_s. The same is true for f', and hence we can assume that both S and S' are local.

We can also assume $X \to S$ (and hence X) to be affine. [INCLUDE EXPLANATION].

Finally, writing the coordinate ring of X as a colimit of finitely presented sub-algebras, we can reduce to the case where f is of finite presentation. Even better, doing the same to S we can suppose that S is the strict Henselization of a finite type $\mathbf{Z}$-algebra, and hence reduce to the Noetherian setting.

To prove the theorem we assumem freely proper base change. This means that the result is already proven for $X \to S$ proper. In particular using Nagata's compactification we can reduce to the case of an open immersion. We start with a key case.

Lemma 2. In the situation of the Theorem, assume that $X = s \to S$ is a geometric point of S. Then the base change holds for the sheaf Λ.

Proof. Let Y be the normalization of the closure of the image of s in S. By the proper base change, we need only prove base change for the diagram

$$\begin{tikzcd} Y'_\eta \arrow[r, "g'"] \arrow[d, "f'"] & \eta \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

where $\eta \to Y$ is the generic point (the residue is already a closed field!) and Y is an integral, normal scheme. We note that Y is not necessarily noetherian, and that, by the properties mentioned above, all its local rings are already strictly henselian. The morphism $Y' \to Y$ is also locally acyclic by quasi-finite base change.

Unraveling what the theorem says we have to show that

$$g^*\eta_*\Lambda \xrightarrow{\sim} Rf'_*\Lambda$$

as sheaves on Y'. We claim that the left hand side is actually just Λ as a constant sheaf on Y'. Indeed, this is follows from the fundamental fact that each connected étale scheme over a irreducible, normal (or unibranch) scheme is itself irreducible and normal. Hence, we conclude that $\eta_*\Lambda = \Lambda$.

Now we simply note that we can check the desired claim at stalks. Let $y' \to Y'$ be a geometric point, and again the stalk of $Rf'\Lambda$ at this point is just

$$Rf'_*\Lambda = R\Gamma(Y'_s \times_{Y'} Y'_{y'}, \Lambda) = R\Gamma(\widetilde{Y'}^{y'}_\eta, \Lambda) = \Lambda$$

by local acyclicity. $\square$

Remark. Consider X to be the nodal curve, an integral but not unbranch scheme. Then one can construct an étale map $Y \to X$

given by gluing two $\mathbf{P}^1$'s at 0 and ∞. Therefore if $\eta \to X$ denotes the geometric generic point, one sees that

$$\Lambda(Y) = \Lambda \to \Lambda^2 = R\eta_* \Lambda(Y)$$

is not an isomorphism. Therefore it is really necessary to pass to the normalization for the argument above to work so smoothly.

Proof of Theorem. We have reduced the proof to the Noetherian open immersion case. The result is now known for every sheaf which is the pushfoward of a point by the lemma above. Indeed, this is a simple diagram chase from

$$\begin{tikzcd}
U'_s \arrow[r] \arrow[d] & s \arrow[d] \\
U' \arrow[r] \arrow[d] & U \arrow[d] \\
S' \arrow[r] & S
\end{tikzcd}$$

by noting that $U' \to U$ is locally acyclic.

Now, it is a deep theorem that this class of sheaves generates the category $D(U_{\text{ét}})$ for Noetherian U. This finishes the proof by dévissage. □

Example. Let X/k be any k-scheme for a closed field k. Let K be another closed field and $k \subset K$ an extension. Then there is a canonical isomorphism

$$R\Gamma(X, \Lambda) \xrightarrow{\sim} R\Gamma(X_K, \Lambda).$$

This can fail without characteristic assumptions. Indeed, just take $X = \mathbf{A}^1_k$ for k of characteristic p and use Artin-Schreier.

In particular, we need not assume our geometric points to be the closures of the residues of X. Any closed extension, independent of trancendence degree, would do.

A different base change result

Proposition 3. Let $f : X \to S$ be a locally acyclic morphism, with S locally Noetherian and assume all cospecialization maps are isomorphisms. Then

$$(Rf_* f^* K)_s \cong R\Gamma(X(s), f^* K(s))$$

is an isomorphism for all geometric points $s \to S$. In particular $(Rf_*\Lambda)_s \cong R\Gamma(X(s),\Lambda)$.

Remark. The above theorem implies that the base change transformation is an isomorphism for all sheaves which are pulled back from S and all $S' \to S$. However, to conclude this we need to show that $X' \to S'$ is also locally acyclic, which is not straightfoward (cf. last section).

Proof. The category of sheaves on S is generated by $s_*\Lambda$, where $s \to S$ is a geometric point (here is where the Noetherian assumption comes in). Now by the Lemma above, we can write

$$f^*s_*\Lambda \cong Rj_*\Lambda$$

where $j\colon X(s) \to X$ is the inclusion of the fiber. Hence, $R\Gamma(X, Rj_*\Lambda) = R\Gamma(X(s),\Lambda)$ and the base change map is easily seen to be the cospecialization map, which is an isomorphism by assumption. □

Corollary 1. Composition of locally acyclic morphisms between locally noetherian schemes is locally acyclic.

