Characterisation of finite quotients of Abelian varieties via Chern class conditions

Masterarbeit
an der Fakultät für Mathematik
der Universität Duisburg-Essen

vorgelegt von

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August 2016
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Introduction

One of the major goals in modern geometry is the classification of geometric objects up to some equivalence relation between these objects. Whereas the classification of all complex manifolds of a given dimension seems to be a hopeless task, there are promising results that suggest that the classification of the more restrictive class of compact complex manifolds and projective complex manifolds might be achievable. We shall give a brief introduction to the classification of algebraic varieties to motivate the content of this thesis.

It is well-known that algebraic varieties are, in contrast to complex projective manifolds, not necessary smooth. However, many different ways of desingularisation of algebraic varieties have been established, all of them work at least in dimension one and two. In the case of algebraic curves, desingularisation of an irreducible curve C gives a unique non-singular curve $\widehat{C}$, birational equivalent to C, and one may study $\widehat{C}$ instead of C. The situation becomes more complicated in dimension two, i.e. for algebraic surfaces, since each irreducible surface S is birational equivalent to an infinite amount of smooth projective surfaces. Therefore, one aims to choose one of those smooth projective surfaces that is as simple as possible and to study those in order to study the complete birational equivalence class. Such a surface $\widehat{S}$ is called minimal model for the birational equivalence class of a given algebraic surface S and the key criterion for a surface S to be minimal is the canonical divisor K_S being nef. This theory has been established by Enriques and Castelnuovo arround 1914. It should be pointed out that minimal models are not only existent, they moreover can be calculated via the following procedure: Given an algebraic surface S, then either the canonical divisor K_S of the surface is nef and the surface is already a minimal model, or one of two cases holds:

- S contains a curve with self-intersection number (-1). Then we may contract this curve to a point, obtain another surface S' and restart the algorithm using S'.
- There is no such curve in S. Then, S can be described very explicitly.

An important point to mention is that the classification yields ten different classes of algebraic surfaces and that each of these classes can be parametrised by a moduli space which is – except for one class of surfaces, the surfaces of general type – nowadays well understood.

Some examples indicated that a generalisation of this method to higher dimensions would not be possible. For example, Ueno showed the following ([Uen74, Proposition 16.17]): given an Abelian threefold A and considering the algebraic variety X obtained as the quotient of A by the involution, then X has 64 singularities. These singularities are mild in the sense that they are isolated and each singularity admits a neighborhood that is analytically isomorphic to $\mathbb{C}^3/G$, where G is some group, but there is no non-singular variety $\widehat{X}$ birational equivalent to X. In other words, X admits no non-singular minimal model. Beside many other contributions, the works of Kollár and Mori in the 1980s hinted that, allowing singular minimal models, a higher-dimensional generalisation of the procedure above might be achievable. Nowadays, this approach is known as the minimal model program or Mori's program and was for dimension three completed by Mori in 1988.

The aim of this thesis is to find a characterisation of algebraic varieties that occur as quotients of Abelian varieties via conditions on the Chern classes of the variety. A particular example of such a variety is the example of Ueno above. We will generalise the classical result for smooth varieties, stating that every complex algebraic variety with trivial canonical divisor and trivial second chern class is an étale quotient of an Abelian variety.

Some recent works proved our main statement under additional assumptions. We want to give a brief consitution of which results have been established so far.

In 1994, Shepherd-Barron and Wilson in [SW94] proved the result for threefolds by using results from the two-dimensional case after restriction to a complete intersection surface. The first generalisation to arbitrary dimension was established 2011 by Greb, Kebekus and Peternell who proved in [GKP16b] that the statement is true under the additional assumption that the variety is smooth in codimension two. Based on their paper, the result was established in full generality by Lu and Taji in their 2013 paper [LT14].

We give a new proof of the result, following the original ideas of Shepherd-Barron and Wilson and the methods established by Greb, Kebekus and Peternell, by generalising some differential-geometric results to surfaces with mild singularities. More precisely, this thesis is organised as follows.

Chapter one contains some of the classical theory of compact complex manifolds, in particular the differential geometry of holomorphic vector bundles. We start by introducing the fundamental notions and develop local descriptions for our objects. At the end of the first section we recall the definition of Chern classes. Independent from that discussion, the notion of a flat vector bundle is introduced. The proof and discussion of the inequality of Kobayashi and Lübke establishes the connection between flat bundles and vanishing of Chern classes. Restricting ourselfs to Kähler manifolds we will obtain a stronger inequality for the Chern classes of the tangent bundle, namely the Guggenheimer-Yau inequality. A direct consequence of this result is a condition on the Chern classes implying that a compact complex Kähler manifold is a torus quotient. To obtain an algebro-geometric analogue of this result, the notion of slope stability of a coherent sheaf is introduced. We will discuss one direction of the Kobayashi-Hitchin correspondence and finish our discussion on smooth manifolds by proving that a smooth complex projectiv variety with trivial canonical divisor and trivial second Chern class is an étale quotient of an Abelian variety. In other words, we will establish our main result for smooth complex projective varieties.

We start in chapter two with the discussion of a mild type of singularities, quotient singularities. They are mild in the sense that one is able to perform differential-geometry on spaces with those singularities. Such spaces can analytically be described by complex orbifolds. Most important for the proof of the main result is the study of orbifold surfaces, i.e. two-dimensional orbifolds. Two different principles for defining objects on orbifold surfaces via smooth objects will be introduced and we will illustrate the equivalence of both principles which then leads to a generalisation of the Kobayashi-Lübke inequality for orbifolds. The result is slightly weaker than the one in chapter one but still strong enough for our purposes in chapter three.

Chapter three proves our main result. Starting from the fact that klt spaces admit orbifold structure in codimension two, and the existence of a suitable quasi-étale cover, we can restrict ourselfs to orbifold surfaces by means of a carefully constructed complete intersection surfaces. Using the Kobayashi-Lübke inequality for orbifolds we obtain the flatness of the tangent sheaf on the regular locus of such an intersection surface. Using the characterisation of flat sheaves as representations of the fundamental group, the Lefschetz theorem on hyperplane sections implies that we can extend flatness first to the regular locus and then, involving another result, to the whole space. In particular it follows that the tangent sheaf of the quasi-étale cover mentioned above is locally free. Therefore, using the confirmation of the Lipman-Zariski conjecture for klt spaces due to Greb, Kebekus, Kovács and Peternell in [GKK11], the quasi-étale cover is smooth and the application of the results in chapter one yields the main result.

1 Characterisation of smooth quotients of Abelian Varieties

1.1 Differential geometry of holomorphic vector bundles

In this section we start by introducing the basic notions and objects that we will use in the following. We use this opportunity to develop our notations and to deduce local descriptions for our differential-geometric objects. We refer to [Wel08, Gei13, Kob87] for further aspects on the differential geometry of holomorphic and complex vector bundles. One should also notice that most of the theory presented in this section can also be deduced from global calculations. However, we prefer the local point of view which will allow us to perform very explicit calculations.

Throughout this chapter, X will always denote a compact, complex manifold of dimension $n \in \mathbb{N}$. We often will not specify sets of indices, either if they are clear or unimportant in the current situation. Moreover in all of this thesis, varieties are always defined over $\mathbb{C}$ and sheaves are sheaves of $\mathcal{O}_X$-modules where not otherwise stated. All tensor products will be denoted by the usual tensor symbol ($\otimes$) and only have a subscript if we want to accentuate the respective ring or module or to avoid confusion. If we are working with matrices, a superscript index indicates rows, a subscript index columns.

1.1.1 Notations and local description of the basic objects

The local theory in this and the subsequent two subsections closely follows [Gei13] with some additions from [Huy05],[Kob87] and [Wel08].

As usual, a holomorphic vector bundle $p : E \to X$ of rank r over X is a surjective holomorphic map

$$p : E \longrightarrow X$$

of complex manifolds that satisfies the following conditions:

(i) For every point $x \in X$, the p-fibre over x, denoted $E_x := p^{-1}(x)$, is a complex vector space of dimension r.

(ii) Every point $x \in X$ possesses a trivialising neighborhood $U = U(x)$, i.e. a neighborhood

U of x together with a biholomorphic map $f : p^{-1}(U) \to U \times \mathbb{C}^r$ such that the diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "f"] \arrow[dr, "p"'] & & U \times \mathbb{C}^r \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

commutes, where pr_1 denotes the projection onto the first factor, and moreover such that for every point $u \in U$ there is an isomorphism $f_u : E_u \to \mathbb{C}^r$ of complex vector spaces induced by f, i.e. there is a factorisation

$$\begin{tikzcd} E_u \arrow[r, "f|_{E_u}"] \arrow[rr, bend right, "f_u"'] & \{u\} \times \mathbb{C}^r \arrow[r, "\mathrm{pr}_2"] & \mathbb{C}^r. \end{tikzcd}$$

If $\{U_i\}_{i\in I}$ is an open cover of X that also trivialises the holomorphic vector bundle $p : E \to X$ via the holomorphic functions $f_i : p^{-1}(U_i) \to U_i \times \mathbb{C}^r$, then we may consider each of the maps f_i as isomorphisms from the restricted bundle $E|_{U_i}$ to the trivial bundle $U_i \times \mathbb{C}^r$. Given a holomorphic section of the trivial bundle $U_i \times \mathbb{C}^r$, i.e. a holomorphic map $\varphi_i : U_i \to U_i \times \mathbb{C}^r$ such that $\mathrm{pr}_1 \circ \varphi_i = \mathrm{id}_{U_i}$, we obtain a holomorphic section of the bundle E over U_i by $f_i^{-1} \circ \varphi_i$, i.e we have $f_i^{-1} \circ \varphi_i \in \Gamma(U_i, \mathcal{O}(E))$, where $\mathcal{O}(E)$ denotes the sheaf of holomorphic sections of E. The sections $f_i^{-1} \circ \varphi_i$ are exactly those obtained by restriction of a holomorphic section $\varphi \in \Gamma(M, \mathcal{O}(E))$ to the respective sets U_i. Consequently, if those sections coincide on all non-empty overlapses $U_{ij} := U_i \cap U_j$, that is, if

$$\left(f_i^{-1} \circ \varphi_i\right)\Big|_{U_{ij}} = \left(f_j^{-1} \circ \varphi_j\right)\Big|_{U_{ij}} \tag{1.1}$$

holds, then φ is totally described by the collection $\{f_i^{-1} \circ \varphi_i\}_{i\in I}$ of local sections. By the equation above, the relation $\varphi_i = f_i \circ f_j^{-1} \circ \varphi_j$ holds on every non empty intersection U_{ij}. This gives rise to the definition of the holomorphic transition functions

$$f_{ij} : U_{ij} \to \mathrm{GL}(r, \mathbb{C}) \tag{1.2}$$

of E with respect to the trivialisation, defined as

$$f_{ij}(p)(\cdot) := (\mathrm{pr}_2 \circ f_i \circ f_j^{-1})(p, \cdot)$$

for $p \in U_{ij}$. Using this definition we are able to reformulate the compatibility condition (1.1) for the local sections into the following:

$$(\mathrm{pr}_2 \circ \varphi_i)(p) = f_{ij}(p) \cdot (\mathrm{pr}_2 \circ \varphi_j)(p), \quad \forall p \in U_{ij}.$$

So we can summarise that a holomorphic section $\varphi \in \Gamma(M, \mathcal{O}(E))$ of E is determined by a collection of holomorphic functions $\{\varphi_i' : U_i \to \mathbb{C}^r\}_i$ that satisfy the relation

$$\varphi_i'(p) = f_{ij}(p) \cdot \varphi_j'(p), \quad \forall p \in U_{ij}. \tag{1.3}$$

Moreover, given a collection of such functions that fulfill this compatibility relation, (1.1) together with the sheaf axioms for $\mathcal{O}(E)$ then gives the existence of a holomorphic section described by these data.

The transition functions are also satisfying a cocycle condition

$$\begin{aligned} f_{ij} \circ f_{jk} \circ f_{ki} &= \mathrm{id}_{U_{ijk}}, \\ f_{ii} &= \mathrm{id}_{U_i}, \end{aligned} \tag{1.4}$$

that can easily be verified. It is well known that a holomorphic vector bundle $p : E \to X$ is, up to isomorphy, uniquely determined by a trivialising covering together with respective transition functions.

From now on, we only will speak of vector bundles and only mention the property of being holomorphic if we especially want to point at this property. Moreover, we usually speak of a vector bundle E and only will write the holomorphic map $p : E \to X$ if we want to make use of it.

1.1 Definition (Hermitian bundle). *A hermitian structure on a vector bundle $E \to X$ is a hermitian scalar product h_x on every fibre E_x such that for every choice of differentiable sections $\varphi, \psi \in \Gamma(U, \mathcal{A}(E))$ over an open set $U \subset X$ the map $x \mapsto h_x(\varphi(x), \psi(x))$ is differentiable. A vector bundle together with a hermitian structure is called hermitian vector bundle.*

We will write (E, h) if we want to accentuate the choice of hermitian structure h on a given vector bundle E but usually only speak of an hermitian vector bundle E. Our formalism for the local description of sections of a vector bundle can now be used to deduce local descriptions of hermitian structures and to obtain their transition behaviour.

Using the notations above and let E be described by a trivialising cover $\{U_i\}_{i \in I}$ and transition functions $\{f_{ij}\}_{i,j \in I}$, the hermitian metric h induces an hermitian scalar product in each fibre E_x over $x \in X$ that can be described by a hermitian matrix in $\mathbb{C}^{r \times r}$. Thus we can define for every index $i \in I$ a differentiable function $h_i : U_i \to \mathbb{C}^{r \times r}$ that assigns to every point a hermitian, positive definite matrix. Now, given two sections φ, ψ of E, described by local sections φ_i, ψ_i on U_i, then for every point $x \in U_i$

$$h_x(\varphi(x), \psi(x)) = (\varphi_i(x))^t \cdot h_i(x) \cdot \overline{\psi_i(x)}.$$

From this we can calculate the relation on non-empty overlapses U_{ij},

$$\begin{aligned} (\varphi_j(x))^t \cdot h_j(x) \cdot \overline{\psi_j(x)} &= h_x(\varphi(x), \psi(x)) = (\varphi_i(x))^t \cdot h_i(x) \cdot \overline{\psi_i(x)} \\ &= (f_{ij}(x) \cdot \varphi_j(x))^t \cdot h_j(x) \cdot \overline{f_{ij}(x) \cdot \psi_j(x)} \\ &= (\varphi_j(x))^t \cdot \left(f_{ij}(x)^t \cdot h_j(x) \cdot \overline{f_{ij}(x)} \right) \cdot \overline{\psi_j(x)}, \end{aligned}$$

and conclude that the local description of a hermitian metric h has the following transition behaviour:

$$h_j = f_{ij}^t \cdot h_i \cdot \overline{f_{ij}}. \tag{1.5}$$

Conversely, it is easily seen that a collection of functions $\{h_i : U_i \to \mathbb{C}^{r \times r}\}_{i \in I}$ with this transformation property defines a hermitian metric on the bundle E.

1.2 Remark. *Using a partition of unity subordinate to the trivialising cover, one can always define a hermitian structure on a given holomorphic vector bundle.*

We will now briefly explain how to describe bundle valued differential forms locally. We restrict ourselfs to differentiable (p,q)-type forms at this point but the discussion is also valid for other bundle valued differential forms, in particular holomorphic ones. Let $\Omega_X := (TM)^\vee$ denote the holomorphic cotangent bundle. The sheaf of forms of (p,q)-type (or (p,q)-forms) with values in the bundle E is given by

$$\mathcal{A}^{p,q}(E) := \mathcal{A}\left(\bigwedge^{p,q}\Omega_X \otimes_{\mathbb{C}} E\right).$$

For our convenience is the following isomorphy

$$\mathcal{A}^{p,q}(E) = \mathcal{A}\left(\bigwedge^{p,q}\Omega_X \otimes_{\mathbb{C}} E\right) \cong \mathcal{A}^{p,q}(X) \otimes_{\mathcal{A}_X} \mathcal{A}(E), \tag{1.6}$$

that is obtained after choosing a local frame for E and from which we will now obtain the desired local description. If $U \subset X$ is a trivialising open set for E, then for the restricted bundle,

$$\mathcal{A}^{p,q}(E|_U) \cong \mathcal{A}^{p,q}(U) \otimes_{\mathcal{A}_U} \mathcal{A}(E|_U) \cong \mathcal{A}^{p,q}(U) \otimes_{\mathcal{A}_U} \mathcal{A}(U \times \mathbb{C}^r) \cong (\mathcal{A}^{p,q}(U))^r. \tag{1.7}$$

This means that forms of (p,q)-type, say $\varphi \in \Gamma(X, \mathcal{A}^{p,q}(E))$, are locally described by (p,q)-type form vectors, that is we have for every index $i \in I$

$$\varphi_i = (\varphi_i^1, \dots, \varphi_i^r) \quad \text{with} \quad \varphi_i^l \in \Gamma(U_i, \mathcal{A}^{p,q}(X)) \quad \text{and} \quad 1 \leq l \leq r. \tag{1.8}$$

Following the isomorphisms in (1.7), one deduces that the condition for these local objects to define a global one is

$$\varphi_i = f_{ij} \cdot \varphi_j \quad \text{on non-empty } U_{ij}, \tag{1.9}$$

where the product is a matrix product between differentiable functions and forms of (p,q)-type. This local description is, however, not sufficient for our latter calculations. We will also need to trivialise the tangent bundle of X in order to obtain a more precise description. At first we may assume that every set from our trivialising cover is also a coordinate neighborhood and denote the respective coordinates for $i \in I$ by $z_i : U_i \to \mathbb{C}^n$ with $z_i = (z_i^1, \dots, z_i^n)^t$. Our cover then also trivialises the holomorphic tangent bundle and hence the holomorphic cotangent bundle, for which $dz_i^1, \dots, dz_i^n$ defines a frame over U_i. Now, given a component φ_i^l from the local description (1.8) of bundle valued forms of (p,q)-type with skew-symmetric coefficients, it can be written as a linear combination

$$\varphi_i^l = \sum_{\alpha,\beta} \varphi^l_{i,\alpha_1\dots\alpha_p\overline{\beta}_1\dots\overline{\beta}_q} dz_i^{\alpha_1} \wedge \dots \wedge dz_i^{\alpha_p} \wedge dz_i^{\overline{\beta}_1} \wedge \dots \wedge dz_i^{\overline{\beta}_q}, \quad i \in I,\ 1 \leq l \leq r, \tag{1.10}$$

where the conjugation of an index indicates the conjugation of the corresponding term. It is not difficult to derive a formula for the transition behaviour of the $\varphi^l_{i,\alpha_1\dots\alpha_p\overline{\beta}_1\dots\overline{\beta}_q}$ but since we will never make explicit use of it, we omit those.

1.1.2 Connections

Since we are working with holomorphic vector bundles, there is a natural extension of the $\overline{\partial}$-operator to a bundle valued operator: Given any $\varphi \in \Gamma(X, \mathcal{A}^k(E))$ we may use (1.8) to define locally for $i \in I$,

$$\overline{\partial}_E \varphi_i := (\overline{\partial}\varphi_i^1, \dots, \overline{\partial}\varphi_i^r).$$

Equation (1.9) now implies that on every non-empty intersection U_{ij},

$$\overline{\partial}_E \varphi_i - \overline{\partial}_E \varphi_j = \overline{\partial}_E (f_{ij} \cdot \varphi_j) - \overline{\partial}_E \varphi_j = \varphi_j \cdot \overline{\partial}(f_{ij}) = 0,$$

since the transition functions f_{ij} are holomorphic. Moreover we see that we are not able to extend the ∂-operator to holomorphic vector bundles, since ∂f_{ij} will not vanish in general. Henceforth the exterior differential $d = \partial + \overline{\partial}$ does not extend to E either. Therefore, we introduce connections as a substitute for exterior differentiation in bundles.

1.3 Definition (Connection)**.** *A $\mathbb{C}$-linear homomorphism of sheaves $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ is called a connection of the bundle E if for every open subset $U \subset X$, any local function $f \in \Gamma(U, \mathcal{A}(U))$ on X and every local section $\varphi \in \Gamma(U, \mathcal{A}(E))$ of E the Leibniz-rule*

$$\nabla(f \cdot \varphi) = d(f) \cdot \varphi + f \cdot \nabla(\varphi) \tag{1.11}$$

is satisfied. Here we use the isomorphisms (1.6)*, resp.* (1.7)*, to interpret ∇ as a map $\mathcal{A}^0(E) \to \mathcal{A}^1(E)$.*

A section φ of E is called flat (or parallel) with respect to the connection ∇ on E if $\nabla\varphi = 0$.

We will briefly describe the local action of connections on a section φ of E. For this purpose we are using the local description on any of the trivialising neighborhoods U_i, i.e. that we can write φ as $\varphi_i = (\varphi_i^1, \dots, \varphi_i^r)^t$. For $1 \leq l \leq r$ we define

$$e'_l : U_i \to U_i \times \mathbb{C}^r, \quad x \mapsto (x, e_l), \tag{1.12}$$

where e_l denotes the l-th standard unit vector in $\mathbb{C}^r$. The e'_l are sections of the trivial bundle $U_i \times \mathbb{C}^r$. Now, defining $\sigma_l := f_i^{-1} \circ e'_l$, we obtain a frame $\{\sigma_l\}_{l=1}^r$ for E over U_i. Since $\varphi_i = \sum_{l=1}^r \varphi_i^l \cdot e'_l$ and $\varphi|_{U_i} = f_i^{-1} \circ \varphi_i$ we obtain

$$\varphi|_{U_i} = \sum_{l=1}^r \varphi_i^l \cdot \sigma_l.$$

This enables us to define the matrix $\theta_i \in (\Gamma(U_i, \mathcal{A}^1(U_i)))^{r \times r}$ of one-forms with respect to the frame above by

$$\theta_i = \left(\theta_{i,\beta}^\alpha\right)_{\alpha,\beta=1}^r \quad \text{where} \quad \nabla(\sigma_\beta) = \sum_{\alpha=1}^r \theta_{i,\beta}^\alpha \cdot \sigma_\alpha. \tag{1.13}$$

This matrix is called the connection matrix of ∇ over U_i. Using the connection matrix we are able to calculate the local action of ∇ on a section φ of E to

$$\nabla(\varphi|_{U_i}) = \nabla\left(\sum_{\alpha=1}^r \varphi_i^\alpha \cdot \sigma_\alpha\right) = \sum_{\beta=1}^r d\varphi_i^\beta \cdot \sigma_\beta + \sum_{\alpha=1}^r \varphi_i^\alpha \nabla(\sigma_\alpha) = \sum_{\beta=1}^r \left(d\varphi_i^\beta + \sum_{\alpha=1}^r \varphi_i^\alpha \theta_{i,\beta}^\alpha\right) \cdot \sigma_\beta.$$

Writing $d\varphi_i = (d\varphi_i^1, \dots, d\varphi_i^r)^t$ we can rephrase this in terms of the following formula

$$\nabla(\varphi_i) = \sum_{\beta=1}^{r} (d\varphi_i + \theta_i \varphi_i) \cdot \sigma_\beta,$$

where the product is to be understood as a matrix product between matrices of forms. The local action of ∇ on a section φ is consequently described by

$$(\nabla(\varphi))_i = (d + \theta_i) \cdot \varphi_i, \quad i \in I. \tag{1.14}$$

1.4 Proposition (cf. [Huy05, Proposition 4.2.3 and Corollary 4.2.4])**.** *If ∇ and ∇' are two connections, then their difference can be considered as an element in $\Gamma(X, \mathcal{A}^1(\mathrm{End}(E)))$. Conversely, if ∇ is a connection in E and $a \in \Gamma(X, \mathcal{A}^1(\mathrm{End}(E)))$, then $\nabla' = \nabla + a$ again is a connection in E. In other words, the set of all connections in E is an affine space over the vector space $\Gamma(X, \mathcal{A}^1(\mathrm{End}(E)))$.*

Proof. We calculate, using the Leibniz rule, that for every local function f on X and section φ of E,

$$(\nabla - \nabla')(f \cdot \varphi) = d(f) \cdot \varphi + f \cdot \nabla(\varphi) - d(f) \cdot \varphi - f \cdot \nabla'(\varphi) = f \cdot (\nabla - \nabla')(\varphi)$$

holds. Therefore, $\nabla - \nabla'$ is $\mathcal{A}^0(X)$-linear and can be considered as an element in $\Gamma(X, \mathcal{A}^1(\mathrm{End}(E)))$.

