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Master Thesis

Non-Kähler complex structures on $\mathbb{R}^4$ after Di Scala-Kasuya-Zuddas

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1. Introduction

It is a basic fact that there are no other compact complex submanifolds in any positive dimensional complex vector space than points. This is an immediate consequence of the maximum principle (see [FG02, I 4.11.]). Therefore it is natural to ask for complex structures on this spaces admitting positive dimensional compact complex submanifolds. In the 1950's Eugenio Calabi and Beno Eckmann already constructed complex structures on the product of two odd-dimensional spheres

$$M_{p,q} = S^{2p+1} \times S^{2q+1},$$

with remarkable properties. According to [CE53, Theorem II] $M_{p,q}$ with $p > 0$ admits a holomorphic fibering

$$M_{p,q} \to \mathbb{P}^p_{\mathbb{C}} \times \mathbb{P}^q_{\mathbb{C}}$$

with two-dimensional, pairwise biholomorphically equivalent, complex tori as fibers. Here $\mathbb{P}^n_{\mathbb{C}}$ denotes the complex projective space of dimension $n \in \mathbb{Z}_{>0}$. Furthermore [CE53, Theorem VI] states that the submanifold

$$E_{p,q} = S^{2p+1} \setminus \{\mathrm{pt}\} \times S^{2q+1} \setminus \{\mathrm{pt}\}$$

which arises by removing any pair of points in $M_{p,q}$ is homeomorphically equivalent to $\mathbb{C}^{p+q+1}$. In contrast it does not admit a holomorphic atlas with one coordinate system. Therefore it is not biholomorphically equivalent to $\mathbb{C}^{p+q+1}$. And even more important most of the fibers of $M_{p,q}$ are contained in $E_{p,q}$. Hence there exist compact complex submanifolds of $E_{p,q}$ of positive dimension. In [DKZ15a] and [DKZ15b] Antonio Di-Scala, Naohiko Kasuya and Daniele Zuddas refer to these results and improve them. More precisely they construct a family $E = E(\varrho_1, \varrho_2)_{\varrho_1 \varrho_2}$ of two-dimensional complex manifolds parameterized by

$$\{(\varrho_1, \varrho_2) \mid 1 < \varrho_2 < \varrho_1^{-1} < \infty\}$$

each of whom being diffeomorphically equivalent to $\mathbb{C}^2$. By construction they get a holomorphic fibering map

$$E(\varrho_1, \varrho_2) \to \mathbb{P}^1_{\mathbb{C}}$$

with fibers being either embedded one-dimensional complex tori, a cubic curve with one node or embedded one-dimensional annulli (see [DKZ15b, Theorem III]). We will see that all one dimensional complex tori can be embedded in such a manifold. The goal of this work is to give a more explicit construction of the manifolds $E(\varrho_1, \varrho_2)$. Moreover we will see that the Picard groups of these manifolds carry the structure of complex vector spaces of infinite dimension.
We will start this work with an agreement on the used notations. Then we will construct the manifolds $E(\varrho_1, \varrho_2)$.In order to do that we will make use of the theory of moduli spaces of elliptic curves. In the last section we will give a short review on the relation between cohomology

and line bundles to get in the position to consider the Picard groups as complex vector spaces. Finally we will prove these that vector spaces are of infinite dimension. The idea for this proof goes back to Dr. Tim Kirschner and represents the starting of this work.

2. Notations

In this section we will reach an agreement on the notations we will always use in this work.

2.1. **Discs and annuli.** For non-negative numbers $0 \leq r_1 < r_2 \leq \infty$ and $0 < r \leq \infty$ we will always denote the disc with center 0 and radius r by

$$\mathrm{D}(r) = \{z \in \mathbb{C} \mid |z| < r\}$$

and the annulus with center 0, inner radius r_1 and outer radius r_2 by

$$\mathrm{D}(r_1, r_2) = \{z \in \mathbb{C} \mid r_1 < |z| < r_2\}.$$

Furthermore we define

$$\mathrm{D} \coloneqq \mathrm{D}(1) \quad \text{and} \quad \mathrm{D}^* \coloneqq \mathrm{D}(0,1) \quad \text{and} \quad \mathrm{D}(r)^* := \mathrm{D}(0,r).$$

In order to allow discs with center z_0 different to 0 we also denote

$$D_r(z_0) := \{z \in \mathbb{C} \mid |z - z_0| < r\}.$$

2.2. **Complex tori.** For real linearly independent complex numbers ω_1, $\omega_2 \in \mathbb{C}$ we denote the associated lattice in the complex plane by

$$\Lambda \coloneqq \Lambda(\omega_1, \omega_2) = \omega_1\mathbb{Z} + \omega_2\mathbb{Z},$$

which is a subgroup of $(\mathbb{C}, +)$. According to [FG02, page 206] Λ acts freely and properly discontinuously on $\mathbb{C}$ by translation. Therefore [FG02, IV 5.5.] provides a one-dimensional complex structure on $\mathbb{C}/\Lambda$ such that the natural projection $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ is an unbranched holomorphic covering. In particular for every $[z_0] \in \mathbb{C}/\Lambda$ there exists an open neighborhood $U = U([z_0]) \subset \mathbb{C}/\Lambda$ of $[z_0]$ in $\mathbb{C}/\Lambda$ so that

$$\begin{aligned} \varphi_{[z_0]} : U &\to \mathbb{C} \\ [z] &\mapsto z \end{aligned}$$

is a local chart for $\mathbb{C}/\Lambda$ at $[z_0]$.

2.3. **Projective space.** Let $n \in \mathbb{Z}_{\geq 0}$ be a positive integer. Then we denote the n-dimensional complex projective space by

$$\mathbb{P}^n_{\mathbb{C}} := (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*,$$

where $\mathbb{C}^*$ acts on $\mathbb{C}^{n+1} \setminus \{0\}$ by multiplication with scalars. According to [FG02, p.208-210] $\mathbb{P}^n_{\mathbb{C}}$ is a topological Hausdorff space which becomes an n-dimensional complex manifold by virtue of the holomorphic atlas

$$\{(\psi_i, U_i) \mid i = 0, \ldots, n\}$$

with

$$\begin{aligned} \psi_i : U_i &\to \mathbb{C}^n \\ [z_0 : \cdots : z_n] &\mapsto \left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i}\right) \end{aligned}$$

and

$$U_i := \{[z_0 : \cdots : z_n] \mid z_i \neq 0\}$$

for $i \in \{0, \ldots, n\}$. For $n \in \{1, 2\}$ we will also write $[x : y]$ and $[x : y : z]$ instead of $[z_0 : z_1]$ and $[z_0 : z_1 : z_2]$. In this case it is also useful to write $\psi_x := \psi_0$ as well as $\psi_y := \psi_1$ and $\psi_z := \psi_2$. Analogously $U_x := U_0$ as well as $U_y := U_1$ and $U_z := U_2$. Usually projective algebraic subsets $A \subset \mathbb{P}^n_{\mathbb{C}}$ are defined as the zero set of homogeneous polynomials P in $n+1$ variables. For $i \in \{0, \ldots, n\}$ the dehomogenization of P with respect to i is the polynomial with complex coefficients in n variables defined by

$$P_i(z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n) := P(z_0, \ldots, z_{i-1}, 1, z_{i+1}, \ldots, z_n).$$

Its zero set

$$\{z \in \mathbb{C}^n \mid P_i(z) = 0\} = \psi_i(A) \cong \{[z_0 : \ldots : z_n] \in A \mid z_i \neq 0\}$$

is called the affine part of A with respect to i.