Proof. Indeed if $X \to Y \to Z$ are locally acyclic morphisms with points x, y, z then the morphism

$$\widetilde{X}^x_z \to \widetilde{Y}^y_z$$

is locally acyclic with fibers $\widetilde{X}^x_{y'}$ acyclic also. By last proposition, it follows that

$$R\Gamma(\widetilde{X}^x_z,\Lambda) = R\Gamma(\widetilde{Y}^y_z,\Lambda) = 0$$

by Leray, and hence we're done. □

4 Smooth morphisms and local acyclicity

In this section we will show the following very important class of examples of (universal) local acyclicity.

Theorem 3. *Let $f\colon X \to S$ be a smooth morphism. Then it is locally acyclic.*

Proof. By passing to a Zariski cover we can assumme $X \to S$ to be étale over $\mathbf{A}^n_S \to S$. We've proven last section that locally acyclic morphisms compose, hence it is enough to show that for every S $\mathbf{A}^1_S \to S$ is locally acyclic. By the proof of lemma 2, we can also assume S to be normal domain with algebraically closed generic point $\eta \to S$. Considering the cartesian diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d, "f"] & \eta \arrow[d] \\ \mathbf{A}^1_S \arrow[r] & S \end{tikzcd}$$

we must show that $\Lambda \xrightarrow{\sim} Rf_*\Lambda$.

We follow the ideas of Stacks project tag [0EYU].

The derived pushfoward can be seen as the sheafification of

$$U \mapsto R\Gamma(U \times_S \eta, \Lambda).$$

Now we need a strong input about the étale site of a smooth morphism:

> If $X \to S$ is smooth, then any étale morphism $U \to X$ locally factors, over S, as a morphism $U \to V \to S$ where $V \to S$ is étale and $U \to V$ is a smooth morphism of affine schemes with geometrically connected fibers and admiting a section. (Tag [0EY4].)

Now, the étale site of S is essentially trivial, in the sense that each (separated) étale maps $V \to S$ are (a disjoint union of) open subsets of S. This means we can assume that $U \to S$ has a section and geometrically connected fibers.

Now the map in question is already an equivalence in degrees > 1 for dimension reasons. In degree 0 it is an equivalence as follows from the connectedness above. In degree 1, we must see that each class $\xi \in \mathrm{H}^1(U \times_S \eta, \Lambda)$ becomes trivial after passing to an étale map $P \to U$. The finite étale Λ-torsor $\widetilde{P}$ representing ξ trivializes ξ. Now the results follows by another deep result of étale covering spaces:

> Let S be a qcqs integral normal scheme with closed generic point η. Let $X \to S$ be a smooth morphism with geometrically connected fibers and a section. Then any étale Λ-torsor (with order invertible on S) of X_η can be extended to X. (Tag [0EZJ].)

$\square$

Remark. The proof given above is desceptively simple. All proofs of this result need to involve some kind of tame purity statement as this last step in the above. The original involved Nagata-Zariski purity. Smooth morphisms are not acyclic if the order of Λ is not invertible in S.

5 Proper and locally acyclic base change

Theorem 4. *Let $X \to S$ be a morphism which is both proper and locally acyclic. Then for all specializations $t \to S_s$ of the base the cospecialization maps*

$$R\Gamma(X(t), K) \xrightarrow{\sim} R\Gamma(X(s), K)$$

are isomorphisms for all locally constant K. Furthermore the push-foward Rf_*K is also locally constant.

Proof. The proper base change theorem tells us that Rf_*K is constructible, and the stalk at $s \in S$ is $R\Gamma(X(s), \Lambda)$. The cospecialization morphisms are isomorphisms because we can identify them with the base change maps with respect to the proper and locally acyclic morphism $X_s \to S_s$ along the generalization $ttoS_s$. Finally, a constructible sheaf whose specialization maps are isomorphisms is locally constant. $\square$

Corollary 2. Let X/S be smooth. Then all fibers have the same ℓ-adic cohomology. In particular, if K is a local field and X/K has good reduction then the ℓ-adic cohomology of X and of the special fiber $\mathscr{X} \otimes k$ agree.

6 Gabber's Theorem on universal local acyclicity

We finish this document by showing Gabber's powerful theorem on universal local acyclicity.

Theorem 5 (Gabber). *Let S be a noetherian scheme, $X \to S$ finite type. If X/S is locally acyclic, then it is univeresally locally acyclic.*

Proof. Fill in later... $\square$

Remark. Note that this implies some interesting results: For example, over a field we have that $X \times Y \to Y$ is always locally acyclic. Similarly, this also reproves that $\mathbf{A}^1_S \to S$ is locally acyclic if S is an algebraic variety.