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MASTER'S THESIS

Fine analysis of the degeneration of Frölicher spectral sequences

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1 Introduction

If M is a smooth manifold, one introduces the de Rham cohomology of M, which is an important geometrical invariant of M. It is defined as the quotient

$$\mathrm{H}_{\mathrm{dR}}(M,E)=\frac{\ker(\mathrm{d})}{\mathrm{im}(\mathrm{d})},$$

where d denotes the exterior derivative on the space of smooth differential forms on M with values in some vector space E over the real or complex numbers.
In the case of a complex manifold X, there is an almost complex structure on the tangent bundle of X. Hence every differential form of degree k splits uniquely into a sum of forms of degree (p,q), where $p+q=k$.
Since the almost complex structure on a complex manifold is integrable, the exterior derivative splits into a sum

$$\mathrm{d}=\partial+\bar{\partial},$$

where $\bar{\partial}$ and ∂ denote the Dolbeault operator and the conjugate Dolbeault operator (see [Wel08]). They satisfy

$$\partial\circ\partial=\bar{\partial}\circ\bar{\partial}=\partial\circ\bar{\partial}+\bar{\partial}\circ\partial=0.$$

So, in addition to the de Rham cohomology, we can consider the so-called Dolbeault cohomologies of X, which are defined to be

$$\mathrm{H}_{\bar{\partial}}(X,E)=\frac{\ker(\bar{\partial})}{\mathrm{im}(\bar{\partial})}\ \text{and}\ \mathrm{H}_{\partial}(X,E)=\frac{\ker(\partial)}{\mathrm{im}(\partial)}.$$

While complex conjugation induces an isomorphism between the two Dolbeault cohomologies, in general, there is no natural map between $\mathrm{H}_{\mathrm{dR}}(M,E)$ and $\mathrm{H}_{\bar{\partial}}(M,E)$. But, one can introduce two cohomologies of X, which connect the de Rham and the Dolbeault cohomology. The first one is the Bott-Chern cohomology [BC65], defined as

$$\mathrm{H}_{\mathrm{BC}}(X)=\frac{\ker(\partial)\cap\ker(\bar{\partial})}{\mathrm{im}(\partial\bar{\partial})},$$

and the other one is the Aeppli cohomology [Aep65], which is set to be

$$\mathrm{H}_{\mathrm{A}}(X)=\frac{\ker(\partial\bar{\partial})}{\mathrm{im}(\partial)+\mathrm{im}(\bar{\partial})}.$$

The connection that they provide is expressed in the following commutative diagram of (bi-)graded complex vector spaces

$$\begin{tikzcd}
 & \mathrm{H}_{\mathrm{BC}}(X) \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathrm{H}_{\bar{\partial}}(X) \arrow[dr] & \mathrm{H}_{\mathrm{dR}}(X) \arrow[d] & \mathrm{H}_{\partial}(X) \arrow[dl] \\
 & \mathrm{H}_{\mathrm{A}}(X) &
\end{tikzcd},$$

where the maps are all induced by the identity on the space of differential forms on X. The natural question arises whether the maps in the diagram are injective or surjective. It turns out that in general neither of them is. If X is compact, one knows that all of the above cohomologies are finite-dimensional. Even in this case, no injectivity or surjectivity is guaranteed.

The map from Bott-Chern to the de Rham cohomology is injective if and only if every form which is exact with respect to ∂ and $\bar{\partial}$ and closed with respect to d is $\partial\bar{\partial}$-exact. Manifolds that fulfil this property are said to satisfy the $\partial\bar{\partial}$-Lemma [Ang14]. P. Deligne, Ph. A. Griffiths, J. Morgan, and D. P. Sullivan proved in [DGMS75] that this is the case for compact Kähler manifolds. They also proved that all maps are isomorphisms whenever X satisfies the $\partial\bar{\partial}$-Lemma. In section 3 we will further investigate the relation of the injectivity and surjectivity of the maps in the above diagram. Special attention will be given to the case, where the diagram is restricted to one degree.

Another connection between the Dolbeault and the de Rham cohomology is the Frölicher spectral sequence. It is a spectral sequence whose first sheet is isomorphic to the Dolbeault cohomology and which abuts to the de Rham cohomology. This spectral sequence yields the Frölicher inequality for compact complex manifolds [Frö55]

$$\dim_{\mathbb{C}} \mathrm{H}^k_{\mathrm{dR}}(X) \leq \sum_{p+q=k} \dim_{\mathbb{C}} \mathrm{H}^{p,q}_{\bar{\partial}}(X) < \infty.$$

Also, the spectral sequence degenerates at the first sheet if and only if equality holds in the Frölicher inequality.

In [DGMS75] it was proved that a complex manifold satisfies the $\partial\bar{\partial}$-Lemma if and only if the Frölicher spectral sequence degenerates at sheet 1 and, in addition, the natural filtration on the space of differential forms induces some Hodge structure on the de Rham cohomology. In section 4 we will recall this result and its proof.

The question arises what happens if the $\partial\bar{\partial}$-Lemma holds only in one degree. This means, we will fix an integer k such that every differential form of degree k that is exact with respect to ∂ and $\bar{\partial}$ and closed with respect to d is $\partial\bar{\partial}$-exact. In section 5 we will see an equivalent formulation of this property in terms of the Frölicher spectral sequence and the filtration on the de Rham cohomology. The last section will give some applications of this theorem.

We will state and prove everything in an abstract homological setting. The statements about complex manifolds will be simple corollaries of their homological versions. For this purpose, section 2 will introduce the homological concepts we will need. We will explain the basics of the theory of complexes and their cohomologies as well as spectral sequences.

2 Definitions

2.1 Complexes and cohomologies

In this section we introduce the basic notions in cohomology of complexes. For more details we refer to [Wei95] and [Bou98].

2.1.1 Complexes

If M is a smooth manifold, we can consider the space $\mathcal{A}^k(M)$ of complex valued smooth differential forms of degree k on M. On this space we have the exterior derivative

$$\mathrm{d}^k : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$$

satisfying $\mathrm{d}^{k+1} \circ \mathrm{d}^k = 0$. Thanks to this property one can introduce the de Rham cohomology $\mathrm{H}^k_{\mathrm{dR}}(M)$ of M, which is defined to be

$$\mathrm{H}^k_{\mathrm{dR}}(M) = \ker(\mathrm{d}^k)/\operatorname{im}(\mathrm{d}^{k-1}).$$

This notion, of course, can be viewed in a more general context, which leads to the notion of a cochain complex of vector spaces and its cohomology.

Definition 2.1 (Complex)**.** A cochain complex or simple complex $(K; \mathrm{d})$ of vector spaces over a field F is a $\mathbb{Z}$-graded vector space

$$K = \bigoplus_{n \in \mathbb{Z}} K^n$$

together with a linear map

$$\mathrm{d} : K \to K$$

satisfying $\mathrm{d} \circ \mathrm{d} = 0$, which is homogeneous of degree one. That is,

$$\mathrm{d}^n = \mathrm{d}|_{K^n} : K^n \to K^{n+1}$$

is a linear map for every $n \in \mathbb{Z}$. We call d the differential of K.

Let $(K; \mathrm{d})$ be a cochain complex. We denote the image of d by B, B_d or $\mathrm{B}_\mathrm{d}(K)$ and call its elements exact. Similarly, the kernel of d is denoted by Z, Z_d or $\mathrm{Z}_\mathrm{d}(K)$ and the elements in the kernel are called closed. Since $\mathrm{d} \circ \mathrm{d} = 0$, every exact element is closed. This leads to the following definition.

Definition 2.2 (Cohomology of a complex)**.** Let $(K; \mathrm{d})$ be a cochain complex. We set the cohomology of $(K; \mathrm{d})$ to be

$$\mathrm{H}_\mathrm{d}(K) = \mathrm{Z}_\mathrm{d} / \mathrm{B}_\mathrm{d}\,.$$

Since d is a homogeneous homomorphism, we have

$$\mathrm{Z}_{\mathrm{d}}^n = \mathrm{Z}_{\mathrm{d}}^n(K) = \ker(\mathrm{d}) \cap K^n = \ker(\mathrm{d}^n)$$

and

$$\mathrm{B}_{\mathrm{d}}^n = \mathrm{B}_{\mathrm{d}}^n(K) = \mathrm{im}(\mathrm{d}) \cap K^n = \mathrm{im}(\mathrm{d}^{n-1}).$$

This defines a grading on $\mathrm{H}_{\mathrm{d}}(K)$ by

$$\mathrm{H}_{\mathrm{d}}(K) = \bigoplus_{n\in\mathbb{Z}} \mathrm{H}_{\mathrm{d}}^n(K),$$

where $\mathrm{H}_{\mathrm{d}}^n(K) = \mathrm{Z}_{\mathrm{d}}^n(K) / \mathrm{B}_{\mathrm{d}}^n(K)$.

2.1.2 Double complexes

Let X be a complex manifold. Then every smooth k-form can be decomposed uniquely into a sum of forms of degree (p, q), where $p + q = k$. We have

$$\mathcal{A}^k(X) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(X).$$

Also, the exterior derivative splits into two parts

$$\mathrm{d}^k = \partial^k + \bar{\partial}^k,$$

where

$$\partial^{p,q} = \partial^k|_{\mathcal{A}^{p,q}(X)} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p+1,q}(X)$$

and

$$\bar{\partial}^{p,q} = \bar{\partial}^k|_{\mathcal{A}^{p,q}(X)} : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X).$$

The maps ∂ and $\bar{\partial}$ are the so-called Dolbeault operators. They satisfy $\partial \circ \partial = 0$, $\bar{\partial} \circ \bar{\partial} = 0$ and $\partial \circ \bar{\partial} + \bar{\partial} \circ \partial = 0$. So, we can define several cohomology vector spaces as the Dolbeault, Bott-Chern and Aeppli cohomology. As before, we want to introduce these concepts in a more general setting.

Definition 2.3 (Double complex)**.** A double complex $(K; \partial, \bar{\partial})$ of vector spaces over a field is a $\mathbb{Z}$-bigraded vector space

$$K = \bigoplus_{(p,q)\in\mathbb{Z}^2} K^{p,q}$$

together with two linear maps

$$\partial : K \to K \text{ and } \bar{\partial} : K \to K$$

satisfying

$$\begin{aligned}
&\partial \circ \partial = 0,\\
&\bar\partial \circ \bar\partial = 0,\\
&\bar\partial \circ \partial + \partial \circ \bar\partial = 0,
\end{aligned}$$

which are homogeneous of degree $(1,0)$ and $(0,1)$, respectively. That is,

$$\partial^{p,q} = \partial|_{K^{p,q}} : K^{p,q} \to K^{p+1,q}$$

and

$$\bar\partial^{p,q} = \bar\partial|_{K^{p,q}} : K^{p,q} \to K^{p,q+1}$$

are linear maps for each $(p,q) \in \mathbb{Z}^2$.
We say a double complex is bounded if there are integers $a \le b$ such that $K^{p,q} = 0$ whenever $p \ge b$, $q \ge b$, $p \le a$ or $q \le a$.

Let $(K; \partial, \bar\partial)$ be a double complex. By setting $\mathrm{d} = \partial + \bar\partial$ and

$$K^n = \bigoplus_{p+q=n} K^{p,q},$$

we get a cochain complex, called the associated simple complex of $(K; \partial, \bar\partial)$. Indeed, by linearity of ∂ and $\bar\partial$,

$$\begin{aligned}
\mathrm{d} \circ \mathrm{d} &= (\partial + \bar\partial) \circ (\partial + \bar\partial)\\
&= \partial \circ \partial + (\partial \circ \bar\partial + \bar\partial \circ \partial) + \bar\partial \circ \bar\partial\\
&= 0,
\end{aligned}$$

and for $x_n = \sum_{p+q=n} x_{p,q} \in K^n$ we have

$$\begin{aligned}
\mathrm{d}(x_n) &= \sum_{p+q=n} \mathrm{d}(x_{p,q})\\
&= \sum_{p+q=n} \partial(x_{p,q}) + \sum_{p+q=n} \bar\partial(x_{p,q}),
\end{aligned}$$

which is indeed an element of K^{n+1}.
For $n \in \mathbb{Z}$ we also set

$$\partial^n = \partial|_{K^n} : K^n \to K^{n+1} \text{ and } \bar\partial^n = \bar\partial|_{K^n} : K^n \to K^{n+1}.$$

As in the case of a simple complex, one sets $\mathrm{B}_\partial = \mathrm{B}_\partial(K) = \mathrm{im}(\partial)$ and $\mathrm{B}_{\bar\partial} = \mathrm{B}_{\bar\partial}(K) = \mathrm{im}(\bar\partial)$ as well as $\mathrm{Z}_\partial = \mathrm{Z}_\partial(K) = \ker(\partial)$ and $\mathrm{Z}_{\bar\partial} = \mathrm{Z}_{\bar\partial}(K) = \ker(\bar\partial)$. We define the cohomologies by

$$\mathrm{H}_\partial(K) = \mathrm{Z}_\partial \,/\, \mathrm{B}_\partial$$

and

$$\mathrm{H}_{\bar\partial}(K) = \mathrm{Z}_{\bar\partial} \,/\, \mathrm{B}_{\bar\partial}$$

and call the first one the conjugate Dolbeault cohomology and the second one the Dolbeault cohomology of $(K;\partial,\bar\partial)$.
Also, we will also use the notations

$$\begin{aligned}
\mathrm{B}^n_\partial = \mathrm{B}^n_\partial(K) &= \operatorname{im}(\partial) \cap K^n = \operatorname{im} \partial^{n-1}, \\
\mathrm{B}^{p,q}_\partial = \mathrm{B}^{p,q}_\partial(K) &= \operatorname{im}(\partial) \cap K^{p,q} = \operatorname{im} \partial^{p-1,q}, \\
\mathrm{Z}^n_\partial = \mathrm{Z}^n_\partial(K) &= \ker(\partial) \cap K^n = \ker \partial^n, \\
\mathrm{Z}^{p,q}_\partial = \mathrm{Z}^{p,q}_\partial(K) &= \ker(\partial) \cap K^{p,q} = \ker \partial^{p,q}
\end{aligned}$$

and the analogous notation for $\bar\partial$. This leads to the cohomologies

$$\begin{aligned}
\mathrm{H}^n_\partial(K) &= \mathrm{Z}^n_\partial \,/\, \mathrm{B}^n_\partial, \\
\mathrm{H}^{p,q}_\partial(K) &= \mathrm{Z}^{p,q}_\partial \,/\, \mathrm{B}^{p,q}_\partial, \\
\mathrm{H}^n_{\bar\partial}(K) &= \mathrm{Z}^n_{\bar\partial} \,/\, \mathrm{B}^n_{\bar\partial}, \\
\mathrm{H}^{p,q}_{\bar\partial}(K) &= \mathrm{Z}^{p,q}_{\bar\partial} \,/\, \mathrm{B}^{p,q}_{\bar\partial},
\end{aligned}$$

which define (bi-)gradings on $\mathrm{H}_\partial(K)$ and $\mathrm{H}_{\bar\partial}(K)$, respectively.
If no confusion can arise, we will always write $x_{p,q}$ for the component of an element $x \in K$ in $K^{p,q}$ and x_n for the component in K^n.
In the case of a complex manifold complex conjugation induces an isomorphism between its Dolbeault cohomology in some degree (p,q) and its conjugate Dolbeault cohomology in degree (q,p). However, there is no natural map between the Dolbeault and the de Rham cohomology.
Also in the case of an arbitrary double complex there are no natural maps between the introduced cohomologies at all. To get a connection between them we define the Bott-Chern and the Aeppli cohomology of a double complex as for complex manifolds.