2.4. **Partial derivatives and jacobians.** Let $n, m \in \mathbb{Z}_{\geq 0}$ and $U \subset \mathbb{C}^n$ an open domain in the n-dimensional complex space. Moreover let

$$f = (f_1, \ldots, f_m) : U \to \mathbb{C}^m$$

be a holomorphic map. We denote the partial derivative of f in a point $z_0 \in \mathbb{C}$ by

$$(f_j)_{z_i}(z_0) := D_i f_j(z_0) = \lim_{0 \neq h \to 0} \frac{1}{h} f_j(z_0 + he_i) - f_j(z_0),$$

where $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. Then [FG02, I 7.1.] implies that the complex total derivative of f in z_0 is the complex linear map $Df(z_0) : \mathbb{C}^n \to \mathbb{C}^m$ given by

$$Df(z_0)(v) = J_f(z_0)v,$$

where $J_f(z_0)$ denotes the jacobian of f in z_0

$$J_f(z_0) := \begin{pmatrix} D_1 f_1(z_0) & \cdots & D_n f_1(z_0) \\ \vdots & & \vdots \\ D_1 f_m(z_0) & \cdots & D_n f_m(z_0) \end{pmatrix}.$$

2.5. **Tangent space.** For a complex manifold X of dimension $n \in \mathbb{N}$ we denote the set of charts at a given point $x \in X$ by $\mathcal{C}_x = \mathcal{C}_{X,x}$. The tangent space of X at x is defined to be

$$T_x X := \mathcal{C}_x \times \mathbb{C}^n / \sim,$$

where $(\varphi, v) \sim (\psi, w)$ if and only if

$$J_{\psi \circ \varphi^{-1}}(\varphi(x))v = w.$$

Note that [FG02, p.165, 166] shows that $\sim$ is an equivalence relation and that T_xX becomes an n-dimensional complex vector space by virtue of

$$[\varphi, v] + [\varphi, w] := [\varphi, v + w] \quad \text{for v,w} \in \mathbb{C}^n$$
$$\lambda[\varphi, v] := [\varphi, \lambda v] \quad \text{for } \lambda \in \mathbb{C} \text{ and } v \in \mathbb{C}^n,$$

with a fixed chart $\varphi \in \mathcal{C}_x$. Every holomorphic map $f : X \to Y$ induces a complex vector space homomorphism, the so called tangent map

$$T_xf : T_xX \to T_{f(x)}Y$$
$$[\varphi, v] \mapsto [\psi, J_{\psi\circ\varphi^{-1}}(\varphi(x))v].$$

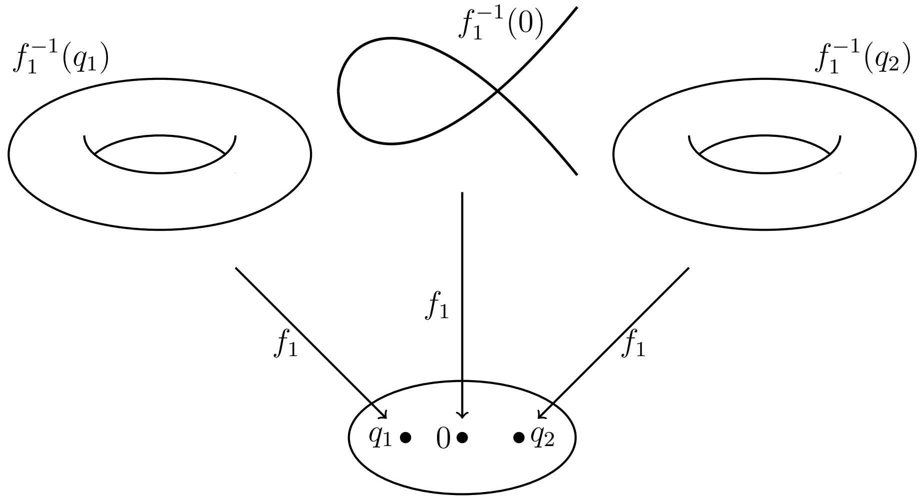

FIGURE 1. A universal family of cubic curves

3. DKZ-Manifolds

In this section we will discuss the construction of a family

$$E(\varrho_1, \varrho_2)_{(\varrho_1,\varrho_2)\in P}$$

of complex manifolds parameterized by

$$P := \{(\varrho_1, \varrho_2) \in (\mathbb{R}_{>0})^2 \mid 1 < \varrho_2 < \varrho_1^{-1} < \infty\}$$

such that each manifold $E(\varrho_1, \varrho_2)$ is diffeomorphically but not biholomorphically equivalent to $\mathbb{C}^2$. Since these manifolds have first been described by Di Scala Kasuya and Zuddas (see [DKZ15b]) we will call them Di Scala-Kasuya-Zuddas manifolds or DKZ-manifolds, shortly. Our goal in this section is to increase understanding of the DKZ manifolds by making the construction more explicit. Note that [DKZ15b] starts by fixing a relatively minimal elliptic holomorphic Lefschetz fibration

$$f_1 : W_1 \to \mathrm{D}$$

with one singular fiber. The existence of the fibration f_1 is referred to [Kod60] and [Kod63]. It can be characterized as a holomorphic function such that the fibers of non-zero base points correspond to elliptic curves and $f_1^{-1}(0)$ corresponds to a cubic curve with one node. This situation is illustrated in figure (1). We will not assume the existence of f_1 but give an explicit description of a manifold W_1 having the latter properties. In order to do this we will refer to [Hai08]. According to [DKZ15b] we start with positive real numbers ϱ_0, ϱ_1 and $\varrho_2 > 0$ satisfying

