Examples and applications

Recall from the previous talks:

Definition: (Pure Hodge structure)

A pure Hodge structure of weight n is a $\mathbb{Z}$-module H of finite type (finitely generated) together with a decreasing filtration F^p on $H_{\mathbb{C}} = H \otimes_{\mathbb{Z}} \mathbb{C}$ such that for all $p+q = n+1$ we have

1) $F^p H_{\mathbb{C}} \oplus \overline{F^q H_{\mathbb{C}}} = H_{\mathbb{C}}$
2) $F^p H_{\mathbb{C}} \cap \overline{F^q H_{\mathbb{C}}} = 0$.

Definition: (Mixed Hodge structure)

A mixed Hodge structure is a finitely generated $\mathbb{Z}$-module H endowed with

- an increasing "weight" filtration $W_\bullet$ on $H_{\mathbb{Q}} = H \otimes_{\mathbb{Z}} \mathbb{Q}$;
- a decreasing "Hodge" filtration $F^\bullet$ on H

such that for every $m \in \mathbb{Z}$ the group $gr_m^W H_{\mathbb{Q}}$ endowed with the induced filtration from $F^\bullet$ is a pure Hodge structure of weight m over $\mathbb{Q}$.

Theorem (Deligne) Let X be a complex quasi-projective variety.
Then $H^m(X, \mathbb{Z})$ carries a canonical mixed Hodge structure.

This canonical mixed Hodge structure satisfies several properties:

1) If X is smooth and projective, then $H^m(X,\mathbb{Z})$ is pure of weight m.
This means that the weight filtration looks like: $0 = W_{m-1} \subseteq W_m = H^m(X,\mathbb{Z})$
2) If X is projective (not necessarily smooth), then
$0 = W_{-1} \subseteq W_0 \subseteq W_1 \subseteq \dots \subseteq W_m = H^m(X)$ (so the only weights are $\{0, \dots, m\}$)
3) If X is smooth (not necessarily projective), then
$0 = W_{m-1} \subseteq W_m \subseteq W_{m+1} \subseteq \dots \subseteq W_{2m} = H^m(X)$ (so the only weights are $\{m, \dots, 2m\}$)
4) If X is smooth and $\overline{X}$ is a smooth compactification of X, $i: X \hookrightarrow \overline{X}$, then
$W_m = i^* H^m(\overline{X})$.
5) Functoriality: for every morphism $f: X \to Y$ of quasi projective varieties, the induced map $f^*: H^m(Y) \to H^m(X)$ preserves weight and Hodge filtration (it is even strict with respect to both filtrations)

§ Mixed Hodge structures on cohomology with compact support

Let X be a complex variety. Classically, one can define the cohomology with compact support $H^*_c(X)$ and the relative cohomology $H^*(X, Z)$ with respect to a subvariety $Z \subseteq X$.

If $\overline{X}$ is a compactification of X with complement Z, then $H^*_c(X) = H^*_c(\overline{X}, Z)$.

In particular, one obtains a long exact sequence:

$$\dots \to H^n_c(X) \to H^n(\overline{X}) \to H^n(Z) \to H^{n+1}_c(X) \to \dots$$

Goal: there is a canonical MHS on $H^*_c(X)$ such that the above sequence is an exact sequence of MHS.

We treat the case when X is smooth. Take a smooth compactification $\overline{X}$ of X such that $D = \overline{X} \setminus X$ is a simple normal crossings divisor. Wrote $j : X \hookrightarrow \overline{X}$.

Take X smooth, $\overline{X}$ a compactification of X with $D = \overline{X} \setminus X$ a SNC divisor.

de Rham: $(\Omega^*_{\overline{X}}(\log D), W, F)$ $\qquad F^p = \tau_{\geq p}$

W looking at poles.

Betti: $(Rj_*\underline{\mathbb{Q}}, F)$ $\qquad F^p = \tau_{\geq p}$

This is a cohomological mixed Hodge complex.

Taking $R\Gamma \rightsquigarrow$ mixed Hodge complex.

Let us denote this mixed Hodge complex by $A_{\overline{X}}(\log D)$

If X is proper, then we take $\overline{X} = X$, $D = \emptyset$. The construction simplifies: we consider Ω^*_X instead of $\Omega^*_{\overline{X}}(\log D)$ and there is no more W.