Definition 2.4 (Bott-Chern cohomology and Aeppli cohomology)**.** Let $(K;\partial,\bar\partial)$ be a double complex. The Bott-Chern cohomology is the vector space

$$\mathrm{H}_{\mathrm{BC}}(K) = \frac{\ker(\partial) \cap \ker(\bar\partial)}{\operatorname{im}(\partial\bar\partial)}.$$

The Aeppli cohomology is the vector space

$$\mathrm{H}_{\mathrm{A}}(K) = \frac{\ker(\partial\bar\partial)}{\operatorname{im}(\partial) + \operatorname{im}(\bar\partial)}.$$

One easily checks that the defined quotients make sense. Furthermore, we define $\mathrm{H}^n_{\mathrm{BC}}(K)$, $\mathrm{H}^{p,q}_{\mathrm{BC}}(K)$, $\mathrm{H}^n_{\mathrm{A}}(K)$ and $\mathrm{H}^{p,q}_{\mathrm{A}}(K)$ as in the case of the Dolbeault cohomologies.
The next statement explains how the Bott-Chern and the Aeppli cohomology connect the Dolbeault cohomologies.

Proposition 2.5. Let $(K;\partial,\bar\partial)$ be a double complex. The identity on K induces the following commutative diagram

$$\begin{tikzcd} & \mathrm{H}_{\mathrm{BC}}(K) \arrow[dl, "\varphi_{\mathrm{BC}-\bar\partial}"'] \arrow[dr, "\varphi_{\mathrm{BC}-\partial}"] & \\ \mathrm{H}_{\bar\partial}(K) \arrow[dr, "\varphi_{\bar\partial-\mathrm{A}}"'] & & \mathrm{H}_{\partial}(K) \arrow[dl, "\varphi_{\partial-\mathrm{A}}"] \\ & \mathrm{H}_{\mathrm{A}}(K) & \end{tikzcd} \tag{1}$$

Proof. To show that the maps are well-defined we have to show that each, the denominator and numerator, of the left side is contained in the one of the right side. But this is obvious for all four maps. The commutativity is clear, since all maps are induced by the identity. □

In the same manner the identity on K^n or $K^{p,q}$ induces maps between the (bi-)graded cohomologies as in diagram (1). For instance, $\varphi^n_{\mathrm{BC}-\partial}$ will denote the map from $\mathrm{H}^n_{\mathrm{BC}}(K)$ to $\mathrm{H}^n_{\partial}(K)$. Furthermore we set

$$\varphi_{\mathrm{BC}-\mathrm{A}} = \varphi_{\partial-\mathrm{A}} \circ \varphi_{\mathrm{BC}-\partial} = \varphi_{\bar\partial-\mathrm{A}} \circ \varphi_{\mathrm{BC}-\bar\partial}.$$

Also the cohomology of the associated simple complex can be fitted in this diagram and hence be connected with the other cohomologies of the double complex.

Proposition 2.6. Let $(K;\partial,\bar\partial)$ be a double complex and $(K;\mathrm{d})$ its associated simple complex. The identity on K induces linear maps

1. $\varphi_{\mathrm{BC}-\mathrm{d}} : \mathrm{H}_{\mathrm{BC}}(K) \to \mathrm{H}_{\mathrm{d}}(K)$,
2. $\varphi_{\mathrm{d}-\mathrm{A}} : \mathrm{H}_{\mathrm{d}}(K) \to \mathrm{H}_{\mathrm{A}}(K)$.

These maps extend diagram (1) to the commutative diagram

$$\begin{tikzcd} & \mathrm{H}_{\mathrm{BC}}(K) \arrow[dl, "\varphi_{\mathrm{BC}-\bar\partial}"'] \arrow[d, "\varphi_{\mathrm{BC}-\mathrm{d}}"] \arrow[dr, "\varphi_{\mathrm{BC}-\partial}"] & \\ \mathrm{H}_{\bar\partial}(K) \arrow[dr, "\varphi_{\bar\partial-\mathrm{A}}"'] & \mathrm{H}_{\mathrm{d}}(K) \arrow[d, "\varphi_{\mathrm{d}-\mathrm{A}}"] & \mathrm{H}_{\partial}(K) \arrow[dl, "\varphi_{\partial-\mathrm{A}}"] \\ & \mathrm{H}_{\mathrm{A}}(K) & \end{tikzcd} \tag{2}$$

Proof. It is clear that $\ker(\partial)\cap\ker(\bar\partial) \subseteq \ker(\mathrm{d})$. If $x \in \mathrm{im}(\partial\bar\partial)$, say $x = \partial(\bar\partial(y))$, then

$$\begin{aligned} \mathrm{d}(\bar\partial(y)) &= \partial(\bar\partial(y)) + \bar\partial(\bar\partial(y)) \\ &= \partial(\bar\partial(y)) \\ &= x. \end{aligned}$$

This shows $x \in \operatorname{im}(\mathrm{d})$ so the map $\varphi_{\mathrm{BC}\,-\,\mathrm{d}}$ exists.
Now let $x \in \ker(\mathrm{d})$. Then we can write $x = \sum_{n \in \mathbb{Z}} x_n$ with $x_n \in K^n$. Since d is a homogeneous homomorphism, we have

$$\mathrm{d}(x_n) = 0$$

for all $n \in \mathbb{Z}$. For fixed $n \in \mathbb{Z}$ we can write

$$x_n = \sum_{p+q=n} x_{p,q}.$$

Introducing $x_{-1,n+1} = 0$ and $x_{n+1,-1} = 0$ in K^n we get

$$\begin{aligned} 0 &= \mathrm{d}(x_n) \\ &= \sum_{p+q=n} \partial(x_{p,q}) + \bar{\partial}(x_{p,q}) \\ &= \sum_{p+q=n+1} \bar{\partial}(x_{p,q-1}) + \partial(x_{p-1,q}). \end{aligned} \tag{3}$$

But we always have

$$\bar{\partial}(x_{p,q-1}) + \partial(x_{p-1,q}) \in K^{p,q},$$

and because

$$K^{n+1} = \bigoplus_{p+q=n+1} K^{p,q},$$

every summand in equation (3) is zero. We deduce that for $p + q = n + 1$

$$\begin{aligned} \partial(\bar{\partial}(x_{p,q-1})) &= \partial(\bar{\partial}(x_{p,q-1}) + \partial(x_{p-1,q})) \\ &= \partial(0) \\ &= 0. \end{aligned}$$

But this means that for arbitrary p and q

$$x_{p,q-1} \in \ker(\partial\bar{\partial}),$$

hence also

$$x \in \ker(\partial\bar{\partial}),$$

which had to be shown.
The last thing which has to be checked is that $\operatorname{im}(\partial) + \operatorname{im}(\bar{\partial})$ contains $\operatorname{im}(\mathrm{d})$. This clearly holds, since

$$\mathrm{d}(y) = \partial(y) + \bar{\partial}(y) \in \operatorname{im}(\partial) + \operatorname{im}(\bar{\partial})$$

for $y \in K$ arbitrary. This shows the existence of the map $\varphi_{\mathrm{d}\,-\,\mathrm{A}}$.
The commutativity of the diagram follows again, because all maps are induced by the identity on K. □

Note that this diagram can be restricted to subquotients of K^n instead of K.
The natural question arises if the maps in diagram (2) are injective or surjective. The answer is that for an arbitrary double complex this has not to be the case. In section 3 we will discuss this question in greater detail.

2.2 Spectral sequences

Let X be a complex manifold. If X is compact, one has the Frölicher inequality

$$\dim_{\mathbb{C}}(\mathrm{H}^k_{\mathrm{dR}}(X)) \leq \sum_{p+q=k} \dim_{\mathbb{C}}(\mathrm{H}^{p,q}_{\bar{\partial}}(X)),$$

which gives a relation between the dimensions of the Dolbeault cohomology and the de Rham cohomology of X ([Frö55]).
The Frölicher inequality can be proved using the so-called Frölicher spectral sequence. In this section we introduce spectral sequences in general and pay special attention to spectral sequences associated to a filtered complex. The main example will be the natural filtration(s) of a double complex. For a deeper discussion of this topic we refer to [GH14] and [Wei95].

Definition 2.7 (Spectral sequence)**.** A spectral sequence is a sequence $(E_r; \mathrm{d}_r)_{r\geq 0}$ of $\mathbb{Z}$-bigraded vector spaces

$$E_r = \bigoplus_{p,q} E_r^{p,q},$$

together with maps $\mathrm{d}_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ satisfying $\mathrm{d}_r^{p+r,q-r+1} \circ \mathrm{d}_r^{p,q} = 0$ such that

$$E_{r+1}^{p,q} = \mathrm{H}^{p,q}_{\mathrm{d}_r}(E_r) = \frac{\ker(\mathrm{d}_r^{p,q})}{\mathrm{im}(\mathrm{d}_r^{p-r,q+r-1})}.$$

If there is $r_0 \geq 0$ such that $E_{r_0} = E_{r_0+1} = \ldots$, we say that the spectral sequence degenerates at E_{r_0} or at sheet r_0 and write E_∞ for this limit term. Note that the degeneration at sheet r_0 is equivalent to the maps $\mathrm{d}_r^{p,q}$ being the zero map whenever $r \geq r_0$.

Note that a spectreal sequence degenerates at sheet r_0 if and only if $\mathrm{d}_r^{p,q} = 0$ for all (p,q) and $r \geq r_0$.
One example of a spectral sequence is the spectral sequence that is induced by a so-called filtration of a cochain complex.

Definition 2.8 (Filtered complex)**.** Let $(K; \mathrm{d})$ be a cochain complex. A filtration of $(K; \mathrm{d})$ is a sequence of subcomplexes $F^p(K) \subseteq K$, $p \in \mathbb{Z}$ such that $F^p(K) \subseteq F^{p-1}(K)$ for all p, $\bigcap_p F^p(K) = 0$, $\bigcup_p F^p(K) = K$ and $\mathrm{d}(F^p(K)) \subseteq F^p(K)$.
The associated graded complex is

$$G(K) = \bigoplus_{p\in\mathbb{Z}} G^p(K),$$

where

$$G^p(K) = F^p(K)/F^{p+1}(K).$$

This filtration on K induces a filtration on the cohomology $\mathrm{H}^n_{\mathrm{d}}(K)$ by

$$F^p\,\mathrm{H}^n_{\mathrm{d}}(K) = \frac{F^p(K^n) \cap \ker(\mathrm{d})}{F^p(K^n) \cap \mathrm{im}(\mathrm{d})}.$$

We sometimes say that $(K; \mathrm{d})$ is a filtered complex.
From now on we will always assume that the filtration is finite.

Definition 2.9. We say the differential is strictly compatible with the filtration or strict relative to the filtration if for all p we have

$$\mathrm{d}(F^p(K)) = \mathrm{im}(\mathrm{d}) \cap F^p(K)$$

or equivalently

$$\mathrm{d}(F^p(K^{n-1})) = \mathrm{im}(\mathrm{d}^{n-1}) \cap F^p(K^n)$$

for all p and n.

We can assign a spectral sequence to a filtered complex in the following way.

Proposition 2.10. Let $(K; \mathrm{d})$ be a complex together with a filtration F. Then there exists a spectral sequence $(E_r; \mathrm{d}_r)_{r\geq 0}$ with

$$\begin{aligned} E_0^{p,q} &= F^p(K^{p+q})/F^{p+1}(K^{p+q}), \\ E_1^{p,q} &= \mathrm{H}^{p+q}_{\mathrm{d}}(G^p(K)), \\ E_\infty^{p,q} &= G^p(\mathrm{H}^{p+q}_{\mathrm{d}}(K)). \end{aligned}$$

Remark 2.11. We say that the spectral sequence abuts to $\mathrm{H}_{\mathrm{d}}(K)$.

Proof. For the proof see [GH14] and [Wei95]. We here just note that the terms for $E_r^{p,q}$ are given by

$$E_r^{p,q} = \frac{F^p(K^{p+q}) \cap \mathrm{d}^{-1}(F^{p+r}(K^{p+q+1}))}{(F^{p+1}(K^{p+q-1}) + \mathrm{d}(F^{p-r+1}(K^{p+q-1}))) \cap (F^p(K^{p+q}) \cap \mathrm{d}^{-1}(F^{p+r}(K^{p+q+1})))}$$

and d_r is the map that is induced by d on the quotients. $\square$

In our case we will have a double complex $(K; \partial, \bar{\partial})$ with two filtrations defined to be

$$'F^p(K^n) = \bigoplus_{\substack{r+s=n \\ r\geq p}} K^{r,s}, \tag{4}$$

$$''F^q(K^n) = \bigoplus_{\substack{r+s=n \\ s\geq q}} K^{r,s}. \tag{5}$$

This induces two spectral sequences, which are denoted by $({}'E_r; {}'\mathrm{d}_r)$ and $({}''E_r; {}''\mathrm{d}_r)$, respectively. Their terms are the following.

$$
\begin{aligned}
{}'E_0^{p,q} = {}''E_0^{q,p} &= K^{p,q} && (6)\\
{}'\,\mathrm{d}_0^{p,q} &= \bar{\partial}^{p,q} && (7)\\
{}''\,\mathrm{d}_0^{p,q} &= \partial^{p,q} && (8)\\
{}'E_1^{p,q} &= \mathrm{H}_{\bar{\partial}}^{p,q}(K) && (9)\\
{}''E_1^{p,q} &= \mathrm{H}_{\partial}^{q,p}(K) && (10)\\
{}'E_\infty^{p,q} &= F^p(\mathrm{H}_{\mathrm{d}}^{p+q}(K))/F^{p+1}(\mathrm{H}_{\mathrm{d}}^{p+q}(K)) && (11)
\end{aligned}
$$

Again, we refer to [GH14] and [Wei95] for more details.
In the case of a complex manifold and its double complex, one has that

$$
{}''F^q(\mathcal{A}^k(X)) = \bigoplus_{\substack{r+s=n\\ s\geq q}} \mathcal{A}^{r,s}(X) = \bigoplus_{\substack{r+s=n\\ s\geq q}} \overline{\mathcal{A}^{s,r}(X)} = \overline{{}'F^q(\mathcal{A}^k(X))}.
$$

It follows that also the two induced spectral sequences and the filtrations on the cohomology are conjugate to each other

$$
\begin{aligned}
\overline{{}'F^p(\mathrm{H}_{\mathrm{dR}}^k(X))} &= {}''F^p(\mathrm{H}_{\mathrm{dR}}^k(X)),\\
\overline{{}'E_r^{p,q}} &= {}''E_r^{p,q}.
\end{aligned}
$$

In this context the spectral sequence induced by ${}'F$ is called the Frölicher spectral sequence.