$$1 < \varrho_2 < \varrho_1^{-1} < \varrho_0^{-1}.$$

Then we will define a two-dimensional complex submanifold of $\mathbb{P}^2_{\mathbb{C}} \times \mathrm{D}$ as the zero set of a family of homogeneous polynomials $(P_q(x,y,z))_{q\in\mathrm{D}}$ parameterized by D. In order to understand the coefficients of P_q we will learn some elliptic function theory. It turns out that the projection to the coefficient q is of the desired form. In order to increase understanding of W we will follow [DKZ15b] and identify the restriction of $f : W \to \mathrm{D}$ to D with a Hopf manifold $W' := \mathbb{C}^* \times \mathrm{D}^*/\mathbb{Z}$, where $\mathbb{Z}$ acts freely and properly discontinuously on $\mathbb{C}^* \times \mathrm{D}^*$. Finally, embedding $\mathrm{D}(1,\varrho_2) \times \mathrm{D}(\varrho_0,\varrho_1)$ into $W_1' = W'_{|\mathrm{D}(\varrho_1)^*}$ will allow us to glue $W_1 = f^{-1}(\mathrm{D}(\varrho_1))$ and $W_2 := \mathrm{D}(1,\varrho_2) \times \mathrm{D}(\varrho_0^{-1})$ along the biholomorphism corresponding to

$$\begin{aligned} \mathrm{D}(1,\varrho_2) \times \mathrm{D}(\varrho_0,\varrho_1) &\to V_2 := \mathrm{D}(1,\varrho_2) \times \mathrm{D}(\varrho_1^{-1},\varrho_0^{-1}) \\ (z,w) &\mapsto (z,w^{-1}). \end{aligned}$$

This yields the DKZ-manifolds.

3.1. **Elliptic functions and curves.** In this subsection we will define the coefficients of the polynomials mentioned in the introduction. Throughout this subsection let $\omega_1,\omega_2 \in \mathbb{C}$ be real linearly independent complex numbers.

Definition 3.1 (Elliptic function)**.** An elliptic function for the lattice $\Lambda = \Lambda(\omega_1,\omega_2)$ is a meromorphic function on the complex plane being invariant under translation by Λ.

Note that an elliptic function can be viewed as an element of $\mathrm{Hol}(\mathbb{C}/\Lambda,\mathbb{C}_\infty)$, where $\mathrm{Hol}(\mathbb{C}/\Lambda,\mathbb{C}_\infty)$ denotes the complex vector space of holomorphic maps $\mathbb{C}/\Lambda \to \mathbb{C}_\infty$ and $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere. According to [Fre06] the first example for an elliptic function is the Weierstrass $\wp$-function.

Definition 3.2 (Weierstrass's $\wp$ function)**.** The Weierstrass $\wp$-function $\wp_\Lambda := \wp : \mathbb{C} \to \mathbb{C}_\infty$ for Λ is defined by

$$(1) \qquad \wp_\Lambda(z) := \wp(z) := \begin{cases} \dfrac{1}{z^2} + \displaystyle\sum_{\omega\in\Lambda\setminus\{0\}} \left[\dfrac{1}{(z-\omega)^2} - \dfrac{1}{\omega^2}\right] & \text{for } z \notin \Lambda \\ \infty & \text{for } z \in \Lambda \end{cases}$$

As stated in [Fre06, 5.4.] the Weierstrass $\wp$-function is a surjective elliptic function whose only poles are of order two and are located in the points of Λ. In order to calculate the Laurent expansion of $\wp$ with center $z_0 = 0$ one defines the so-called Eisenstein series.

Definition 3.3 (Eisenstein series)**.** For a positive integer $k \in \mathbb{Z}_{\geq 0}$ with $k \geq 3$ the Eisenstein series of weight k for Λ is defined to be

$$G_k = G_k(\Lambda) = \sum_{\omega\in\Lambda\setminus\{0\}} \omega^{-k}.$$

According to [Fre06, p.271] this series are absolutely convergent and they provide the desired Laurent expansion by virtue of

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)G_{2(k+1)}z^2.$$

As described in chapter I paragraph 4 of [KK98, I.4.] the Eisenstein series of weight at least 4 can be considered as 1-periodic holomorphic functions on the upper half-plane $\mathbb{H}$ by virtue of

$$G_k(\tau) := G_k(\mathbb{Z} + \tau\mathbb{Z}) = 2\zeta(k) + \frac{(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}\sigma_{k-1}(m)e^{2\pi i m\tau}$$

for $\tau \in \mathbb{H}$, where

$$\sigma_k(m) := \sum_{d|m} d^k$$

is the divisor sum function and ζ is the Riemann zeta function. This gives rise to the definition

$$\tilde{G}_k(q) := 2\zeta(k) + \frac{(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}\sigma_{k-1}(m)q^m, \tag{2}$$

for $q \in \mathrm{D}^*$. Since $e^{2\pi i\mathbb{H}} = \mathrm{D}^*$ this functions are holomorphic on D^* with Laurentexpansion as above and hence with obvious liftable singularity in 0 (see for example [Fre06, p.145]). The Weierstrass $\wp$ function satisfies a differential equation which helps relating one-dimensional complex tori with elliptic curves.

Proposition 3.4. *We define the quantities*

$$g_2 = g_2(\Lambda) = 60G_4 \quad \textit{and} \quad g_3 = g_3(\Lambda) = 140G_6$$

as well as the discriminant for Λ

$$\Delta = \Delta(\Lambda) = g_2^3 - 27g_3^2.$$

Then Δ *is not zero and the Weierstrass* $\wp$*-function for* Λ *satisfies the differential equation*

$$\wp'(z)^2 = 4\wp(z) - g_2\wp(z) - g_3.$$

The proof is just a combination of [KK98, I.3.Korollar C] and [Fre06, 5.3. Thm 3.4.]. This yields the classical parameterization of the elliptic curve[1]

$$E_\Lambda := \{[x:y:z] \in \mathbb{P}^2_{\mathbb{C}} \mid y^2z = 4x^3 - g_2xz^2 - g_3z^3\}$$

by the torus $\mathbb{C}/\Lambda$. More precisely:

[1]In general an elliptic curve is a compact Riemann surface of genus 1 with the choice of an endowed point (see[Hai08, Def 1.1.]). According to [Hai08, Prop.5.2.] this is the same as the projective solutions of equations $y^2 = 4x^3 - ax - b^2$ with $a^3 - 27b \neq 0$.

Proposition 3.5. *The elliptic curve E_Λ is a one-dimensional complex submanifold of $\mathbb{P}^2_{\mathbb{C}}$. Furthermore the Weierstrass $\wp$-function induces an isomorphism*

$$\mathbb{C}/\Lambda \to E_\Lambda \quad [z] \mapsto \begin{cases} [\wp(z) : \wp'(z) : 1] & ; z \neq [0] \\ [0 : 1 : 0] & ; z = [0], \end{cases}$$

of complex manifolds.