We write simply $A_{\overline{X}}$ instead of $A_{\overline{X}}(\log D)$

There is also a cohomological mixed Hodge complex that computes the MHS on $H^*(D)$. Let us denote by A_D^{SNC} the mixed Hodge complex associated with this. We then have a morphism $f : A_{\overline{X}} \to A_D^{SNC}$ of mixed Hodge complexes.

More precisely:

- Write $D_1, \dots, D_r$ for the irred. comp. of D, $D_I = \bigcap_{i \in I} D_i$, $D^p = \bigsqcup_{|I|=p} D_I$.
- We consider A_{D^p} for all p. Since D^p is proper, there is no need to take logarithmic differentials. A_{D^p} is a mixed Hodge complex
- We make a double complex $(A_D)^{p,q} =$ term of A_{D^p} in degree q. This is a dg-mixed Hodge complex
- We can take the total complex of a dg-mixed Hodge complex (with diagonal filtration and all that). This gives:
  $A_D^{SNC} = \mathrm{Tot}\left((A_D)^{*,*} \right)$, a mixed Hodge complex
- The embeddings $D_I \hookrightarrow \overline{X}$ for all I induce a morphism $A_{\overline{X}} \longrightarrow A_D^{SNC}$ of mixed Hodge complexes. This follows by functoriality of the construction.

We look at this morphism in the derived category of MHCs (which is equivalent to the derived category of complexes of mixed Hodge structures). Here we can form the cone $A_{\overline{X}} \xrightarrow{f} A_D \to \mathrm{cone}(f) \xrightarrow{+1}$

Taking cohomology, this leads to an exact sequence

$$\to H^n(\overline{X}) \to H^n(D) \to H^n(\mathrm{cone}(f)) \to H^{n+1}(X)$$

Thus, we define $H^n_c(X) = H^{n-1}(\mathrm{cone}(f))$.

In this way, we obtain the desired long exact sequence for $H^n_c(X)$.

If X is not smooth $\rightsquigarrow$ similar argument, but introduce hypercoverings.

§ Mixed Hodge structure on relative cohomology

A similar construction gives us a way to define relative cohomology as a mixed Hodge structure. We deal with the smooth case.

We start with a smooth variety X and a SNC divisor D on X (watch out: D is a divisor on X now, not on $\overline{X}$!).

We can find a compactification $\overline{X}$ such that the Zariski closure $\overline{D}$ of D in $\overline{X}$ has the following property:

- $\overline{D}_I$ is a smooth compactification of D_I
- $\overline{D}_I \setminus D_I$ is a SNC divisor.

We may then form the mixed Hodge complexes $A_{\overline{X}}(\log \overline{X} \setminus X)$ and $A^{SNC}_{\overline{D}}(\log \overline{D} \setminus D)$. These come with a map $f: A_{\overline{X}}(\log \overline{X} \setminus X) \to A^{SNC}_{\overline{D}}(\log \overline{D} \setminus D)$

We consider the cone $\mathrm{cone}(f)$ in the derived category of mixed Hodge complexes and define $H^n(X,D) = H^{n-1}(\mathrm{cone}(f))$.

This gives the long exact sequence for relative cohomology

$$\dots \to H^n(X,D) \to H^n(X) \to H^n(D) \to H^{n+1}(X,D) \to \dots$$

§ Cup product, Künneth formula, Poincaré duality, Gysin

- Cup product: Let X, Y be complex varieties. Then the external product $H^n(X) \otimes H^m(Y) \to H^{n+m}(X \times Y)$ is a morphism of MHS. In particular, the cup-product $H^n(X) \otimes H^m(X) \to H^{n+m}(X)$ is a morphism of MHS.
- Künneth formula passes to Hodge structures.
- X smooth, proper, irreducible, $\dim X = d$. Then $H^{2d}(X) \simeq \mathbb{Q}(-d)$
- Poincaré duality: for X smooth of dimension d, we have:
  $H^n(X) \cong \mathrm{Hom}(H^{2d-n}_c(X), \mathbb{Q}(-d))$