3 The $\partial\bar{\partial}$-Lemma

In this section we wish to investigate the injectivity and surjectivity of the maps in diagram (2). In general, none of the maps has to be injective or surjective. Even if the double complex is the double complex of a complex manifold, no injectivity or surjectivity is guaranteed. The injectivity of the map $\varphi_{\mathrm{BC}-\mathrm{d}}$ is encoded in the so-called $\partial\bar{\partial}$-Lemma. P. Deligne, Ph. A. Griffiths, J. Morgan, and D. P. Sullivan stated in [DGMS75] that this injectivity has strong connections to the injectivity and surjectivity of the other maps in diagram (2). They stated that if $\varphi_{\mathrm{BC}-\mathrm{d}}$ is injective, then automatically all maps in this diagram are isomorphisms. In this section we will see what the injectivity of $\varphi^n_{\mathrm{BC}-\mathrm{d}}$ for some integer n means for the injectivity and surjectivity of the other maps between the cohomology spaces.
The next proposition gives equivalent formulations of the injectivity of $\varphi^n_{\mathrm{BC}-\mathrm{d}}$.

Theorem 3.1. Let $(K;\partial,\bar{\partial})$ be a double complex and $(K;\mathrm{d})$ its associated simple complex. Let $n\in\mathbb{Z}$. Then the following conditions are equivalent.

(1) $\varphi^n_{\mathrm{BC}-\mathrm{d}}:\mathrm{H}^n_{\mathrm{BC}}(K)\to\mathrm{H}^n_{\mathrm{d}}(K)$ is injective.

(2) $\varphi^n_{\mathrm{BC}-\partial}:\mathrm{H}^n_{\mathrm{BC}}(K)\to\mathrm{H}^n_{\partial}(K)$ and $\varphi^n_{\mathrm{BC}-\bar{\partial}}:\mathrm{H}^n_{\mathrm{BC}}(K)\to\mathrm{H}^n_{\bar{\partial}}(K)$ are injective.

(3) $\varphi^n_{\mathrm{BC}-\mathrm{A}}:\mathrm{H}^n_{\mathrm{BC}}(K)\to\mathrm{H}^n_{\mathrm{A}}(K)$ is injective.

(1*) $\varphi^{n-1}_{\mathrm{d}-\mathrm{A}}:\mathrm{H}^{n-1}_{\mathrm{d}}(K)\to\mathrm{H}^{n-1}_{\mathrm{A}}(K)$ is surjective.

(2*) $\varphi^{n-1}_{\partial-\mathrm{A}}:\mathrm{H}^{n-1}_{\partial}(K)\to\mathrm{H}^{n-1}_{\mathrm{A}}(K)$ and $\varphi^{n-1}_{\bar{\partial}-\mathrm{A}}:\mathrm{H}^{n-1}_{\bar{\partial}}(K)\to\mathrm{H}^{n-1}_{\mathrm{A}}(K)$ are surjective.

(3*) $\varphi^{n-1}_{\mathrm{BC}-\mathrm{A}}:\mathrm{H}^{n-1}_{\mathrm{BC}}(K)\to\mathrm{H}^{n-1}_{\mathrm{A}}(K)$ is surjective.

To prove theorem 3.1 we will use the following theorem.

Theorem 3.2. Let $(K;\partial,\bar{\partial})$ be a double complex and $(K;\mathrm{d})$ its associated simple complex. Let $n\in\mathbb{Z}$. Then the following conditions are equivalent.

(1) $\ker(\partial^n)\cap\ker(\bar{\partial}^n)\cap\mathrm{im}(\mathrm{d}^{n-1})=\mathrm{im}(\partial^{n-1}\bar{\partial}^{n-2})\subseteq K^n$

(2) (i) $\ker(\bar{\partial}^n)\cap\mathrm{im}(\partial^{n-1})=\mathrm{im}(\partial^{n-1}\bar{\partial}^{n-2})\subseteq K^n$ and
(ii) $\ker(\partial^n)\cap\mathrm{im}(\bar{\partial}^{n-1})=\mathrm{im}(\partial^{n-1}\bar{\partial}^{n-2})\subseteq K^n$

(3) $\ker(\partial^n)\cap\ker(\bar{\partial}^n)\cap(\mathrm{im}(\partial^{n-1})+\mathrm{im}(\bar{\partial}^{n-1}))=\mathrm{im}(\partial^{n-1}\bar{\partial}^{n-2})\subseteq K^n$

(1*) $\mathrm{im}(\partial^{n-2})+\mathrm{im}(\bar{\partial}^{n-2})+\ker(\mathrm{d}^{n-1})=\ker(\partial^n\bar{\partial}^{n-1})\subseteq K^{n-1}$

(2*) (i) $\mathrm{im}(\bar{\partial}^{n-2})+\ker(\partial^{n-1})=\ker(\partial^n\bar{\partial}^{n-1})\subseteq K^{n-1}$ and
(ii) $\mathrm{im}(\partial^{n-2})+\ker(\bar{\partial}^{n-1})=\ker(\partial^n\bar{\partial}^{n-1})\subseteq K^{n-1}$

(3*) $\mathrm{im}(\partial^{n-2})+\mathrm{im}(\bar{\partial}^{n-2})+(\ker(\partial^{n-1})\cap\ker(\bar{\partial}^{n-1}))=\ker(\partial^n\bar{\partial}^{n-1})\subseteq K^{n-1}$

Proof. In the first three statements the inclusion $\supseteq$ is always true for double complexes. In the last three statements the inclusion $\subseteq$ always holds. So, we only have to check the other inclusions.
(1) $\Rightarrow$ (2): To show that (1) implies (2)(i), let $x \in \ker(\bar{\partial}^n) \cap \operatorname{im}(\partial^{n-1})$. Then there is $y \in K^{n-1}$ with $\partial(y) = x$. Now fix $(p,q) \in \mathbb{Z}^2$ with $p+q = n$. The (p,q)-component $x_{p,q}$ of x satisfies $\bar{\partial}(x_{p,q}) = 0 \in K^{p,q+1}$ and $x_{p,q} = \partial(y_{p-1,q})$. Now consider $\mathrm{d}(y_{p-1,q}) \in K^n$. We have

$$\begin{aligned}\partial(\mathrm{d}(y_{p-1,q})) &= \partial(\partial(y_{p-1,q}) + \bar{\partial}(y_{p-1,q})) \\ &= \partial(\bar{\partial}(y_{p-1,q})) \\ &= -\bar{\partial}(\partial(y_{p-1,q})) \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\bar{\partial}(\mathrm{d}(y_{p-1,q})) &= \bar{\partial}(\partial(y_{p-1,q}) + \bar{\partial}(y_{p-1,q})) \\ &= \bar{\partial}(\partial(y_{p-1,q})) \\ &= 0.\end{aligned}$$

By (1), this implies $\mathrm{d}(y_{p-1,q}) \in \ker(\partial^n) \cap \ker(\bar{\partial}^n) \cap \operatorname{im}(\mathrm{d}^{n-1}) = \operatorname{im}(\partial^{n-1}\bar{\partial}^{n-2})$. So there is $z \in K^{n-2}$ with $\partial(\bar{\partial}(z)) = \mathrm{d}(y_{p-1,q})$. Hence $\partial(\bar{\partial}(z_{p-1,q-1}))$ is the (p,q)-component of $\mathrm{d}(y_{p-1,q})$, which is $x_{p,q}$. Since, by this, $x_{p,q} \in \operatorname{im}(\partial^{n-1}\bar{\partial}^{n-2})$ whenever $p+q=n$, also

$$x = \sum_{p+q=n} x_{p,q} \in \operatorname{im}(\partial^{n-1}\bar{\partial}^{n-2}).$$

Analogously one shows that (1) implies (2)(ii).
(2) $\Rightarrow$ (3): Let $x \in \ker(\partial^n) \cap \ker(\bar{\partial}^n) \cap (\operatorname{im}(\partial^{n-1}) + \operatorname{im}(\bar{\partial}^{n-1}))$. Hence, there are $y, z \in K^{n-1}$ with $x = \partial(y) + \bar{\partial}(z)$. Then

$$\bar{\partial}(\partial(y)) = \bar{\partial}(x - \bar{\partial}(z)) = 0.$$

Hence $\partial(y) \in \ker(\bar{\partial}^n) \cap \operatorname{im}(\partial^{n-1}) = \operatorname{im}(\partial^{n-1}\bar{\partial}^{n-2})$ by (2)(i). Similarly, by (2)(ii), $\bar{\partial}(z) \in \ker(\partial^n) \cap \operatorname{im}(\bar{\partial}^{n-1}) = \operatorname{im}(\partial^{n-1}\bar{\partial}^{n-2})$. This implies

$$x = \partial(y) + \bar{\partial}(z) \in \operatorname{im}(\partial^{n-1}\bar{\partial}^{n-2}).$$

(3) $\Rightarrow$ (1): Since $\mathrm{d} = \partial + \bar{\partial}$, we always have $\operatorname{im}(\mathrm{d}) \subseteq \operatorname{im}(\partial) + \operatorname{im}(\bar{\partial})$. Hence,

$$\begin{aligned}\ker(\partial^n) \cap \ker(\bar{\partial}^n) \cap \operatorname{im}(\mathrm{d}^{n-1}) &\subseteq \ker(\partial^n) \cap \ker(\bar{\partial}^n) \cap (\operatorname{im}(\partial^{n-1}) + \operatorname{im}(\bar{\partial}^{n-1})) \\ &= \operatorname{im}(\partial^{n-1}\bar{\partial}^{n-2}),\end{aligned}$$

by (3).
(2) $\Rightarrow$ (2*): Let $x \in \ker(\partial^n\bar{\partial}^{n-1})$. Then $\bar{\partial}(x) \in \ker(\partial^n) \cap \operatorname{im}(\bar{\partial}^{n-1})$. By (2)(ii), there is $y \in K^{n-2}$, which satisfies $\bar{\partial}(x) = \partial(\bar{\partial}(y))$. This yields

$$x = (x + \partial(y)) + (-\partial(y)) \in \ker(\bar{\partial}^{n-1}) + \operatorname{im}(\partial^{n-2}),$$

since $\bar{\partial}(x + \partial(y)) = \bar{\partial}(x) - \partial(\bar{\partial}(y)) = 0$. This shows (2*)(i) and (2*)(ii) works similarly.
(2*) $\Rightarrow$ (2): Let $x \in \ker(\bar{\partial}^n) \cap \operatorname{im}(\partial^{n-1})$ and $y \in K^{n-1}$ such that $\partial(y) = x$. Then $y \in \ker(\partial^n \bar{\partial}^{n-1})$. By (2*)(i), $y \in \operatorname{im}(\bar{\partial}^{n-2}) + \ker(\partial^{n-1})$. Then there are $z \in K^{n-2}$ and $w \in \ker(\partial^{n-1})$ such that $y = \bar{\partial}(z) + w$. Hence

$$x = \partial(y) = \partial(\bar{\partial}(z)) \in \operatorname{im}(\partial^{n-1}\bar{\partial}^{n-2}),$$

which shows (2)(i). Equation (2)(ii) can be proved analogously.
(1*) $\Rightarrow$ (2*): Let $x \in K^{n-1}$ satisfying $\partial(\bar{\partial}(x)) = 0$. Now fix $(p,q) \in \mathbb{Z}^2$ with $p+q = n-1$. The (p,q)-component $x_{p,q}$ of x satisfies $\partial(\bar{\partial}(x_{p,q})) = 0 \in K^{p+1,q+1}$. Then, by (1*), we can write

$$x_{p,q} = \partial(y) + \bar{\partial}(z) + w,$$

where $y \in K^{p-1,q}$, $z \in K^{p,q-1}$ and $w \in \ker(\mathrm{d}^{n-1})$. Note that, by this, $\partial(w) = 0 \in K^{p+1,q}$ and $\bar{\partial}(w) = 0 \in K^{p,q+1}$. It follows that

$$x_{p,q} = \partial(y) + (\bar{\partial}(z) + w) \in \operatorname{im}(\partial^{n-2}) + \ker(\bar{\partial}^{n-1}).$$

and

$$x_{p,q} = (\partial(y) + w) + \bar{\partial}(z) \in \operatorname{im}(\bar{\partial}^{n-2}) + \ker(\partial^{n-1}),$$

which yields equations (2*)(i) and (2*)(ii).
(2*) $\Rightarrow$ (3*): Let $x \in K^{n-1}$ satisfying $\partial(\bar{\partial}(x)) = 0$. Then, by (2*)(ii), we can write $x = \partial(a) + b$ with $a \in K^{n-2}$ and $b \in \ker(\bar{\partial}^{n-1})$. Since $\ker(\bar{\partial}^{n-1}) \subseteq \ker(\partial^n \bar{\partial}^{n-1})$, we may write $b = \bar{\partial}(y) + z$ with $y \in K^{n-2}$ and $z \in \ker(\partial^{n-1})$, by equation (2*)(i). Then also $\bar{\partial}(z) = \bar{\partial}(b - \bar{\partial}(y)) = 0$. This shows that

$$x = \partial(a) + \bar{\partial}(y) + z \in \operatorname{im}(\partial^{n-2}) + \operatorname{im}(\bar{\partial}^{n-2}) + (\ker(\partial^{n-1}) \cap \ker(\bar{\partial}^{n-1})),$$

which we wanted to show.
(3*) $\Rightarrow$ (1*): Since $\mathrm{d} = \partial + \bar{\partial}$, we always have $\ker(\partial) \cap \ker(\bar{\partial}) \subseteq \ker(\mathrm{d})$. So

$$\begin{aligned}\ker(\partial^n \bar{\partial}^{n-1}) &= \operatorname{im}(\partial^{n-2}) + \operatorname{im}(\bar{\partial}^{n-2}) + (\ker(\partial^{n-1}) \cap \ker(\bar{\partial}^{n-1})) \\ &\subseteq \operatorname{im}(\partial^{n-2}) + \operatorname{im}(\bar{\partial}^{n-2}) + \ker(\mathrm{d}^{n-1}),\end{aligned}$$

which proves the claim. $\square$

We can easily prove theorem 3.1 now.

