Research Seminar: Irrationality proofs of zeta values

Organizers: Johannes Sprang and Anneloes Viergever

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Introduction

Arguably one of the most fascinating objects in mathematics is the Riemann zeta function ζ. An interesting question which one might ask: what happens if we plug in positive integer values? Euler found a remarkable, explicit formula for the values of the Riemann zeta function at the positive even integers. But for odd numbers, not much is known. There is an intriguing conjecture stating that

$$2\pi i, \zeta(3), \dots, \zeta(2n+1), \dots$$

are algebraically independent over $\mathbb{Q}$. But it is surprisingly hard to prove. However, some things have been done in this direction. For example, we know that π is transcendental (proven by Lindemann in 1882), that $\zeta(3)$ is irrational (shown by Apéry in 1978) and that the $\mathbb{Q}$-vector space spanned by the odd zeta values is infinite dimensional (proven by Rivoal and Ball–Rivoal around 2000).

All these results can be proven by constructing small linear forms in zeta values coming from elementary integrals. More precisely, the basic structure of all these irrationality proofs is as follows:

1. Construct a sequence $(I_n)_n$ of $\mathbb{Q}$-linear forms

   $$I_n \in \mathbb{Q}\zeta_1 + \cdots + \mathbb{Q}\zeta_s,$$

   in fixed zeta values $\zeta_1, \dots, \zeta_s$.

2. Prove bounds on the linear forms, i.e. find inequalities of the form

   $$0 < |I_n| < \epsilon^n,$$

   where ϵ is a small positive integer.

3. Bound the denominators of the linear forms, i.e. prove there exists a sequence $(d_n)_n$ of integers $d_n \in \mathbb{Z}$ such that

   $$d_n I_n \in \mathbb{Z}\zeta_1 + \cdots + \mathbb{Z}\zeta_s$$

   for all n, and such that $d_n \epsilon^n \to 0$ as $n \to \infty$.

4. Use these to deduce the desired result (by deriving a contradiction in some way).

So how does one find suitable linear forms? In practice, they often seem to come from elementary integrals. It was the insight of Francis Brown that there is a more conceptional construction of such linear forms which recovers many of the known irrationality proofs. The motivation of his approach is the following theorem:

Theorem (Brown, 2009)**.** *The periods of the moduli space $\overline{M}_{0,n}$ of stable n-punctured curves of genus* 0 *are all linear combinations of multiple zeta values.*

Remark. For any tuple of integers $(s_1, \cdots, s_k) \in \mathbb{Z}^k$ such that $s_1 > 1$ and $s_i \in \mathbb{Z}_{\geq 1}$ for $i \in \{2, \cdots, k\}$, there is a corresponding *multiple zeta value* defined by the formula

$$\zeta(s_1, \cdots, s_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}.$$

One can show that this sum converges. The *weight* of the above multiple zeta value is defined to be $s_1 + \cdots + s_k$. These multiple zeta values have quite some relations between them, often also involving 'single' zeta values. A famous example is the fact that $\zeta(2, 1) = \zeta(3)$ (shown by Euler). Therefore, the above theorem gives one some hope that computing periods of $\overline{M}_{0,n}$ will produce some linear forms to use in irrationality proofs.

In [4], Brown constructs fast converging sequences of period integrals on the moduli space of stable n-punctured curves of genus 0. Furthermore, he gives a motivic criterion for the vanishing of multiple zeta values of certain weight. His families of period integrals recover essentially all of the above irrationality results on zeta values. Furthermore, they provide promising candidates for new interesting sequences of linear forms in multiple zeta values.

The aim of this seminar is to understand Brown's paper [4]. In the first two talks, we summarize the 'classical' proofs of Apéry and Ball–Rivoal. Afterwards, we recall some facts about periods of smooth algebraic varieties. As an example, we will realize $\zeta(2)$ as a period. In the following talks, we give a brief introduction to moduli spaces of stable n-punctured curves. We will not be able to give full proofs in this part of the seminar. Instead, we focus on the key ideas and an explicit description of the moduli space $\overline{M}_{0,n}$. Finally, we will study Brown's paper [4] and prove that it recovers the irrationality proofs of Apéry and Ball-Rivoal.

List of talks

In the following, we give a detailed summary of the content of the talks.

1 Introduction and overview, October 14

We give an introduction to the topic and summarize the goals of this seminar.

2 Irrationality of zeta values, October 21

In the first part of this talk, we will prove Apéry's theorem on the irrationality of $\zeta(3)$ following Beuker's proof in [2].

- Prove the upper bound $\mathrm{lcm}(1,\ldots,n) < 3^n$, see [2, §2].
- State and prove statements (b) and (d) of Lemma 1 in [2] and explain how they are used to construct linear forms in 1 and $\zeta(3)$.
- Deduce the irrationality of $\zeta(3)$, see [2, Theorem 2].