•) *Gysin morphism:* X smooth, $\dim X = d$, $Z \subseteq X$ smooth, $\operatorname{codim}_X Z = p$.
There is a Gysin map $\gamma : H^n(Z)(-p) \to H^{n+2p}(X)$, defined by

$$
\begin{aligned}
H^n(Z)(-p) \xrightarrow{\sim} \; & \mathrm{Hom}\left(H_c^{2d-2p-n}(Z), \mathbb{Q}(p-d)\right)(-p) \\
& \qquad \| \\
& \mathrm{Hom}\left(H_c^{2d-2p-n}(Z), \mathbb{Q}(-d)\right) \\
& \qquad \downarrow \\
& \mathrm{Hom}\left(H_c^{2d-2p-n}(X), \mathbb{Q}(-d)\right) \\
& \qquad \cong \\
& H^{n+2p}(X).
\end{aligned}
$$

The Gysin map fits into the following long exact sequence: $(U = X \setminus Z)$

$$\cdots \longrightarrow H^{n-1}(X) \to H^{n-1}(U) \to H^{n-2p}(Z)(-p) \xrightarrow{\gamma} H^n(X) \to \cdots$$

§ <u>Example: cohomology of the projective space</u>

$X = \mathbb{P}^d$, Z hyperplane, $Z \cong \mathbb{P}^{d-1}$; $U = X \setminus Z$. We apply Gysin

$$\cdots \to H^n(X) \to H^n(U) \to H^{n+1-2}(Z)(-1) \to H^{n+1}(X) \to \cdots$$

$$\cdots \to H^n(\mathbb{P}^d) \to 0 \to H^{n+1-2}(\mathbb{P}^{d-1})(-1) \to H^{n+1}(\mathbb{P}^d) \to 0 \to \cdots$$

Thus: $\quad H^{n+1}(\mathbb{P}^d) \cong H^{n+1-2}(\mathbb{P}^{d-1})(-1)$.

This allows us to compute inductively. For $d = 1$, the sequence reads:

$$0 \to H^0(\mathbb{P}^1) \to H^0(\mathbb{A}^1) \to 0 \to H^1(\mathbb{P}^1) \to 0 \to H^0(pt)(-1) \to H^2(\mathbb{P}^1) \to 0$$

This gives:

$$H^0(\mathbb{P}^1) \cong \mathbb{Q}(0), \qquad H^1(\mathbb{P}^1) = 0, \qquad H^2(\mathbb{P}^1) = \mathbb{Q}(-1).$$

Inductively:

$$H^0(\mathbb{P}^2) \cong \mathbb{Q}(0),\ H^1(\mathbb{P}^2) = 0,\ H^2(\mathbb{P}^2) = \mathbb{Q}(-2),\ H^3(\mathbb{P}^2) = 0,\ H^4(\mathbb{P}^2) = \mathbb{Q}(-2)$$

Hence:

$$H^n(\mathbb{P}^d) = \begin{cases} \mathbb{Q}(-\frac{n}{2}) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \qquad \text{for } 0 \le n \le 2d$$

<u>Remark</u>: We have exploited the fact that $H^n(\mathbb{A}^d) = 0$ for $n \geq 1$. However, we still need to know that the Gysin sequence is exact in the category of MHS to conclude.

§ Smooth non-proper, proper non-smooth

Let X be a smooth variety, $\overline{X}$ a smooth compactification of X, $D = \overline{X} \setminus X$ SNC divisor.
To construct a MHS over $H^n(X)$, we consider the cohomological mixed Hodge complex
$((Rj_* \underline{\mathbb{Q}}_X, \tau^\cdot), (\Omega^\cdot_{\overline{X}}(\log D), W_\cdot, \tau^\cdot)$, comparison isos$)$
The weight filtration on $\Omega^\cdot_{\overline{X}}(\log D)$ takes into account the poles of differential forms:

$$W_m \Omega^p_{\overline{X}}(\log D) = \begin{cases} 0 & \text{if } m<0 \\ \Omega^{p-m}_{\overline{X}} \wedge \Omega^m_{\overline{X}}(\log D) & \text{if } 0 \le m \le p \\ \Omega^p_{\overline{X}}(\log D) & \text{if } m \ge p \end{cases}$$

Applying $R\Gamma$ we obtain a mixed Hodge complex, whose cohomology therefore carries a mixed Hodge structure. It follows that:
$W_m H^n(X) = \operatorname{im}\left(H^n(X, W_{m-n} \Omega^\cdot_{\overline{X}}(\log D)) \to H^n(X) \right)$
In particular, if $m<n$, we have $W_{m-n} \Omega^\cdot_{\overline{X}}(\log D) = 0$, so $W_m H^n(X) = 0$.
This implies that the weights of $H^n(X)$ lie in $\{n, n+1, \dots, 2n\}$ whenever X is smooth.