Proof (of theorem 3.1). For the proof we note what the statements mean in terms of kernels and images.

$$\begin{array}{ll}
\varphi^n_{\mathrm{BC}-\mathrm{d}} & \text{injective} \Leftrightarrow \ker(\partial^n)\cap\ker(\bar\partial^n)\cap\operatorname{im}(\mathrm{d}^{n-1}) = \operatorname{im}(\partial^{n-1}\bar\partial^{n-2})\\
\varphi^n_{\mathrm{BC}-\partial} & \text{injective} \Leftrightarrow \ker(\bar\partial^n)\cap\operatorname{im}(\partial^{n-1}) = \operatorname{im}(\partial^{n-1}\bar\partial^{n-2})\\
\varphi^n_{\mathrm{BC}-\bar\partial} & \text{injective} \Leftrightarrow \ker(\partial^n)\cap\operatorname{im}(\bar\partial^{n-1}) = \operatorname{im}(\partial^{n-1}\bar\partial^{n-2})\\
\varphi^n_{\mathrm{BC}-\mathrm{A}} & \text{injective} \Leftrightarrow \ker(\partial^n)\cap\ker(\bar\partial^n)\cap(\operatorname{im}(\partial^{n-1})+\operatorname{im}(\bar\partial^{n-1})) = \operatorname{im}(\partial^{n-1}\bar\partial^{n-2})\\
\varphi^{n-1}_{\mathrm{d}-\mathrm{A}} & \text{surjective} \Leftrightarrow \operatorname{im}(\partial^{n-2})+\operatorname{im}(\bar\partial^{n-2})+\ker(\mathrm{d}^{n-1}) = \ker(\partial^n\bar\partial^{n-1})\\
\varphi^{n-1}_{\partial-\mathrm{A}} & \text{surjective} \Leftrightarrow \operatorname{im}(\bar\partial^{n-2})+\ker(\partial^{n-1}) = \ker(\partial^n\bar\partial^{n-1})\\
\varphi^{n-1}_{\bar\partial-\mathrm{A}} & \text{surjective} \Leftrightarrow \operatorname{im}(\partial^{n-2})+\ker(\bar\partial^{n-1}) = \ker(\partial^n\bar\partial^{n-1})\\
\varphi^{n-1}_{\mathrm{BC}-\mathrm{A}} & \text{surjective} \Leftrightarrow \operatorname{im}(\partial^{n-2})+\operatorname{im}(\bar\partial^{n-2})+(\ker(\partial^{n-1})\cap\ker(\bar\partial^{n-1})) = \ker(\partial^n\bar\partial^{n-1})
\end{array}$$

Now theorem 3.2 yields the claim. □

If one of the properties of theorem 3.2 is satisfied, then we say that the double complex satisfies the $\partial\bar\partial$-Lemma in degree n.
If one property is fulfilled for all n, the double complex is said to satisfy the $\partial\bar\partial$-Lemma. Similarly, we say a complex manifold X satisfies the $\partial\bar\partial$-Lemma (in degree n) if its associated double complex does.
Besides these equivalences, we have several other consequences of satisfying the $\partial\bar\partial$-Lemma, which we state in the following proposition. But first we will give a lemma which will be helpful for its proof.

Lemma 3.3. Let X be a vector space over some field and $U, V, W \subseteq X$ linear subspaces such that $U \subseteq V$. Then $V \cap (U + W) \subseteq U + (V \cap W)$.

Proof. Let $x \in V \cap (U + W)$. Then we may write $x = u + w$ with $u \in U$ und $w \in W$. This means $w = x - u \in V$. Since, by assumption, already $w \in W$, this shows that $x \in U + (V \cap W)$. □

Proposition 3.4. Let $n \in \mathbb{Z}$ and suppose that the double complex satisfies the $\partial\bar\partial$-Lemma in degree n. Then

1. all maps in diagram (2) are injective in degree n and
2. all maps in diagram (2) are surjective in degree $n - 1$.

Proof. Theorem 3.1 yields the injectivity of the maps with domain $\mathrm{H}^n_{\mathrm{BC}}(K)$ and the surjectivity of the maps that have $\mathrm{H}^{n-1}_{\mathrm{A}}(K)$ as codomain.
We start by showing that $\varphi^{n-1}_{\mathrm{BC}-\mathrm{d}}$ is surjective. We have to check that

$$\ker(\mathrm{d}^{n-1}) = \operatorname{im}(\mathrm{d}^{n-2}) + (\ker(\partial^{n-1}) \cap \ker(\bar\partial^{n-1})).$$

The inclusion from right to left is obvious. For the other inclusion let $x \in \ker(\mathrm{d}^{n-1})$. Then

$$\bar\partial(\partial(x)) = \bar\partial(-\bar\partial(x)) = 0$$

and hence

$$\partial(x) \in \ker(\bar{\partial}^n) \cap \operatorname{im}(\partial^{n-1}).$$

By equation (2)(i), we have $\partial(x) \in \operatorname{im}(\partial^{n-1}\bar{\partial}^{n-2})$. So let $a \in K^{n-2}$ such that

$$\partial(\bar{\partial}(a)) = \partial(x). \tag{12}$$

Then also

$$\bar{\partial}(x) = -\partial(x) = -\partial(\bar{\partial}(a)) = \bar{\partial}(\partial(a)).$$

Hence, we can write

$$x = \mathrm{d}(a) + (x - \mathrm{d}(a))$$

with $x - \mathrm{d}(a) \in \ker(\partial^{n-1}) \cap \ker(\bar{\partial}^{n-1})$. Indeed

$$\partial(x - \mathrm{d}(a)) = \partial(x - \bar{\partial}(a)) - \partial(\partial(a)) = 0$$

by equation (12) and also

$$\begin{aligned} \bar{\partial}(x - \mathrm{d}(a)) &= \mathrm{d}(x - \mathrm{d}(a)) - \partial(x - \mathrm{d}(a)) \\ &= \mathrm{d}(x) - \mathrm{d}(\mathrm{d}(a)) \\ &= 0. \end{aligned}$$

Now we want to prove that $\varphi^{n-1}_{\mathrm{BC}-\partial}$ is surjective. This map is surjective if and only if

$$\ker(\partial^{n-1}) = (\ker(\partial^{n-1}) \cap \ker(\bar{\partial}^{n-1})) + \operatorname{im}(\partial^{n-2}).$$

The inclusion from right to left always holds. Furthermore

$$\begin{aligned} \ker(\partial^{n-1}) &= \ker(\partial^{n-1}) \cap \ker(\partial^n\bar{\partial}^{n-1}) \\ &= \ker(\partial^{n-1}) \cap (\operatorname{im}(\partial^{n-2}) \cap \ker(\bar{\partial}^{n-1})) \\ &\subseteq \operatorname{im}(\partial^{n-2}) + (\ker(\partial^{n-1}) \cap \ker(\bar{\partial}^{n-1})), \end{aligned}$$

where we use equation (2*)(ii) in the second line and lemma 3.3 in the third one. The surjectivity of $\varphi^{n-1}_{\mathrm{BC}-\bar{\partial}}$ is proved analogously.
Next, we want to show that $\varphi^n_{\mathrm{d}-\mathrm{A}}$ is injective, that is

$$\operatorname{im}(\mathrm{d}^{n-1}) = (\operatorname{im}(\partial^{n-1}) + \operatorname{im}(\bar{\partial}^{n-1})) \cap \ker(\mathrm{d}^n).$$

The inclusion $\subseteq$ is clear. So let $x \in \ker(\mathrm{d}^n) \cap (\operatorname{im}(\partial^{n-1}) + \operatorname{im}(\bar{\partial}^{n-1}))$. Let $a, b \in K^{n-1}$ such that

$$x = \partial(a) + \bar{\partial}(b).$$

Then $b - a \in \ker(\partial^n \partial^{n-1})$, because

$$\begin{aligned} 0 &= \mathrm{d}(x) \\ &= \mathrm{d}(\partial(a) + \bar{\partial}(b)) \\ &= \bar{\partial}(\partial(a)) + \partial(\bar{\partial}(b)) \\ &= \partial(\bar{\partial}(b - a)). \end{aligned}$$

By equation (2*)(ii), $x \in \operatorname{im}(\partial^{n-2}) + \ker(\bar{\partial}^{n-1})$. So there are $r \in K^{n-2}$ and $s \in \ker(\bar{\partial}^{n-1})$ such that

$$b - a = \partial(r) + s.$$

We infer that

$$\begin{aligned} x &= \partial(a) + \bar{\partial}(b) \\ &= \partial(a) + \bar{\partial}(a) + \bar{\partial}(b - a) \\ &= \mathrm{d}(a) + \bar{\partial}(\partial(r) + s) \\ &= \mathrm{d}(a) + \bar{\partial}(\partial(r)) \\ &= \mathrm{d}(a + \partial(r)), \end{aligned}$$

which is an element of $\operatorname{im}(\mathrm{d}^{n-1})$.
The last thing we have to check is that $\operatorname{im}(\partial^{n-1}) = \ker(\partial^n) \cap (\operatorname{im}(\partial^{n-1}) + \operatorname{im}(\bar{\partial}^{n-1}))$, which yields the injectivity of $\varphi^n_{\partial-\mathrm{A}}$ and, by analogy, the injectivity of $\varphi^n_{\bar{\partial}-\mathrm{A}}$. Again, the inclusion from left to right is clear. For the other inlusion we compute, using lemma 3.3 and equation (2)(ii),

$$\begin{aligned} \ker(\partial^n) \cap (\operatorname{im}(\partial^{n-1}) + \operatorname{im}(\bar{\partial}^{n-1})) &\subseteq \operatorname{im}(\partial^{n-1}) + (\ker(\partial^n) \cap \operatorname{im}(\bar{\partial}^{n-1})) \\ &= \operatorname{im}(\partial^{n-1}) + \operatorname{im}(\partial^{n-1}\bar{\partial}^{n-2}) \\ &= \operatorname{im}(\partial^{n-1}), \end{aligned}$$

which finishes the proof. □

One gets several easy corollaries from this if one assumes that the $\partial\bar{\partial}$-Lemma holds in several degrees. For instance, we get the following statement from [DGMS75] easily.

Corollary 3.5. If a double complex satisfies the $\partial\bar{\partial}$-Lemma, then all maps in diagram (2) are isomorphisms.

4 The $\partial\bar{\partial}$-Lemma and the Frölicher spectral sequence

In this section we want to recall the result from [DGMS75] that satisfying the $\partial\bar{\partial}$-Lemma is equivalent to the degeneration of the Frölicher spectral sequence at the first sheet and the fact that the two induced filtrations on the cohomology are n-opposite. We will improve this statement in section 5. For the proof of this theorem we will need the following proposition.

Proposition 4.1. Let $(K; \mathrm{d})$ be a cochain complex with a filtration F. Then the following are equivalent.

1. The corresponding spectral sequence degenerates at E_1.
2. The differential d is strictly compatible with the filtration.

Proof. We refer to [Del72]. In section 5 we will prove a stronger, degreewise version of this result. ☐

Theorem 4.2. Let $(K; \partial, \bar{\partial})$ be a bounded double complex and $(K; \mathrm{d})$ its associated simple complex. Then the following conditions are equivalent.

(1) $(K; \partial, \bar{\partial})$ satisfies the $\partial\bar{\partial}$-Lemma.

(2) The double complex is a possibly infinite sum of double complexes of the following types:

- (α) There is a pair (r, s) such that $K^{p,q} = 0$ if $(p, q) \neq (r, s)$, and $\partial = \bar{\partial} = 0$.
- (β) Complexes which are a square of isomorphisms. This means that there is a pair (r, s) such that $K^{p,q} = 0$ if $(p, q) \notin \{(r, s), (r+1, s), (r, s+1), (r+1, s+1)\}$ and

$$\begin{tikzcd} K^{r,s+1} \arrow[r, "\partial^{r,s+1}", "\cong"'] & K^{r+1,s+1} \\ K^{r,s} \arrow[u, "\bar{\partial}^{r,s}", "\cong"'] \arrow[r, "\cong", "\partial^{r,s}"'] & K^{r+1,s} \arrow[u, "\cong", "\bar{\partial}^{r+1,s}"'] \end{tikzcd} .$$

  is a square of isomorphisms.

(3) The two spectral sequences induced by $'F(K)$ and $''F(K)$ (cf. (4) and (5)) degenerate both at sheet 1 and the two induced filtrations on $\mathrm{H}^n_{\mathrm{d}}(K)$ are n-opposite, that is

$$'F^p(\mathrm{H}^n_{\mathrm{d}}(K)) \oplus {}''F^q(\mathrm{H}^n_{\mathrm{d}}(K)) \cong \mathrm{H}^n_{\mathrm{d}}(K)$$

for $n = p + q - 1$.

Proof. We will prove the equivalence of statements (1) and (3) later as a degreewise result, which gives this as an easy corollary. Nevertheless, we give the complete proof from [DGMS75] here. On the one hand because we have the decomposition property (2), and on the other hand the later proof of (3) $\Rightarrow$ (1) is motivated by this one.

(1) $\Rightarrow$ (2): Let $(r,s) \in \mathbb{Z}^2$ and $S^{r,s} \subseteq K^{r,s}$ be such that

$$S^{r,s} \oplus \ker(\partial^{r,s+1}\bar{\partial}^{r,s}) = K^{r,s},$$

and $T^{r,s} \subseteq \ker(\partial^{r,s}) \cap \ker(\bar{\partial}^{r,s})$ such that

$$T^{r,s} \oplus \mathrm{im}(\partial^{r-1,s}\bar{\partial}^{r-1,s-1}) = \ker(\partial^{r,s}) \cap \ker(\bar{\partial}^{r,s}) \subseteq \ker(\partial^{r,s+1}\bar{\partial}^{r,s}).$$

By corollary 3.5, the map $\varphi^{r,s}_{\mathrm{BC-A}}$ is an isomorphism. This yields that

$$T^{r,s} \oplus (\mathrm{im}(\partial^{r-1,s}) + \mathrm{im}(\bar{\partial}^{r,s-1})) = \ker(\partial^{r,s+1}\bar{\partial}^{r,s}).$$

So we get the three following properties

(*a*) $K^{r,s} = S^{r,s} \oplus T^{r,s} \oplus (\mathrm{im}(\partial^{r-1,s}) + \mathrm{im}(\bar{\partial}^{r,s-1}))$,

(*b*) $\partial^{r,s}(T^{r,s}) = 0 = \bar{\partial}^{p,q}(T^{r,s})$ and

(*c*) $S^{r,s} \cap \ker(\partial^{r,s+1}\bar{\partial}^{r,s}) = 0$,

which hold for all $(r,s) \in \mathbb{Z}^2$.