The goal of the second part of this talk is to sketch the proof of the theorem of Rivoal and Ball–Rivoal, see [1] and [7, §2.2] for a nice summary.

- Introduce the series $S_n(z)$, see [1, p. 195] or [7, equation (19)].
- State [7, Proposition 2.5] and sketch the proof of (a), see [7, §2.3.2 and §2.3.3] and [1, Lemme 1].
- State Nesterenko's linear independence criterion ([7, Théorème 2.8]) which will be proven in the next talk and deduce the theorem of Rivoal and Ball-Rivoal.
- State (without proof) the alternative formula for the value $S_n(1)$ given at the end of §2 in [7, p. 31]:

  $$S_n(1) = \frac{(rn)!^2}{n!^{2r}} \int_{[0,1]^{a-1}} \frac{\prod_{j=1}^{a-1} x_j^{rn}(1-x_j)^n dx_j}{(1-x_1\ldots x_{a-1})^{rn+1}\prod_{2\le 2j\le a-2}(1-x_1\ldots x_{2j})^{n+1}}.$$

  This formula will be used in a later talk.

3 Nesterenko's linear independence criterion, October 28

The aim of this talk is to prove Nesterenko's linear independence criterion. We will follow the simplified proof of Fischler and Zudilin, see [8].

- Recall Minkowski's convex body theorem.
- Formulate Nesterenko's linear independence criterion, see [8, Theorem A].
- Prove Nesterenko's criterion, see [8, §2.1]. The statement proven in [8, §2.1] is slightly stronger than Nesterenko's criterion (Theorem A). For our purposes it is enough to prove Nesterenko's criterion in its original form. You may assume $\delta_{i,n} = 1$, which simplifies the formulation and the proof.

4 The comparison isomorphism, November 4

The aim of this talk is to sketch the proof of the comparison isomorphism between de Rham and Betti cohomology for smooth varieties over a field of characteristic zero.

- Sketch the proof of the comparison isomorphism [6, Theorem 2.90] between Betti and de Rham cohomology for smooth varieties over a field of characteristic zero. A nice exposition is given in [6, §2.3].

5 Periods, November 11

In this talk, we discuss algebraic de Rham cohomology relative to a normal crossing divisor. We extend the comparison isomorphism to this relative situation and define periods as matrix coefficients of this period isomorphism. Finally, we give a detailed example realizing $\zeta(2)$ as a period. This computation will serve as a guiding example for realizing zeta values as period integrals.

- Recall the definition of de Rham comomology relative to a normal crossing divisor, see [6, §2.2.7] or [9, §3.3.1.2].
- State [6, Theorem 2.127] and explain [9, Remark 2.129].
- Define periods, see [6, Definition 2.133] and give some Examples, e.g. [6, Example 2.135].
- Explain how to realize $\zeta(2)$ as a period, see [6, §2.5.1]. This example is important, so please take some time to discuss it. You can put this example into perspective by observing that the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at $(0,0)$, $(1,1)$ and (∞,∞) is the moduli space $\overline{M}_{0,5}$, see [12, p. 41]. We will discuss these moduli spaces in the following talks.

6 Moduli functors and moduli spaces, November 18

The aim of this talk is to recall basic facts about moduli spaces.

- Define moduli functors, fine moduli spaces and coarse moduli spaces, for example using [12, §2].
- Give examples and non-examples, e.g. projective space, the j-invariant, etc. See [12, §1 and §2.3].
- Prove [13, Proposition 2.4.23] and deduce [12, Proposition 2.10].

7 Moduli of stable curves, November 25

In this talk, we introduce the moduli problem of stable curves (with marked points) and discuss its basic properties.

- Recall some basic facts about (complex) curves, see [12, Definition 3.2, Proposition 3.3, Exercise 3.4, Definition 3.6, Exercise 3.8].
- Explain [12, Fact 3.9].
- Define stable curves and stable n-pointed curves, see [12, Definition 3.10].
- Prove [12, Proposition 3.13].
- Define n-pointed families of smooth/stable curves and introduce their moduli functors, [12, Definition 3.16 and 3.17].
- State (without proof) [12, Theorem 3.19].

8 The construction of $\overline{M}_{0,n}$, December 2

In this talk, we sketch the construction of the moduli space of stable curves of genus 0 with n marked points.

- Explain the combinatorics of the boundary divisors, e.g., using 'stable graphs', see [12, Definitions 4.5 and 4.8].
- Sketch the construction of the 'gluing, forgetful and stabilization morphisms', see [12, Proposition 4.15], [13, §4.5.].
- Explain the key ideas behind the inductive construction of $\overline{M}_{0,n}$, see [11] or [13, §4.5.4].