On the other hand, given $a \in \Gamma(X, \mathcal{A}^1(\mathrm{End}(E)))$, we can easily verify the Leibniz rule for $\nabla' = \nabla + a$,

$$\nabla'(f \cdot s) = \nabla(f \cdot s) + a(f \cdot s) = d(f) \cdot s + f \cdot \nabla(s) + f \cdot a(s) = d(f) \cdot s + f \cdot \nabla'(s),$$

where we have used that a acts on $\mathcal{A}(E)$ by multiplication in the form part and by evaluation on the bundle component. □

Every holomorphic vector bundle admits a connection as is easily seen by gluing the trivial connection, given by the exterior differentials on every set of a respective trivialisation, by means of a partition of unity subordinate to the trivialising cover. Therefore, Proposition 1.4 implies that a bundle in fact possesses many different connections but none of them is canonically associated to the bundle. In contrast, when we are working with hermitian vector bundles, there is a natural condition on the compatibility of connections with the chosen hermitian structure.

1.5 Definition. *Let (E, h) be a hermitian vector bundle. A connection ∇ in E is called compatible with the hermitian structure h, or hermitian connection with respect to h, if for any local sections φ, ψ of E the following relation holds at every point $x \in X$:*

$$d(h_x(\varphi(x), \psi(x))) = h_x(\nabla(\varphi(x)), \psi(x)) + h_x(\varphi(x), \nabla(\psi(x))). \tag{1.15}$$

We may not only require compatibility with the hermitian structure but also compatibility with the holomorphic structure, i.e. with the natural operator $\overline{\partial}_E$. There is the usual natural decomposition $\mathcal{A}^1(E) = \mathcal{A}^{1,0}(E) \oplus \mathcal{A}^{0,1}(E)$, from which we also may decompose a connection

∇ in E into $(1,0)$- and $(0,1)$-part, that is, we may write $\nabla = \nabla^{1,0} \oplus \nabla^{0,1}$. Applying (1.11), $\nabla^{0,1}$ satisfies the relation

$$\nabla^{0,1}(f \cdot \varphi) = \overline{\partial}(f) \cdot \varphi + f \cdot \nabla^{0,1}(\varphi),$$

for any local function f on X and local section φ of E. We see that $\nabla^{0,1}$ thus behaves very similar to $\overline{\partial}_E$ and say that a connection ∇ is compatible with the holomorphic structure, if they in fact coincide, i.e. if $\nabla^{0,1} = \overline{\partial}_E$. In other words, a connection is compatible with the holomorphic structure, if it is compatible with the natural extension of the $\overline{\partial}$-operator.

These two possible compatibility conditions for connections on hermitian vector bundles are always achievable and restrictive enough to determine a unique connection in the bundle, which then is canonically associated after choosing holomorphic and hermitian structure. We will prove this in the following lemma.

1.6 Lemma (cf. [Wel08, Theorem 2.1])**.** *Let (E,h) be a hermitian vector bundle. Then, there is a unique hermitian connection ∇ on E that is also compatible with the holomorphic structure. This connection is called the Chern connection of E with respect to h.*

Proof. Uniqueness: We start by proving uniqueness. As this is a local statement we may assume $E = X \times \mathbb{C}^r$. By our local calculations above, $\nabla = d + \theta_i$ with $\theta \in (\Gamma(X, \mathcal{A}^1(X)))^{r\times r}$. For an arbitrary point $x \in X$ let $(h_{i,\mu\overline{\nu}}) := h_i(x)$ be the coefficient matrix describing the hermitian structure on E. ∇ being holomorphically compatible, we obtain, using our notations from (1.12),

$$\begin{aligned} dh_{i,\overline{\nu}\mu} &= d(h_x(e'_\mu(x), e'_\nu(x))) = h_x(\nabla e'_\mu(x), e'_\nu(x)) + h_x(e'_\mu(x), \nabla e'_\nu(x)) \\ &= h\left(\sum_{\alpha=1}^{r} \theta^{\alpha}_{i,\mu} e_\alpha(x), e_\nu(x)\right) + h_x\left(e_\mu(x), \sum_{\beta=1}^{r} \theta^{\beta}_{i,\nu} e_\beta(x)\right) = \sum_{\alpha=1}^{r} \theta^{\alpha}_{i,\mu} h_{i,\nu\alpha} + \sum_{\beta=1}^{r} \overline{\theta}^{\beta}_{i,\nu} h_{i,\beta\mu} \\ &= h_{i,\overline{\nu}\mu}\theta^{\nu}_{i,\mu} + \overline{\theta}^{\mu}_{i,\nu} h_{i,\overline{\nu}\mu}. \end{aligned}$$

Rewriting this in terms of a matrix product gives the relation

$$dh_i = h_i\theta_i + \overline{\theta}^t_i h_i. \tag{1.16}$$

We will now rephrase the condition on the compatibility with the holomorphic structure. If $\varphi \in \Gamma(U_i, \mathcal{O}(E))$, then

$$\nabla(\varphi) = (d + \theta_I)\varphi = (\partial + \theta^{1,0}_i)\varphi + (\overline{\partial} + \theta^{0,1}_i)\varphi,$$

where $\theta_i = \theta^{1,0}_i + \theta^{0,1}_i$ is the natural type decomposition. Now it follows immediately from the above that

$$\nabla^{0,1}\varphi = (\overline{\partial} + \theta^{0,1}_i)\varphi = \theta^{0,1}_i\varphi,$$

since φ is holomorphic. Therefore, a connection ∇ is compatible with the holomorphic structure if and only if its connection matrix θ_i is an element of $(\Gamma(U_i, \mathcal{A}^{1,0}(U_i)))^{r\times r}$. But now, examining types in (1.16), $\overline{\partial}h_i = \overline{\theta}^t_i h_i$ and $\partial h_i = h_i\theta_i$, which implies

$$\theta_i = h_i^{-1}\partial h_i \tag{1.17}$$

and proves uniqueness since we are left with no choice in θ_i after fixing a hermitian structure.

Existence: We may locally define a connection ∇ by (1.17). Then it is clear that ∇ is compatible with the holomorphic and the hermitian structure. Moreover, using a partition of unity subordinate to the trivialising cover, this defines a global connection with the same properties. □

1.1.3 Curvature and Chern classes

Our intention for defining connections was to get a replacement for the exterior differential. We now start by extending connections to differential forms of higher degree.

Let $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ be a connection in E. Define $\nabla : \mathcal{A}^k(E) \to \mathcal{A}^{k+1}(E)$ by the following assignment for any local section $\varphi \in \Gamma(U, \mathcal{A}(E))$ and local k-form $\eta \in \Gamma(U, \mathcal{A}^k(U))$

$$\nabla(\eta \cdot \varphi) := d(\eta) \cdot \varphi + (-1)^k \eta \wedge \nabla(\varphi). \tag{1.18}$$

Again, we are using the identifications (1.6), resp (1.7), to interpret ∇ as a map $\mathcal{A}^k(E) \to \mathcal{A}^{k+1}(E)$. In the case of $k = 0$, (1.18) and (1.11) coincide and we have thus in fact defined an extension of the initial connection. Since in general $\nabla^2 \neq 0$, connections are actually no differentials. But the study of the obstruction of a given connection to be a differential provides much information about the bundle. This obstruction is defined to be the curvature of the connection. More precisely we make the following definition.

1.7 Definition (Curvature)**.** *The curvature Ω_∇ of a connection ∇ in E is defined as $\Omega_\nabla := \nabla \circ \nabla : \mathcal{A}^0(E) \to \mathcal{A}^2(E)$.*

Given any local section φ of E and local function f on X the following simple calculation

$$\begin{aligned} \Omega_\nabla(f \cdot s) &= \nabla(\nabla(f \cdot s)) \\ &= d^2 f \cdot s - df \wedge \nabla(s) + df \wedge \nabla(s) + f \cdot \nabla(\nabla(s)) \\ &= f \cdot \Omega_\nabla(s) \end{aligned}$$

shows that Ω_∇ is not just $\mathbb{C}$-linear but $\mathcal{A}^0(X)$-linear and therefore may be considered as an element of $\Gamma(X, \mathcal{A}^2(\mathrm{End}(E)))$. Since the Chern connection is the most important for our considerations, we will shorten our notion and denote by Ω the curvature of the Chern connection in E. As for connections, the action of Ω can locally on trivialising sets U_i, $i \in I$, be described by a matrix $\Omega_i \in (\Gamma(U_i, \mathcal{A}^2(U_i)))^{r \times r}$. An easy calculation, analogue to those we did before for connections, shows

$$\Omega_i = d\theta_i + \theta_i \wedge \theta_i. \tag{1.19}$$

According to (1.17), θ_i is of type $(1, 0)$. Moreover, since $\partial h_i^{-1} = -h_i^{-1} \cdot \partial h_i \cdot h_i^{-1}$, we calculate

$$\partial\theta_i = \partial(h_i^{-1}\partial h_i) = -h_i^{-1} \cdot \partial h_i \cdot \partial h_i^{-1} \wedge \partial h_i = -(h_i^{-1}\partial h_i) \wedge (h_i^{-1}\partial h_i) = -\theta_i \wedge \theta_i.$$

We thus obtain immediately

$$\Omega_i = \partial\theta_i + \overline{\partial}\theta_i + \theta_i \wedge \theta_i = \overline{\partial}\theta_i. \tag{1.20}$$

In particular, Ω_i only consists of $(1, 1)$-type forms. This will be used to obtain very explicit local descriptions of connections and curvature. At first we may write the coefficients of the connection matrix θ_i as a linear combination

$$\theta^\alpha_{i\beta} = \sum_{\mu=1}^{r} \Gamma^\alpha_{i,\mu\beta} dz^\mu_i \tag{1.21}$$

with some coefficients $\Gamma^\alpha_{i,\mu\beta}$. As it did before, $(h_{i,\mu\overline{\nu}})_{\mu,\nu}$ will denote the coefficient matrix corresponding to the hermitian structure in E. We further denote its inverse by $(h_i^{\overline{\nu}\mu})_{\mu,\nu}$ to shorten the notation. With this we calculate

$$\theta^\alpha_{i,\beta} = \sum_{\nu=1}^r \partial h_{i\beta\overline{\nu}} h_i^{\overline{\nu}\alpha} = \sum_{\nu=1}^r h_i^{\overline{\nu}\alpha} \sum_{\mu=1}^r \partial_\mu h_{i,\beta\overline{\nu}} dz_i^\mu = \sum_{\mu=1}^r \left(\sum_{\nu=1}^r h_i^{\overline{\nu}\alpha} \partial_\mu h_{i,\beta\overline{\nu}} \right) dz_i^\mu,$$

which yields

$$\Gamma^\alpha_{i,\mu\beta} = \sum_\nu h_i^{\overline{\nu}\alpha} \partial_\mu h_{i,\beta\overline{\nu}}, \tag{1.22}$$

and shows that the $\Gamma^\rho_{i,\alpha\sigma}$ are the Christoffel symbols of the Chern connection. Furthermore, for the $(1,1)$-type forms from the local representation of the curvature Ω we can write

$$\Omega^\alpha_{i,\beta} = \sum_{\mu,\nu} R^\alpha_{i,\beta\mu\overline{\nu}} dz_i^\mu \wedge dz_i^{\overline{\nu}} \tag{1.23}$$

and then, using our previous identities,

$$\Omega^\alpha_{i,\beta} = \overline{\partial}\theta^\alpha_{i,\beta} = \overline{\partial}\left(\sum_\mu \Gamma^\alpha_{i,\mu\beta} dz_i^\mu \right) = \sum_{\mu,\nu} \overline{\partial}_{\overline{\nu}} \Gamma^\alpha_{i,\mu\beta} dz_i^{\overline{\nu}} \wedge dz_i^\mu = \sum_{\mu,\nu} \left(-\overline{\partial}_{\overline{\nu}} \Gamma^\alpha_{i,\mu\beta} \right) dz_i^\mu \wedge dz_i^{\overline{\nu}},$$

which yields

$$R^\alpha_{i,\beta\mu\overline{\nu}} = -\overline{\partial}_{\overline{\nu}} \Gamma^\alpha_{i,\mu\beta}. \tag{1.24}$$

From now on we will drop the index that indicates the trivialising set, whenever we know a priori that our objects are globally well defined, and tacitly work on some open set $U \in \{U_i\}_{i\in I}$. In particular, this is the case, if we work with the local description of global objects such as connections and curvature.

1.8 Example. *Let (L,h) be a line bundle and $x \in X$ a point in a trivialising neighborhood U_i. Define the hermitian matrix h_i as above, i.e.*

$$h_i = h_x(e_1'(x), e_1'(x)),$$

which is a non-zero scalar, since $\operatorname{rank}(L) = 1$. *Then,*

$$\theta_i = h_i^{-1}\partial h_i = h_i^{-2}(\overline{\partial}\partial h_i - h_i \wedge \partial h_i)$$

is the connection matrix, $\theta_i \wedge \theta_i = 0$ and

$$-\partial\overline{\partial}\log(h_i) = -\partial(h_i^{-1}\overline{\partial}h_i) = -h_i^{-1}\partial\overline{\partial}h_i + h_i^{-2}\cdot\partial h_i \wedge \overline{\partial}h_i = d(h_i^{-1}\partial h_i) = d\theta_i = \Omega_i$$

is the curvature matrix.

1.9 Remark. *We have always just considered one specific frame for our vector bundle E. For a in-depth discussion on arbitrary frames, the behaviour of the formulas above under a change of frames and certain compatibility discussions that we omit, we refer to [Wel08].*

We now define Chern classes of hermitian holomorphic vector bundles using Chern-Weil theory. The discussion roughly follows [Wel08, Section 3]. Denote by $\mathrm{GL}(r, \mathbb{C})$ the Lie group of invertable $r \times r$ matrices with complex entries and by $\mathfrak{gl}(r, \mathbb{C}) = \mathrm{Mat}(r, \mathbb{C})$ its associated Lie algebra. For every fixed element $g \in \mathfrak{gl}(r, \mathbb{C})$ define polynomials $f_k : \mathfrak{gl}(r, \mathbb{C}) \to \mathbb{C}$ by the identity

$$\det\left(\mathbb{I}_n - tg\right) = \sum_{k=0}^{n} f_k(g) t^{n-k}, \quad t \in \mathbb{C},$$

wherein $\mathbb{I}_r$ is the unit matrix in $\mathfrak{gl}(r, \mathbb{C})$. Since for every $a \in \mathrm{GL}(r, \mathbb{C})$,

$$\det\left(\mathbb{I}_r - atga^{-1}\right) = \det\left(a\left(\mathbb{I}_r - tg\right) a^{-1}\right) = \det\left(\mathbb{I}_r - tg\right),$$

we obtain the $\mathrm{GL}(r, \mathbb{C})$-invariance of the f_k. Now, if we insert the curvature matrix Ω in the equation above, define $c_k(E, h) := f_k(\Omega)$. In other words,

$$\det\left(\mathbb{I}_r - t\Omega\right) = \sum_{k=0}^{r} f_k(\Omega) t^{r-k} =: t^r + \sum_{k=1}^{r} c_k(E, h) t^{r-k}, \quad t \in \mathbb{C}. \tag{1.25}$$

The $c_k(E, h) \in \mathcal{A}^{2k}(X, \mathbb{C})$ are called Chern forms of E with respect to h. Note that the $\mathrm{GL}(r, \mathbb{C})$-invariance of the Chern forms implies that they are defined independent from the choice of frame, see [Wel08]. Since Ω is a matrix of $(1, 1)$-type forms, the Chern forms are actually forms of (k, k)-type. Using our local formulas one verifies easily that the Chern forms can locally be calculated as

$$c_k(E, h) = \sum \delta^{j_1 \cdots j_l}_{k_1 \cdots k_l} \Omega^{j_1}_{k_1} \wedge \cdots \wedge \Omega^{j_l}_{k_l}. \tag{1.26}$$

In particular, for the first and second Chern form the formulas

$$c_1(E, h) = \sum_{\mu} \Omega^{\mu}_{\mu} = \sum_{k,l} R_{kl} dz^k \wedge d\bar{z}^l, \tag{1.27}$$

and

$$c_2(E, h) = \sum_{j,k} (\Omega^j_j \wedge \Omega^k_k - \Omega^j_k \wedge \Omega^k_j), \tag{1.28}$$

hold locally.

1.10 Theorem (Chern forms induce well defined cohomology classes)**.** *For every k, the cohomology class of $c_k(E, h)$ does not depend on the specific choice of a hermitian metric h, that is,*

$$c_k(E) := [c_k(E, h)] \in H^{2k}(X, \mathbb{C}) \tag{1.29}$$

is independent of h. We call $c_k(E)$ the k-th Chern class of E.

Proof. Step 1: Chern forms are closed, cf. the proof of [Wel08, Theorem 3.2 (a)]. For fixed $k \in \mathbb{N}$ consider the $f_k : \mathfrak{gl}(r, \mathbb{C}) \to \mathbb{C}$ with $f_k(\Omega) = c_k(E, h)$ from above. We start by defining the following map

$$F_k : \mathfrak{gl}(r, \mathbb{C})^{\times k} \to \mathbb{C}, \quad (g_1, \ldots, g_k) \mapsto \frac{(-1)^k}{k!} \sum_{i=1}^{k} \sum_{j_1 < \cdots < j_i} (-1)^i f_k(g_{j_1} + \cdots + g_{j_i}), \tag{1.30}$$

which is an extension of $f_k : \mathfrak{gl}(r,\mathbb{C}) \to \mathbb{C}$ to a k-multilinear form, where extension means, that the restriction of F_k to the diagonal yields f_k, i.e.

$$F_k(g,\dots,g) = f_k(g) \quad \forall g \in \mathfrak{gl}(r,\mathbb{C}).$$

Now, let $a \in \mathrm{GL}(r,\mathbb{C})$. Then,

$$\begin{aligned} F_k(ag_1a^{-1},\dots,ag_ka^{-1}) &= \frac{(-1)^k}{k!}\sum_{i=1}^{k}\sum_{j_1<\cdots<j_i}(-1)^i f_k(ag_{j_1}a^{-1}+\cdots+ag_{j_i}a^{-1}) \\ &= \frac{(-1)^k}{k!}\sum_{i=1}^{k}\sum_{j_1<\cdots<j_i}(-1)^i f_k(a(g_{j_1}+\cdots+g_{j_i})a^{-1}) \\ &= F_k(g_1,\dots,g_r), \end{aligned}$$

which shows that F_k is also $\mathrm{GL}(r,\mathbb{C})$-invariant. A simple calculation shows that with respect to our standard frame the local formula

$$d(f_k(\Omega)) = \sum_{i=1}^{k} F_k(\Omega,\dots,d\Omega,\dots,\Omega) \tag{1.31}$$

holds, where the $d\Omega$ is the i-th agument in F_k by summation over the index i. Since we already know that $\Omega = d\theta + \theta\wedge\theta$, it follows that

$$d\Omega = d^2\theta + d\theta\wedge\theta - \theta\wedge d\theta$$

and moreover the following version of the Bianchi identity holds,

$$[\Omega,\theta] = [d\theta+\theta\wedge\theta,\theta] = d\theta\wedge\theta+\theta\wedge\theta\wedge\theta-(-1)^2(\theta\wedge d\theta+\theta\wedge\theta\wedge\theta) = d\Omega, \tag{1.32}$$

so that (1.31) becomes

$$d(f_k(\Omega)) = \sum F_k(\Omega,\dots,[\Omega,\theta],\dots,\Omega),$$

where $[\cdot,\cdot]$ is the Lie bracket on the endomorphism bundle. Thus it is sufficient to prove that for all $g, g_1, \dots g_k \in \mathfrak{gl}(r,\mathbb{C})$

$$\sum_{i=1}^{k} F_k(g_1,\dots,g_{i-1},[g,g_i],g_{i+1},\dots,g_k) = 0. \tag{1.33}$$

Define for $s \in \mathbb{R}$ the matrix $\exp(sg) \in \mathrm{GL}(r,\mathbb{C})$. Because of the $\mathrm{GL}(r,\mathbb{C})$-invariance of F_k,

$$F_k(g_1,\dots,g_k) = F_k(\exp(sg)g_1\exp(-sg),\dots,\exp(sg)g_k\exp(-sg))$$

and differentiating this equation at $s=0$ yields (1.33). Thus we have proved that for every $k\in\mathbb{N}$ the form $f_k(\Omega) = c_k(E,h)$ is closed.
Step 2: Independence from the connection, cf. [Ji10, Theorem 18.1]. Let θ' be the connection matrix associated to another hermitian metric h' in E. We have to show that for every $k\in\mathbb{N}$ there is a $(2k-1)$-form $\omega = \omega_k$ such that

$$f_k(\Omega) - f_k(\Omega') = d\omega, \tag{1.34}$$

where Ω' is the curvature matrix associated to θ'. Let $\xi := \theta - \theta'$ and define for $t \in [0,1] \subset \mathbb{R}$

$$\theta_t := \theta + t\xi,$$

which we shall interpret as path from θ to θ' in the set of all connections in E. Each θ_t in fact is a connection in E as one may verify. We can calculate the associated curvature matrices for $t \in [0,1]$ to

$$\Omega_t = d\theta_t - \theta_t \wedge \theta_t = \Omega + tD\xi - t^2\xi \wedge \xi, \tag{1.35}$$

where $D\xi := d\xi - \theta \wedge \xi - \xi \wedge \theta$ denotes the covariant derivative of ξ. Consequently we obtain

$$\frac{d}{dt}\Omega_t = D\xi - 2t\xi \wedge \xi. \tag{1.36}$$

Using this equation together with (1.31) and denoting for $g, h \in \mathfrak{gl}(r, \mathbb{C})$ the map $r \cdot F_k(g, h, \ldots, h)$ by $W(g, h)$ we have, using that F_k is multilinear,

$$\frac{d}{dt}f_k(\Omega_t) = rF_k\left(\frac{d}{dt}\Omega_t, \Omega_t, \ldots, \Omega_t\right) = W\left(\frac{d}{dt}\Omega_t, \Omega\right).$$

We claim that

$$\omega := \int_0^1 W(\xi, \Omega_t)dt$$

is our desired form ω from (1.34). To see this we will need some identities obtained from simple calculations. At first,

$$\begin{aligned}
D\Omega_t &= D\Omega + tD^2\xi - t^2D(\xi \wedge \xi) && \text{by (1.35)}\\
&= tD^2\xi - t^2D(\xi \wedge \xi) && \text{by the Bianchi-identity (1.32)}\\
&= t(\xi \wedge \Omega - \Omega \wedge \xi) + t^2(\xi \wedge D\xi - D\xi \wedge \xi) && \text{again by the Bianchi-identity (1.32)}\\
&= t[\xi, \Omega] + t^2[\xi, D\xi] &&\\
&= t[\xi, \Omega] && \text{by (1.35).}
\end{aligned}$$

Now, (1.33) with $g = g_1 = \xi$ and $g_2 = \cdots = g_k = \Omega_t$ becomes

$$2W(\xi \wedge \xi, \Omega_t) = (r-1)F_k(\xi, [\xi, \Omega_t], \Omega_t, \ldots, \Omega_t). \tag{1.37}$$

Those two equations finally yield

$$\begin{aligned}
dW(\xi, \Omega_t) &= r \cdot dF_k(\xi, \Omega_t, \ldots, \Omega_t)\\
&= rF_k(d\xi, \Omega_t, \ldots, \Omega_t) - r(r-1)F_k(\xi, d\Omega_t, \Omega_t, \ldots, \Omega_t)\\
&= rF_k(D\xi, \Omega_t, \ldots, \Omega_t) - r(r-1)F_k(\xi, D\Omega_t, \Omega_t, \ldots, \Omega_t)\\
&= W(D\xi, \Omega_t) - r(r-1)t \cdot F_k(\xi, [\xi, \Omega_t], \Omega_t, \ldots, \Omega_t)\\
&= W(D\xi, \Omega_t) - 2tW(\xi \wedge \xi\Omega_t)\\
&= \frac{d}{dt}F_k(\Omega_t).
\end{aligned}$$

Integrating this equation with respect to t we obtain

$$d\omega = F_k(\Omega_0) - F_k(\Omega_1) = F_k(\Omega) - F_k(\Omega'),$$

which was to be shown. $\square$

1.11 Example. *Consider a line bundle (L, h). In the previous example 1.8 we have seen that $\Omega = -\partial\overline{\partial}\log(h_i)$. Then, by (1.27), $c_1(L, h) = \Omega = -\partial\overline{\partial}\log(h_i)$.*

We finish our preliminary discussion on the differential geometry of vector bundles by proving some essential properties of Chern classes.