Proof. See [Hai08, Proposition 5.4.] □

Analogously to what we did above we also consider g_2, g_3 and Δ as holomorphic functions on the upper half-plane and moreover $\tilde{g}_2 := 60\tilde{G}_4$ and $\tilde{g}_3 := 140\tilde{G}_6$ and $\tilde{\Delta} := \tilde{g}_2^3 - 27\tilde{g}_3^2$ as holomorphic functions on the unit disc, respectively. Keep in mind, that

$$\begin{aligned} \tilde{g}_2(0) &= \frac{4}{3}\pi^4, \\ \tilde{g}_3(0) &= \frac{8}{27}\pi^6, \\ \tilde{g}_2'(0) &= 320\pi^4, \\ \tilde{g}_3'(0) &= -\frac{448}{3}\pi^6. \end{aligned}$$

3.2. **A universal family of cubic curves.** In this subsection we want to reach an explicit description of the family $f : W \to \mathrm{D}$ mentioned in the introduction of this section. For every $q \in \mathrm{D}$ we define the homogeneous polynomial

$$P_q(x, y, z) := -y^2 z + 4x^3 - \tilde{g}_2(q)xz^2 - \tilde{g}_3(q)z^3,$$

and the analytic subset of $\mathbb{P}^2_{\mathbb{C}} \times \mathrm{D}$

$$W := \{([x : y : z], q) \in \mathbb{P}^2_{\mathbb{C}} \times \mathrm{D} \mid P_q(x, y, z) = 0\}.$$

Further more let

$$\begin{aligned} f : W &\to \mathrm{D} \\ ([x : y : z], q) &\mapsto q. \end{aligned}$$

Remark 3.6. Note that the family $f : W \to \mathrm{D}$ naturally arises in theory of moduli spaces of elliptic curves (see [Hai08, 5.6.]). We will come back to it later.

In the chart $U_x \times \mathrm{D}$ the total space W is given as the zero set of the holomorphic function

$$g_x : U_x \times \mathrm{D} \to \mathbb{C} \quad ; \quad ([x : y : z], q) \mapsto P_q\left(1, \frac{y}{x}, \frac{z}{x}\right).$$

Analogously for U_y and U_z. Note that $\mathbb{P}^2_{\mathbb{C}} \times \mathrm{D}$ is a three-dimensional complex manifold with holomorphic atlas $\{\phi_x, \phi_y, \phi_z\}$, where

$$\phi_a := \psi_a \times \mathrm{Id}_{\mathrm{D}} \quad \text{for } a \in \{x, y, z\}.$$

In Order to show that W is a two-dimensional submanifold of $\mathbb{P}^2_{\mathbb{C}} \times \mathrm{D}$ and that f is a submersion in every but one exceptional point, we will use the following purely technical lemmas.

Lemma 3.7. *Let $p \in W$ with $p \neq ([\pi^2 : 0 : -3], 0)$. Then there exists a coefficient $a \in \{x, y, z\}$ so that the first two entries of the jacobian*

$$J_{g_a \circ \phi_a^{-1}}(\phi_a(p))$$

do not vanish simultaneously.

Proof. For $p = ([x : y : z], q)$ with $x \neq 0$ we consider

$$g_x \circ \phi_x^{-1} : (Y, Z, q) \mapsto -Y^2 Z + 4 - \tilde{g_2}(q)Z^2 - \tilde{g_3}(q)Z^3.$$

For $(Y, Z, q) := \phi_x(p) = (y/x, z/x, q)$ it follows that

$$J_{g_x \circ \phi_x^{-1}}(\phi(p)) = \begin{pmatrix} -2YZ \\ -Y^2 - 2\tilde{g_2}(q)Z - 3\tilde{g_3}(q)Z^2 \\ -\tilde{g_2}'(q)Z^2 - \tilde{g_3}'(q)Z^3 \end{pmatrix}^t$$

and $g_x \circ \phi_x^{-1}(Y, Z, q) = P_q(1, Y, Z) = 0$. Note that $Z \neq 0$, since $Z = z = 0$ implies the contradiction

$$0 = P_q(x, y, 0) = 4x^3.$$

Since $Y \neq 0$ implies $(g_x \circ \phi^{-1})_Y(\phi(p)) \neq 0$ it remains to consider $Y = 0$. For $q = 0$ it follows $(g_x \circ \phi_x^{-1})_Z(\phi(p)) \neq 0$, because $(g_x \circ \phi_x^{-1})_Z(\phi(p)) = 0$ implies

$$0 = \frac{8}{3}\pi^4 + \frac{8}{9}\pi^6 Z$$

and hence $p = ([\pi^2 : 0 : -3], q)$. For $q \neq 0$ assume $(g_x \circ \phi^{-1})_Z(\phi(p)) = 0$. Then

$$Z = -\frac{2\tilde{g_2}(q)}{3\tilde{g_3}(q)}$$

implies

$$0 = P_q(1, Y, Z) = 4 - \frac{4\tilde{g_2}(q)^3}{9\tilde{g_3}(q)^2} + \frac{8\tilde{g_2}(q)^3}{27\tilde{g_3}(q)} = 4 - \frac{4\tilde{g_2}(q)}{27\tilde{g_3}(q)},$$

which is equivalent to the contradiction

$$\tilde{\Delta}(q) = \tilde{g_2}(q)^3 - 27\tilde{g_3}(q) = 0.$$

For $p = ([x : y : z], q)$ with $x = 0$ and $y \neq 0$ we consider

$$g_y \circ \phi_y^{-1} : (X, Z, q) \mapsto -Z + 4X^3 - \tilde{g_2}(q)XZ^2 - \tilde{g_3}(q)Z^3.$$

For $(X, Z, q) := \phi_y(p) = (x/y, z/y, q)$ it follows

$$D_{g_y \circ \phi_y^{-1}}(\phi(p)) = \begin{pmatrix} 12X^2 - \tilde{g}_2(q)Z^2 \\ -1 - 2\tilde{g}_2(q)XZ - 3\tilde{g}_3(q)Z^2 \\ -\tilde{g}_2{}'(q)XZ^2 - \tilde{g}_3{}'(q)Z^3 \end{pmatrix}^t$$
$$= \begin{pmatrix} -\tilde{g}_2(q)Z^2 \\ -1 - 3\tilde{g}_3(q)Z^2 \\ -\tilde{g}_3{}'(q)Z^3 \end{pmatrix}^t$$

and $g_y \circ \phi_y^{-1}(X, Z, q) = P_q(X, 1, Z) = 0$. Assume

$$-\tilde{g}_2(q)Z^2 = -1 - 3\tilde{g}_3(q)Z^2 = 0$$

and in particular $Z \neq 0$. Then the equality

$$-\frac{1}{3} = \tilde{g}_3(q)Z^2$$

implies the contradiction

$$P_q(0, 1, Z) = -\frac{2}{3}Z = 0.$$

For $p = ([x : y : z], q)$ with $x = y = 0$ and $z \neq 0$ we consider

$$g_z \circ \phi_z^{-1} : (X, Y, q) \mapsto -Y^2 + 4X^3 - \tilde{g}_2(q)X - \tilde{g}_3(q).$$