9 Dihedral extensions, December 9

The aim of the following two talks is to give an explicit description of the moduli space $\overline{M}_{0,n}$ and its boundary divisors $\overline{M}_{0,n} \setminus M_{0,n}$.

- Define the cross ratio and introduce simplicial and cubical coordinates, see [3, §2.1].
- Introduce dihedral structures, dihedral coordinates and prove Lemma 2.2 and Corollary 2.3 from [3].
- Discuss the $\mathbb{R}$-valued points of the moduli space, define the open cells $X_{S,\delta}$ and explain the simplicial and cubical coordinates in the example $M_{0,5}$. Discuss [3, Lemma 2.6].
- Introduce dihedral extensions and discuss their basic properties, see [3, Definition 2.4, Lemma 2.5, Lemma 2.6, Definition 2.7]. If time permits sketch the proofs of Lemma 2.5 and Lemma 2.6. In the next talk, we will see that the dihedral extensions form explicit affine charts for the compactification $\overline{M}_{0,n}$.

10 Explicit description of $\overline{M}_{0,n}$, December 16

In this talk, we will prove that dihedral extensions give us explicit charts for the moduli space $\overline{M}_{0,n}$. Using these charts, we will be able to check that $\overline{M}_{0,n}$ is smooth and irreducible, and that $\overline{M}_{0,n} \setminus M_{0,n}$ is a normal crossing divisor.

- Recall the definition of dihedral coordinates and dihedral extensions from the last talk.
- Introduce the forgetful maps and prove that they extend to the dihedral extensions, see [3, Lemma 2.9].
- State [3, Proposition 2.12], you do not have to prove it.
- Prove Lemma 2.13 and deduce Theorem 2.15 in [3].
- Introduce Brown's compactification $\overline{M}_{0,n}$ and state that it coincides with the Deligne-Mumford-Knudsen compactification, see [3, §2.8].
- Prove Lemma 2.30 and Lemma 2.31 and deduce Corollary 2.32, see [3, §2.8].
- State Definition 2.34, Proposition 2.35 and deduce Corollary 2.36, see [3].

11 Configurations and cellular integrals, January 13

The aim of this talk is to introduce cellular integrals, which are the first examples of period integrals that give interesting linear forms in zeta values. We will introduce cellular integrals and discuss their basic properties. Their convergence is proven in the next talk. This talk covers essentially §3.1 -§3.3 of [4].

- Recall the notion of being 'of finite distance' from Talk 10.
- Define convergent configurations, see [4, Definition 3.1].
- Define cellular forms and show that they descent to $M_{0,S}$, see [4, §3.2].
- Prove [4, Lemma 3.2], i.e. [5, Proposition 2.7]. The proof of equation (2.7) of [5] will be given in the next talk, so you can take it for granted.
- Define basic cellular integrals, see [4, Definition 3.5]. Observe Remark 3.7 and equation (3.8). These observations imply that $0 < |I_{\delta/\delta'}(N)| \to 0$ as $N \to \infty$.

12 Convergence of cellular integrals, January 20

In this talk, we will prove the convergence of the cellular integrals introduced in the previous talk. We follow [4, §3.4].

- Recall [3, Corollary 2.36] from Talk 10.
- Prove Lemma 3.8 and Corollary 3.9 of [4] and state Remark 3.10.

- Prove [4, Lemma 3.10], i.e. [5, equation (2.7)]. The proof uses [3, Proposition 7.5].
- Sketch the proof of [3, Proposition 7.5]. For the proof, you will need to recall [3, Proposition 2.35] from Talk 10.

13 Generalized cellular integrals, January 27

The aim of this talk is to generalize the period integrals of the previous two talks, i.e. to introduce generalized cellular integrals. This family of integrals allows us to recover many irrationality proofs for zeta values. For example, we will see in the next talk that the linear forms of Apéry and Ball–Rivoal are special cases of generalized cellular integrals. The main source of this talk is [4, §5.1 and §5.2].

- Define generalized cellular integrals and parametrize them by sequences of integers, see [4, §5.1 and §5.2].
- Discuss the polar locus of the generalized cellular forms and the convergence properties of generalized cellular integrals, see [4, Proposition 5.2] and [4, Corollary 5.3].

14 Back to irrationality proofs, February 3

In this final talk, we will close the circle and come back to irrationality proofs. We will recover the linear forms of Apéry and Ball–Rivoal as period integrals on certain moduli spaces of stable curves.

- Sketch the proof of [4, Proposition 7.2] and relate the corresponding generalized cellular integral to the linear forms appearing in the proof of Ball–Rivoal from Talk 2.
- Show that the basic cellular integrals for $n = 6$ recover Apéry's proof from Talk 2. If time permits show that the vanishing of $\zeta(2)$ in these linear forms has a 'motivic' explanation, see [4, §11, Appendix 3].

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