1.12 Lemma (Properties of the Chern classes, cf. [Ji10, Proposition 18.2])**.** *For any holomorphic vector bundle E over X the Chern classes satisfy the following:*

(i) For another complex manifold Y and a smooth map $f : Y \to X$,

$$c_k(f^*E) = f^*c_k(E) \quad \forall k.$$

(ii) If F is another holomorphic vector bundle over X, then

$$c_k(E \oplus F) = \sum_{m+n=k} c_m(E) \cdot c_n(F).$$

(iii) $c_k(E^\vee) = (-1)^k c_k(E)$.

Proof. (i) The open sets $\{f^{-1}(U_i)\}_{i\in I}$ are known to trivialise the pullback bundle f^*E. Moreover, pulling back our frame $\{\sigma_l\}_{l=1}^r$ over U_i we obtain a frame $\{f^*\sigma_l\}_{l=1}^r$ for f^*E over $f^{-1}(U_i)$. The pullback-connection $f^*\nabla$ has the connection matrix $f^*\theta$ and the associated curvature matrix is because of

$$df^*\theta - f^*\theta \wedge f^*\theta = f^*(d\theta - \theta \wedge \theta) = f^*\Omega,$$

exactly the pullback of the connection matrix. Consequently,

$$\sum_{k=0}^{r} t^{r-k} c_k(f^*E, f^*h) = \det(\mathbb{I}_r - tf^*\Omega) = f^*\det(\mathbb{I}_r - t\Omega) = \sum_{k=0}^{r} t^{r-k} f^*c_k(E, h),$$

and therefore $c_k(f^*E) = f^*c_k(E)$.

(ii) Denote the connection matrix of F by θ' and its associated curvature matrix by Ω'. One verifies easily that the connection matrix and curvature matrix of $E \oplus F$ then are given by

$$\begin{pmatrix} \theta & 0 \\ 0 & \theta' \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \Omega & 0 \\ 0 & \Omega' \end{pmatrix},$$

respectively. Consequently,

$$\det\left(\begin{pmatrix} \mathbb{I}_r - t\Omega & 0 \\ 0 & \mathbb{I}_r - t\Omega' \end{pmatrix}\right) = \det(\mathbb{I}_r - t\Omega)\det(\mathbb{I}_r - t\Omega'),$$

which implies the assertion.

(iii) The induced connection and curvature matrix in the dual bundle are $\theta^\vee = -\theta^t$ and $\Omega^\vee = -\Omega^t$, respectively. Therefore,

$$\sum_{k=0}^{r} t^{r-k} f^*c_k(E^\vee, h^\vee) = \det(\mathbb{I}_r + t\Omega^\vee) = \det(\mathbb{I}_r - t\Omega^t) = \sum_{k=0}^{r} t^{r-k}(-1)^k f^*c_k(E, h)$$

and thus $c_k(E^\vee) = (-1)^k c_k(E)$.

$\square$

1.2 Flatness of vector bundles

Chern classes were constructed out of the curvature of a connection which is its obstruction to be a differential. This is closely related to the concept of flatness of vector bundles as the following definition shows.

1.13 Definition (Flatness). *A holomorphic vector bundle $\pi : E \to X$ is said to be*

(i) flat, if there is a connection ∇ in E such that the curvature of ∇ vanishes, $\Omega_\nabla = 0$;

(ii) unitary flat, if it is hermitian and the curvature of the Chern connection associated to the hermitian metric vanishes, $\Omega = 0$.

Before discussing the connection between Chern classes and flatness of bundles in more detail, we will prove a characterisation of flat bundles in terms of representations of the fundamental group. This characterisation in particular is important when we start to consider singular spaces in chapter three. The main result of this section is the following theorem.

1.14 Theorem (Flat bundles and representations of the fundamental group). *For a holomorphic vector bundle E over X the following are equivalent:*

(i) E is (unitary) flat.

(ii) The transition functions $f_{jk} : U_{jk} \to \mathrm{GL}(\mathrm{rank}(E), \mathbb{C})$ may be taken as locally constant (and take values in the unitary group $U(\mathrm{rank}(E))$).

(iii) E is induced by a (unitary) representation of the fundamental group, i.e. a homomorphism of groups

$$\rho : \pi_1(X) \to \mathrm{GL}(\mathrm{rank}(E), \mathbb{C}), \tag{1.38}$$

such that

$$E = (X^{un} \times \mathbb{C}^r)/\pi_1(X),$$

where X^{un} denotes the universal covering of X and the action of $\pi_1(X)$ is given by

$$\pi_1(X) \times (X^{un} \times \mathbb{C}^r) \to X^{un} \times \mathbb{C}^r, \quad (\gamma, (x, c)) \mapsto (\gamma(x), \rho(\gamma) \cdot c),$$

where we regarded $\pi_1(X)$ as covering transformation group $X^{un} \to X$.

One usally proves this theorem by considering $X^{un} \times \mathbb{C}^r$ as $\pi_1(X)$-principal bundle and considering the holonomy group of a connection, cf. [KN63, Wel08]. However, technical details are usually omitted for the sake of brevity. We want to give a proof based upon the correspondence between local systems, i.e. locally constant sheaves, and representations of the fundamental group. Our proof follows Achar [Ach07], and we start with two little lemmas for later reference.

1.15 Lemma. *Let $\mathcal{F}$ be a locally constant sheaf of complex vector spaces on X and $K \subset X$ a connected set such that $K \subset V$ for some V on which $\mathcal{F}|_V$ is constant. Let $\{x_1, \dots, x_n\} \subset K$ be a set of points in K.*

(i) For all $i, j \in \{1, \dots, n\}$ we have $\mathcal{F}_{x_i} \cong \mathcal{F}_{x_j}$.

(ii) For all $i, j, k \in \{1, \dots, n\}$ the isomorphisms $\mathcal{F}_{x_i} \xrightarrow{\varphi} \mathcal{F}_{x_j} \xrightarrow{\psi} \mathcal{F}_{x_k}$ and $\mathcal{F}_{x_i} \xrightarrow{\alpha} \mathcal{F}_{x_j}$ are compatible, i.e. the relation $\psi \circ \varphi = \alpha$ holds.

Proof. Take K and V as in the statement. Since $\mathcal{F}|_V$ is constant, the morphism $\varphi_i : \mathcal{F}(V) \to \mathcal{F}_{x_i}$ is an isomorphism for all $i \in \{1, \ldots, n\}$ and we obtain a commutative diagram:

$$\begin{tikzcd} & \mathcal{F}(V) \arrow[dl, "\varphi_j"'] \arrow[dr, "\varphi_i"] & \\ \mathcal{F}_{x_j} \arrow[rr, "\varphi_i \circ \varphi_j^{-1}"'] & & \mathcal{F}_{x_i}. \end{tikzcd}$$

The statements of the lemma now are clear. □

1.16 Lemma. *Let $\mathcal{F}$ be a locally constant sheaf of complex vector spaces on X and $\gamma : [0,1] \to X$ a path (resp. $H : [0,1]^2 \to X$ a homotopy). Then there exist $0 = a_0 < a_1 < \cdots < a_n = 1$ (resp. also $0 = b_0 < b_1 < \cdots < b_m = 1$), such that for all $i \in \{1, \ldots, n\}$ (resp. also $j \in \{1, \ldots, m\}$) also $\gamma([a_i, a_{i+1}])$ (resp. $H([a_i, a_{i+1}] \times [b_j, b_{j+1}])$) is connected and contained in a connected open set V, such that $\mathcal{F}|_V$ is constant.*

Proof. We only will prove the statement for paths. The statement for homotopies follows analogous. Take $t \in [0,1]$ arbitrary. There are $U = U(\gamma(t)) \subset X$ and $V \subset X$, such that $U \subset V$ and $\mathcal{F}|_V$ is constant. Since γ is continuous, we further find a neighborhood $W = W(t) \subset [0,1]$ with $\gamma(W) \subset V$. Now, we take an intervall $[a, a'] \subset W$ that contains t. The interior of this interval is clearly open and its image $\gamma([a, a']^\circ) \subset V$ is also. By varying t in the intervall $[0,1]$ we obtain a covering of $[0,1]$ by the interiors of intervalls $[a, a']$ constructed in this way. Now, $[0,1]$ is compact so we may choose a finite number of sets from our covering that are also covering $[0,1]$. By ordering the boundary points of this intervalls we obtain an increasing sequence

$$a_0 < a_1 < \cdots < a_n.$$

From the construction follows that for all $i \in \{0, \ldots, n\}$ every of the intervalls $[a_i, a_{i+1}]$ is contained in a set W_i, such that $\gamma(W_i) \subset V_i \subset X$ and $\mathcal{F}|_{V_i}$ is constant. □

1.17 Theorem (Correspondence between locally constant sheaves and representations of the fundamental group.)**.** *There is a bijection*

$$\left\{ \begin{array}{c} \textit{locally constant sheaves of } \mathbb{C}\textit{-vector spaces on } X \\ \textit{up to isomorphisms} \end{array} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{representations of } \pi_1(X) \\ \textit{up to isomorphisms} \end{array} \right\}.$$

Proof. Organisation of the proof. Our proof is divided into five steps. In step one we construct a possible candidate for a representation of the fundamental group out of a given locally constant sheaf. Step two shows that this construction in fact defines a representation of the fundamental group. In the third step we go backwards and construct a locally constant sheaf out of a given representation of the fundamental group. The remaining two steps are showing that these two constructions define bijective mappings which are inverse to each other.
Step 1: Construction of a candidate. Take $\gamma : [0,1] \to X$ to be a path and $0 = a_0 < a_1 < \cdots < a_n = 1$ as in Lemma 1.16. According to Lemma 1.15 there is a sequence of isomorphisms

$$\mathcal{F}_{\gamma(0)} \xrightarrow{\varphi_0} \mathcal{F}_{\gamma(a_1)} \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{n-1}} \mathcal{F}_{\gamma(1)}.$$

We define $\rho(\gamma) := \varphi_{n-1} \circ \cdots \circ \varphi_0$ and claim that this is independent from the choice of the $a_1, \ldots a_n$. To prove this, take $a' \in [0,1]$ with $a_i < a' < a_{i+1}$ for some $i \in \{0, \ldots, n\}$. It is clear that $\gamma([a_i, a']) \subset V_i$ and $\gamma([a', a_{i+1}]) \subset V_i$ for a set $V_i \subset X$ that is open and connected, such that $\mathcal{F}|_{V_i}$ is constant. Due to 1.15 we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{F}_{\gamma(a_i)} \arrow[rr] \arrow[dr] & & \mathcal{F}_{\gamma(a_{i+1})} \\ & \mathcal{F}_{\gamma(a')} \arrow[ur] & \end{tikzcd}$$

with all the arrows being isomorphisms. Iterating this argument we see that $\rho(\gamma)$ is independent of addition of a finite number of points to $\{a_0, \dots, a_n\}$. Now, passing from two different partitions of $[0,1]$ to the common partition, this shows the independence of $\rho(\gamma)$ from the choice of such a partition.

Step 2: ρ is actually defined on $\pi_1(X)$. Take γ and γ' to be homotopically equivalent paths. We want to show $\rho(\gamma) = \rho(\gamma')$, so that ρ is actually defined on $\pi_1(X)$. For this purpose, let $H : [0,1]^2 \to X$ be a homotopy between γ and γ'. Further take $0 < a_0 < \cdots < a_n = 1$ and $0 = b_0 < b_1 < \cdots < b_m = 1$ as in Lemma 1.16 and define $\gamma_j(t) := H(t, b_j)$ with $j \in \{1, \dots, m\}$. By Lemma 1.15 and Lemma 1.16 we obtain the following commutative diagram

$$\begin{tikzcd} \mathcal{F}_{H(0,b_{j+1})} \arrow[r] \arrow[d, equal] & \mathcal{F}_{H(a_1,b_{j+1})} \arrow[r] \arrow[d, leftrightarrow, "\text{isom.}"] & \cdots \arrow[r] & \mathcal{F}_{H(a_{n-1},b_{j+1})} \arrow[r] \arrow[d, leftrightarrow, "\text{isom.}"] & \mathcal{F}_{H(1,b_{j+1})} \arrow[d, equal] \\ \mathcal{F}_{H(0,b_j)} \arrow[r] & \mathcal{F}_{H(a_1,b_j)} \arrow[r] & \cdots \arrow[r] & \mathcal{F}_{H(a_{n-1},b_j)} \arrow[r] & \mathcal{F}_{H(1,b_j)} \end{tikzcd}$$

with all the horizontal arrows being isomorphism. Composing all mappings in the first row of the diagram is exactly $\rho(\gamma)$ and composing all mappings in the second row is $\rho(\gamma')$. But since the diagram is commutative, we obtain $\rho(\gamma) = \rho(\gamma')$. Therefore, we in fact have constructed a homomorphism of groups $\rho : \pi_1(X, x_0) \to \mathrm{GL}(\mathcal{F}_{x_0})$ for every fixed basepoint $x_0 \in X$, that is, a representation of the fundamental group.

Step 3: Construction of a sheaf. Going backwards, take any representation $\tau : \pi_1(X, x_0) \to \mathrm{GL}(E)$ of the fundamental group with a complex vector space E of dimension r. For $x \in X$ arbitrary denote by $\alpha_x : [0,1] \to X$ a path from x_0 to x that we will fix after choosing once. (α_{x_0} is defined to be the constant map x_0.) Further define for any open subset $U \subset X$

$$\mathcal{G}(U) := \{f : U \to E \mid \forall \gamma : [0,1] \to U : f(\gamma(1)) = \beta \cdot f(\gamma(0))\},$$

where $\beta := [\alpha^{-1}_{\gamma(1)} \cdot \gamma \cdot \alpha_{\gamma(0)}] \in \pi_1(X, x_0)$ acts on $f(\gamma(0))$ via τ, i.e. the action of the fundamental group on $f(\gamma(0))$ is given by

$$\pi_1(X, x_0) \times f(\gamma(0)) \to f(\gamma(0)), \quad (\beta, v) \mapsto \tau(\beta) \cdot v.$$

It is easy to see that $\mathcal{G}$ actually defines a sheaf on X and we claim it to be locally constant. Take $V \subset X$ non-empty and with trivial fundamental group (which is possible since X is locally the simply connected space $\mathbb{C}^r$) and $x \in V$, we may define a map $\varphi_x : \mathcal{G}(V) \to E$ by evaluation, i.e. by setting $f \mapsto f(x)$. For every $y \in V$ and every path γ from x to y we obtain

$$f(y) = [\alpha_y^{-1} \cdot \gamma \cdot \alpha_x] \cdot f(x). \tag{1.39}$$

Hence, if $f(x) = \varphi_x(f) = \varphi_x(g) = g(x)$ holds for $f, g \in \mathcal{G}(V)$, then, by (1.39), this holds on the whole of Vwhich proves injectivity of φ_x.

For surjectivity, take $e \in E$ and define $f(y) = [\alpha_y^{-1} \cdot \gamma \cdot \alpha_x] \cdot e$ for a path γ from x to y. Then, $f \in \mathcal{G}(V)$ and $\varphi_x(f) = f(x) = e$. It remains to show that this is independent from the choice of path γ. Hence let γ' be another path joining x and y. Then, $\gamma^{-1} \cdot \gamma$ is null-homotopic in X, since V is simply connected. But this means $\gamma \sim \gamma'$ in X and therefore we have

$$[\alpha_y^{-1} \cdot \gamma \cdot \alpha_x] = [\alpha_y^{-1} \cdot \gamma' \cdot \alpha_x] \in \pi_1(X, x),$$

which shows that f does not depend on our choice of path and φ_x is surjective. Thus we obtained $\mathcal{G}|_V = E$ which shows that $\mathcal{G}$ is locally constant.

Step 4: Reversing the constructions, I. Now, take as representation ρ constructed as in steps one and two and $\mathcal{G}$ as constructed in step three. We will show that $\mathcal{F} \cong \mathcal{G}$. First notice, that the vector space E from step three becomes the vector space $\mathcal{F}_{x_0}$. Furthermore we already know that $\mathcal{F}$ is a sheaf, hence is isomorphic to its sheafication $\mathcal{F}^+$. Thus it is sufficient to prove $\mathcal{G} \cong \mathcal{F}^+$.

We define a morphism $\Phi : \mathcal{F}^+ \to \mathcal{G}$ as follows: For a connected open set U such that $\mathcal{F}^+|_U$ is locally constant define

$$\Phi_U : \begin{cases} \mathcal{F}^+(U) & \to \mathcal{G}(U) \\ s & \mapsto (\Phi_U(s) : U \to \mathcal{F}_x,\ x \mapsto \rho(\alpha_x^{-1}) \cdot s(x)) \end{cases}.$$

This is well defined since $s(x) \in \mathcal{F}_x$ and α_x^{-1} is a path joining x and x_0 implies that $\rho(\alpha_x^{-1}) : \mathcal{F}_x \to \mathcal{F}_{x_0}$ is a linear transformation. Next, we claim $\Phi_U(s)$ to be a section of $\mathcal{G}$. Let $\gamma : [0,1] \to U$ be a path. Then $\gamma([0,1])$ is obviously contained in U and $\mathcal{F}^+|_U$ is locally constant from which we see that $\rho(\gamma)$, constructed as in step one, is precisely the canonical isomorphism $\mathcal{F}_x \to \mathcal{F}_{x_0}$ from Lemma 1.15. But this was defined via germs of a given section which means that we have $\rho(\gamma) \cdot s(\gamma(0)) = s(\gamma(1))$. From this we obtain

$$\begin{aligned} \Phi_U(s)(\gamma(1)) &= \rho(\alpha_{\gamma(1)}^{-1}) \cdot s(\gamma(1)) = \rho(\alpha_{\gamma(1)}^{-1}) \cdot \rho(\gamma) s(\gamma(0)) \\ &= \rho(\alpha_{\gamma(1)}^{-1} \cdot \gamma \cdot \alpha_{\gamma(0)}) \rho(\alpha_{\gamma(0)}^{-1}) s(\gamma(0)) = \rho([\alpha_{\gamma(1)}^{-1} \cdot \gamma \cdot \alpha_{\gamma(0)}]) \Phi_U(s)(\gamma(0)). \end{aligned}$$

Thus, $\Phi_U(s)$ is a section of $\mathcal{G}(U)$. Gluing the maps Φ_U together to a global map we obtain the desired morphism Φ.

Let us now construct a morphism $\Psi : \mathcal{G} \to \mathcal{F}^+$. For an open subset $V \subset X$ with trivial fundamental group take $k \in \mathcal{G}(V)$, i.e. a mapping $k : U \to \mathcal{F}_{x_0}$, and define $\Psi_V(k) \in \mathcal{F}^+(V)$ by

$$\Psi_V(k) : \begin{cases} U & \to \coprod_{y \in V} \mathcal{F}_y \\ x & \mapsto \rho(\alpha_x) \cdot k(x). \end{cases}$$

Again, we have to show that $\Psi_V(k)$ in fact is a section of $\mathcal{F}^+(V)$. Take $U \subset V$ connected such that $\mathcal{F}|_U$ is constant and further $s \in \mathcal{F}(U)$ such that $s_x = \Psi_V(k)(x)$ holds at some point $x \in U$. We have to show that this extends from the point x to a neighborhood of x. Take another point $y \in U$. Arguing as before, there is a path γ from x to y in U such that $s_y = \rho(\gamma) \cdot s_x$. By definition of $\mathcal{G}$, k satisfies

$$k(y) = \rho(\alpha_y^{-1} \cdot \gamma \cdot \alpha_x) \cdot k(x).$$

Using this, we obtain

$$\begin{aligned} \Psi_V(k)(y) &= \rho(\alpha_y) \cdot k(y) = \rho(\alpha_y) \cdot \rho(\alpha_y^{-1} \cdot \gamma \cdot \alpha_x) \cdot k(x) \\ &= \rho(\gamma) \cdot \rho(\alpha_x) \cdot k(x) = \rho(\gamma) \cdot s_x \\ &= s_y, \end{aligned}$$

which shows that $\Psi_V(k)$ in fact is a section of $\mathcal{F}^+(V)$ and therefore we can glue it to a global morphism Ψ. A straightforward calculation from the formulas above now shows that

$$\Psi \circ \Phi = \mathrm{id}_{\mathcal{F}+} \quad \text{and} \quad \Phi \circ \Psi = \mathrm{id}_{\mathcal{G}}.$$

Step 5: Reversing the constructions, II, and end of the proof. Finally, take τ and $\mathcal{G}$ as in step three and ρ as above. It remains to show that if $\mathcal{F} = \mathcal{G}$, then $\rho = \tau$. To see this, note at first that every stalk of $\mathcal{G}$ is a copy of the same vector space E and that the germ at x of any section k is just $k(x)$. Take U open such that $\mathcal{G}|_U$ is constant and $\gamma : [0,1] \to U$ a path in U. Analogous to the proof of Lemma 1.15 we can construct $\rho(\gamma) : E \to E$ by the following: Take $e \in E$ and $k \in \mathcal{G}(U)$ with $k(x) = e$. Then, $\rho(\gamma) \cdot e = k(y)$ for some y. Since we have that $k(y) = \tau([\alpha_y^{-1} \cdot \gamma \cdot \alpha_x]) \cdot k(x)$, it follows $\rho(\gamma) = \tau([\alpha_y^{-1} \cdot \gamma \cdot \alpha_x])$. Now, if γ is closed with initial point x_0, we may construct $\rho(\gamma)$ as in step one: For $0 = a_0 < a_1 < \cdots < a_n = 1$ as in Lemma 1.16 define the action of $\gamma|_{[a_i,a_{i+1}]}$ as before and $\rho(\gamma)$ as the composition. We obtain

$$\begin{aligned}
\rho(\gamma) &= \tau([\alpha_{x_0}^{-1} \cdot \gamma|_{[a_{n-1},1]} \cdot \alpha_{\gamma(a_{n-1})}]) \cdot \tau([\alpha_{\gamma(a_{n-1})}^{-1} \cdot \gamma|_{[a_{n-2},a_{n-1}]} \cdot \alpha_{\gamma(a_{n-2})}]) \cdot \ldots \\
&\qquad \cdots \cdot \tau([\alpha_{\gamma(a_1)}^{-1} \cdot \gamma|_{[0,a_1]} \cdot \alpha_{x_0}]) \\
&= \tau([\gamma|_{[a_{n-1},1]} \cdot \gamma|_{[a_{n-2},a_{n-1}]} \cdot \cdots \cdot \gamma|_{[0,a_1]}]) = \tau([\gamma]),
\end{aligned}$$

so $\rho = \tau$. This ends the proof of the theorem. □

It is not without reason that we called this a correspondence. There holds in fact a categorial correspondence, but we do not prove this stronger statement and finish this section by proving Theorem 1.14.