For $(X, Y, q) := \phi_y(p) = (x/z, y/z, q)$ it follows

$$D_{g_y \circ \phi_y^{-1}}(\phi(p)) = \begin{pmatrix} 12X^2 + \tilde{g}_2(q) \\ -2Y \\ -\tilde{g}_2{}'(q)X \end{pmatrix}^t = \begin{pmatrix} \tilde{g}_2(q) \\ 0 \\ 0 \end{pmatrix}^t$$

and $g_z \circ \phi_z^{-1}(X, Y, q) = P_q(X, Y, 1) = 0$. For $q \neq 0$ assume $\tilde{g}_2(q) = 0$. Then the equality

$$P_q(X, Y, q) = \tilde{g}_3(q) = 0$$

implies the contradiction $\tilde{\Delta}(q) = 0$. Since $\tilde{g}_2(0) \neq 0$ this proves the lemma. □

Lemma 3.8. *Let X be a complex manifold of dimension $n \in \mathbb{Z}_{>0}$ and $k \in \{1, \dots, n\}$ and*

$$f = (f_1, \dots, f_{n-k}) : X \to \mathbb{C}^{n-k}$$

holomorphic, with

$$rk_f(p) = rk(T_p f) = n - k \text{ for all } p \in A := \{p \in X \mid f(p) = 0\}.$$

Then

$$T_p\iota(T_pA) = Ker(T_pf) \subset T_pX,$$

where $\iota : A \hookrightarrow X$ denotes the embedding of the submanifold A into X.

Proof. Let $p \in A$, $[\varphi, v] \in T_pA$ and ψ a chart of X at p. Then

$$\begin{aligned} T_pf(T_p\iota([\varphi, v]) &= T_pf([\psi, D_{\psi\circ\varphi^{-1}}(\varphi(p))]) \\ &= [id, D_{f\circ\psi^{-1}}(\psi(p))D_{\psi\circ\varphi^{-1}}(\varphi(p))v]) \\ &= [id, D_{f\circ\varphi^{-1}}(\varphi(p))v] \\ &= [id, 0]. \end{aligned}$$

Therefore $T_p\iota(T_pA) \subset Ker(T_pf)$. Since

$$dim(T_p\iota(T_pA)) = dim(Ker(T_pf)) = k,$$

this proves the proposition. □

Proposition 3.9. *The analytic set W is a two-dimensional complex submanifold of $\mathbb{P}^2_{\mathbb{C}} \times \mathrm{D}$.*

Proof. As a result of lemma 3.7, W is regular of codimension one at every point but $p = ([\pi^2 : 0 : -3], 0)$. For p

$$\begin{aligned} (g_x \circ \phi_x^{-1})_q(\phi(p)) &= -\tilde{g}_2'(0)\left(-\frac{3}{\pi^2}\right)^2 - \tilde{g}_3'(0)\left(-\frac{3}{\pi^2}\right)^3 \\ &= -320 \cdot 9 - 448 \cdot 9 \neq 0 \end{aligned}$$

implies that W is regular of codimension one at p, which proves the proposition. □

Proposition 3.10. *The projection map*

$$f : W \to \mathrm{D} \qquad ([x : y : z], q) \mapsto q$$

is a proper holomorphic surjection. Furthermore for $p = ([x : y : z], q) \neq ([\pi^2 : 0 : -3], 0)$ it is a submersion at p.

Proof. Obviously f is a holomorphic surjection. Since $\mathbb{P}^2_{\mathbb{C}}$ is compact (see [FG02, p.209]) the projection to the disc

$$Pr = Pr_{\mathrm{D}} : \mathbb{P}^2_{\mathbb{C}} \times \mathrm{D} \to \mathrm{D}$$

is proper. Since $W \subset \mathbb{P}^2_{\mathbb{C}} \times \mathrm{D}$ is closed, [FG02, IV 6.1.] implies that $f = Pr_{|W}$ is proper. Now let $\iota : W_1 \hookrightarrow \mathbb{P}^2_{\mathbb{C}} \times \mathrm{D}$ be the obvious embedding. Note that for every $p \in \mathbb{P}^2_{\mathbb{C}} \times \mathrm{D}$, we have

$$T_pPr : [\phi, (w_1, w_2, w_3)] \mapsto [id, w_3].$$

By definition f is a submersion at p if T_pf is surjective. Since the tangent space of D at q is of dimension one, we only have to check that T_pf is not the zero map. The chain rule for jacobians implies that $T_pf = T_pPr \circ T_p\iota$, so that we only have to show that $T_p\iota(T_pW) \not\subset Ker(T_pPr)$. According to lemma 3.7 there exists a coefficient $a \in \{x, y, z\}$ such that the first two entries of the jacobian

$$J_{g_a\circ\phi_a^{-1}}(\phi_a(p))$$

do not vanish simultaneously. Applying

$$T_p g_a : [\phi, v] \mapsto [id, J_{g_a \circ \phi_a^{-1}}(\phi_a(p))v]$$

and Lemma3.8 yields

$$T_p\iota(T_p W_1) = \ker(T_p g_a) \not\subset \ker(T_p Pr).$$

□

The fundamental theorem of algebra implies that no fiber of f is empty. Therefore [FG02, IV 1.17] implies that the fibers of f over non-zero base points are complex submanifolds of W. In remark 3.6 we have already mentioned that these fibers are elliptic curves or complex tori, if you will. More precisely we are able to determine the equivalence class of the elliptic curve.

Proposition 3.11. *For $f^{-1}(q)$ with $q \in \mathrm{D}$, there are two possibilities:*

(1) For $q \neq 0$ the fiber $f^{-1}(q)$ corresponds to the elliptic curve E_{Λ_τ}, where $\tau \in \mathbb{H}$ is any element of the upper half-plane with $e^{2\pi i \tau} = q$.

(2) The fiber $f^{-1}(0)$ is a cubic curve with one node.