Instead, if X is proper, we have $\overline{X} = X$, $D = \phi$, so $\Omega^m_{\overline{X}}(\log D) = \Omega^m_{\overline{X}} = \Omega^m_X$.
This implies that $W_{m-n} \Omega^\cdot_{\overline{X}}(\log D) = \Omega^\cdot_X$ for all $m>n$. Hence the map $H^n(X, W_{m-n}\Omega^\cdot_{\overline{X}}(\log D)) \cong H^n(X, \Omega^\cdot_X) \to H^n(X) = H^n(X, \Omega^\cdot_X)$ is the identity, so $H^n(X)$ has weights in $\{0, \dots, n\}$.
Combining these two properties, if X is smooth and projective, then $H^n(X)$ carries a pure Hodge structure of weight n.

§ Example 1: Union of two smooth varieties

Let X be the union of two smooth projective varieties X_1, X_2 that intersect transversally. We want to describe the MHS on $H^m(X)$.

We look at Mayer-Vietoris:

$$\cdots \xrightarrow{\beta_{m-1}} \underbrace{H^{m-1}(X_1 \cap X_2)}_{\text{pure, weight } m-1} \xrightarrow{\delta} H^m(X) \xrightarrow{\alpha_m} \underbrace{H^m(X_1) \oplus H^m(X_2)}_{\text{pure, weight } m} \xrightarrow{\beta_m} \underbrace{H^m(X_1 \cap X_2)}_{\text{pure, weight } m} \longrightarrow \cdots$$

We observe that:

- $\alpha_m : H^m(X) \to H^m(X_1) \oplus H^m(X_2)$ is induced by $X_1 \hookrightarrow X$, $X_2 \hookrightarrow X$.
  $\beta_m : H^m(X_1) \oplus H^m(X_2) \to H^m(X_1 \cap X_2)$ is induced by $X_1 \cap X_2 \hookrightarrow X_1$, $X_1 \cap X_2 \hookrightarrow X_2$.
  Thus, both α_m and β_m are morphisms of MHS.
- δ is a connecting homomorphism, so it is not induced from a morphism of varieties. To prove that δ is a morphism of MHS, an argument is needed. We will assume this fact.

Looking at the weights of the exact sequence above, the weights of $H^m(X)$ can only belong to $\{m-1, m\}$. The weight filtration looks like:

$$0 = W_{m-2} \subseteq W_{m-1} \subseteq W_m = H^m(X).$$

Since $\delta : H^m(X_1 \cap X_2) \to H^m(X)$ is strict, we have $W_{m-1} H^m(X) = \operatorname{im} \delta$.

In particular: $gr^W_{m-1} H^m(X) = W_{m-1}/W_{m-2} = \operatorname{im} \delta \cong \operatorname{coker} \beta_{m-1}$

This is the cokernel of a map between pure Hodge structures of weight $m-1$, so (as expected) it is pure of weight $m-1$.

Similarly: $gr^W_m H^m(X) = W_m/W_{m-1} = H^m(X)/\operatorname{im}\delta \cong \ker \beta_m$.

Suppose for example that $\dim X_1 = \dim X_2 = 1$. Then the above sequence reads:

$$0 \to \tilde{H}^0(X_1 \cap X_2) \longrightarrow H^1(X) \to H^1(X_1) \oplus H^1(X_2) \longrightarrow 0, \quad \text{so}$$

$$gr^W_0 H^1(X) \cong \tilde{H}^0(X_1 \cap X_2) \quad \text{and} \quad gr^W_1 H^1(X) \cong H^1(X_1) \oplus H^1(X_2).$$

If X is the union of several varieties that intersect transversally, a spectral sequence is needed.