Proof of Theorem 1.14. (i)⇒(iii): We refer to [Wel08, Pages 258 and 259].
(iii)⇒(i): The natural connection d on $X^{un} \times \mathbb{C}^r$ is invariant under the action of $\pi_1(X)$ and therefore descends to a flat connection on E.
(ii)⇒(iii): Since the transition functions are locally constant, the sheaf of holomorphic sections of E, $\mathcal{O}(E)$, has to be locally constant. Applying Theorem 1.17 to $\mathcal{O}(E)$ we see that $\mathcal{O}(E)$ is induced by a representation of the fundamental group

$$\rho : \pi_1(X) \to \mathrm{GL}(r, \mathbb{C}),$$

which satisfies our assertion.
(iii)⇒(ii): If E comes from a representation of the fundamental group, so does $\mathcal{O}(E)$ which then is locally constant by Theorem 1.17. Using for example (1.13) we see that the transition functions are locally constant. □

1.3 Kobayashi-Lübke inequality

This section establishes an important connection between flatness of vector bundles and Chern number inequalities that is useful to prove a version of our later main result for smooth varieties and even for the bigger class of compact Kähler manifolds.

From now on we will assume that our manifold is Kähler and denote the Kähler form by ω. It is well known that we locally may write

$$\omega = \frac{\sqrt{-1}}{2} \sum_{\mu,\nu} h_{\mu\overline{\nu}}(z) dz^{\mu} \wedge dz^{\overline{\nu}}, \tag{1.40}$$

where h is the Kähler metric on T_X. To avoid confusion we want to point out that the form above can always be defined but a complex manifold is called Kähler if and only if this form is closed.

Given a Kähler manifold X with Kähler form ω defined as above, we remark for later reference that we obtain a well-defined volume form on X,

$$dX := \frac{\omega^n}{n!} := \frac{1}{n!} \underbrace{\omega \wedge \cdots \wedge \omega}_{n-times}, \tag{1.41}$$

and we define the volume of X to be

$$\mathrm{vol}(X) := \int_X dX. \tag{1.42}$$

A simple calculation using local coordinates shows that for every matrix $\Xi = (\Xi^{\alpha}_{\beta})_{\alpha,\beta}$, consisting of (k,k)-type forms,

$$\Xi \wedge \frac{\omega^{n-k}}{(n-k)!} = \sum_{\mu=1}^{r} \Xi^{\alpha}_{\beta\mu\overline{\mu}} \cdot \frac{\omega^n}{n!}, \tag{1.43}$$

holds in local coordinates, where

$$\Xi^{\alpha}_{\beta} = \sum_{\mu,\nu} \Xi^{\alpha}_{\beta\mu\overline{\nu}} dz^{\mu} \wedge dz^{\overline{\nu}}$$

is the representation of Ξ in local coordinates. We define

$$\mathrm{Tr}_{\omega}\Xi := \sum_{\mu} \Xi^{\alpha}_{\beta\mu\overline{\mu}}. \tag{1.44}$$

1.18 Definition (Hermitian-Einstein bundle)**.** *A hermitian vector bundle (E,h) is said to satisfy the Einstein condition with respect to the Kähler form ω, if there is a constant $\lambda > 0$ such that $\mathrm{Tr}_{\omega}\Omega = \lambda \cdot \mathrm{id}_E$. In this case we call E a Hermitian-Einstein vector bundle and h a Hermitian-Einstein metric.*

If our bundle is Hermitian-Einstein, we are able to use the Hermitian-Einstein condition as well as our local formula for the first chern class to simplify (1.44) to

$$c_1(E,h) \wedge \frac{\omega^{n-1}}{(n-1)!} = \lambda r \cdot \frac{\omega^n}{n!}. \tag{1.45}$$

In particular, integration of this equation yields

$$n \cdot \int_X c_1(E) \wedge [\omega]^{n-1} = \lambda r \cdot \mathrm{vol}(X) \tag{1.46}$$

and we will make use of this equation later when we are talking about stability of bundles.

The following theorem is essential and establishes a link between flat bundles and Chern classes that can be generalised to singular varieties and also complex orbifolds as we will see in chapters two and three.

1.19 Theorem (Kobayashi-Lübke inequality, cf. [Dem])**.** *For every Hermitian-Einstein vector bundle (E,h) the inequality*

$$[(r-1)\cdot c_1(E,h)^2 - 2r\cdot c_2(E,h)]\wedge\omega^{n-2}\le 0 \tag{1.47}$$

holds at every point of X. Moreover, equality occurs exactly if

$$\Omega = \frac{1}{r}c_1(E,h)\otimes \mathrm{id}_E. \tag{1.48}$$

Proof. The statement is local so we may work on a neighborhood of a given point in X and use our local description of Chern classes in terms of the curvature which yields

$$\begin{aligned}
&(r-1)\cdot c_1(E,h)^2 - 2r\cdot c_2(E,h)\\
&\qquad= (r-1)\cdot\left(\left(\sum_{\mu=1}^{r}\Omega_\mu^\mu\right)\wedge\left(\sum_{\mu=1}^{r}\Omega_\mu^\mu\right)\right) - 2r\cdot\sum_{\mu,\nu=1}^{r,r}\left(\Omega_\mu^\mu\wedge\Omega_\nu^\nu - \Omega_\nu^\mu\wedge\Omega_\mu^\nu\right)\\
&\qquad= -r\cdot\sum_{\mu,\nu}\left(\Omega_\mu^\mu\wedge\Omega_\nu^\nu - \Omega_\nu^\mu\wedge\Omega_\mu^\nu\right) + (r-1)\cdot\sum_{\mu,\nu}\Omega_\mu^\mu\wedge\Omega_\nu^\nu\\
&\qquad= \sum_{\mu,\nu}(-r+r-1)\cdot(\Omega_\mu^\mu\wedge\Omega_\nu^\nu) + r\cdot(\Omega_\nu^\mu\wedge\Omega_\mu^\nu)\\
&\qquad= \sum_{\alpha,\beta} -\Omega_\alpha^\alpha\wedge\Omega_\beta^\beta + r\cdot(\Omega_\beta^\alpha\wedge\Omega_\alpha^\beta).
\end{aligned} \tag{1.49}$$

Taking the wedge product with $\omega^{n-2}/(n-2)!$ means taking the trace and thus we obtain

$$\begin{aligned}
&[(r-1)c_1(E,h)^2 - 2rc_2(E,h)]\wedge\frac{\omega^{n-2}}{(n-2)!}\\
&\qquad= \frac{1}{2}\sum_{\alpha,\beta,\mu,\nu} -(R^\alpha_{\alpha\mu\mu}R^\beta_{\beta\nu\nu} - R^\alpha_{\alpha\mu\nu}R^\beta_{\beta\nu\mu}) + r(R^\alpha_{\beta\mu\mu}R^\beta_{\alpha\nu\nu} - R^\alpha_{\beta\mu\nu}R^\beta_{\alpha\nu\mu}).
\end{aligned} \tag{1.50}$$

Due to the Einstein condition on our metric, $\sum_\mu R^\alpha_{\beta\mu\mu} = \lambda\delta_{\alpha\beta}$, and therefore

$$\sum_{\alpha,\beta,\mu,\nu} -R^\alpha_{\alpha\mu\mu}R^\beta_{\beta\nu\nu} + rR^\alpha_{\beta\mu\mu}R^\beta_{\alpha\nu\nu} = \sum_{\alpha,\beta} -\lambda^2\delta_{\alpha\alpha}\delta_{\beta\beta} + r\lambda^2\delta_{\alpha\beta}\delta_{\beta\alpha} = -r^2\lambda^2 + r^2\lambda^2 = 0.$$

The curvature matrix is hermitian, i.e. $\overline{\Omega^\alpha_\beta} = \Omega^\beta_\alpha$. This translates to the coefficients of the curvature matrix as the symmetry relation

$$\overline{R}^\alpha_{\beta\mu\nu} = R^\beta_{\alpha\nu\mu}. \tag{1.51}$$

Applying this relation to our equation above we obtain

$$\sum_{\alpha,\beta,\mu,\nu} -rR^\alpha_{\beta\mu\nu}R^\beta_{\alpha\nu\mu} = \sum_{\alpha,\beta,\mu,\nu} -r|R^\alpha_{\beta\mu\nu}|^2$$

and thus

$$\begin{aligned}
&[(r-1)c_1(E)^2 - 2rc_2(E)] \wedge \frac{\omega^{n-2}}{(n-2)!} \\
&\quad = \frac{1}{2} \sum_{\alpha,\beta,\mu,\nu} R^{\alpha}_{\alpha\mu\nu} \overline{R}^{\beta}_{\beta\mu\nu} - r|R^{\alpha}_{\beta\mu\nu}|^2 \\
&\quad = -\frac{r}{2} \sum_{\alpha,\beta,\mu,\nu,\alpha\neq\beta} |R^{\alpha}_{\beta\mu\nu}|^2 + \frac{1}{2}\sum_{\mu,\nu}\left(\sum_{\alpha,\beta} R^{\alpha}_{\alpha\mu\nu}\overline{R}^{\beta}_{\beta\mu\nu} - r\sum_{\alpha}|R^{\alpha}_{\alpha\mu\nu}|^2\right) \\
&\quad = -\frac{r}{2} \sum_{\alpha,\beta,\mu,\nu,\alpha\neq\beta} |R^{\alpha}_{\beta\mu\nu}|^2 - \frac{1}{4}\sum_{\alpha,\beta,\mu,\nu} |R^{\alpha}_{\alpha\mu\nu} - R^{\beta}_{\beta\mu\nu}|^2 \leq 0,
\end{aligned}$$

which proves the inequality. Equality now holds exactly if for all $\alpha \neq \beta$ we have $R^{\alpha}_{\beta\mu\nu} = 0$ and furthermore if for every index of summation α the collection $\{R^{\alpha}_{\alpha\mu\nu}\}_{\mu,\nu}$ defines a form of $(1,1)$-type with this coefficients; in other words, if η defined by

$$\eta = \sqrt{-1}\sum_{\mu,\nu} \eta_{\mu\nu} dz^{\mu} \wedge dz^{\overline{\nu}}, \qquad \eta_{\mu\nu} := R^{\alpha}_{\alpha\mu\nu}$$

is a well defined $(1,1)$-type form on X. In this case $\Omega = \eta \otimes \mathrm{id}_E$. Taking the trace in this equality we obtain $c_1(E) = r\eta$, which tells us that equality is equivalent to

$$\Omega = \frac{1}{r} c_1(E) \otimes \mathrm{id}_E,$$

which was our claim and had to be shown. □

One should note that the equality holds pointwise already if the following integral exists and the numerical equality

$$\int_X [(r-1)c_1(E)^2 - 2rc_2(E)] \wedge \omega^{n-2} = 0 \tag{1.52}$$

is satisfied. We will usually check this condition in order to prove flatness of a vector bundle.

1.20 Corollary (cf. [Dem])**.** *Every Hermitian-Einstein bundle (E,h) with $c_1(E) = 0$ and $c_2(E) = 0$ is unitary flat.*

Proof. We need to find a hermitian metric h' such that the curvature $\Omega_{h'}$ of its Chern connection vanishes, $\Omega_{h'} = 0$. Since $c_1(E) = 0$, it follows from the $\partial\overline{\partial}$-lemma that there is a global function ψ on X satisfying $c_1(E) = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\psi$. Moreover the vanishing of both, $c_1(E)$ and $c_2(E)$, implies equality in the Bogomolov inequality. Hence,

$$\Omega_{h'}(E) = \Omega_h(E) - \frac{\sqrt{-1}}{2\pi r}\partial\overline{\partial}\psi \otimes \mathrm{id}_E = 0,$$

where $h' = h\exp(\psi)^{\frac{1}{r}}$. □

1.4 Guggenheimer-Yau inequality and characterisation of torus quotients

We now prove a slightly stronger Chern number inequality for the special case of the tangent bundle of a compact Kähler-Einstein manifold (X, ω). In this case, since $\mathrm{Tr}_\omega \Omega(T_X) = \lambda \mathrm{id}_{T_X}$, the tangent bundle T_X is Hermitian-Einstein. Then it follows that we obtain additional symmetry relations between the coefficients of the curvature tensor, namely

$$R^{\alpha}_{\beta\mu\nu} = R^{\mu}_{\beta\alpha\nu} = R^{\alpha}_{\nu\mu\beta} = R^{\mu}_{\nu\alpha\beta}, \tag{1.53}$$

which follow after applying the Einstein condition to (1.22) and (1.24).

1.21 Theorem (Guggenheimer-Yau inequality, cf. [Dem])**.** *Let (X, w) be a compact Kähler-Einstein manifold. Then,*

$$[n \cdot c_1(T_X, h)^2 - (2n+2) \cdot c_2(T_X, h)] \wedge \omega^{n-2} \leq 0, \tag{1.54}$$

and if equality holds, then, denoting the Einstein constant by λ, X is a finite unramified quotient of a torus, if $\lambda = 0$.

Proof. If we replace r by $n+1$ in the Kobayashi-Lübke inequality, Theorem 1.19, formula (1.49) becomes

$$n \cdot c_1(T_X, h)^2 - (2n+2) \cdot c_2(T_X, h) = \sum_{\alpha,\beta} -\Omega^{\alpha}_{\alpha} \wedge \Omega^{\beta}_{\beta} + (n+1)\Omega^{\alpha}_{\beta} \wedge \Omega^{\beta}_{\alpha}$$

and, moreover, formula (1.50) becomes

$$\begin{aligned}
&[r \cdot c_1(E)^2 - (2n+2) \cdot c_2(E)] \wedge \frac{\omega^{n-2}}{(n-2)!} \\
&\quad = \frac{1}{2} \sum_{\alpha,\beta,\mu,\nu} -(R^{\alpha}_{\alpha\mu\mu} R^{\beta}_{\beta\nu\nu} - R^{\alpha}_{\alpha\mu\nu} R^{\beta}_{\beta\nu\mu}) + (2n+2) \cdot (R^{\alpha}_{\beta\mu\mu} R^{\beta}_{\alpha\nu\nu} - R^{\alpha}_{\beta\mu\nu} R^{\beta}_{\alpha\nu\mu}).
\end{aligned}$$

We also have seen in the proof of the Kobayashi-Lübke inequality that terms with factor n cancel, so it remains only the following:

$$\begin{aligned}
&[r \cdot c_1(E)^2 - (2n+2) \cdot c_2(E)] \wedge \frac{\omega^{n-2}}{(n-2)!} \\
&\quad = \frac{1}{2} \sum_{\alpha,\beta,\mu,\nu} R^{\alpha}_{\alpha\mu\nu} R^{\beta}_{\beta\nu\mu} + R^{\alpha}_{\beta\mu\mu} R^{\beta}_{\alpha\nu\nu} - (n+1) \cdot R^{\alpha}_{\beta\mu\nu} R^{\beta}_{\alpha\nu\mu} \\
&\quad = \frac{1}{2} \sum_{\alpha,\beta,\mu,\nu} 2 \cdot (R^{\alpha}_{\alpha\mu\nu} R^{\beta}_{\beta\nu\mu} - (n+1) \cdot R^{\alpha}_{\beta\mu\nu} R^{\beta}_{\alpha\nu\mu} \\
&\quad = \frac{1}{2} \sum_{\alpha,\beta,\mu,\nu} \left(|R^{\alpha}_{\alpha\mu\nu} - R^{\beta}_{\beta\mu\nu}|^2 - (n+1) \cdot |R^{\alpha}_{\beta\mu\nu}|^2 \right) + n \cdot \sum_{\alpha,\mu,\nu} |R^{\alpha}_{\alpha\mu\nu}|^2 \\
&\quad =: \Sigma,
\end{aligned}$$

where we used the symmetry relations (1.53) in the second equality and the hermitian symmetry relation (1.51) in the last equality. Reorganising the sums yields

$$\begin{aligned}
\Sigma = &-\frac{1}{2}\sum_{\alpha,\beta,\mu,\nu}|R^{\alpha}_{\alpha\mu\nu}-R^{\beta}_{\beta\mu\nu}|^2\\
&-\frac{n+1}{2}\Bigg(\sum_{\substack{\alpha,\beta,\mu,\nu\\ \text{pairwise}\neq}}|R^{\alpha}_{\beta\mu\nu}|^2+8\sum_{\substack{\alpha,\mu,\nu\\ \mu<\nu\\ \mu,\nu\neq\alpha}}|R^{\alpha}_{\alpha\mu\nu}|^2+4\sum_{\substack{\alpha,\mu\\ \alpha\neq\mu}}|R^{\alpha}_{\alpha\alpha\mu}|^2+4\sum_{\substack{\alpha,\mu\\ \alpha<\mu}}|R^{\alpha}_{\alpha\mu\mu}|^2+\sum_{\alpha}|R^{\alpha}_{\alpha\alpha\alpha}|^2\Bigg)\\
&+n\Bigg(2\sum_{\substack{\alpha,\mu,\nu\\ \mu<\nu\\ \mu,\nu\neq\alpha}}|R^{\alpha}_{\alpha\mu\nu}|^2+2\sum_{\substack{\alpha,\mu\\ \alpha\neq\mu}}|R^{\alpha}_{\alpha\alpha\mu}|^2+2\sum_{\substack{\alpha,\mu\\ \alpha<\mu}}|R^{\alpha}_{\alpha\mu\mu}|^2+\sum_{\alpha}|R^{\alpha}_{\alpha\alpha\alpha}|^2\Bigg)\\
= &-\frac{1}{2}\sum_{\substack{\alpha,\beta,\mu,\nu\\ \alpha\neq\beta}}|R^{\alpha}_{\alpha\mu\nu}-R^{\beta}_{\beta\mu\nu}|^2-\frac{n+1}{2}\sum_{\substack{\alpha,\beta,\mu,\nu\\ \text{pairwise}\neq}}|R^{\alpha}_{\beta\mu\nu}|^2-(2n+4)\sum_{\substack{\alpha,\mu,\nu\\ \mu<\nu\\ \mu,\nu\neq\alpha}}|R^{\alpha}_{\alpha\mu\nu}|^2\\
&-2\sum_{\substack{\alpha,\mu\\ \alpha\neq\mu}}|R^{\alpha}_{\alpha\alpha\mu}|^2-2\sum_{\substack{\alpha,\mu\\ \alpha<\mu}}|R^{\alpha}_{\alpha\mu\mu}|^2+\frac{n+1}{2}\sum_{\alpha}|R^{\alpha}_{\alpha\alpha\alpha}|^2.
\end{aligned}$$

We see that every term in Σ is negative, except for $\sum\limits_{\alpha}|R^{\alpha}_{\alpha\alpha\alpha}|^2$. Therefore we are trying to absorb this term in every sum in Σ that sums over $R^{\alpha}_{\alpha\mu\mu}$ and obtain

$$\begin{aligned}
\Sigma = &-\frac{1}{2}\sum_{\substack{\alpha,\beta,\mu,\nu\\ \alpha\neq\beta,\mu\neq\nu}}|R^{\alpha}_{\alpha\mu\nu}-R^{\beta}_{\beta\mu\nu}|^2-\frac{1}{2}\sum_{\substack{\alpha,\beta,\mu\\ \alpha\neq\beta}}|R^{\alpha}_{\alpha\mu\nu}-R^{\beta}_{\beta\mu\nu}|^2\\
&-\frac{n+1}{2}\sum_{\substack{\alpha,\beta,\mu,\nu\\ \text{pairwise}\neq}}|R^{\alpha}_{\beta\mu\nu}|^2-(2n+4)\sum_{\substack{\alpha,\mu,\nu\\ \mu<\nu\\ \mu,\nu\neq\alpha}}|R^{\alpha}_{\alpha\mu\nu}|^2\\
&-2\sum_{\substack{\alpha,\mu\\ \alpha\neq\mu}}|R^{\alpha}_{\alpha\alpha\mu}|^2-2\sum_{\substack{\alpha,\mu\\ \alpha<\mu}}|R^{\alpha}_{\alpha\mu\mu}|^2+\frac{n+1}{2}\sum_{\alpha}|R^{\alpha}_{\alpha\alpha\alpha}|^2.
\end{aligned}$$

Considering

$$\Sigma' = -\frac{1}{2}\sum_{\substack{\alpha,\beta,\mu\\ \alpha\neq\beta}}|R^{\alpha}_{\alpha\mu\nu}-R^{\beta}_{\beta\mu\nu}|^2-2\sum_{\substack{\alpha,\mu\\ \alpha<\mu}}|R^{\alpha}_{\alpha\mu\mu}|^2+\frac{n+1}{2}\sum_{\alpha}|R^{\alpha}_{\alpha\alpha\alpha}|^2$$

and absorbing the $\mu = \beta$ term in the first sum in Σ', we obtain

$$\begin{aligned}
\Sigma' &= -\frac{1}{2} \sum_{\substack{\alpha,\beta,\mu \\ \text{pairwise} \neq}} |R^{\alpha}_{\alpha\mu\nu} - R^{\beta}_{\beta\mu\nu}|^2 - \frac{1}{2} \sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} |R^{\alpha}_{\alpha\mu\nu} - R^{\beta}_{\beta\mu\nu}|^2 - 2 \sum_{\substack{\alpha,\mu \\ \alpha<\mu}} |R^{\alpha}_{\alpha\mu\mu}|^2 + \frac{n+1}{2} \sum_{\alpha} |R^{\alpha}_{\alpha\alpha\alpha}|^2 \\
&= -\frac{1}{2} \sum_{\substack{\alpha,\beta,\mu \\ \text{pairwise} \neq}} |R^{\alpha}_{\alpha\mu\nu} - R^{\beta}_{\beta\mu\nu}|^2 - \frac{n-1}{2} \sum_{\alpha} |R^{\alpha}_{\alpha\alpha\alpha}|^2 - 4 \sum_{\substack{\alpha,\beta \\ \alpha<\beta}} |R^{\alpha}_{\alpha\beta\beta}|^2 \\
&\quad + \sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} \left(R^{\alpha}_{\alpha\beta\beta} \overline{R}^{\beta}_{\beta\beta\beta} + \overline{R}^{\alpha}_{\alpha\beta\beta} R^{\beta}_{\beta\beta\beta} \right) \\
&= -\frac{1}{2} \sum_{\substack{\alpha,\beta,\mu \\ \text{pairwise} \neq}} |R^{\alpha}_{\alpha\mu\nu} - R^{\beta}_{\beta\mu\nu}|^2 - \sum_{\substack{\alpha,\beta \\ \alpha\neq\beta}} |R^{\alpha}_{\alpha\alpha\alpha} - 2R^{\alpha}_{\alpha\beta\beta}|^2 \leq 0,
\end{aligned}$$

which shows $\Sigma \leq 0$.

Now, $\Sigma = 0$ if and only if there is a scalar μ such that

$$R^{\alpha}_{\alpha\beta\beta} = R^{\alpha}_{\beta\beta\alpha} = R^{\alpha}_{\beta\alpha\beta} = \mu \quad \text{for all } \alpha \neq \beta, \quad R^{\alpha}_{\alpha\alpha\alpha} = 2\mu$$

and all other coefficients $R^{\alpha}_{\beta\mu\nu}$ vanish identically. Then,

$$\mathrm{Tr}_{\omega}\Omega(T_X) = \sum_{\mu} R^{\alpha}_{\beta\mu\overline{\mu}} = \sum_{\mu} R^{\alpha}_{\alpha\mu\mu} = (n+1)\mu \cdot \mathrm{id},$$

thus $\lambda = (n+1)\mu$. Therefore, X has constant holomorphic sectional curvature. It is classical, see for example [Tia00], that the universal cover X^{un} of X is isometric to

(i) $\mathbb{C}^n$, if $\lambda = 0$;

(ii) $\mathbb{P}^n$, if $\lambda > 0$;

(iii) $\mathbb{B}^n$, if $\lambda < 0$.