Now fix $(p,q) \in \mathbb{Z}$. Then equation (a) applied first for $(r,s) = (p-1,q)$ and then for $(r,s) = (p-1,q-1)$ yields

$$\begin{aligned}
\mathrm{im}(\partial^{p-1,q}) &= \partial(K^{p-1,q}) \\
&= \partial(S^{p-1,q} + T^{p-1,q} + \mathrm{im}(\partial^{p-2,q}) + \mathrm{im}(\bar{\partial}^{p-1,q-1})) \\
&= \partial(S^{p-1,q}) + \partial(T^{p-1,q}) + \partial(\partial(K^{p-2,q})) + \\
&\qquad \partial(\bar{\partial}(S^{p-1,q-1} + T^{p-1,q-1} + \mathrm{im}(\partial^{p-2,q-1}) + \mathrm{im}(\bar{\partial}^{p-1,q-2}))) \\
&= \partial(S^{p-1,q}) + \partial(\bar{\partial}(S^{p-1,q-1})).
\end{aligned}$$

In the same way we get

$$\mathrm{im}(\bar{\partial}^{p,q-1}) = \bar{\partial}(S^{p,q-1}) + \bar{\partial}(\partial(S^{p-1,q-1})).$$

So going back to equation (a) for $(r,s) = (p,q)$, we see

$$K^{p,q} = S^{p,q} \oplus T^{p,q} \oplus (\partial(S^{p-1,q}) + \bar{\partial}(S^{p,q-1}) + \partial(\bar{\partial}(S^{p-1,q-1}))).$$

Now we want to show that the sum in parentheses is in fact a direct sum. So, suppose there are $a \in S^{p-1,q}$, $b \in S^{p,q-1}$ and $c \in S^{p-1,q-1}$ such that

$$\partial(a) + \bar{\partial}(b) + \partial(\bar{\partial}(c)) = 0.$$

Applying ∂ and $\bar{\partial}$ to this equation yields

$$0 = \partial(\bar{\partial}(b)) \text{ and } 0 = \bar{\partial}(\partial(a)).$$

But by (c), this shows $a = b = 0$. This implies $\partial(\bar{\partial}(c)) = 0$, which again, by (c), yields that $c = 0$. Hence we get

$$K^{p,q} = T^{p,q} \oplus S^{p,q} \oplus \partial(S^{p-1,q}) \oplus \bar{\partial}(S^{p,q-1}) \oplus \partial(\bar{\partial}(S^{p-1,q-1})). \tag{13}$$

If we now define for each (p, q) the double complexes $L_{p,q}$ as

$$\begin{tikzcd}
 & 0 & 0 & \\
0 \arrow[r] & \bar{\partial}(S^{p,q}) \arrow[r] \arrow[u] & \partial(\bar{\partial}(S^{p,q})) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & S^{p,q} \arrow[r] \arrow[u] & \partial(S^{p,q}) \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

and $M_{p,q}$ as

$$\begin{tikzcd}
 & 0 & \\
0 \arrow[r] & T^{p,q} \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] &
\end{tikzcd}$$

then we get, by equation (13), that

$$K = \bigoplus_{p,q} L_{p,q} \oplus \bigoplus_{p,q} M_{p,q},$$

where $M_{p,q}$ is of type (α) and $L_{p,q}$ of type (β).
(2) $\Rightarrow$ (3): It is enough to prove the claim for double complexes that have one of the types (α) or (β). So, first consider the double complex

$$\begin{tikzcd}
 & 0 & \\
0 \arrow[r] & K^{p,q} \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] &
\end{tikzcd}$$

for some $(p,q) \in \mathbb{Z}$. Then, by equation (6), we have

$$'E_1^{r,s} = \mathrm{H}_{\bar\partial}^{r,s}(K) = \begin{cases} 0 & (r,s) \neq (p,q) \\ K^{p,q} & (r,s) = (p,q). \end{cases}$$

It follows that

$$'E_2^{r,s} = \mathrm{H}_{\mathrm{d}_1}^{r,s}('E_1) = {}'E_1^{r,s}$$

for all $(r,s) \in \mathbb{Z}^2$. This shows by induction that the spectral sequence degenerates at $'E_1$. In the same way one shows that the other spectral sequence degenerates at $''E_1$. Now we want to show that the filtrations on the cohomology are n-opposite. In our case B^n is zero for all $n \in \mathbb{Z}$, so

$$'F^r\,\mathrm{H}_{\mathrm{d}}^m(K) = \bigoplus_{\substack{a+b=m \\ a \geq r}} K^{a,b} \cap \mathrm{Z}^m$$

and

$$''F^s\,\mathrm{H}_{\mathrm{d}}^m(K) = \bigoplus_{\substack{a+b=m \\ b \geq s}} K^{a,b} \cap \mathrm{Z}^m$$

for all $r, s \in \mathbb{N}$ and $m \in \mathbb{Z}$. If now $m = r+s-1 \neq p+q$, then

$$\mathrm{H}_{\mathrm{d}}^m(K) = 0,$$

since $\mathrm{Z}^m \subseteq K^m = 0$ here. Hence

$$'F^r\,\mathrm{H}_{\mathrm{d}}^m(K) \oplus {}''F^s\,\mathrm{H}_{\mathrm{d}}^m(K) = \mathrm{H}_{\mathrm{d}}^m(K).$$

If $m = r+s+1 = p+q$, then

$$\begin{aligned} 'F^r\,\mathrm{H}_{\mathrm{d}}^m(K) + {}''F^s\,\mathrm{H}_{\mathrm{d}}^m(K) &= (\bigoplus_{\substack{a+b=m \\ a \geq r}} K^{a,b} \cap \mathrm{Z}^m) + (\bigoplus_{\substack{a+b=m \\ b \geq s}} K^{a,b} \cap \mathrm{Z}^m) \\ &= (\bigoplus_{\substack{a+b=m \\ a \geq r}} K^{a,b} \cap \mathrm{Z}^m) + (\bigoplus_{\substack{a+b=m \\ a \leq r-1}} K^{a,b} \cap \mathrm{Z}^m) \\ &= (\bigoplus_{\substack{a+b=m \\ a \geq r}} K^{a,b} \cap K^{p,q}) + (\bigoplus_{\substack{a+b=m \\ a \leq r-1}} K^{a,b} \cap K^{p,q}) \\ &= K^{p,q} \\ &= \mathrm{H}_{\mathrm{d}}^m(K). \end{aligned}$$

Furthermore,

$$
\begin{aligned}
{}'F^r \,\mathrm{H}^m_{\mathrm{d}}(K) \cap {}''F^s \,\mathrm{H}^m_{\mathrm{d}}(K) &= \Big(\bigoplus_{\substack{a+b=m\\ a\geq r}} K^{a,b} \cap \mathrm{Z}^m\Big) \cap \Big(\bigoplus_{\substack{a+b=m\\ b\geq s}} K^{a,b} \cap \mathrm{Z}^m\Big)\\
&= \Big(\bigoplus_{\substack{a+b=m\\ a\geq r}} K^{a,b} \cap \mathrm{Z}^m\Big) \cap \Big(\bigoplus_{\substack{a+b=m\\ a\leq r-1}} K^{a,b} \cap \mathrm{Z}^m\Big)\\
&= \Big(\bigoplus_{\substack{a+b=m\\ a\geq r}} K^{a,b} \cap K^{p,q}\Big) \cap \Big(\bigoplus_{\substack{a+b=m\\ a\leq r-1}} K^{a,b} \cap K^{p,q}\Big)\\
&= 0,
\end{aligned}
$$

which proves the claim in case (α).
Now consider the double complex

$$
\begin{tikzcd}
 & 0 & 0 & \\
0 \arrow[r] & K^{p,q+1} \arrow[u] \arrow[r, "\cong"] & K^{p+1,q+1} \arrow[u] \arrow[r] & 0\\
0 \arrow[r] & K^{p,q} \arrow[u, "\cong"] \arrow[r, "\cong"'] & K^{p+1,q} \arrow[u, "\cong"'] \arrow[r] & 0\\
 & 0 \arrow[u] & 0 \arrow[u] & ,
\end{tikzcd}
$$

which is of type (β). Here we have that

$$
\begin{aligned}
{}'E_1^{r,s} &= \mathrm{H}^{r,s}_{\bar\partial}(K) = 0,\\
{}''E_1^{r,s} &= \mathrm{H}^{r,s}_{\partial}(K) = 0
\end{aligned}
$$

for all $(r,s) \in \mathbb{Z}^2$. It follows that both spectral sequences degenerate at the first sheet. Now we will show for all n that $\mathrm{H}^n_{\mathrm{d}}(K) = 0$. Hence the filtration will be n-opposite in case (β). For this purpose, let $n \in \mathbb{Z}$. If $n \notin \{p+q, p+q+1, p+q+2\}$, this is obvious, since K^n is trivial. For $x \in \mathrm{Z}^{p+q} \subseteq K^{p,q}$ we have

$$0 = \mathrm{d}(x) = \partial(x) + \bar\partial(x).$$

But this is a bidegree decomposition in degree $p+q+1$, since x is of pure bidegree. So $\partial(x) = \bar\partial(x) = 0$. With the injectivity of ∂ and $\bar\partial$ it follows that $x = 0$ and in particular

$$\mathrm{H}^{p+q}_{\mathrm{d}}(K) = \mathrm{Z}^{p+q} \,/\, \mathrm{B}^{p+q} = 0.$$

Now we choose $z \in \mathrm{Z}^{p+q+1}$. We can write $z = x + y$ with $x \in K^{p+1,q}$ and $y \in K^{p,q+1}$. Since $\partial^{p,q}$ and $\bar\partial^{p,q}$ are both surjective, there are $a, b \in K^{p,q}$ satisfying $\partial(a) = x$ and

$\bar{\partial}(b) = y$. We compute

$$\begin{aligned} 0 &= \mathrm{d}^{p+q+1}(x+y) \\ &= \bar{\partial}^{p+1,q}(x) + \partial^{p,q+1}(y) \\ &= \bar{\partial}^{p+1,q}(\partial^{p,q}(a)) + \partial^{p,q+1}(\bar{\partial}^{p,q}(b)) \\ &= \bar{\partial}^{p+1,q}(\partial^{p,q}(a-b)). \end{aligned}$$

Since $\partial^{p,q}$ and $\bar{\partial}^{p+1,q}$ are injective, it follows that $a = b$ and hence $z = \mathrm{d}(a)$. We infer that $\mathrm{Z}^{p+q+1} = \mathrm{B}^{p+q+1}$, which proves the claim in this case.
Now take $x \in \mathrm{Z}^{p+q+2} \subseteq K^{p+1,q+1}$. By surjectivity of $\partial^{p,q+1}$, there is $a \in K^{p,q+1}$ such that $x = \partial(a)$. But then also $x = \mathrm{d}(a)$, because $\bar{\partial}^{p,q+1} = 0$. This finishes the case of $\mathrm{H}_{\mathrm{d}}^{p+q+2}(K)$.
Hence $\mathrm{H}_{\mathrm{d}}(K) = 0$, so the filtrations are n-opposite.
(3) $\Rightarrow$ (1): Let $(p,q) \in \mathbb{Z}^2$. We want to show that equation (2)(i) in theorem 3.2 holds in degree $n = p+q$. For this assume $x \in \ker(\bar{\partial}^{p,q}) \cap \mathrm{im}(\partial^{p-1,q})$. We have to prove that $x \in \mathrm{im}(\partial^{p-1,q}\bar{\partial}^{p-1,q-1})$. Since $\mathrm{im}(\partial^{p-1,q}) \subseteq \ker(\partial^{p,q})$, it follows that

$$x \in \ker(\partial^{p,q}) \cap \ker(\bar{\partial}^{p,q}) \subseteq \ker(\mathrm{d}^{p+q}).$$

Now we choose $y \in K^{p-1,q}$ such that $x = \partial(y)$. Then the class of x in $\mathrm{H}_{\mathrm{d}}(K)$ is the same as the class of

$$x - \mathrm{d}(y) = -\bar{\partial}(y).$$

But, by assumption, the filtration on $\mathrm{H}_{\mathrm{d}}^n(K)$ is n-opposite, that is

$${}'F^p(\mathrm{H}_{\mathrm{d}}^n(K)) \oplus {}''F^{q+1}(\mathrm{H}_{\mathrm{d}}^n(K)) = \mathrm{H}_{\mathrm{d}}^n(K).$$

But the class of x is in the first and the class of $-\bar{\partial}(y)$ is in the second summand. It follows that the class of x is zero, hence $x \in \mathrm{im}(\mathrm{d}^{p+q-1})$.
By proposition 4.1, the differential d is strict relative to both filtrations. In our case this means that there are $a \in {}'F^p(K^{n-1})$ and $b \in {}''F^q(K^{n-1})$ such that $x = \mathrm{d}(a) = \mathrm{d}(b)$. It follows that $a - b \in \ker(\mathrm{d}^{n-1})$ and since

$$\mathrm{H}_{\mathrm{d}}^{n-1}(K) = {}'F^p(\mathrm{H}_{\mathrm{d}}^{n-1}(K)) \oplus {}''F^q(\mathrm{H}_{\mathrm{d}}^{n-1}(K)),$$

the class of $a-b$ in $\mathrm{H}_{\mathrm{d}}^{n-1}(K)$ is the sum of an element in ${}'F^p(\mathrm{H}_{\mathrm{d}}^{n-1}(K))$ and an element in ${}''F^q(\mathrm{H}_{\mathrm{d}}^{n-1}(K))$. Say

$$a - b = u + v + \mathrm{d}(w)$$

with $u \in {}'F^p(K^{n-1}) \cap \ker(\mathrm{d}^{n-1})$, $v \in {}''F^q(K^{n-1}) \cap \ker(\mathrm{d}^{n-1})$ and $w \in K^{n-2}$. If we write

$$v = \sum_{r \geq p} v_{r,s},$$

then we see that $\bar{\partial}(v_{p,q-1}) = 0$, since

$$\begin{aligned} 0 &= \mathrm{d}(v) \\ &= \bar{\partial}(v_{p,q-1}) + (\partial(v_{p,q-1}) + \bar{\partial}(v_{p+1,q-2})) + \ldots + \partial(v_{p+q-1,0}) \end{aligned}$$

is the bidegree decomposition of $\mathrm{d}(v)$. It follows that

$$\begin{aligned} x &= \bar{\partial}(a_{p,q-1}) \\ &= \bar{\partial}(v_{p,q-1} + \partial(w_{p-1,q-1}) + \bar{\partial}(w_{p-2,q})) \\ &= -\partial(\bar{\partial}(w_{p-1,q-1})) \in \mathrm{im}(\partial^{p-1,q}\bar{\partial}^{p-1,q-1}), \end{aligned}$$

which finishes the proof. □

4.1 Hodge structures

In the case that the double complex is the double complex of a complex manifold we can rephrase the condition that the two induced filtrations on $\mathrm{H}^n_{\mathrm{d}}(K)$ are n-opposite by the existence of a so-called Hodge structure on the cohomology. We want to introduce the basic concept here and refer to [PS08] for more details.