Example 2: Blow-up.

Take $\pi : Y \to X$, $Z \subseteq X$ closed, $U = X \setminus Z$. Suppose that π restricts to an isomorphism $Y \setminus E \to X \setminus Z$, where $E = \pi^{-1}(Z)$

There is a long exact sequence:

$$\dots \to H^m(X) \to H^m(Y) \oplus H^m(Z) \to H^m(E) \to H^{m+1}(X) \to \dots$$

To understand why, notice that π give a map of pairs $(Y,E) \to (X,Z)$, so we have a map between the cohomology sequences of the pairs (Y,E), (X,Z)

$$\begin{tikzcd}
\dots \arrow[r] & H^n(Y,E) \arrow[r] & H^n Y \arrow[r] & H^n E \arrow[r] & H^{n+1}(Y,E) \arrow[r] & \dots \\
\dots \arrow[r] & H^n(X,Z) \arrow[r] \arrow[u] & H^n X \arrow[r] \arrow[u] & H^n Z \arrow[r] \arrow[u] & H^{n+1}(X,Z) \arrow[r] \arrow[u] & \dots
\end{tikzcd}$$

It is a standard fact (Aeppli 1957) that $H^n(X,Z) \xrightarrow{\sim} H^n(Y,E)$ (essentially an instance of proper base change). Let us give some names:

$$\begin{tikzcd}
\arrow[r] & H^{n-1}(E) \arrow[r, "\delta_{n-1}"] & H^n(Y,E) \arrow[r, "a'_n"] & H^n Y \arrow[r, "b'_n"] & H^n E \arrow[r, "\delta'_n"] & H^{n+1}(Y,E) \arrow[r] & \dots \\
\arrow[r] & H^{n-1}(Z) \arrow[r, "\delta_{n-1}"] \arrow[u, "\pi^*"] & H^n(X,Z) \arrow[r, "a_n"] \arrow[u, "\wr\ \pi^*"] & H^n X \arrow[r, "b_n"] \arrow[u, "\pi^*"] & H^n Z \arrow[r, "\delta_n"] \arrow[u, "\pi^*"] & H^{n+1}(X,Z) \arrow[r] \arrow[u, "\wr\ \pi^*"] & \dots
\end{tikzcd}$$

We build a sequence of the form:

$$\dots \to H^{n-1}(E) \xrightarrow{f_{n-1}} H^n(X) \xrightarrow{g_n} H^n(Y) \oplus H^n(Z) \xrightarrow{h_n} H^n(E) \to H^{n+1}(X) \to \dots$$

$$x \longmapsto (\pi^*(x),\ b_n(x))$$

$$(y,z) \longmapsto b'_n(y) - \pi^*(z)$$

$$e \longmapsto (a_n \circ (\pi^*)^{-1} \circ \delta_{n-1})(e)$$

To make it more colourful, this sequence looks like this:

$$\begin{tikzcd}
\arrow[r] & H^{n-1}(E) \arrow[r, "\delta_{n-1}"] & H^n(Y,E) \arrow[r] & H^n Y \arrow[r, "b'_n"] \arrow[dr, phantom, "-"] & H^n E \arrow[r] & H^{n+1}(Y,E) \arrow[r] & \dots \\
\arrow[r] & H^{n-1}(Z) \arrow[r] \arrow[u] & H^n(X,Z) \arrow[r, "a_n"] \arrow[u, "\wr\ \pi^*"] & H^n X \arrow[r, "b_n"] \arrow[u, "\pi^*"] \arrow[ur, phantom, "\oplus"] & H^n Z \arrow[r] \arrow[u, "\pi^*"] & H^{n+1}(X,Z) \arrow[r] \arrow[u, "\wr"] & \dots
\end{tikzcd}$$

$$x \longmapsto 0$$

Exactness can be checked by diagram chasing.