If $\lambda = 0$, then $\pi_1(X)$ acts on $X^{un} \cong \mathbb{C}^n$ by isometries. The classification of subgroups of affine transformations acting freely and having a compact quotient, cf. [KN63, Chapter VI, section 4], shows that $\pi_1(X)$ is a semidirect product of a finite group G of isometries by a translation group associated to a lattice $\Lambda \subset \mathbb{C}^n$. The splitting lemma for groups, cf. page 147 and the discussion on top of page 148 of [Hat02], then gives an exact sequence

$$0 \longrightarrow \Lambda \longrightarrow \pi_1(X) \longrightarrow G \longrightarrow 0. \tag{1.55}$$

Consequently, there is a finite unramified covering map $\eta : \mathbb{C}^n/\Lambda \to X^{un}$ and we obtain a diagram

$$\begin{tikzcd}
\mathbb{C}^n/\Lambda \arrow[r, "\text{finite, unramified}"] \arrow[rr, bend right, "\text{finite, unramified}"'] & X^{un} \arrow[r, "\text{universal cover}"] & X
\end{tikzcd}$$

which implies $(\mathbb{C}^n/\Lambda)/G \cong X^{un}$ as claimed. □

1.5 Slope stability and existence of Hermitian-Einstein metrics

Up to this point, we did not bother too much with the actual existence of Hermitian-Einstein metrics. In fact, constructing such a metric on a given bundle can be very challenging. However, if our manifold is projective, then there is a nice criterion for the existence of Hermitian-Einstein metrics and even if we don't restrict ourselves to projective manifolds, there is a condition on the bundle that implies the existence of a Hermitian-Einstein metric, namely the stability of the bundle and this condition is in many situations easier to verify. Throughout this section, the language of sheaves is prefered and we introduce the notions for slightly more general objects than just locally free sheaves. We follow chapter V of [Kob87].

For direct comparison recall that a sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules on X is called locally free of rank r, if for every point $x \in X$ we find an open neighborhood $U = U(x)$ of x in X together with an exact sequence

$$0 \longrightarrow \mathcal{O}_X^{\oplus r}|_U \longrightarrow \mathcal{E}|_U \longrightarrow 0. \tag{1.56}$$

In other words, $\mathcal{E}$ is locally free of rank r, if it is locally isomorphic to $\mathcal{O}_X^{\oplus r}$. As a generalisation of this concept we call a sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules on X coherent, if for every point $x \in X$ there is an open neighborhood $U = U(x) \subset X$ of x and an exact sequence

$$\mathcal{O}_X^{\oplus p}|_U \longrightarrow \mathcal{O}_X^{\oplus q}|_U \longrightarrow \mathcal{E}|_U \longrightarrow 0. \tag{1.57}$$

It is well known, see for example [Kob87], that the set $S_{\mathcal{E}}$ of all points in $x \in X$ such that $\mathcal{E}_x$ is not free, forms a closed, analytic subset of X of at least codimension one. We define the rank of a coherent sheaf $\mathcal{E}$ to be

$$\operatorname{rank}(\mathcal{E}) := \operatorname{rank}(\mathcal{E}_x) \quad \text{ for any } x \notin S_{\mathcal{E}}. \tag{1.58}$$

Note that a coherent sheaf $\mathcal{E}$ is locally free on $X \setminus S_{\mathcal{E}}$. It is possible to define determinant bundles for general coherent sheaves, cf. [Kob87] and in the case that a coherent sheaf $\mathcal{E}$ is torsion-free, i.e. that every stalk of $\mathcal{E}$ is torsion-free, the determinant bundle of $\mathcal{E}$ can be calculated in the same way as for locally free sheaves via

$$\det(\mathcal{E}) = \left(\bigwedge^{\operatorname{rank}(\mathcal{E})} \mathcal{E} \right)^{\vee\vee}. \tag{1.59}$$

Note that since we are taking the reflexive hull, $\det(\mathcal{E})$ is reflexive, hence torsion-free and therefore in fact locally free of rank one. Thus we may identify $\det(\mathcal{E})$ with a line bundle on X and define the first Chern class of a coherent sheaf $\mathcal{E}$ to be

$$c_1(\mathcal{E}) := c_1(\det(\mathcal{E})). \tag{1.60}$$

If ω is a Kähler form on X with respect to the hermitian metric h, we further define the degree of $\mathcal{E}$ with respect to ω to be

$$\deg_\omega(\mathcal{E}) := \int_X c_1(\mathcal{E}) \wedge [\omega]^{n-1} \tag{1.61}$$

and the slope of $\mathcal{E}$ with respect to ω as

$$\mu_\omega(\mathcal{E}) := \frac{\deg_\omega(\mathcal{E})}{\operatorname{rank}(\mathcal{E})}. \tag{1.62}$$

1.22 Remark. *In the case that we are working on a normal, projective variety X, the slope can also be defined with respect to an ample divisor H as follows: Take general elements $D_1, \dots, D_{n-1} \in |H|$ and consider the complete intersection curve $C := D_1 \cap \dots \cap D_{n-1}$, which we can assume to be smooth since H is ample and normal varieties have singularities only in codimension at least two. Then the degree of a coherent sheaf $\mathcal{E}$ on X with respect to H is defined as*

$$\deg_H(\mathcal{E}) := c_1(\mathcal{E}) \cdot H^{n-1} = \int_C c_1(\mathcal{E}|_C), \tag{1.63}$$

wherein we may integrate since $\mathcal{E}|_C$ is a locally free sheaf on the smooth and compact complex curve C. Furthermore, the slope with respect to H, or the H-slope, is defined as

$$\mu_H(\mathcal{E}) := \frac{\deg_H(\mathcal{E})}{\operatorname{rank}(\mathcal{E})}. \tag{1.64}$$

1.23 Definition (Slope stability)**.** *Let $\mathcal{E}$ be a torsion-free, coherent sheaf on the compact Kähler manifold X.*

(i) $\mathcal{E}$ is called (slope) semistable, if for every coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ with $0 < \operatorname{rank}(\mathcal{F}) < \operatorname{rank}(\mathcal{E})$ the inequality

$$\mu_\omega(\mathcal{F}) \le \mu_\omega(\mathcal{E}) \tag{1.65}$$

holds.

(ii) $\mathcal{E}$ is called (slope) stable, if for any subsheaf as in (i), the inequality (1.65) *is strict.*

(iii) $\mathcal{E}$ is called (slope) polystable, if there exist stable subsheaves $\mathcal{E}_1, \dots, \mathcal{E}_k$ of $\mathcal{E}$ with $\mu_\omega(\mathcal{E}_i) = \mu_\omega(\mathcal{E}_j)$ for all $i, j \in \{1, \dots, k\}$ such that

$$\mathcal{E} = \bigoplus_{i=1}^{k} \mathcal{E}_i. \tag{1.66}$$

(iv) A vector bundle E on X is called semistable, stable or polystable, if $\mathcal{O}(E)$ has the respective property.

H-stability, -semistability and -polystability are defined analogous for projective X.

There are other notions of stability than just slope stability but since none of them is important for our considerations, we will usually talk of stable, semistable and polystable coherent sheaves and omit to accentuate that we are working with this specific notion of stability.

1.24 Remark. *If X is a normal, projective variety,* (1.63) *implies that one only needs to check stability on curves in order to deduce that a sheaf is stable. Mehta and Ramanathan proved in [MR82] that, given an ample divisor H on X, there always is an integer $m \gg 0$ such that the restriction of a stable sheaf to a complete intersection surface constructed from general elements in $|mH|$ is stable again. The analogous result holds for semi- and polystability.*

The slope of a coherent sheaf can also be defined for not necessary compact manifolds, respectively quasi-projective varieties, by replacing the degree with $d(\mathcal{E}) := c_1(\mathcal{E}) \wedge [\omega]^{n-1}$, respectively $d(\mathcal{E}) := c_1(\mathcal{E}) \cdot H^{n-1}$. For details, in particular for the fact that this is well-defined, we refer to [Kob87, Chapter V, section 8]. However, at this point we only use this to state the following proposition.

1.25 Proposition ([Kob87, Chapter V, Proposition 8.2])**.** *Let (E, h) be a Hermitian-Einstein vector bundle on a not necessary compact Kähler manifold (X, ω) and denote the Einstein constant of E by λ. Moreover, let*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

be an exact sequence of vector bundles. Then,

$$\frac{d(E')}{\mathrm{rank}(E')} \leq \frac{d(E)}{\mathrm{rank}(E)},$$

and if equality holds, the sequence splits and E', E'' are also Hermitian-Einstein with Einstein constant λ.

One should notice that splitting of the sequence means splitting as a sequence of holomorphic vector bundles which is a stronger statement than the splitting as a sequence of complex vector bundles.

Before discussing the connection between stability and the existence of Hermitian-Einstein metrics we need the following helpful lemma telling us that, in order to prove stability, we only need to check subsheaves with torsion-free quotient.

1.26 Lemma ([Kob87, Chapter V, Proposition 7.6])**.** *For a torsion-free, coherent sheaf $\mathcal{E}$ on a compact Kähler manifold (X, ω), the following are equivalent.*

(i) *$\mathcal{E}$ is semistable.*

(ii) *$\mu_\omega(\mathcal{F}) \leq \mu_\omega(\mathcal{E})$ for every subsheaf $\mathcal{F}$ with $0 < \mathrm{rank}(\mathcal{F}) < \mathrm{rank}(\mathcal{E})$ and torsion-free quotient sheaf $\mathcal{E}/\mathcal{F}$.*

And the analogous result holds for stable sheaves.

Proof. Take a subsheaf $\mathcal{F}$ of $\mathcal{E}$ such that $0 < \mathrm{rank}(\mathcal{F}) < \mathrm{rank}(\mathcal{E})$. Then we have an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{G} \longrightarrow 0.$$

Let $\mathrm{tor}(\mathcal{G})$ be the torsion subsheaf of $\mathcal{G}$, define $\mathcal{G}' := \mathcal{G}/\mathrm{tor}(\mathcal{G})$ and further $\mathcal{F}'$ by exactness of the following sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{G}' \longrightarrow 0.$$

Then, $\mathcal{F}$ is a subsheaf of $\mathcal{F}'$ and $\mathcal{F}'/\mathcal{F}$ is isomorphic to $\mathrm{tor}(\mathcal{G})$. Applying [Kob87, Chapter V, items (6.9) and (7.5)] we obtain the inequalities

$$\mu_\omega(\mathcal{G}') \leq \mu_\omega(\mathcal{G}) \quad \text{and} \quad \mu_\omega(\mathcal{F}) \leq \mu_\omega(\mathcal{F}')$$

from which the assertion follows. □

The link between the existence of Hermitian-Einstein metrics in vector bundles and slope stability of vector bundles was originally established by Kobayashi in [Kob82]; Lübke simplified the original proof in [Lü83]. They proved the following theorem:

1.27 Theorem (Every Hermitian-Einstein bundle is semistable, cf. [Kob87, Chapter V, Theorem 8.3])**.** *Every Hermitian-Einstein bundle (E, h) is semistable and a direct sum of stable Hermitian-Einstein bundles (E_i, h_i) with the same Einstein constants as the original bundle (E, h).*

Proof. Step 1: Semistability. First, we will prove the semistability E. Write $\mathcal{E} = \mathcal{O}(E)$ and let $\mathcal{F}$ be a subsheaf of $\mathcal{E}$ with $0 < \operatorname{rank}(\mathcal{F}) = s < r = \operatorname{rank}(\mathcal{E})$ such that $\mathcal{E}/\mathcal{F}$ is torsion-free. Then the inclusion map $j : \mathcal{F} \to \mathcal{E}$ induces a homomorphism of sheaves

$$\det(j) : \det(\mathcal{F}) \longrightarrow \bigwedge^{s} \mathcal{E}. \tag{1.67}$$

$\det(j)$ is injective away from $S_{\mathcal{F}}$; therefore its kernel is a torsion subsheaf of the reflexive sheaf $\det(\mathcal{F})$ and thus trivial, since reflexive sheaves are torsion-free. Tensoring (1.67) with $\det(\mathcal{F})^\vee$ we obtain a non-trivial homomorphism

$$f : \mathcal{O}_X \to \bigwedge^{s} \mathcal{E} \otimes \det(\mathcal{F})^\vee.$$

We may consider f as non-trivial section of the vector bundle $\bigwedge^{s} E \otimes \det(\mathcal{F})^\vee$. Recall from (1.46) that the Hermitian-Einstein constant for (E, h) can be calculated via the formula

$$\lambda = \frac{\mu_\omega(E)}{\operatorname{vol}(X)}.$$

Now, since every line bundle possesses an Hermitian-Einstein metric, we may choose one for $\det(\mathcal{F})$ and calculate its Einstein constant λ' as before via the formula

$$\lambda' = \frac{\mu_\omega(\det(\mathcal{F}))}{\operatorname{vol}(X)} = \frac{s \cdot \mu_\omega(\mathcal{F})}{\operatorname{vol}(X)}.$$

Consequently, the vector bundle $\bigwedge^{s} E \otimes \det(\mathcal{F})^\vee$ is Hermitian-Einstein with Einstein constant $s\lambda - \lambda'$. Since this bundle admits a non-trivial holomorphic section, we may apply the Kobayashi vanishing theorem, [Kob87, Chapter III, Theorem 1.9], and obtain the inequality

$$s\lambda - \lambda' \geq 0, \tag{1.68}$$

which is equivalent to $\mu_\omega(\mathcal{F}) \leq \mu_\omega(\mathcal{E})$, i.e. the semistability of $\mathcal{E}$.
Step 2: Polystability. If $\mu_\omega(\mathcal{F}) < \mu_\omega(\mathcal{E})$ holds, then $\mathcal{E}$ is stable, since $\mathcal{F}$ was arbitrary and there would be nothing more to prove. We may thus assume

$$\mu_\omega(\mathcal{F}) = \mu_\omega(\mathcal{E}). \tag{1.69}$$

Let $X_0 := X \backslash S_{\mathcal{F}}$. Then it is clear that $\mathcal{E}$ and $\mathcal{F}$ are locally free on X_0 and therefore correspond to vector bundles E, respectively F, on X_0. Since λ and λ' are constant, Proposition 1.25 implies that

$$E|_{X_0} = F \oplus G,$$

where $\mathcal{O}(G) := \mathcal{G}|_{X_0} := (\mathcal{E}/\mathcal{F})|_{X_0}$. Since $\mathcal{E}$ is reflexive, $\mathcal{F}$ is reflexive as a subsheaf of a reflexive sheaf. Therefore, also $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ and $\mathcal{H}om(\mathcal{F}, \mathcal{F})$ are reflexive. The restricted sequence

$$0 \longrightarrow \mathcal{F}|_{X_0} \xrightarrow{j_0 := j|_{\mathcal{F}|_{X_0}}} \mathcal{E}|_{X_0} \longrightarrow \mathcal{G}|_{X_0} \longrightarrow 0 \tag{1.70}$$

splits. But now the splitting homomorphism $s_0 \in \Gamma(X_0, \mathcal{H}om(\mathcal{E}, \mathcal{F}))$ satisfying $s_0 \circ j_0 = \mathrm{id}_{\mathcal{F}|_{X_0}} \in \Gamma(X_0, \mathcal{H}om(\mathcal{F}, \mathcal{F}))$ extends to a unique splitting homomorphism $s \in$

$\Gamma(X, \mathcal{H}om(\mathcal{E}, \mathcal{F}))$, satisfying $s \circ j = \mathrm{id}_{\mathcal{F}}$, since $\mathcal{H}om(\mathcal{F}, \mathcal{F})$ and $\mathcal{H}om(\mathcal{F}, \mathcal{F})$ are reflexive and therefore determined by their restriction to X_0. Thus we have

$$\mathcal{E} = \mathcal{F} \oplus \mathcal{G}.$$

Therefore, also the sheaves $\mathcal{E}$ and $\mathcal{G}$ are locally free and $E = F \oplus G$. Now, recalling from [Kob87, Chapter IV, Item 1.4] that a direct sum $E = E_1 \oplus E_2$ of vector bundles is Hermitian-Einstein if and only if E_1 and E_2 are Hermitian-Einstein with the same Einstein constant, the claim follows. □

In fact, the converse is also true:

1.28 Theorem (Kobayashi [Kob82], Lübke [Lü83], Donaldson [Don85] and Uhlenbeck-Yau [UY86])**.** *A holomorphic vector bundle over a compact Kähler manifold is polystable if and only if it admits a Hermitian-Einstein metric.*

The one-dimensional case is due to a theorem of Narasimhan and Seshadri on projectively flat bundles on Riemann surfaces, see [Kob87, Chapter V, Theorem 2.7] for a proof. Donaldson has proved the existence of Hermitian-Einstein metrics in stable bundles first for algebraic surfaces and later for smooth algebraic varieties of arbitrary dimension. His proof relies on induction on the dimension of the variety and uses the result from Mehta and Ramanathan on the stability of bundles after restricting to complete intersection surfaces we mentioned in Remark 1.24. The general case then was established by Uhlenbeck and Yau using real analysis and the theory of partial differential equations.

There also is a categorial version of this theorem, known as the Kobayashi-Hitchin correspondence.

Bogomolov has shown that the inequality (1.47) holds for stable sheaves, which is a slightly more general statement than the one we obtained, i.e. the Kobayashi-Lübke-inequality. Therefore, when meaning the inequality for stable sheaves we will speak of the Bogomolov inequality.

One should notice that this notion of stability is purely analytic. If X is projective, then there is the following criterion for the tangent sheaf to be polystable:

1.29 Theorem (Algebro-geometric criterion for polystability, cf. [Gue15, Corollary on page 2])**.** *Let X be a smooth, projective variety. Assume that $K_X \equiv 0$. Then the tangent sheaf $\mathcal{T}_X$ is polystable with respect to any Kähler class.*

Using this theorem we may reformulate our result on the characterisation of finite quotients of Abelian varieties in the projective case and obtain the following theorem.

1.30 Theorem (Characterisation of smooth quotients of Abelian varieties)**.** *Let X be a smooth, projective variety. Then the following are equivalent:*

(i) $K_X \equiv 0$ *and* $c_2(\mathcal{T}_X) = 0$.

(ii) *X is a finite, unramified quotient of an Abelian variety. In other words, there is a surjective, Galois morphism $\eta : A \to X$ from an Abelian variety A that is étale.*

Proof. (i)⇒(ii): Since $K_X \equiv 0$, $\mathcal{T}_X$ is polystable with respect to any Kähler class, Theorem 1.29, and therefore admits a Hermitian-Einstein metric by Theorem 1.28. Moreover, $c_1(\mathcal{T}_X) = c_1(K_X) = 0$ and $c_2(\mathcal{T}_X) = 0$ hold by assumption. Assertion (ii) now follows by applying the

Guggenheimer-Yau inequality, Theorem 1.21.
(ii)⇒(i): Since $\eta : A \to X$ is étale, there is an linear equivalence $\eta^* K_X \sim K_A \equiv 0$ and thus $K_X \equiv 0$. Moreover, $\eta^* c_2(\mathcal{T}_X) = c_2(\eta^* \mathcal{T}_X) = c_2(\mathcal{T}_A) = 0$. $\square$

2 Differential geometry of complex orbifold surfaces

Orbifolds were originally introduced by Satake in [Sat56]; he called them V-manifolds. Whereas complex manifolds are locally modelled by domains in $\mathbb{C}^n$, the local model of a complex orbifold is a domain in $\mathbb{C}^n$ modulo some group, i.e. they are locally modelled by the orbit space of some group operation on $\mathbb{C}^n$. It is well known that such orbit spaces are smooth, i.e. themselves a complex manifold, only if the respective group action is free. Therefore, orbifolds are not necessary smooth; they have a special kind of singularities, namely quotient singularities, and are the complex analytic analogue of a $\mathbb{Q}$-variety. Nowadays, the theory of orbifolds is applied to many different problems and, unfortunately, different authors often mean different objects when talking about orbifolds, depending on their purposes. Our notion of orbifold is the one originally introduced by Satake, altough we change some notions compared to Satakes original paper to accentuate the analogies to manifolds and more likely use the language of sheaves to describe for example vector bundles on orbifolds. For an in-depth discussion on orbifolds we refer to [Bla96] for complex- and algebro-geometric aspects of the theory and to [CR02] for general differential-geometric aspects. The definitions in both of the references are different from the one we give below but they are easily seen to be equivalent to ours.

2.1 Definition and basic facts

We start by defining orbifolds and prove some basic facts that we will use throughout this chapter. Our definition is the same Satake gave in [Sat56], but we follow those of Biswas and Schumacher in [BS15], since they consider complex orbifolds from the start. For we introduce a new category of objects, we will, in contrast to chapter one, take care of distinguishing between the underlying topological space of an orbifold and the space together with the orbifold structure.

Let $\mathfrak{X}$ be a topological Hausdorff space, satisfying the second countability axiom. A tripel (V, Γ, φ) consisting of

(i) a domain $V \subset \mathbb{C}^n$,

(ii) a finite subgroup $\Gamma < \mathrm{Aut}_{\mathcal{O}}(V)$, and

(iii) a continuous map $\varphi : V \to U \subset \mathfrak{X}$ onto a connected open subset U inducing a homeomorphism $\widetilde{\varphi}$ that factors via the quotient map $\pi : V \to V/\Gamma$, i.e.

$$\begin{tikzcd} V \arrow[r, "\pi"] \arrow[rr, bend right, "\varphi"'] & V/\Gamma \arrow[r, "\widetilde{\varphi}"] & U \end{tikzcd}$$

is called n-dimensional complex orbifold chart for $U \subset \mathfrak{X}$.

If (V', Γ', φ') is another orbifold chart for an open subset $U' \subset \mathfrak{X}$ that satisfies $U' \subset U$, then an embedding of orbifold charts

$$(V', \Gamma', \varphi') \hookrightarrow (V, \Gamma, \varphi)$$

is a biholomorphic map $\lambda : V' \to V$ such that for all $\gamma' \in \Gamma'$ there exists some $\gamma \in \Gamma$ such that

$$\lambda \circ \gamma' = \gamma \circ \lambda. \tag{2.1}$$

A collection $\mathfrak{A} = \{(V_i, \Gamma_i, \varphi_i)\}_{i \in I}$ of n-dimensional orbifold charts is called n-dimensional orbifold atlas for X, if

(i) the open sets $\{\varphi(V_i)\}_{i \in I}$ form a basis for the topology of $\mathfrak{X}$, i.e. every open subset of $\mathfrak{X}$ can be written as a union of sets in $\{\varphi(V_i)\}_{i \in I}$;

(ii) given orbifold charts $(V_i, \Gamma_i, \varphi_i)$ for $U_i = \varphi_i(V_i)$, respectively $(V_j, \Gamma_j, \varphi_j)$ for $U_j = \varphi_j(V_j)$ with $U_i \subset U_j$, then there exists an embedding of orbifold charts $(V_i, \Gamma_i, \varphi_i) \hookrightarrow (V_j, \Gamma_j, \varphi_j)$, given by a biholomorphic map λ_{ji}. Moreover, for every other embedding of orbifold charts $(V_i, \Gamma_i, \varphi_i) \hookrightarrow (V_j, \Gamma_j, \varphi_j)$, given by a biholomorphic map λ'_{ji}, there is a unique element $\gamma_{ji} \in \Gamma_j$, such that $\lambda'_{ji} = \gamma_{ji} \circ \lambda_{ji}$;

(iii) for λ_{ji} is as in (ii), i.e. a biholomorphic map representing an embedding of orbifold charts, and any $\gamma_i \in \Gamma_i$, the composition $\lambda_{ji} \circ \gamma_i$ gives rise to another embedding of orbifold charts;

(iv) the composition of two biholomorphic maps, representing injections of orbifold charts, defines another injection of orbifold charts.