Definition 4.3 (Hodge structure)**.** Let V be a real vector space and $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ its complexification. A Hodge structure of weight k on V is a decomposition

$$V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q}$$

of $V_{\mathbb{C}}$, satisfying

$$V^{p,q} = \overline{V^{q,p}}.$$

One way to obtain a Hodge structure of weight k on V is via a decreasing filtration $F(V_{\mathbb{C}})$ on $V_{\mathbb{C}}$. It has to satisfy

$$F^p(V_{\mathbb{C}}) \cap \overline{F^q(V_{\mathbb{C}})} = 0$$

whenever $p + q - 1 = k$. If $p + q = k$, we set

$$V^{p,q} = F^p(V_{\mathbb{C}}) \cap \overline{F^q(V_{\mathbb{C}})}.$$

Then the $V^{p,q}$ induce a Hodge structure of weight k on V.
Note that conversely one can obtain a filtration on $V_{\mathbb{C}}$ from a given Hodge structure by setting

$$F^p(V_{\mathbb{C}}) = \bigoplus_{\substack{r+s=k \\ r \geq p}} V^{r,s}.$$

Furthermore, these two methods are inverse to each other.
With this notion introduced, we can reformulate the equivalence of statements (1) and (3) of theorem 4.2 in the following way.

Proposition 4.4. Let X be a complex manifold. Then X satisfies the $\partial\bar{\partial}$-Lemma if and only if

(a) the Frölicher spectral sequence degenerates at the first sheet and

(b) the filtration on $\mathrm{H}_{\mathrm{dR}}(X)$, which is induced by the natural filtration on the space of differential forms on X, induces a Hodge structure of weight k on $\mathrm{H}_{\mathrm{dR}}^{k}(X)$.

5 Degreewise results

In the previous section we have seen the result of P. Deligne, Ph. A. Griffiths, J. Morgan, and D. P. Sullivan that the $\partial\bar{\partial}$-Lemma holds if and only if the Frölicher spectral sequence degenerates at the first sheet and the filtrations on the cohomology are n-opposite. Now we want to study this relation degreewise. The starting point will be to assume that the $\partial\bar{\partial}$-Lemma does not hold for all degrees but only for one fixed degree n. In section 3 we have seen what this means for the surjectivity and injectivity of the maps between the cohomologies. In this section we want to study what the $\partial\bar{\partial}$-Lemma in one degree means in terms of the degeneration of the spectral sequences and filtrations on the cohomology. It turns out that the $\partial\bar{\partial}$-Lemma holds in degree n if and only if the maps $\mathrm{d}_r^{k,n-k-1}$ are zero for all k and the induced filtrations on the cohomology have trivial intersection in degree n and they span the whole cohomology space in degree $n-1$.
To prove this result we also need a degreewise formulation of proposition 4.1, with which we start.

Proposition 5.1. Let $(K; \mathrm{d})$ be a cochain complex with a filtration F and let $(E_r, \mathrm{d}_r)_{r\in\mathbb{N}}$ be the induced spectral sequence. Let $n, p \in \mathbb{Z}$. Assume

$$\mathrm{d}_r^{k,n-k} : E_r^{k,n-k} \to E_r^{k+r,n-k-r+1}$$

is the zero map for all $r \in \mathbb{N}$ and $k < p$. Then

$$\mathrm{d}\, F^p(K^n) = \mathrm{im}(\mathrm{d}^n) \cap F^p(K^{n+1}).$$

Remark 5.2. This yields the implication (1) $\Rightarrow$ (2) of proposition 4.1.

Proof. First fix $k < p$. We want to prove that

$$\mathrm{d}\, F^k(K^n) \cap F^{k+1}(K^{n+1}) = \mathrm{d}\, F^{k+1}(K^n).$$

It is clear that $\mathrm{d}\, F^{k+1}(K^n) \subseteq \mathrm{d}\, F^k(K^n) \cap F^{k+1}(K^{n+1})$. Conversely, let $x \in F^k(K^n)$ with $\mathrm{d}(x) \in F^{k+1}(K^{n+1})$. We want to find an element of $F^{k+1}(K^n)$ with same differential as x. Certainly x defines a class in $E_1^{k,n-k}$. By assumption,

$$0 = \mathrm{d}_1^{k,n-k}(x) = [\mathrm{d}^n(x)] \in E_1^{k+1,n-k},$$

which implies that

$$\mathrm{d}(x) \in \mathrm{d}\, F^{k+1}(K^n) + F^{k+2}(K^{n+1}).$$

So we can write

$$\mathrm{d}(x) = \mathrm{d}(a) + b$$

with $a \in F^{k+1}(K^n)$ and $b \in F^{k+2}(K^{n+1})$. If we set $c = x - a$, then $c \in F^k(K^n)$ and

$$\mathrm{d}(c) = \mathrm{d}(x) - \mathrm{d}(a) = b \in F^{k+2}(K^{n+1}).$$

Furthermore c and x define the same class in $E_1^{k,n-k}$.
Now we can apply a similar argument as before to c. We have $c \in F^k(K^n)$ and $\mathrm{d}(c) \in F^{k+2}(K^{n+1}) \subseteq F^{k+1}(K^{n+1})$. So c defines a class in $E_2^{k,n-k}$, and again

$$\mathrm{d}(c) \in \mathrm{d}\, F^{k+1}(K^n) + F^{k+3}(K^{n+1})$$

since $\mathrm{d}_2^{k,n-k}$ is zero. So we can write

$$\mathrm{d}(c) = \mathrm{d}(e) + f,$$

where $e \in F^{k+1}(K^n)$ and $f \in F^{k+3}(K^{n+1})$. Let $g = c - e$. Then $g \in F^k(K^n)$ and g, c and x have the same class in $E_1^{k,n-k}$. Moreover

$$\mathrm{d}(g) = \mathrm{d}(c) - \mathrm{d}(e) \in F^{k+3}(K^{n+1}).$$

Since the filtration of the complex is finite, there is $N \in \mathbb{N}$ such that

$$F^{k+N}(K^{n+1}) = 0.$$

So, repeating the argument above we find an element $y \in F^k(K^n)$, which has the same class in $E_1^{k,n-k}$ as x, and

$$\mathrm{d}(y) \in F^{k+N}(K^{n+1}) = 0.$$

The fact that the class of x and y is the same in $E_1^{k,n-k}$ yields

$$x - y \in \mathrm{d}\, F^k(K^{n-1}) + F^{k+1}(K^n).$$

So there are $z \in F^{k+1}(K^n)$ and $s \in \mathrm{d}\, F^k(K^{n-1})$ such that $x - y = s + z$. It follows that

$$\mathrm{d}(z) = \mathrm{d}(x) - \mathrm{d}(y) - \mathrm{d}(s) = \mathrm{d}(x).$$

We infer that

$$\mathrm{d}\, F^k(K^n) \cap F^{k+1}(K^{n+1}) = \mathrm{d}\, F^{k+1}(K^n)$$

for all $k < p$.
By induction, we get

$$\mathrm{d}\, F^l(K^n) = \mathrm{d}\, F^k(K^n) \cap F^l(K^{n+1})$$

for all $k < l \leq p$.
Again by finiteness of the filtration, there is $M \in \mathbb{Z}$, $M < p$ such that $F^M(K^n) = K^n$.
We conclude

$$\begin{aligned} \mathrm{d}\, F^p(K^n) &= \mathrm{d}\, F^M(K^n) \cap F^p(K^{n+1}) \\ &= \mathrm{im}(\mathrm{d}^n) \cap F^p(K^{n+1}), \end{aligned}$$

which was to be shown. $\square$

As before also a converse statement holds.

Proposition 5.3. Let $(K; \mathrm{d})$ be a cochain complex with a filtration F and let $(E_r, \mathrm{d}_r)_{r\in\mathbb{N}}$ be the induced spectral sequence. Let $n \in \mathbb{Z}$. Assume

$$\mathrm{d}\, F^p(K^n) = \mathrm{im}(\mathrm{d}^n) \cap F^p(K^{n+1})$$

for all $p \in \mathbb{Z}$. Then

$$\mathrm{d}_r^{k,n-k} : E_r^{k,n-k} \to E_r^{k+r,n-k-r+1}$$

is the zero map for all $r \in \mathbb{N}$ and $k \in \mathbb{Z}$.

Remark 5.4. This implies the implication (2) $\Rightarrow$ (1) of proposition 4.1.

Proof. Let $r \in \mathbb{N}$ and $k \in \mathbb{Z}$. In order to show that $\mathrm{d}_r^{k,n-k}$ is the zero map, we have to prove that

$$\mathrm{d}(F^k(K^n) \cap \mathrm{d}^{-1}(F^{k+r}(K^{n+1}))) \subseteq F^{k+r+1}(K^{n+1}) + \mathrm{d}(F^{k+1}(K^n)).$$

This follows from

$$\begin{aligned}
\mathrm{d}(F^k(K^n) \cap \mathrm{d}^{-1}(F^{k+r}(K^{n+1}))) &\subseteq \mathrm{d}(F^k(K^n)) \cap \mathrm{d}(\mathrm{d}^{-1}(F^{k+r}(K^{n+1}))) \\
&\subseteq \mathrm{d}(F^k(K^n)) \cap F^{k+r}(K^{n+1}) \\
&\subseteq \mathrm{im}(\mathrm{d}^n) \cap F^{k+r}(K^{n+1}) \\
&= \mathrm{d}(F^{k+r}(K^n)) \\
&\subseteq \mathrm{d}(F^{k+1}(K^n)) \\
&\subseteq F^{k+r+1}(K^{n+1}) + \mathrm{d}(F^{k+1}(K^n)),
\end{aligned}$$

where we used the assumption in line 4. □

The two previous propositions yield the following corollary, which is a degreewise version of proposition 4.1.

Corollary 5.5. Let $(K; \mathrm{d})$ be a cochain complex with a filtration F and let $(E_r, \mathrm{d}_r)_{r\in\mathbb{N}}$ be the induced spectral sequence. Let $n \in \mathbb{Z}$. Then the following statements are equivalent:

a) $\mathrm{d}\, F^p(K^n) = \mathrm{im}(\mathrm{d}^n) \cap F^p(K^{n+1})$ for all $p \in \mathbb{Z}$

b) $\mathrm{d}_r^{k,n-k} : E_r^{k,n-k} \to E_r^{k+r,n-k-r+1}$ is the zero map for all $r \in \mathbb{N}$ and $k \in \mathbb{Z}$.

Now we want to obtain a degreewise version of theorem 4.2. We start with the following refinement of the implication (3) $\Rightarrow$ (1) in theorem 4.2.

Proposition 5.6. Let $(K; \partial, \bar{\partial})$ be a bounded double complex and $(K; \mathrm{d})$ its associated simple complex. Denote by $({}'E_r; {}'\mathrm{d}_r)_{r\in\mathbb{N}}$ and $({}''E_r; {}''\mathrm{d}_r)_{r\in\mathbb{N}}$ the spectral sequences induced by ${}'F^p(K)$ and ${}''F^p(K)$, respectively. Further let $n \in \mathbb{Z}$ and assume that

(a) ${}'\mathrm{d}_r^{k,n-k-1} = 0$ and ${}''\mathrm{d}_r^{k,n-k-1} = 0$ for all $k \in \mathbb{Z}$ and $r \in \mathbb{N}$, and

(b) the induced filtrations on $\mathrm{H}_\mathrm{d}(K)$ satisfy

$$
\begin{aligned}
{}'F^a(\mathrm{H}_\mathrm{d}^{n-1}(K)) + {}''F^b(\mathrm{H}_\mathrm{d}^{n-1}(K)) &= \mathrm{H}_\mathrm{d}^{n-1}(K) \text{ for } (a,b) \in \mathbb{Z}^2 \text{ with } a+b=n \quad (14)\\
{}'F^a(\mathrm{H}_\mathrm{d}^{n}(K)) \cap {}''F^b(\mathrm{H}_\mathrm{d}^{n}(K)) &= 0 \text{ for } (a,b) \in \mathbb{Z}^2 \text{ with } a+b-1=n \quad (15)
\end{aligned}
$$

Then $(K;\partial,\bar\partial)$ satisfies the $\partial\bar\partial$-Lemma in degree n.

Proof. We show equation (2)(i) of theorem 3.2. For this, take $(p,q) \in \mathbb{Z}$ with $p+q=n$ and let $x \in \ker(\bar\partial^{p,q}) \cap \mathrm{im}(\partial^{p-1,q})$. First, choose $y \in K^{p-1,q}$ with $x = \partial(y)$. Since $\mathrm{im}(\partial^{p-1,q}) \subseteq \ker(\partial^{p,q})$, we have that $x \in \ker(\mathrm{d}^n)$. Hence x defines a class in $\mathrm{H}_\mathrm{d}^n(K)$, which is the same as the class of

$$x - \mathrm{d}(y) = -\bar\partial(y) \in K^{p-1,q+1}.$$

But the class of x is in ${}'F^p(\mathrm{H}_\mathrm{d}^n(K))$ while the class of $-\bar\partial^{p-1,q}(y)$ is an element of ${}''F^{q+1}(\mathrm{H}_\mathrm{d}^n(K))$. Thus, by assumption, the class of x is zero, hence

$$x \in \mathrm{im}(\mathrm{d}^{n-1}).$$

In particular

$$x \in {}'F^p(K^n) \cap \mathrm{im}(\mathrm{d}^{n-1}) \text{ and } x \in {}''F^q(K^n) \cap \mathrm{im}(\mathrm{d}^{n-1}).$$

Using corollary 5.5, we infer that

$$x \in \mathrm{d}({}'F^p(K^{n-1})) \text{ and } x \in \mathrm{d}({}''F^q(K^{n-1})).$$

Note that here we use the boundedness of the complex, since this guarantees that the filtrations are finite.
So, let $a \in {}'F^p(K^{n-1})$ and $b \in {}''F^q(K^{n-1})$ such that $x = \mathrm{d}(a) = \mathrm{d}(b)$. Then $a - b \in \ker(\mathrm{d}^{n-1})$, so $a-b$ defines a class in $\mathrm{H}_\mathrm{d}^{n-1}(K)$. By assumption, there are $e \in {}'F^p(K^{n-1}) \cap \ker(\mathrm{d}^{n-1})$, $f \in {}''F^q(K^{n-1}) \cap \ker(\mathrm{d}^{n-1})$ and $g \in K^{n-2}$ such that

$$a - b = e + f + \mathrm{d}(g).$$

If we write

$$e = \sum_{\substack{r \geq p \\ r+s=n-1}} e_{r,s}$$

as its bidegree composition, then $\bar\partial(e_{p,q-1}) = 0$, because

$$
\begin{aligned}
0 &= \mathrm{d}(e)\\
&= \bar\partial(e_{p,q-1}) + (\partial(e_{p,q-1}) + \bar\partial(e_{p+1,q-2})) + \ldots + \partial(e_{n-1,0})
\end{aligned}
$$

is the bidegree decomposition of $\mathrm{d}(e)$. Writing

$$a = \sum_{\substack{r \geq p \\ r+s=n-1}} a_{r,s}$$

as well as

$$g = \sum_{r \geq p} g_{r,s},$$

we infer that

$$\begin{aligned} x &= \mathrm{d}(a) \\ &= \bar{\partial}(a_{p,q-1}) \\ &= \bar{\partial}(e_{p,q-1} + \partial(g_{p-1,q-1}) + \bar{\partial}(g_{p,q-2})) \\ &= \bar{\partial}(\partial(g_{p-1,q-1})) \in \mathrm{im}(\partial^{n-1}\bar{\partial}^{n-2}). \end{aligned}$$

This proves the claim. $\square$

Before proving that also the converse is true we want to give simple examples, which show that we cannot drop the assumptions (14) and (15) in proposition 5.6.