Suppose now that X has an isolated singular point p and that $Z = \{p\}$.
Suppose Y is smooth and E is a smooth subvariety of Y of codimension 1 (think of a blow-up). Then our sequence reads: $(n \geq 2)$

$$\cdots \to H^{n-1}(E) \to H^n(X) \xrightarrow{\pi^*} H^n(Y) \to H^n(E) \to H^{n+1}(X) \to \cdots$$

Then again the weight filtration on $H^n(X)$ reads $0 = W_{n-2} \subseteq W_{n-1} \subseteq W_n = H^n(X)$

and
$$gr^W_{n-1} H^n(X) = \operatorname{coker}(H^{n-1}(Y) \to H^{n-1}(E))$$
$$gr^W_n H^n(X) = \ker(H^n(Y) \to H^n(E))$$

Example 3: Open subset of a projective variety.

Let X be smooth and projective, $Z \subseteq X$ smooth subvariety of codimension 1 and $U = X \setminus Z$. We want to compute the MHS on $H^m(U)$.

At the level of abelian groups, there is a Gysin long exact sequence:

$$\cdots \to H^{m-2}(Z) \to H^m(X) \to H^m(U) \xrightarrow{R} H^{m-1}(Z) \to H^{m+1}(X) \to \cdots$$

As it is right now, this does not give an exact sequence of MHS: both $H^{m-2}(Z)$, $H^m(X)$ are pure HS, but have different weights, so the map $H^{m-2}(Z) \to H^m(X)$ would be always zero...

Instead, we need to twist it: (tensor with $\mathbb{Q}(-1)$)

$$\to \underbrace{H^{m-2}(Z)(-1)}_{\text{pure weight } m} \to \underbrace{H^m(X)}_{\text{pure weight } m} \to H^m(U) \to \underbrace{H^{m-1}(Z)(-1)}_{\text{pure weight } m+1} \to$$

It follows that the weight filtration is: $0 = W_{m-1} \subseteq W_m \subseteq W_{m+1} = H^m(U)$,

$$gr^W_m H^m(U) = \operatorname{coker}(H^{m-2}(Z)(-1) \to H^m(X))$$
$$gr^W_{m+1} H^m(U) = \ker(H^{m-1}(Z)(-1) \longrightarrow H^{m+1}(X))$$

More concretely, take X to be a smooth curve, Z a finite collection of points. Then

$$0 \to H^1(X) \to H^1(U) \xrightarrow{R} \tilde{H}^0(Z)(-1) \to 0$$

So $gr^W_1 H^1(U) \cong H^1(X)$, $\quad gr^W_2 H^1(U) \cong \tilde{H}^0(Z)(-1)$

The classes of weight 1 are represented by differences of the differential forms $\frac{dz}{z-p_1}, \dots, \frac{dz}{z-p_k}$, where $Z = \{p_1, \dots, p_k\}$

Remark: We can also use MHS to deduce strong geometric properties on varieties. For example, we can prove the existence of smooth varieties which do not admit a smooth compactification by a smooth divisor.
Any such variety, say U, would satisfy that $H^n(U)$ has weights in $\{n, n+1\}$.
It is then enough to give an example of a smooth non-proper variety such that $H^n(U)$ has more weights.
Take $X = \mathbb{P}^2$, $Z \subseteq X$ a singular curve, $U = X \setminus Z$. $(\dim U = 2)$
Relative cohomology sequence of the pair (X,Z):

$$\dots \to H^{n-1}(Z) \to H^n(X,Z) \to H^n(X) \to H^n(Z) \to H^{n+1}(X,Z) \to \dots$$

Compactly supported cohomology: $H^n_c(U) \cong H^n(X,Z)$.
Poincaré duality: $H^n(U) \cong \mathrm{Hom}\left(H^{4-n}_c(U), \mathbb{Q}(-2)\right)$ (U smooth, $\dim U = 2$).

Suppose Z has two singular point, let $\tilde{Z}$ be the blow-up at these points and $E \cong \{pt\}^4$ be the exceptional divisor. I guess the two singular points should be two nodes. The sequence for the blow-up reads:

$$0 \to \underbrace{H^0(Z)}_{\dim 1} \to \underbrace{H^0(\tilde{Z})}_{\dim 1} \to \underbrace{H^0(E)}_{\dim 4} \to H^1(Z) \to H^1(\tilde{Z}) \to \underset{0}{\underset{\parallel}{H^1(E)}} \to H^2(Z) \to \underset{\mathbb{Q}(-1)}{\underset{\cong}{H^2(\tilde{Z})}} \to \underset{0}{\underset{\parallel}{H^2(E)}} \to \underset{0}{\underset{\parallel}{H^3(Z)}} \to \dots$$