These axioms imply the following properties:

2.1 Lemma (compare [BS15, page 3] for the statement)**.** *Let $\mathfrak{A} = \{(V_i, \Gamma_i, \varphi_i)\}_{i \in I}$ be an orbifold atlas for $\mathfrak{X}$.*

(i) For any embedding of orbifold charts $(V_i, \Gamma_i, \varphi_i) \hookrightarrow (V_j, \Gamma_j, \varphi_j)$ with corresponding biholomorphic map λ_{ji} there is an isomorphism of groups $g_{ji} : \Gamma_i \to \Gamma_j$ satisfying

$$\lambda_{ji} \circ \gamma_i = g_{ji}(\gamma_i) \circ \lambda_{ji}. \tag{2.2}$$

(ii) For any $U_i \subset U_j \subset U_k$ and corresponding orbifold charts there is a unique $\gamma_{kji} \in \Gamma_k$ such that for the respective biholomorphic maps we obtain

$$\lambda_{kj} \circ \lambda_{ji} = \gamma_{kji} \circ \lambda_{ki}. \tag{2.3}$$

Proof. (i) Take $\gamma_i \in \Gamma_i$. By (2.1), there is an element $\gamma_j \in \Gamma_j$ such that $\lambda_{ji} \circ \gamma_i = \gamma_j \circ \lambda_{ji}$. Since λ_{ji} is biholomorphic, we may compose with its inverse on the right and obtain the uniqueness of γ_j for given γ_i. Consequently, there is a well-defined map of groups

$$g_{ji} : \Gamma_i \to \Gamma_j, \quad \gamma_i \mapsto \gamma_j = \lambda_{ji} \circ \gamma_i \circ \lambda_{ji}^{-1}.$$

We need to check that g_{ji} is an isomorphism. For this purpose, take $\gamma_i, \gamma_i' \in \Gamma_i$. Using (2.2) repeatedly, we obtain

$$g_{ji}(\gamma_i \circ \gamma_i') \circ \lambda_{ji} = (\lambda_{ji} \circ \gamma_i) \circ \gamma_i' = g_{ji}(\gamma_i) \circ (\lambda_{ji} \circ \gamma_i') = g_{ji}(\gamma_i) \circ g_{ji}(\gamma_i') \circ \lambda_{ji},$$

which shows that g_{ji} is a homomorphism of groups. It is clear that g_{ji} also is bijective.

(ii) By the definition of an orbifold atlas, $\lambda_{kj} \circ \lambda_{ji}$ and λ_{ki} correspond to embeddings of orbifold charts and can therefore be obtained from each other by composing with a unique element $\gamma_{kji} \in \Gamma_k$ as claimed. In fact, a simple calculation shows $\gamma_{kji} = \lambda_{kj} \circ \lambda_{ji} \circ \lambda_{ki}^{-1}$.

□

Two orbifold atlasses $\mathfrak{A}_1$ and $\mathfrak{A}_2$ for $\mathfrak{X}$ are said to be equivalent, if $\mathfrak{A}_1 \cup \mathfrak{A}_2$ again is an orbifold atlas for $\mathfrak{X}$. A pair $X = (\mathfrak{X}, [\mathfrak{A}])$ consisting of such a Hausdorff space X and an equivalence class $[\mathfrak{A}]$ of n-dimensional orbifold atlasses is called n-dimensional complex orbifold. Maximal atlasses exist and equivalence of orbifold atlasses can be rephrased by calling two atlasses equivalent if and only if they are contained in the same maximal atlas. Therefore we shall tacitly work with a maximal atlas. We make the following definition:

2.2 Definition (Standard conditions)**.** *An orbifold atlas $\mathfrak{A} = \{(V_i, \Gamma_i, \varphi_i)\}_{i \in I}$ is said to satisfy the standard conditions, if the following holds for every $i \in I$.*

(i) The set $\mathrm{Fix}(\Gamma_i) = \{z \in V_i \mid \gamma(z) = z \text{ for some } \gamma_i \in \Gamma_i\}$ of fixed points of elements in Γ_i is of codimension at least two.

(ii) The action of Γ_i is linear, i.e. $\Gamma_i < \mathrm{GL}(n, \mathbb{C})$.

2.3 Remark. *We may always assume the standard conditions for an orbifold atlas $\mathfrak{A}$ for $\mathfrak{X}$. See [Pri67, Proposition 6] for condition (i) and [Car57, Lemma 1] for condition (ii) and note that these are conditions on the cover of $\mathfrak{X}$ and we may have to refine the cover in order to obtain the standard conditions, but that each refinement is yet contained in the atlas $\mathfrak{A}$ since we are working with a maximal one.*

Therefore we will assume from now on that our choice of maximal atlas satisfies the standard conditions. Moreover, we will use the following notations throughout this chapter: $\mathfrak{A} = \{(U_i, \Gamma, \varphi_i)\}_{i \in I}$ is a maximal atlas, satisfying the standard conditions. The corresponding open sets are denoted by $U_i = \varphi_i(V_i)$. Moreover, since $U_i \cong V_i/\Gamma_i$, we will usually assume that $\varphi_i = \pi_i$, the projection, so that $U_i = V_i/\Gamma_i$.

Furthermore we want to remark that condition (ii) in our definition of orbifold charts forces our space X to have only isolated singularities.

For the following, denote by X_{reg} the regular locus of X, i.e. the set of all points $x \in \mathfrak{X}$ such that we can find a neighborhood $U = U(x) \subset \mathfrak{X}$ of x with trivial group in the orbifold chart corresponding to U. The set-theoretical complement of X_{reg} in $\mathfrak{X}$, i.e. the set of all points with non-trivial group in the orbifold charts, is denoted by X_{sing}. An orbifold chart with trivial group is a usual chart in the sense of complex manifolds. This implies that X_{reg} in fact is a complex manifold and also proves the existence of complex orbifolds, since every complex manifold X is a complex orbifold with $X = X_{\mathrm{reg}}$.

The aim of this chapter is to find a generalisation of the Kobayashi-Lübke inequality for complex orbifold surfaces. Kawamata has shown in [Kaw92] that Bogomolov's inequality holds for semistable reflexive sheaves on two-dimensional $\mathbb{Q}$-varieties, i.e. projective orbifold surfaces, but without considering the case of equality which is important for the proof of the main result in chapter three. This requires some knowledge on the differential geometry of complex orbifolds. Thus we will generalise our theory from chapter one in the following sections and obtain two equivalent principles for defining differential-geometric objects on orbifolds.

2.2 Differential forms and integration

Since we are mainly interested in the two-dimensional case, from now on only complex orbifold surfaces $S = (\mathfrak{S}, \mathfrak{A})$ will be considered. According to item (i) of our standard conditions, orbifold surfaces only have isolated singularities. Therefore, given any orbifold chart (V, Γ, φ), the isotropy group Γ_v of every point $v \in V,\ v \neq 0$ is trivial.

We start by introducing orbifold differential forms, following the original paper of Satake, [Sat56]; an equivalent definition is obtained later. Let (V, Γ, φ) be an orbifold chart for $U \subset \mathfrak{X}$. We denote by $\mathcal{A}^k(V)^\Gamma$ the module of Γ-invariant differential k-forms on V. If we choose an orbifold chart $(V_i, \Gamma_i, \varphi_i)$ for U_i and $(V_j, \Gamma_j, \varphi_j)$ an orbifold chart for U_j such that $U_i \subset U_j$, and furthermore an embedding $(V_i, \Gamma_i, \varphi_i) \hookrightarrow (V_j, \Gamma_j, \varphi_j)$ of orbifold chart with corresponding biholomorphic map $\lambda_{ji} : V_i \to V_j$, then every $\eta \in \mathcal{A}^k(U_j)^{\Gamma_j}$ pulls back under λ_{ji} to an element $\lambda_{ji}^* \eta \in \mathcal{A}^k(U_i)^{\Gamma_i}$. Since every such η is Γ_j-invariant and biholomorphic maps λ_{ji} corresponding to orbifold charts are only unique up to composition with elements in Γ_j, the asignment $\eta \mapsto \lambda_{ji}^* \eta$ does not depend on the choice of embedding of orbifold chart and corresponding map λ_{ji}. Thus we may define a homomorphism of modules

$$\psi_{ji} : \mathcal{A}^k(V_j)^{\Gamma_j} \to \mathcal{A}^k(V_i)^{\Gamma_i}, \quad \eta \mapsto \lambda_{ji}^* \eta \tag{2.4}$$

independent of the choice of embedding of orbifold charts λ_{ji}.

2.4 Lemma (Cocycle conditions for ψ_{ji})**.** *The following holds for $U_i \subset U_j \subset U_k$ and corresponding orbifold charts:*

(i) $\psi_{ii} = \mathrm{id}_{\mathcal{A}^k(V_i)^{\Gamma_i}}$.

(ii) $\psi_{ji}^{-1} = \psi_{ji}$.

(iii) $\psi_{ji} \circ \psi_{kj} = \psi_{ki}$.

Proof. After choosing embeddings of orbifold charts, the formulas follow directly from those for the biholomorphic functions that correspond to these embeddings. Using the independence of a specific choice of embeddings, the claim follows. □

The lemma above shows that the ψ_{ji} satisfy a cocycle condition and therefore we are able to construct a sheaf on $\mathfrak{S}$:

2.5 Definition (Sheaf of orbifold differential forms)**.** *The sheaf of orbifold k-forms $\mathcal{A}^k_{orb}$ on the topological space $\mathfrak{S}$ is the sheaf which asigns to every open subset $U_i = \varphi_i(V_i) \subset \mathfrak{S}$ the sheaf $\mathcal{A}^k(V_i)^{\Gamma_i}$ and to every inclusion $U_i \hookrightarrow U_j = \varphi_j(V_j)$ the restriction map ψ_{ji}.*

Notice that since the collection $\{U_i\}_{i \in I}$ of open sets $U_i = \varphi_i(V_i)$ form a basis for the topology of $\mathfrak{S}$, it is sufficient to define a sheaf on this sets. Orbifold differential forms are now defined in the usual way as global sections of the sheaf of orbifold differential forms.

The usual operations on differential forms generalise to orbifold differential forms. For example given $\gamma_i \in \Gamma_i$ and Γ_i-invariant differential forms η, ξ on V_i then

$$\gamma_i^*(d\eta) = d\gamma_i^*\eta = d\eta, \quad \text{and} \quad \gamma_i^*(\eta \wedge \xi) = \gamma_i^*\eta \wedge \gamma_i^*\xi = \eta \wedge \xi, \tag{2.5}$$

which shows that d and $\wedge$ are well-defined on invariant forms. Moreover for any biholomorphic map λ_{ji} corresponding to an embedding of orbifold charts,

$$d\lambda_{ji}^*\eta = \lambda_{ji}^* d\eta \tag{2.6}$$

and analogous for the wedge product. Consequently, gluing by means of the ψ_{ji}'s yields well-defined operations $d, \wedge$ on orbifold differential forms. Thus we are able to form the de Rham complex in cohomology for orbifold differential forms which we will denote by $H^*_{orb}(X)$.

2.6 Remark. *Satake proved in his original paper [Sat56] that every open cover of the underlying topological space of an orbifold has a refinement to an open cover with contractible intersections. Therefore the Mayer-Vietoris principle generalises to orbifolds and we may form Čech cohomology, cf. the construction of Čech cohomology in [BT82].*

The construction above illustrates the following principle for defining objects on orbifolds:

2.7 Principle. *Invariant objects on orbifold charts that satisfy a certain compatibility with respect to embeddings of orbifold charts give rise to global objects on orbifolds.*

We now prove a different characterisation of orbifold differential forms, illustrating another principle for defining objects on orbifolds. The subsequent in fact is the key point to generalise the Kobayashi-Lübke inequality to orbifolds as we will see later.

2.8 Theorem (Characterisation of orbifold differential forms via the smooth locus)**.** *The orbifold differential k-forms are exactly the differential k-forms η_{reg} on S_{reg} such that for every open subset $U \subset \mathfrak{S}$ and every orbifold chart (V, Γ, φ) for U, the form $\varphi^*|_{U \cap S_{\mathrm{reg}}}\eta_{reg}$ on $V \setminus \{0\}$ extends to a form on V.*

Proof. Step 1: Setup. Let $\{U_i\}_{i \Delta in I}$ be an open covering of $\mathfrak{S}$, such that for every of the open sets U_i there is an orbifold chart $(V_i, \Gamma_i, \varphi_i)$ and, moreover, every U_i contains exactly one singular point p_i, i.e. $U_i \cap S_{\mathrm{sing}} = \{p_i\}$, and such that the intersection of two such sets does not contain a singular point, $U_i \cap U_j \cap S_{\mathrm{sing}} = \emptyset$ for all $i \neq j$. Note that such a cover can always be constructed since the topology of $\mathfrak{S}$ is Hausdorff, our singularities are isolated and we are working with a maximal atlas. Further we may assume without loss of generality that $U_i = V_i/\Gamma_i$, i.e. that the φ_i are exactly the quotient maps $\pi_i : V_i \to V_i/\Gamma_i$.

Step 2: Local construction. Given any $\eta \in \mathcal{A}^k_{orb}(S)$, defined by a collection $\{\eta_i\}_{i \in I}$ with $\eta_i \in \mathcal{A}^k(V_i)^{\Gamma_i}$, we may fix the index $i \in I$ for this step. For every point $0 \neq v \in V_i$, we have trivial isotropy group $(\Gamma_i)_v$. Therefore, the action of Γ_i on $V_i \setminus \{0\}$ is free and properly discontinuous. Consequently, $\pi_i|_{V_i \setminus \{0\}}$ is a locally biholomorphic, unbranched, $\mathrm{ord}(\Gamma_i)$-sheeted covering map. Take $W \subset V_i \setminus \{0\}$, such that $\pi_i|_W$ is biholomorphic and let $W' := \pi_i|_W(W) \subset S_{\mathrm{reg}}$ be its image. (In fact, $W' = U_i \cap S_{\mathrm{reg}}$.) Then, since $\pi_i|_W$ is biholomorphic, we find a k-form $\eta_{reg,i} \in \mathcal{A}^k(W')$, such that

$$\pi_i|_W^* \eta_{reg,i} = \eta_i|_W.$$

But since η_i is Γ_i-invariant and the action of Γ_i only permutes the connected components of $\pi_i^{-1}(W')$, $\pi_i|_W^*\eta_{reg,i}$ actually defines a form on $V_i \setminus \{0\}$ (defined equally on every connected component of $\pi^{-1}(W')$). Moreover, assigning to $\pi_i^*(\eta_{reg,i})$ the same value in zero as η_i attains, we see that $\pi_i^*\eta_{reg,i}$ extends to a well-defined form $\widehat{\eta_i}$ on V_i.

Step 3: Extension to a global form. Given $U_i \neq U_j$, then $U_i \cap U_j \cap S_{\text{sing}} = \emptyset$ by the assumptions on our cover and therefore

$$\eta_{reg,i}|_{U_i \cap U_j} = \eta_{reg,j}|_{U_i \cap U_j}.$$

Thus, the collection $\{\eta_{reg,i}\}_{i\in I}$ defines a well-defined form $\eta_{reg} \in \mathcal{A}^k(S_{\text{reg}})$.

Step 4: Second implication, end of proof. Conversely, given $\eta_{reg} \in \mathcal{A}^k(S_{\text{reg}})$ such that $\pi_i|_{U_i\cap S_{\text{reg}}}^*\eta_{reg}$ extends to a form $\widehat{\eta_i}$ on V, then $\widehat{\eta_i}$ is Γ_i-invariant as pullback by the quotient map and the collection $\{\widehat{\eta_i}\}_i$ defines a global orbifold differential form. $\square$

This theorem proves, on the level of differential forms, the equivalence of Principle 2.7 and the following principle.

2.9 Principle. *Objects on S_{reg} whose pullback extends on every orbifold chart give rise to well-defined global objects on S.*

We will see more examples illustrating the equivalence between both principles later on. Our next step is to define integration of top forms on orbifolds and to compare it with the usual measure theoretic integration.

2.10 Definition (Orbifold orientation)**.** *An orbifold orientation is a collection of nowhere vanishing invariant differential forms of top degree two on the orbifold charts that define an orbifold differential form.*

As for manifolds, complex orbifolds are always orientable and from now on we will assume that we have fixed some orbifold orientation. Since the underlying topological space of every orbifold is Hausdorff and satisfies the second countability axiom, it is paracompact. Henceforth, partitions of unity exist for orbifolds and enable us to define integration of orbifold differential forms in the following way, cf. [Sat56]: Let $\eta \in \mathcal{A}^4_{orb}(S)$.

(i) If there is an open set $U \subset \mathfrak{S}$ with orbifold chart (V, Γ, φ) such that the support of η is entirely contained in U, we define

$$\int_S^{orb} \eta := \frac{1}{\operatorname{ord}(\Gamma)} \int_V \eta,$$

where we identify η with its image under the canonical isomorphism $\mathcal{A}^4_{orb}(U) \to \mathcal{A}^4(V)^\Gamma$.

(ii) For a general two-form η, take a locally finite partition of unity $\{\rho_i\}_{i\in I}$, subordinate to the cover and define

$$\int_S^{orb} \eta := \sum_i \int_S^{orb} \rho_i \cdot \eta.$$

This is well-defined:

2.11 Lemma. *The value of $\int_S^{orb} \eta$ does not depend on the covering of $\mathfrak{S}$ or the choice of a partition of unity.*

Proof. Same as for manifolds. $\square$

2.12 Remark. *Having defined a notion of integration on orbifolds, most of the classical theorems in homology and cohomology can be proved by using this integral. For example, the orbifold Čech cohomology with values in $\mathbb{C}$ is isomorphic to the orbifold de Rham cohomology. We want to emphasise that, since we are dividing by the group order in the definition of the integral, statements like the Leray-Hirsch theorem only hold with coefficients in $\mathbb{Q}$ and in general fail for coefficients in $\mathbb{Z}$. We refer to [Bla96] for a more detailed discussion on the (co-)homology of orbifolds and proofs of the basic theorems.*

As we have seen in Theorem 2.8, orbifold differential forms correspond to differential forms on the smooth locus that extend after pullback. But on the smooth locus we also have the ordinary measure theoretic integration on manifolds, given via the formula

$$\int_{U\cap S_{\mathrm{reg}}} \eta_{reg} = \int_{V\setminus\{0\}} \pi|^*_{U\cap S_{\mathrm{reg}}} \eta_{reg}.$$

Now, since $\pi|^*_{U\cap S_{\mathrm{reg}}} \eta_{reg}$ extends to a form $\widehat{\eta}$ that can be integrated with finite value over V, the integral above is finite and we may integrate into the singular point to locally obtain

$$\int_{U\cap S_{\mathrm{reg}}} \eta_{reg} = \int_V \widehat{\eta} = \mathrm{ord}(\Gamma) \cdot \int_U^{orb} \eta \tag{2.7}$$

and thus globally with some factor $M > 0$

$$\int_{S_{\mathrm{reg}}} \eta_{reg} = M \cdot \int_S^{orb} \eta, \tag{2.8}$$

i.e. orbifold integration up to a factor coincides with measure theoretic integration in this case. One should note the subtile point that we are only allowed to extend the integral into the singular point after having proved that the differential form extends into this point, but that this also yields finiteness of the integral over S_{reg}, since it coincides with the orbifold integral which is finite.

This observation is of great importance for generalising the Kobayashi-Lübke inequality to orbifold surfaces, since we will be able to restrict ourselves to the smooth locus where we already know that the inequality holds.

2.3 Sheaves and bundles

We now generalise the way we defined the sheaf of orbifold differential forms to define general sheaves on orbifolds. Our basic reference is [Bla96]. Notice that it makes little sense to talk of vector bundles on orbifolds, since they are smooth objects and orbifolds in general have singularities. Therefore we use the language of sheaves that is more flexible and fits our needs due to the fact that sheaves are allowed to be singular. However, we will also give a definition of the right notion of vector bundles on orbifolds using sheaf theory even though such objects are not vector bundles in the usual sense and we only use this terminology to point at the parallels with the theory established in chapter one.

2.13 Definition (Γ-equivariant sheaf, Γ-equivariant section)**.** *Let V be a smooth complex manifold and Γ a finite group acting on V via automorphisms. A Γ-equivariant sheaf, for brevity Γ-sheaf, is a sheaf $\mathcal{E}$ on V such that* *(i) for every $\gamma \in \Gamma$ there is an isomorphism $\Phi_\gamma : \gamma^*\mathcal{E} \to \mathcal{E}$ and*

(ii) for $\gamma, \gamma' \in \Gamma$ the isomorphisms from (i) respect the associativity in the group, i.e.

$$\Phi_\gamma \circ \gamma^*(\Phi_{\gamma'}) = \Phi_{\gamma' \circ \gamma}. \tag{2.9}$$

An equivariant section $s : V \to \mathcal{E}$ is a section satisfying

$$\Phi_\gamma(s(\gamma(v)) = s(v) \tag{2.10}$$

for all $\gamma \in \Gamma$ and $v \in V$.

The set of coherent Γ-sheaves on X is denoted by $\mathrm{Coh}^\Gamma(X)$.

If $\mathcal{E}$ is any Γ-sheaf on V, then Γ acts in an obvious way on $\pi_*(\mathcal{E})$. We denote by $\pi_*^\Gamma\mathcal{E} = \pi_*(\mathcal{E})^\Gamma$ the maximal subsheaf of $\pi_*\mathcal{E}$ on which Γ acts trivially. If the action of Γ on V is free, then $\pi^*\pi_*^\Gamma(\mathcal{E}) = \mathcal{E}$, but since we have non-trivial isotropy group at $0 \in V$ in general, $\pi_*^\Gamma\mathcal{E}$ does not need to be torsion-free.

2.14 Definition (Orbifold sheaf). *An orbifold sheaf $\mathcal{E}$ on the orbifold X is a collection of Γ_i-sheaves $\mathcal{E}_i$ on the V_i such that*

(i) for every biholomorphic map $\lambda_{ji} : V_i \to V_j$, corresponding to an embedding of orbifold charts $(V_i, \Gamma_i, \varphi_i) \hookrightarrow (V_j, \Gamma_j, \varphi_j)$, there is an isomorphism $\psi_{ji} : \lambda_{ji}^*\mathcal{E}_j \to \mathcal{E}_i$;

(ii) the isomorphisms ψ_{ji} satisfy a cocycle condition, $\psi_{ki} = \psi_{ji} \circ \psi_{kj}$.

A section of orbifold sheaf is a choice of Γ_i-invariant sections $s_i \in \Gamma(V_i, \mathcal{E}_i)^{\Gamma_i}$, compatible with the isomorphisms ψ_{ji}, i.e. that satisfy $\lambda_{ji}^*(s_j) = s_i$. In other words, sections of orbifold sheaves are sections of $(\pi_i)_*^{\Gamma_i}\mathcal{E}_i$.

Again, since the open sets $\{U_i\}_{i \in I}$ form a basis for the topology of $\mathfrak{S}$, defining orbifold sheaves in this way is sufficient. Next, we introduce free and locally free orbifold sheaves.

2.15 Definition (Free orbifold sheaf). *A coherent sheaf $\mathcal{E}$ on $U_i \subset \mathfrak{S}$ is called free in the orbifold sense (or orbifold free), if there is a free sheaf $\mathcal{E}_i$ on V_i such that $\mathcal{E} = \pi_*^{\Gamma_i}\mathcal{E}_i$.*

2.16 Proposition (cf. [Bla96, Item 2.4] for the statement). *A coherent sheaf $\mathcal{E}$ on $U \subset \mathfrak{S}$ is orbifold free if and only if it is reflexive.*

Proof. Let $\mathcal{E}$ be a coherent sheaf on $U \subset \mathfrak{S}$ with orbifold chart (V, Γ, φ), orbifold free, and let $\widehat{\mathcal{E}}$ be the free sheaf on V satisfying $\pi_*^\Gamma\widehat{\mathcal{E}} = \mathcal{E}$. Then, since $\pi : V \to V/\Gamma$ is finite and étale over $U \cap S_{\mathrm{reg}}$, π_* is exact and so is π_*^Γ and thus the reflexivity of $\mathcal{E}$ follows from that of $\widehat{\mathcal{E}}$.