Example 5.7. We cannot drop assumption (14). Consider the following double complex. Let V be a nonzero vector space over a field, which does not have characteristic two. We set

$$K^{p,q} = \begin{cases} V & (p,q) \in \{(-1,0),(0,-1),(0,0)\}, \\ 0 & \text{else,} \end{cases}$$

as well as $\partial^{-1,0} = \mathrm{id}_V$ and $\bar{\partial}^{0,-1} = \mathrm{id}_V$.

$$\begin{tikzcd} & 0 & 0 & \\ 0 \arrow[r] & V \arrow[r] \arrow[u] & V \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & 0 \arrow[r] \arrow[u] & V \arrow[r] \arrow[u] & 0 \\ & 0 \arrow[u] & 0 \arrow[u] & \end{tikzcd}$$

We take $n = 0$. Then the filtration is 0-opposite, because $\mathrm{H}^0_{\mathrm{d}}(K)$ is trivial, since d^{-1} is surjective. In particular condition (15) is satisfied.
Furthermore we have that

$${}'E_1^{p,q} \cong \mathrm{H}^{p,q}_{\bar{\partial}}(K) = \begin{cases} K^{-1,0} = V & (p,q) = (-1,0), \\ 0 & \text{else.} \end{cases}$$

It follows that ${}'\mathrm{d}_r^{k,-k-1} = 0$ for all $k \in \mathbb{Z}$ and $r \in \mathbb{N}$. The same argument applies to the second filtration.
But the first condition of theorem 3.2 is not fulfilled in degree 0. Indeed, we have that

$$\ker(\partial^0) = \ker(\bar{\partial}^0) = \mathrm{im}(\mathrm{d}^{-1}) = V$$

whereas

$$\mathrm{im}(\partial^{-1}\bar{\partial}^{-2}) = 0.$$

Example 5.8. We also cannot drop assumption (15). Consider the following double complex. We take V as before and set

$$K^{p,q} = \begin{cases} V & (p,q) \in \{(0,0),(1,0),(0,1)\}, \\ 0 & \text{else,} \end{cases}$$

as well as $\partial^{0,0} = \mathrm{id}_V$ and $\bar{\partial}^{0,0} = \mathrm{id}_V$.

$$\begin{tikzcd}
 & 0 & 0 & \\
0 \arrow[r] & V \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & V \arrow[u] \arrow[r] & V \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Let $n = 1$. Then the filtration is 0-opposite, because $\mathrm{H}^0_{\mathrm{d}}(K)$ is trivial, since d^0 is injective. In particular equation (14) holds.
Furthermore we have that

$${}'E_1^{p,q} \cong \mathrm{H}^{p,q}_{\bar{\partial}}(K) = \begin{cases} K^{1,0} = V & (p,q) = (1,0), \\ 0 & \text{else.} \end{cases}$$

It follows that ${}'\mathrm{d}_r^{k,-k} = 0$ for all $k \in \mathbb{Z}$ and $r \in \mathbb{N}$. The same argument applies to the second filtration.
But the first condition of theorem 3.2 is not fulfilled in degree 1. Indeed, we have that

$$\ker(\partial^1) = \ker(\bar{\partial}^1) = K^1$$

and

$$\mathrm{im}(\mathrm{d}^0) = \{(v,v) \in K^{1,0} \oplus K^{0,1}\} \neq 0$$

whereas

$$\mathrm{im}(\partial^0\bar{\partial}^{-1}) = 0.$$

Proposition 5.9. Let $n \in \mathbb{Z}$ and assume that the double complex $(K; \partial, \bar{\partial})$ satisfies the $\partial\bar{\partial}$-Lemma in degree n. Then the differential

$$\mathrm{d}^{n-1} : K^{n-1} \to K^n$$

is strict relative to both filtrations $'F$ and $''F$, i.e.

$$\mathrm{d}^{n-1}({}'F^p(K^{n-1})) = \mathrm{im}(\mathrm{d}^{n-1}) \cap {}'F^p(K^n),$$
$$\mathrm{d}^{n-1}({}''F^p(K^{n-1})) = \mathrm{im}(\mathrm{d}^{n-1}) \cap {}''F^p(K^n)$$

for all $p \in \mathbb{Z}$

Proof. We prove the claim for $'F$, the other case works analogously. The inclusion

$$\mathrm{d}^{n-1}({}'F^p(K^{n-1})) \subseteq \mathrm{im}(\mathrm{d}^{n-1}) \cap {}'F^p(K^n)$$

is clear. For the other inclusion let $x \in \mathrm{im}(\mathrm{d}^{n-1}) \cap {}'F^p(K^n)$. We write

$$x = \sum_{\substack{r+s=n \\ r \geq p}} x_{r,s}$$

and take $y = \sum_{r+s=n-1} y_{r,s}$ such that $\mathrm{d}(y) = x$. Of course

$$x = \mathrm{d}(y) = \sum_{r+s=n-1} \mathrm{d}(y_{r,s}) = \sum_{\substack{r+s=n-1 \\ r \geq p-1}} \mathrm{d}(y_{r,s}) = \mathrm{d}(\sum_{\substack{r+s=n-1 \\ r \geq p-1}} y_{r,s}),$$

so we may assume that $y \in {}'F^{p-1}(K^{n-1})$. We have that $\bar{\partial}(y_{p-1,n-p}) = 0$. In particular $y_{p-1,n-p} \in \ker(\partial^n \bar{\partial}^{n-1})$. So, by equation (2*)(i) of theorem 3.2,

$$y \in \mathrm{im}(\bar{\partial}^{n-2}) \cap \ker(\partial^{n-1})$$

and we can write

$$y_{p-1,n-p} = \bar{\partial}(a) + b$$

with $\partial(b) = 0$ and $a \in K^{p-1,n-p-1} \subseteq K^{n-1}$. Then

$$\begin{aligned} x_{p,n-p} &= \partial(y_{p-1,n-p}) + \bar{\partial}(y_{p,n-p-1}) \\ &= \bar{\partial}(\partial(a) + y_{p,n-p-1}). \end{aligned}$$

We infer that

$$\begin{aligned} x &= \bar{\partial}(\partial(a)) + \bar{\partial}(y_{p,n-p-1}) + \sum_{\substack{r+s=n \\ r \geq p+1}} x_{r,s} \\ &= \mathrm{d}(\partial(a)) + \mathrm{d}(\sum_{\substack{r+s=n-1 \\ r \geq p}} y_{r,s}), \end{aligned}$$

which is an element of $\mathrm{d}^{n-1}({}'F^p(K^{n-1}))$. This shows the claim. $\square$

We get the following corollary as a consequence of propositions 5.9 and 5.3.

Corollary 5.10. Let $(K; \partial, \bar{\partial})$ be a bounded double complex and $(K; \mathrm{d})$ its associated simple complex. Denote by $('E_r; '\mathrm{d}_r)_{r\in\mathbb{N}}$ and $(''E_r; ''\mathrm{d}_r)_{r\in\mathbb{N}}$ the spectral sequences induced by $'F^p(K)$ and $''F^p(K)$, respectively. Let further $n \in \mathbb{Z}$ and assume that the double complex satisfies the $\partial\bar{\partial}$-Lemma in degree n. Then $'\mathrm{d}_r^{k,n-k-1} = 0$ and $''\mathrm{d}_r^{k,n-k-1} = 0$ for all $k \in \mathbb{Z}$ and $r \in \mathbb{N}$.

The following proposition states what the $\partial\bar{\partial}$-Lemma means for the induced filtrations on the cohomology.

Proposition 5.11. Let $(K; \partial, \bar{\partial})$ be a bounded double complex and $(K; \mathrm{d})$ its associated simple complex. Denote by $('E_r; '\mathrm{d}_r)_{r\in\mathbb{N}}$ and $(''E_r; ''\mathrm{d}_r)_{r\in\mathbb{N}}$ the spectral sequences induced by $'F^p(K)$ and $''F^p(K)$, respectively. Let further $n \in \mathbb{Z}$ and assume that the double complex satisfies the $\partial\bar{\partial}$-Lemma in degree n. Then

$$'F^p(\mathrm{H}_\mathrm{d}^{n-1}(K)) + {''F^q}(\mathrm{H}_\mathrm{d}^{n-1}(K)) = \mathrm{H}_\mathrm{d}^{n-1}(K) \text{ for all } (p,q) \in \mathbb{Z}^2 \text{ with } p+q=n$$
$$'F^p(\mathrm{H}_\mathrm{d}^{n}(K)) \cap {''F^q}(\mathrm{H}_\mathrm{d}^{n}(K)) = 0 \text{ for all } (p,q) \in \mathbb{Z}^2 \text{ with } p+q-1=n.$$

Proof. For the first equation take $\alpha = [z] \in \mathrm{H}_\mathrm{d}^{n-1}(K)$, where $z \in K^{n-1}$ is a representative of α. Fix p and q with $p+q=n$. We want to find $x \in {'F^p}K^{n-1}$ and $y \in {''F^q}K^{n-1}$, both in the kernel of d, such that $[x]+[y]=\alpha$. If we write

$$\begin{aligned} z &= \sum_{r+s=n-1} z_{r,s} \\ &= \sum_{\substack{r+s=n-1\\ r\geq p}} z_{r,s} + \sum_{\substack{r+s=n-1\\ r\leq p-1}} z_{r,s}, \end{aligned}$$

then $\bar{\partial}(z_{p,q-1}) = -\partial(z_{p-1,q})$, because $\mathrm{d}(z)=0$. Hence $\bar{\partial}(z_{p,q-1}) \in \mathrm{im}(\bar{\partial}^{n-1}) \cap \ker(\partial^n)$, which, by equation (2)(ii), is a subset of $\mathrm{im}(\partial^{n-1}\bar{\partial}^{n-2})$. So we find $u \in K^{p-1,q-1} \subseteq K^{n-2}$ such that

$$\partial(\bar{\partial}(u)) = \bar{\partial}(z_{p,q-1}) = -\bar{\partial}(\partial(u)).$$

If we set

$$x = \sum_{\substack{r+s=n-1\\ r\geq p}} z_{r,s} + \partial(u)$$

and

$$y = \sum_{\substack{r+s=n-1\\ r\leq p-1}} z_{r,s} + \bar{\partial}(u),$$

we get

$$x + y = z + \mathrm{d}(u).$$

Furthermore $x \in {}'F^pK^{n-1}$ and $y \in {}''F^qK^{n-1}$. By

$$\bar{\partial}(z_{p,q-1} + \partial(u)) = 0 \text{ and } \partial(z_{p,q-1} + \partial(u)) = \partial(z_{p,q-1})$$

we infer that

$$\begin{aligned} \mathrm{d}(x) &= \mathrm{d}\Big(\sum_{\substack{r+s=n-1 \\ r \geq p}} z_{r,s} + \partial(u)\Big) \\ &= \bar{\partial}(z_{p,q-1} + \partial(u)) \\ &= 0, \end{aligned}$$

and similarly for y. This proves the first equation.
For the second equation let p and q be integers such that $p + q = n + 1$ and $\alpha \in {}'F^p(\mathrm{H}^n_{\mathrm{d}}(K)) \cap {}''F^q(\mathrm{H}^n_{\mathrm{d}}(K))$. This means there is

$$x = \sum_{\substack{r+s=n \\ r \geq p}} x_{r,s} \in {}'F^pK^n \cap \ker(\mathrm{d})$$

as well as

$$y = \sum_{\substack{r+s=n \\ s \geq q}} y_{r,s} \in {}''F^qK^n \cap \ker(\mathrm{d})$$

such that α is the class of x and the class of y. Because their classes are equal, there is $u \in K^{n-1}$ such that $x - y = \mathrm{d}(u)$. It follows that

$$\begin{aligned} x_{p,n-p} &= x_{p,q-1} \\ &= x_{p,q-1} - y_{p,q-1} \\ &= \partial(u_{p-1,q-1}) + \bar{\partial}(u_{p,q-2}) \end{aligned}$$

We have

$$\begin{aligned} 0 &= \mathrm{d}(x) \\ &= \mathrm{d}\Big(\sum_{\substack{r+s=n \\ r \geq p}} x_{r,s}\Big) \\ &= \sum_{\substack{r+s=n \\ r \geq p}} \partial(x_{r,s}) + \bar{\partial}(x_{r,s}). \end{aligned}$$

By considering bidegrees, this gives $\bar{\partial}(x_{p,q-1}) = 0$. Hence

$$\begin{aligned} 0 &= \bar{\partial}(x_{p,q-1}) \\ &= \bar{\partial}(\partial(u_{p-1,q-1})) + \bar{\partial}(\bar{\partial}(u_{p,q-2})) \\ &= \bar{\partial}(\partial(u_{p-1,q-1})). \end{aligned}$$

Applying equation (2*)(ii) to $\partial(u_{p-1,q-1})$ we find $v \in K^{p-1,q-2} \subseteq K^{n-2}$ such that

$$\partial(u_{p-1,q-1}) = \partial(\bar{\partial}(v)) = -\bar{\partial}(\partial(v)).$$

It follows that

$$\mathrm{d}(\sum_{\substack{r+s=n-1 \\ r \geq p}} u_{r,s} - \partial(v)) = (x - \partial(u_{p-1,q-1})) - \bar{\partial}(\partial(v)) = x.$$

So x is exact, and hence α is zero. □

Putting the previous results together, we get the following theorem, which is a degreewise analogue of theorem (5.17) from [DGMS75].