Proof. Ad(1): Since $e^{2\pi i\tau} = q$, we have that

$$g_2(\tau) = \tilde{g_2}(q) \quad \text{and} \quad g_3(\tau) = \tilde{g_3}(q)$$

and hence

$$\begin{aligned} f_1^{-1}(q) &= \{[x : y : z] \in \mathbb{P}^2_{\mathbb{C}} \mid y^2 z = 4x^3 - g_2(\tau)xz^2 - g_3(\tau)z^3\} \times \{q\} \\ &\simeq E_{\Lambda_\tau}. \end{aligned}$$

Ad (2): According to section 3.1 $f^{-1}(0)$ is given by

$$y^2 = 4x^3 - \frac{4}{3}\pi^4 x - \frac{8}{27}\pi^6 = 4x^3 + b_2x^2 + 2b_4x + b_6$$

Using Weierstrass equations along [Sil94, p.42] we compute

$$c_4 = b_2^2 - 24b_4 = 16\pi^4 = (2\pi)^4$$

$$c_6 = b_2^3 + 36b_2b_4 - 216b_6 = 216\frac{8}{27}\pi^6 = 64\pi^6 = (2\pi)^6$$

and

$$\Delta = \frac{c_4^3 - c_6^2}{1728} = \frac{(2\pi)^{12} - (2\pi)^{12}}{1728} = 0.$$

Therefore [Sil94, Prop.1.4.] implies the claim. □

Remark 3.12. According to [Hai08, 1.17.] every elliptic curve in $\mathbb{P}^2_{\mathbb{C}}$ is isomorphic to one of the form E_{Λ_τ} with $\tau \in \mathbb{H}$. Moreover two elliptic curves E_{Λ_τ} and $E_{\Lambda_{\tau'}}$ are isomorphic if and only if there exist complex numbers $a, b, c, d \in \mathbb{C}$ with $ad - bc = 1$ such that

$$\tau' = \frac{a\tau + b}{c\tau + d}.$$

In conclusion every isomorphy class of an elliptic curve is represented in the family $f : W \to \mathrm{D}$.

3.3. **A family of complex tori.** In order to increase understanding of W, [DKZ15b] identifies $f^{-1}(\mathrm{D}^*)$ with the Hopf manifold obtained from the $\mathbb{Z}$-action on $\mathbb{C}^* \times \mathrm{D}^*$ defined by

$$n \circ (z, q) := (zq^n, q). \tag{3}$$

Lemma 3.13. *The group $(\mathbb{Z}, +)$ acts freely and properly discontinuously on $\mathbb{C}^* \times \mathrm{D}^*$ by virtue of 3.*

Proof. Let $n, m \in \mathbb{Z}$ and $(z, q) \in \mathbb{C}^* \times \mathrm{D}^*$ with $n \circ (z, q) = m \circ (z, q)$; that is

$$(q^n z, q) = (q^m z, z).$$

Therefore

$$q^{n-m} = 1,$$

which implies $n = m$. Now let $(z_1, q_1), (z_2, q_2) \in \mathbb{C}^* \times \mathrm{D}^*$ and $\varepsilon > 0$ such that

$$U := \mathrm{D}_\varepsilon(z_1, q_1), V := \mathrm{D}_\varepsilon(z_2, q_2) \subset \mathbb{C}^*,$$

where $\mathrm{D}_\varepsilon(z_i, q_i)$ denotes the open disc around (z_i, q_i) with radius ε for $i = 1, 2$. Furthermore let $\delta > 0$ such that $\mathrm{D}_\delta(q_i) \subset \mathrm{D}^*$ for $i = 1, 2$. For $n \in \mathbb{Z}_{>0}$ every element $(z, q) \in n \circ U$ satisfies

$$|z| = |q_1^n z_1| < (|z_1| + \varepsilon_1)(|q_1| + \delta)^n.$$

Since $(|q_1|+\delta)^n \to 0$ for $n \to \infty$ it follows that $n \circ U$ will not meet V for sufficiently large n. Vice versa for $n \in \mathbb{Z}_{<0}$ every element $(z, q) \in n \circ U$ satisfies

$$(|z_1| - \varepsilon_1)(|q_1| + \delta)^n < |q_1^n z_1| = |z|.$$

Since $(|q_1| + \delta)^n \to \infty$ for $n \to -\infty$ it follows that $n \circ U$ will not meet V for sufficiently small n. □

Applying [FG02, IV 5.5.] to this situation yields the following corollary.

Corollary 3.14. *The topological quotient $W' := \mathbb{C}^* \times \mathrm{D}^* / \mathbb{Z}$ carries the structure of a two-dimensional complex manifold, such that the natural projection map $p : \mathbb{C}^* \times \mathrm{D}^* \to W'$ defines an unbranched holomorphic covering map.*

In particular for every point $[z, q] \in W'$ there exists an open neighborhood $U = U([z, q]) \subset W'$ such that $\varphi : U \to \mathbb{C}^2$ with $p \circ \varphi = id_U$ is a local chart for W' at $[z, q]$. Since the action does not affect the second coordinate, the projection to the punctured unit disc $\mathbb{C}^* \times \mathrm{D}^* \to \mathrm{D}^*$ yields a surjective holomorphic submersion $f' : W' \to \mathrm{D}^*$. Again by [FG02, IV 1.17.] every fiber of f' is a complex submanifold of W'. More precisely we have the following proposition:

Proposition 3.15. *For every $q \in \mathrm{D}^*$ the fiber $(f')^{-1}(q) \subset \mathbb{C}^* \times \mathrm{D}(0,\varrho)/\mathbb{Z}$ is biholomorphically equivalent to the one dimensional complex torus $\mathbb{C}/\Lambda_\tau$, where τ is any complex number in the upper half-plane with $e^{2\pi i\tau}$.*

Proof. Obviously $f^{-1}(q)$ can be identified with the quotient $\mathbb{C}^*/\mathbb{Z}$, where $\mathbb{Z}$ acts freely and properly discontinuously on $\mathbb{C}^*$ by

$$n \cdot \omega = q^n \omega \quad \text{for} \quad \omega \in \mathbb{C}^*.$$

There is a biholomorphic group isomorphism $\mathbb{C}/\mathbb{Z} \to \mathbb{C}^*$ which can be constructed using universal property of abelian groups along the commutative diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[rr, "\exp"] \arrow[dr] & & \mathbb{C}^* \\
& \mathbb{C}/\mathbb{Z}, \arrow[ur, "\widetilde{\exp}"'] &
\end{tikzcd}$$

with $\exp(z) := e^{2\pi i z}$. The preimage of an element $\omega \in \mathbb{C}^*$ is given by $z + \mathbb{Z} \in \mathbb{C}/\mathbb{Z}$ for any $z \in \mathbb{C}$ satisfying $\exp(z) = \omega$. Note that the inverse mapping is holomorphic since $\mathbb{C}^*$ can be covered by open subsets each of whom admitting a holomorphic logarithm. Now fix $\tau \in \exp^{-1}(q)$. Then the action on $\mathbb{C}^*$ translates to $\mathbb{C}/\mathbb{Z}$ as

$$n \cdot (z + \mathbb{Z}) = n\tau + z + \mathbb{Z}.$$

Obviously

$$(\mathbb{C}/\mathbb{Z})/\mathbb{Z} \cong \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}).$$