Conversely, given a reflexive sheaf $\mathcal{E}$ on $U_i \subset \mathfrak{S}$, define $\mathcal{E}_i := \pi^{[*]}\mathcal{E} := (\pi^*\mathcal{E})^{\vee\vee}$. Then, recalling that on smooth surfaces every reflexive sheaf is locally free and reverse, we may assume after shrinking V_i that $\mathcal{E}_i$ is free. Moreover, as it is the pullback by a quotient map, there is an action of Γ_i on $\mathcal{E}_i$ and $\pi_*^{\Gamma_i}\pi^{[*]}\mathcal{E} = \mathcal{E}^{\vee\vee} = \mathcal{E}$, since $\mathcal{E}$ was assumed to be reflexive. □

2.17 Definition (Locally free orbifold sheaf). *An orbifold sheaf $\mathcal{E}$ on $\mathfrak{S}$ is called locally free, if for every point $x \in \mathfrak{S}$ there is a neighborhood $U = U(x) \subset \mathfrak{S}$ of x such that $\mathcal{E}|_U$ is orbifold free.*

Recalling the standard fact that reflexive sheaves coincide if they coincide in codimension two, cf. [Har80, Proposition 1.6], and that $\operatorname{codim}_X X_{\text{sing}} \geq 2$, we obtain the following: If $\iota : S_{\text{reg}} \hookrightarrow \mathfrak{S}$ denotes the inclusion, then every locally free orbifold sheaf $\mathcal{E}$ equals $\iota_* \mathcal{E}|_{S_{\text{reg}}}$. Moreover, since $\mathcal{E}_{reg} := \mathcal{E}|_{S_{\text{reg}}}$ is reflexive, it is a locally free sheaf on the complex manifold S_{reg}. Thus, every locally free orbifold sheaf comes from a locally free sheaf on S_{reg}. Moreover, given any reflexive sheaf $\mathcal{E}_{reg}$ on S_{reg}, then $\iota_* \mathcal{E}_{reg}$ is again reflexive, cf. [Har80], therefore a locally free orbifold sheaf. Consequently, we may have started with locally free sheaves on S_{reg}, which again illustrates the equivalence of our principles 2.7 and 2.9 (note that the pullback of a locally free sheaf to the charts is locally free by Proposition 2.16).

2.18 Definition (Holomorphic orbifold vector bundle)**.** *A holomorphic orbibundle E on S is a holomorphic vector bundle E_{reg} on S_{reg} such that $\iota_*(\mathcal{O}(E_{reg}))$ is a locally free orbifold sheaf, where $\iota : S_{\text{reg}} \to \mathfrak{S}$ denotes the inclusion.*

The following lemma should be seen as a generalisation of Theorem 2.8 to arbitrary orbifold sheaves and now is, using the considerations above, easy to prove.

2.19 Lemma (cf. the remark on page 22 of [Bla96] for the statement)**.** *E is a holomorphic orbifold vector bundle on S if and only if for every point $x \in \mathfrak{S}$ and every neighborhood U of x with orbifold chart (V, Γ, φ) there exists a holomorphic vector bundle $\widehat{E}$ on V such that Γ acts on $\widehat{E}$ and $(\widehat{E}|_{V \setminus \{0\}})/\Gamma \cong E_{reg}|_{U \cap X_{\text{reg}}}$.*

Proof. Let E be a holomorphic vector bundle in the sense of Definition 2.18. For every point $x \in \mathfrak{S}$ we may choose a neighborhood U of x in $\mathfrak{S}$ and an orbifold chart (V, Γ, φ) for U such that $\mathcal{E} := \iota_*(\mathcal{O}(E_{reg}))|_U$ is a free orbifold sheaf. According to Proposition 2.16, there is a free sheaf $\widehat{\mathcal{E}}$ on V, together with an action of Γ, that satisfies $\mathcal{E} = \pi_*^{\Gamma} \widehat{\mathcal{E}}$. Since $\widehat{\mathcal{E}}$ is free, there is a holomorphic vector bundle $\widehat{E}$ on V such that $\widehat{\mathcal{E}} = \mathcal{O}(\widehat{E})$. The equivariant action of Γ on $\widehat{\mathcal{E}}$ induces an equivariant action on $\widehat{E}$. As we have seen before, Γ acts free and properly discontinuous on $V \setminus \{0\}$. Therefore, $\widehat{E}|_{V \setminus \{0\}}$ descends to the quotient $(V \setminus \{0\})/\Gamma \cong U \cap S_{\text{reg}}$, which is a smooth complex manifold, as vector bundle. It is clear by construction that this vector bundle coincides with $E_{reg}|_{U \cap S_{\text{reg}}}$.

Conversely, assume that we are given a holomorphic orbibundle as in the statement of this lemma. We have to check that $\iota_*(\mathcal{O}(E_{reg}))$ is a locally free orbifold sheaf. Take an open neighborhood U of some point $x \in \mathfrak{S}$, an orbifold chart (V, Γ, φ) for U and $\widehat{E}$ such that $(\widehat{E}|_{V \setminus \{0\}})/\Gamma \cong E_{reg}|_{U \cap S_{\text{reg}}}$. Let $\widehat{\mathcal{E}} = \mathcal{O}(\widehat{E})$ be the sheaf of holomorphic sections of $\widehat{E}$. Then the equivariant action of Γ on $\widehat{E}$ induces an action on $\widehat{\mathcal{E}}$. $\mathcal{E} = \pi_*^{\Gamma} \widehat{\mathcal{E}}$ is a free orbifold sheaf and reflexive. Moreover, $\iota_*(\mathcal{O}(E_{reg}))$ and $\mathcal{E}$ coincide on S_{reg}, and since they are both reflexive and the codimension of the singular locus is at least two, they coincide everywhere. Thus, $\iota_*(\mathcal{O}(E_{reg}))$ is a free orbifold sheaf on U and therefore a locally free orbifold sheaf as we claimed. $\square$

2.20 Remark. *According to Lemma 2.19, we could also have defined orbibundles to be a collection of vector bundles on the orbifold charts, equivariant with respect to the corresponding group action, and satisfying a compatibility condition. In this definition, one is also able to speak of transition functions of the bundle and to develop a local theory similiar to those in chapter one.*

For completeness, let us remark that if $\mathcal{E}$ is any locally free orbifold sheaf, there is an orbibundle such that $\mathcal{E}$ is isomorphic to the sheaf of sections of this bundle. Then the orbibundle is unique up to isomorphisms.

2.4 Generalisation of the Kobayashi-Lübke inequality

Now we have established all theory we need to generalise the Kobayashi-Lübke inequality to orbifold surfaces. However, some more notions are needed in order to formulate the statement precisely.

2.21 Definition (Hermitian orbifold metric)**.** *Let E be an orbibundle. A hermitian orbifold metric h on E is a hermitan metric h on E_{reg} such that given any orbifold chart (V, Γ, φ) for $U \subset X$, $\pi|_{U\cap X_{\mathrm{reg}}}^{*} h$ extends to a hermitian metric $\widehat{h}$ on $\widehat{E}$.*

Given any holomorphic orbibundle E and orbifold hermitian metric h on E, all objects discussed in chapter one are defined with respect to the hermitian metric h on the holomorphic vector bundle E_{reg}. In particular, there are Chern forms $c_k(E_{reg}, h) \in \mathcal{A}^{2k}(X_{\mathrm{reg}}, E_{reg})$. In order to define orbifold chern forms out of these data, we have to check that they extend on every orbifold chart. This is done within the following lemma.

2.22 Lemma. *Let (V, Γ, φ) be an arbitrary orbifold chart for some $U \subset X$. Then, $\pi|_{U\cap S_{\mathrm{reg}}}^{*} c_k(E_{reg}, h)|_{U\cap S_{\mathrm{reg}}}$ extends to a form on V, invariant under the action of Γ.*

Proof. Recall from (1.26) that, after possibly shrinking U, we can write

$$c_k(E_{reg}, h)|_{U\cap S_{\mathrm{reg}}} = \sum \delta_{k_1\cdots k_l}^{j_1\cdots j_l} \Omega_{reg,k_1}^{j_1} \wedge \cdots \wedge \Omega_{reg,k_l}^{j_l},$$

where $\Omega_{reg} = (\Omega_{reg,k}^{j})_{j,k}$ is the curvature matrix of the Chern connection associated to h_{reg}. Thus it is sufficient to prove that $\pi|_{U\cap S_{\mathrm{reg}}}^{*} \Omega_{reg,k}^{j}$ extends to V for every j, k or, equivalently, that $\pi|_{U\cap S_{\mathrm{reg}}}^{*} \Omega_{reg}$ extends to V. But this reduces, using (1.22) and (1.24), to extending $\pi|_{U\cap S_{\mathrm{reg}}}^{*} h_{reg}$ to V which holds by definition for the orbifold hermitian metric h. □

Thus we have well-defined and Γ-invariant forms $\widehat{c_k}(\widehat{E}, \widehat{h})$ which define the orbifold Chern forms $c_k^{orb}(E, h)$. In fact, since our methods from chapter one are applicable on S_{reg} as well as on all orbifold charts, one can prove that they define cohomology classes, independent of the choice of metric.

2.23 Theorem (Orbifold Kobayashi-Lübke inequality)**.** *Let E be a holomorphic orbibundle of rank r, Hermite-Einstein with respect to the orbifold hermitian metric h on E and the orbifold Kähler form ω. Then,*

$$\int_S^{orb} (r-1) c_1^{orb}(E, h)^2 + 2r c_2^{orb}(E, h) \leq 0. \tag{2.11}$$

Moreover, in the case of equality, E_{reg} is unitary flat.

Proof. Using (2.8) and the finiteness of the orbifold integral, we obtain

$$M \int_S^{orb} (r-1)c_1^{orb}(E,h)^2 + 2rc_2^{orb}(E,h) = \int_{S_{\mathrm{reg}}} (r-1)c_1(E_{reg},h)^2 + 2rc_2(E,h) \leq 0,$$

since now we are integrating regular Chern classes and therefore the Kobayashi-Lübke inequality, Theorem 1.19, applies. Assume that equality holds. Then equality holds on S_{reg} and, again by Theorem 1.19, E_{reg} is unitary flat. □

One may be wondering why this result is slightly weaker than the result obtained in case of equality in the original Kobayashi-Lübke inequality. Basicly, orbifold Chern forms are a weaker notion than usual Chern forms, since information only is provided by the regular locus and not the singular locus. In fact, when integrating differential forms on orbifolds, the integral does not consider the value in the singular points, which is on one hand the key point for proving this theorem but on the other hand yields a weaker result. Anyways, chapter three will prove that the result is stronger for projective orbifolds.

3 Characterisation of finite quotients of Abelian varieties

This section is dedicated to the proof of the main result. We start by introducing the notion of $\mathbb{Q}$-Cartier divisors and then four different types of singularities.

Let X be a normal, projective variety. We say that a divisor D on X is $\mathbb{Q}$-Cartier, if there exists an $m \in \mathbb{N}$ such that mD is Cartier. Two $\mathbb{Q}$-Cartier divisors D_1 and D_2 are said to be linear equivalent, $D_1 \sim D_2$, if there is an integer $m \in \mathbb{N}$ such that mD_1 and mD_2 are Cartier and linear equivalent.

Let K_X be $\mathbb{Q}$-Cartier and $\pi : \widehat{X} \to X$ be a birational morphism from a smooth projective variety $\widehat{X}$ to X, i.e. a resolution of singularities. If we consider the ramification formula

$$K_{\widehat{X}} \sim \pi^* K_X + \sum_i a_i E_i \quad \text{with prime components } E_i, \tag{3.1}$$

we say that X has at most

(i) canonical singularities if $a_i \geq 0$ holds for all i;

(ii) terminal singularities if $a_i > 0$ holds for all i;

(iii) log canonical singularites if $a_i \geq -1$ holds for all i;

(iv) Kawamata log terminal (klt) singularities if $a_i > -1$ holds for all i.

A normal, complex projective variety X is called a klt-variety, if it has at most klt-singularities. The dimension of a klt-variety will always be doneted by n, i.e. $n = \dim(X)$.

We aim to prove the following generalisation of Theorem 1.30 for klt spaces, wherein notions like Chern classes on klt varieties will be defined later in this chapter.

3.1 Theorem (Characterisation of quotients of Abelian varieties)**.** *Let X be a normal, projective variety. Then the following are equivalent:*

(i) X has at most klt singularities, $K_X \equiv 0$ and $\widehat{c}_2(\mathcal{T}_X) \cdot H^{n-2} = 0$ for all ample divisors H on X.

(ii) There exists an Abelian variety A and a finite, surjective morphism $\gamma : A \to X$ that is étale in codimension one.

The proof consists of two major steps. First, we will construct a complete intersection surface S, having orbifold structure and prove that, under the assumption (i) of the theorem, the restriction of the tangent sheaf to S, $\mathcal{T}_X|_S$ is flat and locally free over S_{reg}. Second, we will prove that flat, locally free sheaves on the regular locus of such a complete intersection surface S extend to flat, locally free sheaves on X after possibly passing to a quasi-étale cover of X. Then, applying the confirmation of the Lipman-Zariski conjecture for klt spaces, X has to be smooth and the result will follow from Theorem 1.30.

Since we are in the following working both with the Zariski and the analytic topology, we denote by X the space equipped with the Zariski topology and by X^{an} the space equipped with the analytic topology. We are consequently considering also algebraic and analytic sheaves, and prepend the following remark, allowing us to interchange those notions fluently in the subsequent discussion.

3.2 Remark (Interchanging analytic and algebraic sheaves)**.** *We argue as follows.*

(i) Flatness of sheaves or vector bundles as defined in chapter one is a purely analytic notion. An algebraic sheaf is called flat, if its associated analytic sheaf is flat, i.e. given by a representation of the topological fundamental group. A fundamental theorem of Deligne, [Del70, II, Corollary 5.8 and Theorem 5.9], implies that every flat analytic sheaf on X^{an}_{reg} *is a flat, algebraic sheaf on* X_{reg}*. This allows us to speak of flat sheaves on* X_{reg}*, respectively* X^{an}_{reg}*, without distinguishing between algebraic and analytic sheaves.*

(ii) When working on X *or on a complete intersection surface* S*, the GAGA theorems of Serre, [Ser56], show that there is no need to distinguish between analytic and algebraic sheaves, either.*

Consequently, we only speak of sheaves and implicitly use the results above to identify algebraic and analytic sheaves.

3.1 Reducing to orbifold surfaces

We will state the following well-known fact that klt spaces have orbifold structure in codimension two. One may consider [GKK11, Proposition 9.3] for a detailed proof of this result.

3.3 Theorem (Klt spaces have orbifold structure in codimension 2)**.** *Let* X *be a klt variety. Then there exists a closed subset* $Z \subset X$ *with* $\mathrm{codim}_X Z \geq 3$ *such that* $X \setminus Z$ *has the structure of a complex orbifold with respect to the Euclidean topology.*

There is a well-defined notion of Chern-classes of reflexive sheaves on klt spaces. The actual construction is technically challenging and we refer to [Mum83] for details and denote the k-th Chern class of a reflexive sheaf $\mathcal{E}$ on X by $\widehat{c}_k(\mathcal{E})$. An important property for our purposes is that Chern classes on klt spaces can be calculated from the orbifold Chern classes introduced in chapter two as follows: Let $\mathcal{E}$ be a reflexive sheaf on X and H a very ample divisor. For $m \gg 0$ such that mH is ample and $|mH|$ is basepoint free, we may take general elements $D_1, \ldots, D_{n-1} \in |mH|$ and define the complete intersection surface

$$S := D_1 \cap \cdots \cap D_{n-2}, \tag{3.2}$$

as well as the complete intersection curve

$$C := D_1 \cap \cdots \cap D_{n-1}.$$

Since H is assumed to be ample, we may assume $S \subset X \setminus Z$, where Z is the non-orbifold locus from Theorem 3.3 and that C is smooth, since X is assumed to be normal and therefore has singularities only in codimension at least two. Therefore, the orbifold structure of $X \setminus Z$ induces an orbifold structure on S and the restriction $\mathcal{E}|_S$ of the reflexive sheaf $\mathcal{E}$ to S is reflexive, [Gro66, Theorem 12.2.1] and thus, by Proposition 2.16 and Lemma 2.19, an orbifold vector bundle. Consequently, we may form the orbifold Chern classes $c_k^{orb}(\mathcal{E}|_S)$. For our convenience are now the following identities,

$$\begin{aligned} \widehat{c}_1(\mathcal{E})^2 \cdot H^{n-2} &= \frac{1}{m^{n-2}} \int_S^{orb} c_1^{orb}(\mathcal{E}|_S)^2 \in \mathbb{Q}, \\ \widehat{c}_2(\mathcal{E}) \cdot H^{n-2} &= \frac{1}{m^{n-2}} \int_S^{orb} c_2^{orb}(\mathcal{E}|_S) \in \mathbb{Q}, \end{aligned} \tag{3.3}$$

and, using smoothness of C in the second equality,

$$\widehat{c}_1(\mathcal{E}) \cdot H^{n-1} = \frac{1}{m^{n-1}} \int_C^{orb} c_1^{orb}(\mathcal{E}|_C) = \frac{1}{m^{n-1}} \int_C c_1(\mathcal{E}|_C), \tag{3.4}$$

see [GKPT15, Theorem 4.1] for a proof. They show that Chern classes on klt spaces can be calculated by integration of orbifold Chern classes. The following corollary is immediate from this property, Theorem 2.23 and the Mehta-Ramanathan theorem for normal spaces, [Fle84, Theorem 1.2].

3.4 Corollary (Bogomolov inequality for stable sheaves on klt spaces)**.** *For every stable reflexive sheaf $\mathcal{E}$ on a klt variety X and every ample divisor H on X the Chern number inequality*

$$\big((\mathrm{rank}(\mathcal{E}) - 1)\widehat{c}_1(\mathcal{E})^2 + 2\mathrm{rank}(\mathcal{E})\widehat{c}_2(\mathcal{E})\big) \cdot H^{n-2} \leq 0$$

holds.

Proof. Let S be a complete intersection surface constructed from general elements in $|mH|$, where m is chosen large enough such that $|mH|$ is basepoint free and the restriction of $\mathcal{E}$ to S remains stable, cf. [Fle84, Theorem 1.2]. The orbifold structure on $X \setminus Z$, where Z is the non-orbifold locus of X, cf. Theorem 3.3, induces an orbifold structure on S. According to [SW01, Theorem 2.3], Theorem 1.28 generalises to orbifolds. Therefore, $\mathcal{E}|_S$ is an orbifold Hermitian-Einstein bundle with respect to the orbifold Kähler form of S and we may use the identities (3.3) together with Theorem 2.23 to calculate

$$\begin{aligned} &((\mathrm{rank}(\mathcal{E}) - 1)\widehat{c}_1(\mathcal{E})^2 + 2\mathrm{rank}(\mathcal{E})\widehat{c}_2(\mathcal{E})) \cdot H^{n-2} \\ &\qquad = \frac{1}{m^{n-2}} \underbrace{\int_S^{orb} (\mathrm{rank}(\mathcal{E}) - 1)c_1^{orb}(\mathcal{E}|_S)^2 + 2\mathrm{rank}(\mathcal{E})c_2^{orb}(\mathcal{E}|_S)}_{\leq 0 \text{ by Theorem 2.23}} \leq 0. \end{aligned}$$

□

3.2 Extension of flat sheaves

For the considerations in this section it is convenient to pass to a specific quasi-étale cover of the original space X. We will argue in the proof of the main theorem later on that this does not limit the generality of our result. We start by briefly recalling the definition of an quasi-étale morphism.

3.5 Definition (Quasi-ètale morphism)**.** *A morphism $\gamma : \widetilde{X} \to X$ between normal varieties is called quasi-ètale, if it is quasi-finite and ètale in codimension one. In other words, γ is quasi-ètale if $\widetilde{X}$ and X have the same dimension and if there exists a closed subset $Z \subset X$ of codimension at least two such that $\gamma|_{\widetilde{X}\setminus Z} : \widetilde{X} \setminus Z \to X$ is ètale.*

We invoke the following theorem on the existence of a quasi-étale cover suitable for our considerations.

3.6 Theorem ([GKP16b, Theorem 1.5])**.** *Let X be normal, complex, quasi-projective and klt. Then, there exists a normal variety $\widetilde{X}$ and a finite, surjective Galois morphism $\gamma : \widetilde{X} \to X$ that is étale in codimension one such that the natural inclusion of the smooth locus $\iota : \widetilde{X}^{an}_{\mathrm{reg}} \to \widetilde{X}^{an}$ induces an isomorphism $\widehat{\iota}_* : \widehat{\pi}_1(\widetilde{X}^{an}_{\mathrm{reg}}) \to \widehat{\pi}_1(\widetilde{X}^{an})$ of étale fundamental groups.*

Using this cover we obtain the following extension theorem for flat, locally free sheaves, see [GKP16b, Theorem 1.14].

3.7 Theorem (Extension of flat, locally free sheaves, I)**.** *Let X be normal, complex, quasi-projective and klt. Then, after passing to the quasi-étale cover $\widetilde{X}$ from Theorem 3.6, the following holds: If $\mathcal{G}$ is any flat, locally free sheaf on $\widetilde{X}^{an}_{\mathrm{reg}}$ then there exists a flat, locally free sheaf $\mathcal{F}$ on $\widetilde{X}^{an}$ such that $\mathcal{G}$ is isomorphic to $\mathcal{F}|_{\widetilde{X}^{an}_{reg}}$.*

Proof, see [GKP16b, Theorem 1.14]. Denote the inclusion of the smooth locus by $\iota : \widetilde{X}^{an}_{reg} \to \widetilde{X}^{an}$. By assumption on $\widetilde{X}$, respectively Theorem 3.6, the induced morphism $\widehat{\iota} : \widehat{\pi}_1(\widetilde{X}^{an}_{\mathrm{reg}}) \to \widehat{\pi}_1(\widetilde{X}^{an})$ is an isomorphism. As $\mathcal{G}$ is flat, it comes from a representaion $\rho : \pi_1(\widetilde{X}^{an}_{reg}) \to \mathrm{GL}(\mathrm{rank}(\mathcal{G}), \mathbb{C})$ of the fundamental group $\pi_1(\widetilde{X}^{an}_{reg})$. Let $G := \mathrm{img}(\rho)$. It is well known that the fundamental group $\pi_1(\widetilde{X}^{an}_{reg})$ is finitely generated and as G is a quotient of $\pi_1(\widetilde{X}^{an}_{reg})$, it is also finetely generated and, as a subgroup of the general linear group, residually finite by Malcev's Theorem, [Weh73, Theorem 4.2]. Hence, if we denote the profinite completion morphism by $a : G \to \widehat{G}$, it is injective by [RZ10, p. 78].