Theorem 5.12. Let $(K; \partial, \bar{\partial})$ be a bounded double complex and $(K; \mathrm{d})$ its associated simple complex. Denote by $({}'E_r; {}'\mathrm{d}_r)_{r\in\mathbb{N}}$ and $({}''E_r; {}''\mathrm{d}_r)_{r\in\mathbb{N}}$ the spectral sequences induced by ${}'F^p(K)$ and ${}''F^p(K)$, respectively. Let further $n \in \mathbb{Z}$. Then $(K; \partial, \bar{\partial})$ satisfies the $\partial\bar{\partial}$-Lemma in degree n if and only if

(a) ${}'\,\mathrm{d}_r^{k,n-k-1} = 0$ and ${}''\,\mathrm{d}_r^{k,n-k-1} = 0$ for all $k \in \mathbb{Z}$ and $r \in \mathbb{N}$, and

(b) the induced filtrations on $\mathrm{H}_{\mathrm{d}}(K)$ satisfy

$$\begin{gathered} {}'F^p(\mathrm{H}_{\mathrm{d}}^{n-1}(K)) + {}''F^q(\mathrm{H}_{\mathrm{d}}^{n-1}(K)) = \mathrm{H}_{\mathrm{d}}^{n-1}(K) \text{ for all } (p,q) \in \mathbb{Z}^2 \text{ with } p+q=n \\ {}'F^p(\mathrm{H}_{\mathrm{d}}^{n}(K)) \cap {}''F^q(\mathrm{H}_{\mathrm{d}}^{n}(K)) = 0 \text{ for all } (p,q) \in \mathbb{Z}^2 \text{ with } p+q-1=n \end{gathered}$$

For complex manifolds this theorem can be reformulated as follows.

Theorem 5.13. Let X be a complex manifold of dimension n. Then X satisfies the $\partial\bar{\partial}$-Lemma in degree k if and only if

(a) the differential maps $\mathrm{d}_r^{k,n-k-1}$ of the Frölicher spectral sequence are zero for all $r \geq 1$, and

(b) the filtration on $\mathrm{H}_{\mathrm{dR}}(X)$ that is induced by the natural filtration on the space of differential forms on X satisfies

$$\begin{gathered} F^p(\mathrm{H}_{\mathrm{dR}}^{k-1}(X)) + \overline{F^q(\mathrm{H}_{\mathrm{dR}}^{k-1}(X))} = \mathrm{H}_{\mathrm{dR}}^{k-1}(X) \text{ for all } (p,q) \in \mathbb{Z}^2 \text{ with } p+q=k \\ F^p(\mathrm{H}_{\mathrm{dR}}^{k}(X)) \cap \overline{F^q(\mathrm{H}_{\mathrm{dR}}^{k}(X))} = 0 \text{ for all } (p,q) \in \mathbb{Z}^2 \text{ with } p+q-1=k. \end{gathered}$$

6 Some applications to complex manifolds

If a complex manifold is compact and kählerian, then it satisfies the $\partial\bar{\partial}$-Lemma and provides a Hodge structure on the de Rham cohomology. In this final section, we will give some applications of theorem 5.13 by dropping one of these assumptions. First, we will consider compact surfaces, and afterwards, we want to consider Kähler manifolds which are not compact but have a certain convexity property.

6.1 The compact case

Compact examples, for which theorem 5.13 can be useful, are connected compact complex surfaces. We refer to [BHPVdV15] for more details about those. Most results, which we will use, can be found in chapter IV.
It is well known that the Frölicher spectral sequence degenerates at E_1 for such manifolds. This allows us to reformulate theorem 5.13 as follows.

Theorem 6.1. Let X be a connected compact complex surface. Then X satisfies the $\partial\bar{\partial}$-Lemma in degree k if and only if the filtration on $\mathrm{H}_{\mathrm{dR}}(X)$ that is induced by the natural filtration on the space of differential forms on X satisfies

$$F^p(\mathrm{H}^{k-1}_{\mathrm{dR}}(X)) + \overline{F^q(\mathrm{H}^{k-1}_{\mathrm{dR}}(X))} = \mathrm{H}^{k-1}_{\mathrm{dR}}(X) \text{ for all } (p,q) \in \mathbb{Z}^2 \text{ with } p+q=k \tag{16}$$

$$F^p(\mathrm{H}^{k}_{\mathrm{dR}}(X)) \cap \overline{F^q(\mathrm{H}^{k}_{\mathrm{dR}}(X))} = 0 \text{ for all } (p,q) \in \mathbb{Z}^2 \text{ with } p+q-1=k. \tag{17}$$

Furthermore, we have equality in the Frölicher inequality. If we denote the Hodge numbers of X by $h^{p,q}$, this yields

$$b_k = \sum_{p+q=k} h^{p,q}.$$

In order to investigate the validity of the $\partial\bar{\partial}$-Lemma here, we have to distinguish if the first Betti number b_1 of X is odd or even. In the even case the following equivalence holds.

Theorem 6.2. A compact complex surface is Kähler if and only if its first Betti number is even.

Proof. See [BHPVdV15]. □

Hence, any compact complex surface with even first Betti number satisfies the $\partial\bar{\partial}$-Lemma in all degrees.
If the first Betti number of X is odd, then $h^{1,0} = h^{0,1} - 1$. In particular, X does not admit a Hodge Structure in degree 1 and is not a Kähler manifold. Now we want to check in which degrees X satisfies the $\partial\bar{\partial}$-Lemma.
Like all complex manifolds, X satisfies the $\partial\bar{\partial}$-Lemma in degree 0.
As a connected compact complex manifold, X satisfies the $\partial\bar{\partial}$-Lemma also in degree 1. To see this, consider equation (2)(i) of theorem 3.2. We have

$$\mathrm{im}(\partial^0\bar{\partial}^{-1}) = 0.$$

So let $x \in \ker(\bar{\partial}^1) \cap \operatorname{im}(\partial^0) = \ker(\bar{\partial}^{1,0}) \cap \operatorname{im}(\partial^{0,0})$. Then there exists a smooth function f on X such that $x = \partial(f)$. Therefore,

$$0 = \bar{\partial}(\partial(f)).$$

Using that X is compact, we infer by the maximum principle for pluriharmonic functions that f is constant. Hence, $x = \partial(f) = 0$. Equation (2)(ii) works the same way.
For degree 2, first note that

$$F^1(\mathrm{H}^1_{\mathrm{dR}}(X)) \cap \overline{F^1(\mathrm{H}^1_{\mathrm{dR}}(X))} = 0,$$

since the $\partial\bar{\partial}$-Lemma holds in degree 1.
Since the Frölicher spectral sequence degenerates at E_1, we have

$$\mathrm{H}^{1,0}_{\bar{\partial}}(X) = E_1^{1,0} = E_\infty^{1,0} = F^1(\mathrm{H}^1_{\mathrm{dR}}(X))/F^2(\mathrm{H}^1_{\mathrm{dR}}(X)) = F^1(\mathrm{H}^1_{\mathrm{dR}}(X)).$$

We deduce

$$\begin{aligned}
\dim_{\mathbb{C}}(F^1(\mathrm{H}^1_{\mathrm{dR}}(X)) + \overline{F^1(\mathrm{H}^1_{\mathrm{dR}}(X))}) &= \dim_{\mathbb{C}}(F^1(\mathrm{H}^1_{\mathrm{dR}}(X))) + \dim_{\mathbb{C}}(\overline{F^1(\mathrm{H}^1_{\mathrm{dR}}(X))}) \\
&= \dim_{\mathbb{C}}(\mathrm{H}^{1,0}_{\bar{\partial}}(X)) + \dim_{\mathbb{C}}(\overline{\mathrm{H}^{1,0}_{\bar{\partial}}(X)}) \\
&= h^{1,0} + h^{1,0} \\
&= h^{1,0} + h^{0,1} - 1 \\
&= b_1 - 1 \\
&= \dim_{\mathbb{C}}(\mathrm{H}^1_{\mathrm{dR}}(X)) - 1.
\end{aligned}$$

Therefore, equation (16) is not fulfilled for $k = 2$, and hence, X does not satisfy the $\partial\bar{\partial}$-Lemma in degree 2.
For degree 3, we consider the filtration on $\mathrm{H}^3_{\mathrm{dR}}(X)$. We have

$$\mathrm{H}^3_{\mathrm{dR}}(X) = F^1(\mathrm{H}^3_{\mathrm{dR}}(X)) \supseteq F^2(\mathrm{H}^3_{\mathrm{dR}}(X)) \supseteq F^3(\mathrm{H}^3_{\mathrm{dR}}(X)) = 0.$$

Note that by Serre duality we have $h^{p,q} = h^{2-p,2-q}$. In particular, $h^{2,1} = h^{1,2} + 1$.
Similar to before, we have

$$\mathrm{H}^{2,1}_{\bar{\partial}}(X) = E_1^{2,1} = E_\infty^{2,1} = F^2(\mathrm{H}^3_{\mathrm{dR}}(X))/F^3(\mathrm{H}^3_{\mathrm{dR}}(X)) = F^2(\mathrm{H}^3_{\mathrm{dR}}(X)).$$

In particular,

$$\dim_{\mathbb{C}}(F^2(\mathrm{H}^3_{\mathrm{dR}}(X))) = h^{2,1} = h^{0,1} = b_1 - h^{1,0} = b_1 - \dim_{\mathbb{C}}(F^1(\mathrm{H}^1_{\mathrm{dR}}(X))). \tag{18}$$

Now consider

$$\begin{aligned}
b : \mathrm{H}^1_{\mathrm{dR}}(X) \times \mathrm{H}^3_{\mathrm{dR}}(X) &\longrightarrow \mathbb{C} \\
([\alpha], [\beta]) &\longmapsto \int_X \alpha \wedge \beta,
\end{aligned}$$

which is a non-degenerate bilinear map, by Poincaré duality. Note that

$$(F^1(\mathrm{H}^1_{\mathrm{dR}}(X)))^{\perp} = \{y \in \mathrm{H}^3_{\mathrm{dR}}(X) \mid b(x,y) = 0 \text{ for all } y \in F^1(\mathrm{H}^1_{\mathrm{dR}}(X)\}$$

contains $F^2(\mathrm{H}^3_{\mathrm{dR}}(X))$. Hence, equation (18) implies that they are equal. The same is true for the conjugated spaces. We infer that

$$\begin{aligned}
\dim_{\mathbb{C}}(F^2(\mathrm{H}^3_{\mathrm{dR}}(X)) \cap \overline{F^2(\mathrm{H}^3_{\mathrm{dR}}(X))}) &= \dim_{\mathbb{C}}((F^1(\mathrm{H}^1_{\mathrm{dR}}(X)))^{\perp} \cap (\overline{F^1(\mathrm{H}^1_{\mathrm{dR}}(X))})^{\perp}) \\
&= \dim_{\mathbb{C}}((F^1(\mathrm{H}^1_{\mathrm{dR}}(X)) + \overline{F^1(\mathrm{H}^1_{\mathrm{dR}}(X))})^{\perp}) \\
&= b_1 - \dim_{\mathbb{C}}(F^1(\mathrm{H}^1_{\mathrm{dR}}(X)) + \overline{F^1(\mathrm{H}^1_{\mathrm{dR}}(X))}) \\
&= 1.
\end{aligned}$$

This implies that equation (17) cannot be satisfied for $k = 3$. But then X does not satisfy the $\partial\bar{\partial}$-Lemma in degree 3.

In degree 4 it is obvious that equation (17) is satisfied for all p and q. Also, equation (16) is fulfilled whenever $p \leq 1$ or $q \leq 1$. For $p = q = 2$ we compute

$$\begin{aligned}
&\dim_{\mathbb{C}}(F^2(\mathrm{H}^3_{\mathrm{dR}}(X)) + \overline{F^2(\mathrm{H}^3_{\mathrm{dR}}(X))}) \\
&= \dim_{\mathbb{C}}(F^2(\mathrm{H}^3_{\mathrm{dR}}(X))) + \dim_{\mathbb{C}}(\overline{F^2(\mathrm{H}^3_{\mathrm{dR}}(X))}) - \dim_{\mathbb{C}}(F^2(\mathrm{H}^3_{\mathrm{dR}}(X)) \cap \overline{F^2(\mathrm{H}^3_{\mathrm{dR}}(X))}) \\
&= h^{2,1} + h^{2,1} - 1 \\
&= b_1 \\
&= \dim_{\mathbb{C}}(\mathrm{H}^3_{\mathrm{dR}}(X)).
\end{aligned}$$

Hence, equation (16) is also satisfied in this case. This implies that X satisfies the $\partial\bar{\partial}$-Lemma in degree 4.

We summarize these considerations in the following theorem.

Theorem 6.3. Let X be a connected compact complex surface. Then the validity of the $\partial\bar{\partial}$-Lemma is expressed as follows.

Table 1: Validity of the $\partial\bar{\partial}$-Lemma for compact complex surfaces

	b_1 even	b_1 odd
degree 0	yes	yes
degree 1	yes	yes
degree 2	yes	no
degree 3	yes	no
degree 4	yes	yes

Note that the arguments we used for degree 3 and 4 can be used for a compact complex manifold of arbitrary dimension as long as its Frölicher spectral sequence degenerates at E_1.

Theorem 6.4. Let X be an n-dimensonal compact complex manifold and suppose that its Frölicher spectral sequence degenerates at E_1. Then X satisfies the $\partial\bar{\partial}$-Lemma in some degree k if and only if X satisfies it in degree $2n - k + 1$.

6.2 The non-compact case

Non-compact manifolds, where theorem 5.13 can be applied, are Kähler manifolds, which satisfy an additional convexity condition. The following results mainly are due to [Ohs81]. We also refer the reader to [BDIP02] and the references given there for more details.

Definition 6.5. A smooth function defined on an n-dimensional complex manifold is called strongly l-convex if its Levi form has $n - l + 1$ positive eigenvalues at every point outside a compact subset of X. Furthermore, we say a complex manifold X is absolutely l-convex if X has a plurisubharmonic exhaustion function, which is strongly l-convex.

Then we have the following theorem.

Theorem 6.6. Let X be an n-dimensional Kähler manifold, which is absolutely l-convex. Then in total degree greater or equal than $n + l$ we have

$$\mathrm{H}^k_{\mathrm{dR}}(X) \cong \bigoplus_{p+q=k} \mathrm{H}^{p,q}_{\bar{\partial}}(X) \text{ and } \overline{\mathrm{H}^{p,q}_{\bar{\partial}}(X)} \cong \mathrm{H}^{q,p}_{\bar{\partial}}(X).$$

Furthermore, all these spaces are finite dimensional.

Proof. See [Ohs81]. □

Hence,

$$\dim_{\mathbb{C}} \mathrm{H}^k_{\mathrm{dR}}(X) = \sum_{p+q=k} \dim_{\mathbb{C}} \mathrm{H}^{p,q}_{\bar{\partial}}(X)$$

whenever $k \geq n + l$, and the Frölicher spectral sequence degenerates at E_1 in total degree at least $n + l$.
In particular, we can apply theorem 5.13 to such manifolds to get the $\partial\bar{\partial}$-Lemma in high degrees.

Theorem 6.7. Let X be an n-dimensional Kähler manifold, which is absolutely l-convex. Then X satisfies the $\partial\bar{\partial}$-Lemma in degrees greater than $n + l$.

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Declaration of Authorship

I hereby declare that this thesis is based on my own unaided work, unless stated otherwise. All references and verbatim extracts have been marked as such and I assure that no other sources have been used. Moreover, this thesis has not been part of any other exam in this or any other form.

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