Therefore the latter diagram extends to the commutative diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\exp"] \arrow[dr] & \mathbb{C}^* \arrow[r] & \mathbb{C}^*/\mathbb{Z} \\
& \mathbb{C}/\mathbb{Z} \arrow[u, "\widetilde{\exp}"'] \arrow[r] & \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}, \arrow[u, "\widetilde{\exp}"']
\end{tikzcd}$$

which gives a biholomorphic map $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z} \to \mathbb{C}^*/\mathbb{Z}$. □

In conclusion we can identify every fiber $(f')^{-1}(q)$ with the fiber $f^{-1}(q)$. Varying this identifications along D^* provides an identification of W' with the restriction $f^{-1}(\mathrm{D}^*)$ of W to D^*. More precisely:

Proposition 3.16. *For $q \in \mathrm{D}^*$ let $\alpha_q \in \mathrm{Hol}((f')^{-1}(q), f^{-1}(q))$ be the biholomorphism identifying $(f_1')^{-1}(q)$ with $f_1^{-1}(q)$. Then*

$$\alpha : W' \hookrightarrow W \qquad [z,q] \mapsto \alpha_q([z,q])$$

embedds W' into W such that the diagram of complex manifolds

$$\begin{tikzcd} W' \arrow[r, "\alpha"] \arrow[d, "f_1'"'] & W \arrow[d, "f_1"] \\ \mathrm{D}^* \arrow[r] & \mathrm{D} \end{tikzcd}$$

commutes.

Proof. Obviously α is weakly holomorphic in complex coordinates. Therefore Osgood's theorem [FG02, I .4.3.] proves that α is holomorphic. Obviously α is injective. Therefore [FG02, I .8.5.] implies that the jacobian of α in complex coordinates does not vanish. Therefore the inverse mapping theorem [FG02, I. 7.5] proves the proposition. $\square$

3.4. **DKZ-Manifolds.** Having the previous sections in mind we will now follow [DKZ15b, Chapter 2] to construct the DKZ-manifolds. We start with non-negative numbers ϱ_0, ϱ_1 and ϱ_2 satisfying

$$1 < \varrho_2 < \varrho_1^{-1} < \varrho_0^{-1} \leq \infty.$$

According to [DKZ15b] the first goal is to embed $\mathrm{D}(1, \varrho_2) \times \mathrm{D}(\varrho_1)^*$ into W' and hence into W. In order to reach this goal we define the binary relations

$$\begin{aligned} \varphi : \mathrm{D}^* &\to \mathbb{C}^* \\ q &\mapsto \exp\left(\frac{(\log q)^2}{4\pi i} - \frac{\log q}{2}\right) \end{aligned}$$

and

$$\begin{aligned} \phi : \mathrm{D}(1, \varrho_2) \times \mathrm{D}^* &\to \mathbb{C}^* \times \mathrm{D}^* \\ (z, q) &\mapsto (z\varphi(q), q), \end{aligned}$$

where $\log q$ is any natural logarithm of q. Note that φ and ϕ are no maps, because the natural logarithm can not be chosen uniquely on D^* as a holomorphic function. Anyhow, the behavior of ϕ under varying the choice of the natural logarithm provides the following lemma.

Lemma 3.17. *The binary relation*

$$\begin{aligned} \pi \circ \phi : \mathrm{D}(1, \varrho_2) \times \mathrm{D}^* &\to W' \\ (z, q) &\mapsto [z\varphi(q), q] \end{aligned}$$

is a holomorphic embedding.

Proof. In order to check that $\pi \circ \phi$ is a map it is necessary and sufficient to check that $\pi \circ \phi$ does not depend on the choice of the natural logarithm. Let $(z, q) \in \mathrm{D}(1, \varrho_2) \times \mathrm{D}^*$ and $\log, \log'$ natural logarithms defined at q. Then there exists an integer $k \in \mathbb{Z}$ such that

$\log(q) = log'(q) + 2\pi ik$. Therefore

$$\begin{aligned} & z\exp\left(\frac{(\log q + 2\pi ik)^2}{4\pi i} - \frac{\log q + 2\pi ik}{2}\right) \\ &= z\exp\left(\frac{(\log q)^2 + 4\pi ik\log q + 4\pi^2 i^2 k^2}{4\pi i} - \frac{\log q + 2\pi ik}{2}\right) \\ &= z\exp\left(\frac{(\log q)^2}{4\pi i} - \frac{\log q}{2} + k\log q + \pi ik^2 - \pi ik\right) \\ &= z\exp\left(\frac{(\log q)^2}{4\pi i} - \frac{\log q}{2} + k\log q\right) \\ &= z\exp\left(\frac{(\log q)^2}{4\pi i} - \frac{\log q}{2}\right)q^k. \end{aligned}$$

In conclusion $[z\varphi(q), q]$ is uniquely determined. Furthermore $\pi \circ \phi$ is holomorphic since $\mathrm{D}(\varrho_1)^*$ can be covered by open subsets each of whom admitting a holomorphic logarithm. Now let $(z_1, q), (z_2, q) \in \mathrm{D}(1, \varrho_2) \times \mathrm{D}^*$ with $\pi \circ \phi(z_1, q) = \pi \circ \phi(z_2, q)$. Then there exists an integer $n \in \mathbb{Z}$ such that $z_1 = z_2 q^n$. Assume that $n \neq 0$. For $n > 0$ we have $|q^n| < \varrho_1^n$ and hence

$$|z_2 q^n| < \varrho_2 \varrho_1^n < 1,$$

where the last estimation follows from

$$\varrho_2 < \varrho_1^{-1} < (\varrho_1^{-1})^n.$$

For $n < 0$ we have that

$$\varrho_2 < \varrho_1^n < |q^n| < |z_2 q^n|.$$

Therefore $n \neq 0$ implies the contradiction $z_1 \notin \mathrm{D}(1, \varrho_2)$. Since the dimensions of the complex manifolds $\mathrm{D}(1, \varrho_2) \times \mathrm{D}^*$ and W' are both two, we are done (again by applying [FG02, I .8.5.] and [FG02, I. 7.5] as in the proof of proposition 3.16). □