To construct an extension of $\mathcal{G}$ to a locally free, flat sheaf on $\widetilde{X}^{an}$, we have to construct a representation $\tau : \pi_1(\widetilde{X}^{an}) \to \mathrm{GL}(\mathrm{rank}(\mathcal{G}), \mathbb{C})$ that, after restricting to $\widetilde{X}^{an}_{reg}$, coincides with ρ. This is, we need to find a factorisation

$$\begin{tikzcd} \pi_1(\widetilde{X}^{an}_{\mathrm{reg}}) \arrow[r, "\iota_*"] \arrow[rr, bend right, "\rho"'] & \pi_1(\widetilde{X}^{an}) \arrow[r, "\tau"] & G. \end{tikzcd} \tag{3.5}$$

First recall that the étale fundamental group is the profinite completion of the topological fundamental group, see for example [Mil80, §5]. Moreover, due to [RZ10, Lemma 3.2.3], taking profinite completion is functorial. Consequently, there is a commutative diagram

$$\begin{tikzcd} \widehat{G} & \widehat{\pi}_1(\widetilde{X}^{an}_{\mathrm{reg}}) \arrow[l, "\widehat{\rho}"'] \arrow[r, "{\widehat{\iota}_*,\ \text{isomorphic}}"] & \widehat{\pi}_1(\widetilde{X}^{an}) \\ G \arrow[u, hook, "{a,\ \text{injective}}"] & \pi_1(\widetilde{X}^{an}_{\mathrm{reg}}) \arrow[l, "\rho"'] \arrow[u, "b"] \arrow[r, two heads, "{\iota_*,\ \text{surjective}}"] & \pi_1(\widetilde{X}^{an}), \arrow[u, "c"'] \end{tikzcd}$$

where the morphisms b and c are the natural profinite completion morphisms; the surjectivity of ι_* follows from [Kol95, Proposition 2.10]. Going through the diagram, we may define

$\tau := \widehat{\rho} \circ \widehat{\iota_*}^{-1} \circ c$. The commutativity of the diagram yields $\operatorname{img}(\tau) \subset \operatorname{img}(a)$, hence by identifying G with its image under a, we have constructed a factorisation as stated in (3.5). $\square$

3.8 Assumption. *Where not otherwise stated, we always assume that our klt variety X already is the quasi-étale cover $\widetilde{X}$ from Theorem 3.6.*

Our next step is to extend flat, locally free sheaves from the regular locus S_{reg} of a complete intersection surface S having orbifold structure to X. The Lefschetz theorem on hyperplane sections implies that the fundamental groups of S_{reg} and X_{reg} are isomorphic. Hence given a reflexive sheaf $\mathcal{E}$ on X, whose restriction to S_{reg} is flat and locally free, there is a flat, locally free sheaf $\mathcal{F}$ on X_{reg} and by Theorem 3.7 on X, such that $\mathcal{F}|_{S_{\mathrm{reg}}} = \mathcal{E}|_{S_{\mathrm{reg}}}$. But since we want to argue that flatness of the tangent sheaf on the regular locus of a complete intersection surface implies flatness of the tangent sheaf and locally freeness everywhere, we need to know in addition that $\mathcal{E} = \mathcal{F}$ holds not just on S_{reg} but on X. The key point to obtain this additional statement is the boundedness of the family of all possible extensions $\mathcal{F}$ of such a sheaf $\mathcal{E}$ that we establish in the following.

More precisely, we want to show that, for fixed r and an ample divisor H on X, the family

$$\mathscr{B}_{r,H} := \{\mathcal{F} \mid \mathcal{F} \text{ is a locally free, flat, } H\text{-semistable sheaf on } X \text{ with } \operatorname{rank}(\mathcal{F}) = r\}$$

is bounded. At first we need the following lemma that holds in a more general situation.

3.9 Lemma (Computing the cohomology on a resolution)**.** *Let X be any variety with rational singularities and $\pi : \widehat{X} \to X$ a resolution of singularities. Then, given a locally free sheaf $\mathcal{E}$ on X, we can compute the cohomology of X with values in $\mathcal{E}$ by pulling back to $\widehat{X}$. More precisely, for any natural number k,*

$$H^k(X, \mathcal{E}) \cong H^k(\widehat{X}, \pi^*\mathcal{E}). \tag{3.6}$$

Proof. First notice that, since X has rational singularities, the direct images $R^q\pi_*\mathcal{O}_{\widetilde{X}}$ vanish for $q > 0$. Therefore, the projection formula yields

$$0 = R^q\pi_*(\mathcal{O}_{\widetilde{X}}) \otimes \mathcal{E} = R^q\pi_*(\mathcal{O}_{\widetilde{X}} \otimes_{\mathcal{O}_{\widetilde{X}}} \pi^*\mathcal{E}) = R^q\pi_*(\pi^*\mathcal{E}). \tag{3.7}$$

Now we may consider the Leray spectral sequence, that can be obtained from the Grothendieck spectral sequence, cf. [Wei94, p. 152], that is, the convergent spectral sequence $\{E_r^{pq}\}$ with

$$E_2^{pq} = H^p(X, R^q\pi_*(\pi^*\mathcal{E})) \quad \text{and} \quad E_r^{pq} \Longrightarrow H^{p+q}(\widetilde{X}, \pi^*\mathcal{E}),$$

where we follow the convention to denote convergence of spectral sequence by $\Rightarrow$. Because of (3.7), the spectral sequence degenerates at E_2^{pq}, so that we have $E_\infty^{pq} = E_2^{pq}$ and hence for all $k \geq 0$,

$$H^k(\widetilde{X}, \pi^*\mathcal{E}) \cong \bigoplus_{p+q=k} E_\infty^{pq} = \bigoplus_{p+q=k} E_2^{pq} = \bigoplus_{p+q=k} H^p(X, R^q\pi_*(\pi^*\mathcal{E})) = H^k(X, \mathcal{E}),$$

where we used the vanishing (3.7) for all $q > 0$ in the last equality. $\square$

We now proof boundedness of the family $\mathscr{B}_{r,H}$ for fixed r and ample divisor H.

3.10 Lemma (Boundedness of the families $\mathscr{B}_{r,H}$, cf. the preprint version of [GKP16b, Proposition 9.1])**.** *For any ample divisor H on a klt variety X and every fixed rank r the family $\mathscr{B}_{r,H}$, defined as above, is bounded.*

Proof. Since every member of the family $\mathscr{B}_{r,H}$ is H-semistable, we may show that the Hilbert polynomial p_H is constant in the family $\mathscr{B}_{r,H}$, that is, $p_H(\mathcal{F}) = p_H(\mathcal{F}')$ for all $\mathcal{F}, \mathcal{F}' \in \mathscr{B}_{r,H}$, and apply [HL10, Corollary 3.3.7] to prove our assertion.

Let $\pi : \widehat{X} \to X$ be a resolution of singularities and $\mathcal{F}$ an arbitrary member of our family $\mathscr{B}_{r,H}$. Then, the pullback $\pi^*(\mathcal{F})$ is flat on $\widetilde{X}$ and hence has vanishing Chern classes. In particular, the Chern character reduces to $\mathrm{ch}(\mathcal{E}) = \mathrm{rank}(\mathcal{E})$. Applying the Hirzebruch-Riemann-Roch theorem, cf. [Ful98, Corollary 15.2.1], we calculate for arbitrary $m \in \mathbb{N}$,

$$\begin{aligned}\mathcal{X}(\widetilde{X}, \pi^*(\mathcal{F}) \otimes \pi^*(\mathcal{O}_X(mH))) &= \int_{\widetilde{X}} \mathrm{ch}(\pi^*(\mathcal{F}) \otimes \pi^*(\mathcal{O}_X(mH))) \cdot \mathrm{td}(X) \\ &= \int_{\widetilde{X}} \mathrm{ch}(\pi^*(\mathcal{F})) \cdot \mathrm{ch}(\pi^*(\mathcal{O}_X(mH))) \cdot \mathrm{td}(X) = \mathrm{rank}(\mathcal{F}) \cdot \mathcal{X}(\widetilde{X}, \pi^*(\mathcal{O}_X(mH))).\end{aligned}$$

Moreover, since X has rational singularities by [KM08, Theorem 5.22], we may apply Lemma 3.9 above to $\pi^*(\mathcal{F} \otimes \mathcal{O}_X(mH))$,

$$H^k(\widetilde{X}, \pi^*(\mathcal{F} \otimes \mathcal{O}_X(mH))) \cong H^k(X, \mathcal{F} \otimes \mathcal{O}_X(mH)) \quad \forall m \in \mathbb{N}, \forall k \in \mathbb{N}.$$

Consequently, we obtain

$$\mathcal{X}(X, \mathcal{F} \otimes \mathcal{O}_X(mH)) = \mathcal{X}(\widetilde{X}, \pi^*(\mathcal{F}) \otimes \pi^*(\mathcal{O}_X(mH))) = \mathrm{rank}(\mathcal{F}) \cdot \mathcal{X}(\widetilde{X}, \pi^*(\mathcal{O}_X(mH)))$$

and since the rank of $\mathcal{F}$ is constant in the family $\mathscr{B}_{r,H}$, the Hilbert polynomial with respect to H does not depend on $\mathcal{F}$, is hence constant in the family $\mathscr{B}_{r,H}$, and this was to be shown. □

Using boundedness of the families $\mathscr{B}_{r,H}$ for every fixed rank r, we are able to demand additional properties for a complete intersection surface S.

3.11 Lemma (Choice of a complete intersection surface)**.** *For every ample divisor H on a klt variety X there is a sufficiently large integer m such that mH is ample and the following conditions are satisfied.*

(i) Given general elements $D_1 \dots, D_{n-2} \in |mH|$, then the complete intersection surface $S := D_1 \cap \dots \cap D_{n-2}$ is entirely contained in $X \setminus Z$, where Z denotes the non-orbifold locus of X, cf. Theorem 3.3. In particular, S has the well defined structure of an orbifold surface.

(ii) For every reflexive sheaf $\mathcal{E}$ on X and every fixed member $\mathcal{F} \in \mathscr{B}_{\mathrm{rank}(\mathcal{E}),H}$,

$$H^1(X, \mathcal{H}om(\mathcal{E}, \mathcal{F}) \otimes \mathcal{O}_X(-mH)) = 0. \tag{3.8}$$

Proof. By [Gro62, §XII, Corollary 1.4], there is an integer m such that $H^1(X, \mathcal{H}om(\mathcal{E}, \mathcal{F}) \otimes \mathcal{O}_X(-mH)) = 0$ holds for every fixed member $\mathcal{F} \in \mathscr{B}_{\mathrm{rank}(\mathcal{E}),H}$. Since the family $\mathscr{B}_{\mathrm{rank}(\mathcal{E}),H}$ is bounded by Lemma 3.10, we can choose m large enough such that mH is ample and the desired vanishing of the cohomology holds for all members in $\mathscr{B}_{\mathrm{rank}(\mathcal{E}),H}$, cf. [GKP16b, page 15]. Property (i) is then immediate from the ampleness of mH. □

We are now able to prove that reflexive sheaves, whose restriction to an intersection surface constructed as in Lemma 3.11 is locally free, semistable and flat, extend to flat, locally free, semistable sheaves on X. One should compare this result to [GKP16b, Proposition 5.1 and Corollary 5.3].

3.12 Theorem (Extension of flat, locally free sheaves, II)**.** *For a klt variety X let S be constructed from the ample divisor H on X as in Lemma 3.11 and $\mathcal{E}$ be a reflexive sheaf on X such that $\mathcal{E}|_{S_{\mathrm{reg}}}$ is locally free, H-semistable and flat. Then, $\mathcal{E}$ extends to a locally free and flat sheaf on X.*

Proof. $\mathcal{E}|_{S_{\mathrm{reg}}}$ is reflexive over S_{reg} and comes from a representation $\rho_S : \pi_1(S^{an}_{reg}) \to \mathrm{GL}(\mathrm{rank}(\mathcal{E}), \mathbb{C})$ which gives rise to a representation $\rho : \pi_1(X^{an}_{reg}) \to \mathrm{GL}(\mathrm{rank}(\mathcal{E}), \mathbb{C})$ by the Lefschetz theorem on hyperplane sections and then by Theorem 3.7 to a representation $\tau : \pi_1(X^{an}) \to \mathrm{GL}(\mathrm{rank}(\mathcal{E}), \mathbb{C})$. Let $\mathcal{F}$ be the sheaf corresponding to this representation. Since $\mathcal{F}|_{S_{\mathrm{reg}}} = \mathcal{E}|_{S_{\mathrm{reg}}}$, which is H-semistable, $\mathcal{F}|_{S_{\mathrm{reg}}}$ is also H-semistable and according to Remark 1.24 and (3.4) also H-semistable on X, i.e. it is a member of $\mathscr{B}_{\mathrm{rank}(\mathcal{E}),H}$. Note that here we used that orbifold Chern classes restricted to the regular locus of an orbifold are the regular Chern classes defined in chapter one.

For the rest of the proof we follow the proof of [GKP16b, Proposition 5.1].

Consider the sheaf of morphisms $\mathcal{H}om(\mathcal{E}, \mathcal{F})$. As $\mathcal{E}$ and $\mathcal{F}$ are coherent, so is $\mathcal{H}om(\mathcal{E}, \mathcal{F})$. Furthermore, since

$$\mathcal{H}om(\mathcal{E}, \mathcal{F})^{\vee\vee} = \mathcal{H}om(\mathcal{E}^{\vee\vee}, \mathcal{F}) \cong \mathcal{H}om(\mathcal{E}, \mathcal{F}),$$

we see that $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ is also reflexive. Every of the sheaves $\mathcal{E}, \mathcal{F}$ and $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ is locally free on X_{reg}, the sheaves $\mathcal{H}om(\mathcal{E}, \mathcal{F})|_H$ and $\mathcal{H}om(\mathcal{E}|_H, \mathcal{F}|_H)$ therefore are locally free and coincide on X_{reg}. Since $\mathrm{codim}_X(X_{\mathrm{sing}}) \geq 2$ and it is sufficient to check equality of reflexive sheaves on sets which have codimension two,

$$\mathcal{H}om(\mathcal{E}, \mathcal{F})|_H = \mathcal{H}om(\mathcal{E}|_H, \mathcal{F}|_H).$$

As H is ample, its ideal sheaf is $\mathcal{O}_X(-H)$ and the ideal sheaf sequence becomes

$$0 \longrightarrow \mathcal{O}_X(-H) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_H \longrightarrow 0. \tag{3.9}$$

The sheaf $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ is reflexive, in particular torsion-free, so tensoring the sequence above with it is exact and we obtain

$$0 \longrightarrow \mathcal{O}_X(-H) \otimes \mathcal{H}om(\mathcal{E}, \mathcal{F}) \longrightarrow \mathcal{H}om(\mathcal{E}, \mathcal{F}) \longrightarrow \mathcal{H}om(\mathcal{E}|_H, \mathcal{F}|_H) \longrightarrow 0. \tag{3.10}$$

The long exact sequence in cohomology contains

$$\begin{aligned} \ldots \longrightarrow H^0(X, \mathcal{H}om(\mathcal{E}, \mathcal{F})) &\longrightarrow H^0(X, \mathcal{H}om(\mathcal{E}|_H, \mathcal{F}|_H)) \\ &\longrightarrow H^1(X, \mathcal{H}om(\mathcal{E}, \mathcal{F}) \otimes \mathcal{O}_X(-H)) \longrightarrow \ldots \end{aligned} \tag{3.11}$$

and the first cohomology group $H^1(X, \mathcal{H}om(\mathcal{E}, \mathcal{F}) \otimes \mathcal{O}_X(-H))$ vanishes by Lemma 3.11, (ii). Using the identifications, cf. [GR84, page 239],

$$H^0(X, \mathcal{H}om(\mathcal{E}, \mathcal{F})) = \mathrm{Hom}(\mathcal{E}, \mathcal{F}), \quad H^0(X, \mathcal{H}om(\mathcal{E}, \mathcal{F})|_H) = \mathrm{Hom}(\mathcal{E}|_H, \mathcal{F}|_H)$$

we see that the restriction map

$$\operatorname{Hom}(\mathcal{E},\mathcal{F}) \xrightarrow{r} \operatorname{Hom}(\mathcal{E}|_H,\mathcal{F}|_H)$$

is surjective. Hence, if we denote by α the isomorphism $\mathcal{E}|_H \xrightarrow{\sim} \mathcal{F}|_H$, it extends to a morphism $\widehat{\alpha} : \mathcal{E} \to \mathcal{F}$ which we claim to be an isomorphism.

To prove injectivity, notice that $\ker(\widehat{\alpha}) \subset \mathcal{E}$ and $\operatorname{img}(\widehat{\alpha}) \subset \mathcal{F}$ are subsheaves, hence torsion-free and thus both locally free on X_{reg}, which implies that the restricted sequence

$$0 \longrightarrow \ker(\widehat{\alpha})|_H \longrightarrow \mathcal{E}|_H \xrightarrow{\widehat{\alpha}|_H} \operatorname{img}(\widehat{\alpha})|_H \longrightarrow 0$$

remains exact. As $\widehat{\alpha}|_H = \alpha$ is an isomorphism, $\ker(\widehat{\alpha})|_H = 0$ or, equivalentely, $\operatorname{supp}(\ker(\widehat{\alpha})|_H) = \emptyset$. It follows that $\ker(\widehat{\alpha})$ has to be a torsion-sheaf, but since $\mathcal{E}$ is torsion-free and $\ker(\widehat{\alpha}) \subset \mathcal{E}$, it has to be zero.

For the surjectivity, notice that $\operatorname{coker}(\widehat{\alpha})|_H = \operatorname{coker}(\widehat{\alpha}|_H) = \operatorname{coker}(\alpha) = 0$ which shows that $\operatorname{supp}(\operatorname{coker}(\widehat{\alpha})) \cap H = \emptyset$. By the ampleness of H, $\operatorname{supp}(\operatorname{coker}(\widehat{\alpha}))$ can only consist of a finite number of points and therefore $\widehat{\alpha}$ is an isomorphism away from this finite number of points. Consequently, $\mathcal{E} \cong \mathcal{F}$, since two reflexive sheaves are isomorphic, if they are isomorphic on the complement of a codimension two set. Successively cutting down with H then yields the assertion for complete intersection surfaces. □

3.3 Proof of the main result

As mentioned before, the proof of our main result, Theorem 3.1, relies on the application of the theorems above to the tangent sheaf of a klt variety X. For completeness we give a definition, following [Gue15, Item 2.1.3].

3.13 Definition (Tangent sheaf of a klt variety)**.** *Let X be any klt variety and denote by $\iota : X_{\mathrm{reg}} \hookrightarrow X$ the inclusion of the smooth locus. The tangent sheaf $\mathcal{T}_X$ of X is defined to be $\iota_*(\mathcal{T}_{X_{\mathrm{reg}}})$, where $\mathcal{T}_{X_{\mathrm{reg}}}$ is the tangent sheaf of the smooth variety X_{reg}.*

This is in fact a meaningful definition: Usually, the tangent sheaf of a singular variety is defined to be the dual of the sheaf of Kähler differentials. If the variety is normal, the tangent sheaf is reflexive, [Gue15, Item 2.1.3], and therefore determined by restriction to the smooth locus. Consequently, using that klt varieties are, by definition, normal, we are able to define the tangent sheaf of a klt variety in this way.

We will now prove Theorem 3.1.

Proof of Theorem 3.1. (i)⇒(ii): Let $\gamma : \widetilde{X} \to X$ be the quasi-étale cover from Theorem 3.6. Then $\gamma^{-1}(X_{\mathrm{sing}}) \subset \widetilde{X}_{\mathrm{sing}}$ and since $\widetilde{X}$ has singularities only in codimension at least two,

$$\operatorname{codim}_{\widetilde{X}} \gamma^{-1}(X_{\mathrm{sing}}) \geq 2. \tag{3.12}$$

Now, $\gamma|_{\widetilde{X}_{\mathrm{reg}}}$ is unramified, thus the equality $\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}) = \mathcal{O}_{\widetilde{X}}(\gamma^* K_X)$ holds on $\widetilde{X}_{\mathrm{reg}}$ and therefore on $\widetilde{X}$, since both sheaves are reflexive and (3.12) implies that they coincide away from a set of codimension at least two. Consequently, $K_{\widetilde{X}} \equiv 0$ and by the functorality of $\widehat{c}_2(\mathcal{T}_X) \cdot H^{n-2}$ under quasi-étale morphisms, cf. [GKPT15, Corollary 4.7], $\widehat{c}_2(\mathcal{T}_{\widetilde{X}}) \cdot \widetilde{H}^{n-2} = 0$ for the divisor

$\widetilde{H} = \gamma^* H$ on $\widetilde{X}$ which is ample since γ is finite. Thus we may assume $X = \widetilde{X}$, see Assumption 3.8, and work with $H = \widetilde{H}$. Restricting ourselfs to ample divisors of this special type does not limit generality of our result, see Remark 3.14 for an explanation.

Since $K_{\widetilde{X}} \equiv 0$, $\mathcal{T}_X$ is polystable by [Gue15, Theorem A]. Let S be a complete intersection surface constructed as in Lemma 3.11. Then $\mathcal{T}_X|_S$ remains polystable by the Mehta-Ramanathan Theorem for normal spaces, [Fle84, Theorem 1.2], and possesses an orbifold Hermite-Einstein metric with respect to the orbifold Kähler form of S by [SW01, Theorem 2.3]. The assumption on the Chern classes combined with formula (3.3) yield equality in the orbifold Kobayashi-Lübke inequality. Therefore, the locally free sheaf $\mathcal{T}_X|_{S_{\mathrm{reg}}}$ is flat and extends by Theorem 3.12 to a flat, locally free sheaf on X which coincides with $\mathcal{T}_X$. Since therefore $\mathcal{T}_X$ is locally free, the confirmation of the Lipman-Zariski conjecture for klt spaces, [GKK11, Theorem 6.1], implies that X is smooth. Applying Theorem 1.30, there is an Abelian variety A and a surjective, étale morphism $\eta : A \to X$. Henceforth we are in the following situation

$$\begin{tikzcd} A \arrow[r, two heads, "\eta,\ \text{étale}"] \arrow[rr, bend right, two heads, "\text{surj., étale in codim. one}"'] & \widetilde{X} \arrow[r, two heads, "\gamma,\ \text{étale in codim. one}"] & X. \end{tikzcd}$$

which gives the first direction of the proof.

(ii)⇒(i): If X is a finite quotient of an Abelian variety A, given by the quasi-étale morphism $\gamma : A \to X$, then due to the functorality of $\widehat{c}_1(X) \cdot H^{n-1}$ and $\widehat{c}_2(X) \cdot H^{n-2}$ with respect to quasi-étale morphisms, cf. [GKPT15, Corollary 4.7], $\widehat{c}_2(X) \cdot H^{n-2} = 0$. Moreover, by arguing as in the first part of the proof and using that $K_A \equiv 0$ holds for every Abelian variety A, we obtain $\mathcal{O}_A(\gamma^* K_X) = \mathcal{O}_A(K_A) = \mathcal{O}_A$, which shows that $K_X \equiv 0$. Now, since X is normal, we may apply [KM08, Proposition 5.20] to deduce that X is klt in this case. □

3.14 Remark. *As we have seen above, in the proof of Theorem 3.1, the assumption that X already is the quasi-étale cover $\widetilde{X}$ from Theorem 3.6 possibly limits the generality of our result, since we had $\widehat{c}_2(\mathcal{T}_X) \cdot H^{n-2} = 0$ for all ample divisors H on X and obtain $\widehat{c}_2(\mathcal{T}_{\widetilde{X}}) \cdot \widetilde{H}^{n-2} = 0$ only for ample divisors $\widetilde{H} = \gamma^* H$ on $\widetilde{X}$ where H is some ample divisor on X. But since the proof shows that $\widetilde{X}$ actually is smooth, applying [GKP16b, Proposition 4.8] gives $\widehat{c}_2(\mathcal{T}_{\widetilde{X}}) \cdot \widetilde{H}^{n-2} = 0$ for all ample divisors $\widetilde{H}$ on $\widetilde{X}$ and shows that generality is not limited by this assumption.*

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Appendix

Statement of authorship

Ich versichere, dass ich die vorliegende Masterarbeit

Characterisation of finite quotients of Abelian varieties via Chern class conditions

selbstständig und ohne fremde Hilfe verfasst habe. Alle verwendeten Quellen und Hilfsmittel habe ich explizit angegeben und wörtliche oder sinngemäße Zitate als solche gekennzeichnet. Die Arbeit in vorliegender oder ähnlicher Form habe ich noch nie zu Prüfungszwecken vorgelegt.

Tobias Heckel
Essen, 04. August 2016

Acknowledgements

I would like to take this opportunity to thank all those who supported me during the work at this thesis, whether through mathematical advice or loving encouragement:

My supervisor Prof. Dr. Daniel Greb for introducing me to the subject of this thesis, for interesting and inspiring discussions, general advice and for reading an earlier version of this thesis with scrupulous care and pointing at inaccuracies,

Annika Heckel, Jutta Heckel, Anna Kahmen and Lena Sauermann for reading the manuscript and correcting my sometimes ineloquent use of the English language,

and finally all my friends, my family, and Lena for your continuous support and love.