It follows that the weight filtrations read:

$0 = W_{-1}H^1(Z) \subseteq W_0 H^1(Z) \subseteq W_1 H^1(Z) = H^1(Z)$ $\qquad gr_0^W H^1(Z) \cong \mathrm{coker}\left(H^0(\tilde{Z}) \to H^0(E)\right) \leadsto \dim 1$

$\qquad gr_1^W H^1(Z) \cong H^1(\tilde{Z})$

$0 = W_1 H^2(Z) \subseteq W_2 H^2(Z) = H^2(Z)$ $\qquad gr_2^W H^2(Z) \cong H^2(E) \cong \mathbb{Q}(-1)$

In particular: $\mathrm{weights}(H^1(Z)) = \{0,1\}$, $\mathrm{weights}(H^2(Z)) \subseteq \{2\}$.

Now we look at $H^2_c(U)$:

$$\begin{tikzcd}
\dots \arrow[r] & H^1(X) \arrow[r] \arrow[d, equal] & H^1(Z) \arrow[r] \arrow[d, squiggly] & H^2_c(U) \arrow[r] & H^2(X) \arrow[r] \arrow[d, equal] & H^2(Z) \arrow[r] \arrow[d, squiggly] & H^3_c(U) \arrow[r] & \dots \\
& H^1(\mathbb{P}^2) \arrow[d, equal] & \text{weights } \{0,1\} & & H^2(\mathbb{P}^2) \arrow[d, "\cong"] & \text{weight } 2 & & \\
& 0 & & & \mathbb{Q}(-1) & & &
\end{tikzcd}$$

It follows that $H^1(Z)$ embeds into $H^2_c(U)$, so $H^2_c(U)$ has non-trivial graded pieces gr^W_0, gr^W_1. Thus the weight filtration reads:

$$0 = W_{-1} \subseteq W_0 \subseteq W_1 \subseteq W_2 = H^2_c(U)$$

$$\begin{array}{ccc}
gr^W_0 & gr^W_1 & gr^W_2 \\
\cong & \cong & \cong \\
gr^W_0 H^1(Z) & gr^W_1 H^1(Z) & \ker\left(H^2(\mathbb{P}^2) \to H^2(Z)\right)
\end{array}$$

Now we apply Poincaré duality:

$$H^2(U) = \mathrm{Hom}\left(H^2_c(U), \mathbb{Q}(-2)\right).$$

Now, if H is a MHS and H' is a pure Hodge structure of weight n, then $\mathrm{Hom}(H,H') \cong H' \otimes H^\vee$, so the weight filtration of $\mathrm{Hom}(H,H')$ will have weights of the form $n-w$ where w is a weight of H.

In our case, $\mathbb{Q}(-2)$ has weight 4, so $H^2(U)$ has weights $4-0, 4-1, 4-2$, i.e. $\{2,3,4\}$. Since $gr^W_0 H^1(Z) \neq 0$, we have $gr^W_0 H^2_c(U) \neq 0$, hence $gr^W_4 H^2(U) \neq 0$.

Since $H^2(U)$ has a non-vanishing graded quotient of weight 4, we conclude that U cannot be compactified by a smooth divisor.

§ Some arithmetic.

Let $\alpha \in \overline{\mathbb{Q}}$, $X = \mathbb{A}^1_{\overline{\mathbb{Q}}} \setminus \{0\}$, $D = \{1, \alpha\}$.

We compute $H^1(X,D)$. We have:

$$\begin{tikzcd}
0 \arrow[r] & H^0(X,D) \arrow[r] & H^0(X) \arrow[r] \arrow[d, equal] & H^0(D) \arrow[r] \arrow[d, "\cong"] & H^1(X,D) \arrow[r] & H^1(X) \arrow[r] \arrow[d, "\cong"] & H^1(D) \arrow[d, equal] \\
& & \mathbb{Q}(0) & \mathbb{Q}(0)^2 & & \mathbb{Q}(-1) & 0
\end{tikzcd}$$

The map $\mathbb{Q}(0) \to \mathbb{Q}(0)^2$ is the diagonal, so we get

$$0 \longrightarrow \mathbb{Q}(0) \longrightarrow H^1(X,D) \longrightarrow \mathbb{Q}(-1) \longrightarrow 0$$

This means that the weight filtration on $H^1(X,D)$ is

$$0 = W_{-1} \subseteq W_0 = W_1 \subseteq W_2 = H^1(X,D)$$

$$gr^W_0 H^1(X,D) \cong \mathbb{Q}(0)$$

$$gr^W_1 H^1(X,D) = 0$$

$$gr^W_2 H^2(X,D) \cong \mathbb{Q}(-1)$$

At the level of singular cohomology, $H_1(X,D)$ is generated by the classes of γ (circle around 0) and δ (segment from 1 to α). Dually, we can understand $W_\bullet H^1(X,D)$ by looking at differentials with log poles. Both X and $D=\{1,\alpha\}$ are affine. Coherent sheaves on affine varieties are acyclic, so we can take global sections without injective resolutions. We compactify X by $\overline{X}=\mathbb{P}^1$ and consider $\Omega^1_{\mathbb{P}^1}(\log\{0,\infty\})$.
We have $\frac{dz}{z} \in F^1\Omega^1_{\mathbb{P}^1}(\log\{0,\infty\})$. This defines an element in $W_2 H^1(X,D)$.

We have a lo exact sequence

$$\begin{tikzcd}
H^0(\Omega_X(X)) \arrow[r] \arrow[d, phantom, "\cong"] & H^0(\Omega^*_D(D)) \arrow[r] \arrow[d, phantom, "\cong"] & H^1(\Omega^*(X,D)) \arrow[r] & H^1(\Omega^*_X(X)) \arrow[r] \arrow[d, phantom, "\cong"] & H^1(\Omega^*_D(D)) \arrow[d, phantom, "="] \\
\overline{\mathbb{Q}} & \overline{\mathbb{Q}}^2 & & \overline{\mathbb{Q}}\cdot\frac{dz}{z} & 0 \\
1 \arrow[r, mapsto] & (1,1) & & &
\end{tikzcd}$$

Thus, we get:

$$\begin{tikzcd}
0 \arrow[r] & \overline{\mathbb{Q}} \arrow[r] & H^1(\Omega^*(X,D)) \arrow[r] & H^1(\Omega^*_X(X)) \arrow[r] \arrow[d, phantom, "\cong"] & 0 \\
& & & \overline{\mathbb{Q}}\cdot\frac{dz}{z} &
\end{tikzcd}$$

A class in $H^1(\Omega^*(X,D))$ is represented by (α,β) with

- $\alpha\in\Omega^1_X(X)$ is closed.
- $\beta\in\Omega^1_D(D)$ satisfies $\alpha|_D=\beta$.

We may then take as representatives $\omega_1=\left(\frac{dz}{1-\alpha},0\right)$, $\omega_2=\left(\frac{dz}{z},0\right)$.
Then ω_1 lies in the image of $\mathbb{Q}(0)\to H^1(\Omega^*(X,D))$, while ω_2 comes from $H^1(\Omega^*_X(X))$.

The comparison isomorphism $H^1_{sing}(X,D) \longrightarrow H^1_{dR}(X,D)$

(basis δ_α, γ) (basis ω_1, ω_2)

has matrix represented by:

$$\begin{array}{c} \\ \omega_1 \\ \omega_2 \end{array} \begin{array}{c} \begin{array}{cc} \delta_\alpha & \gamma \end{array} \\ \begin{pmatrix} \int_{\delta_\alpha}\omega_1 & \int_\gamma \omega_1 \\ \int_{\delta_\alpha}\omega_2 & \int_\gamma \omega_2 \end{pmatrix} \end{array} = \begin{pmatrix} 1 & \frac{1}{1-\alpha}\log\alpha \\ 0 & 2\pi i \end{pmatrix}$$

This means: the extension $0 \longrightarrow \mathbb{Q}(0) \longrightarrow H^1(X,D) \longrightarrow \mathbb{Q}(-1) \to 0$ splits (over $\overline{\mathbb{Q}}$) if and only if $\log\alpha$ and $2\pi i$ are linearly dependent over $\overline{\mathbb{Q}}$.