In particular the image of $\mathrm{D}(1, \varrho_2) \times \mathrm{D}^*$ in W is a two dimensional complex submanifold of W or simply an open subset, if you will. Now we denote the restriction of W to $\mathrm{D}(\varrho_1)$ by

$$W_1 = f^{-1}(\mathrm{D}(\varrho_1))$$

and the restriction of W' to $\mathrm{D}(\varrho_1)^*$ by

$$W_1' = (f')^{-1}(\mathrm{D}(\varrho_1)^*).$$

Note that

$$\pi \circ \phi(\mathrm{D}(1, \varrho_2) \times \mathrm{D}(\varrho_1)^*) \subset W_1'.$$

Furthermore we define V_1 to be the image of $\mathrm{D}(1, \varrho_2) \times \mathrm{D}(\varrho_0, \varrho_1)$ in W_1; that is

$$V_1 = \alpha \circ \pi \circ \phi(\mathrm{D}(1, \varrho_2) \times \mathrm{D}(\varrho_0, \varrho_1)).$$

Finally we define

$$V_2 := \mathrm{D}(1, \varrho_2) \times \mathrm{D}(\varrho_1^{-1}, \varrho_0^{-1})$$

and

$$W_2 := \mathrm{D}(1, \varrho_2) \times \mathrm{D}(\varrho_0^{-1})^*.$$

Now the following lemma will allow us to define the DKZ-manifolds.

Lemma 3.18. *Let $j : V_1 \to V_2$ be the biholomorphic map corresponding to*

$$\begin{aligned} \iota : \mathrm{D}(1, \varrho_2) \times \mathrm{D}(\varrho_0, \varrho_1) &\to V_2 \\ (z, q) &\mapsto (z, q^{-1}). \end{aligned}$$

Then, up to biholomorphism, the gluing

$$W_1 \cup_j W_2$$

only depends on the parameters ϱ_1 and ϱ_2.

As a set $W_1 \cup_j W_2$ is defined to be

$$W_1 \times \{W_1\} \cup W_2 \times \{W_2\} / \sim_j,$$

where $\sim_j$ denotes the equivalence relation induced by the formula

$$(x, W_1) \sim_j (j(x), W_2),$$

whenever this makes sense. It becomes a complex manifold of dimension two by applying [FG02, IV 1.18.].

Proof. Since the embeding $\pi \circ \phi$ does not depend on ϱ_0 it is enough to show that the gluing

$$\mathrm{D}(1, \varrho_2) \times \mathrm{D}(\varrho_1)^* \cup_\iota W_2$$

does not depend on ϱ_0 up to biholomorphism. In order to do so note that every $z \in \mathbb{C} \setminus \mathrm{D}(\varrho_0^{-1})$ satisfies $z^{-1} \in \mathrm{D}(\varrho_1)$, because $|z| \geq \varrho_0^{-1}$ implies $|z| > \varrho_1^{-1}$ and hence $\varrho_1 < |z^{-1}|$. Now let

$$\begin{aligned} h : \mathrm{D}(\varrho_0, \varrho_1) &\to \mathrm{D}(\varrho_1^{-1}, \varrho_0^{-1}) \\ q &\mapsto q^{-1}. \end{aligned}$$

Then we have that

$$\begin{aligned} \mathbb{P}^1\mathbb{C} \setminus \{[1:0]\} &\to \mathrm{D}(\varrho_1)^* \cup_h \mathrm{D}(\varrho_0^{-1}) \\ [x:y] &\mapsto \begin{cases} [xy^{-1}, \mathrm{D}(\varrho_0^{-1})] & \text{if} \quad xy^{-1} \in \mathrm{D}(\varrho_0^{-1}) \\ [xy^{-1}, \mathrm{D}(\varrho_1)^*] & \text{else} \end{cases} \end{aligned}$$

is a biholomorphic map. Note that the inverse map is given by

$$\begin{aligned} [q, \mathrm{D}(\varrho_0^{-1})] &\mapsto [q : 1] \\ [q, \mathrm{D}(\varrho_1)^*] &\mapsto [1 : q]. \end{aligned}$$

In conclusion the gluing $\mathrm{D}(1, \varrho_2) \times \mathrm{D}(\varrho_1)^* \cup_\iota V_2$ is essentially biholomorphically equivalent to

$$\mathrm{D}(1, \varrho_2) \times (\mathbb{P}^1_{\mathbb{C}} \setminus \{[1 : 0]\}).$$

$\square$

Definition 3.19. The DKZ-manifold with respect to ϱ_1 and ϱ_2 is defined to be the gluing

$$E(\varrho_1, \varrho_2) := W_1 \cup_j W_2.$$

Note that the projections $f_1 := f_{|W_1}$ and $f_2 : W_2 \to \mathrm{D}(\varrho_0)$ can be considered as surjective holomorphic submersions on open subsets of $E(\varrho_1, \varrho_2)$. By construction they coincide on the intersection. Therefore we get a surjective holomorphic suurjection

$$\begin{aligned} g : E(\varrho_1, \varrho_2) &\to \mathbb{P}^1_{\mathbb{C}} = \mathrm{D}(\varrho_1) \cup_h \mathrm{D}(\varrho_0^{-1}) \\ [((x, y, z), q), W_1] &\mapsto f_1((x, y, z), q) = [1 : q] \\ [(z, q), W_2] &\mapsto f_2(z, q) = [q : 1], \end{aligned}$$

which is a submersion in all point of $E(\varrho_1, \varrho_2)$ but $([\pi^2 : 0 : -3], 0)$.

Remark 3.20. The fibers of g are the fibers of f_1 or f_2 respectively. We have already seen that every one-dimensional complex torus can be embedded in W, namely as a fiber of f. Since ϱ_1 can be chosen arbitrarily close to 1 we find a manifold W_1 so that the torus is embedded in W_1 and hence in $E(\varrho_1, \varrho_2)$ for any compatible ϱ_2. Recall that all fibers of the manifolds constructed by Calabi and Eckmann are equivalent. Therefore this is a huge difference to the DKZ-manifolds.

As mentioned in the introduction, the DKZ-manifolds are of a very unique kind. A very remarkable property is quoted in the following proposition.

Proposition 3.21. *Every DKZ-manifold $E = E(\varrho_1, \varrho_2)$ is diffeomorphically but not biholomorphically equivalent to $\mathbb{C}^2 = \mathbb{R}^4$.*

Sketch of proof. Note that [DKZ15a, p.7] contains a proof that E is diffeomorphically equivalent to $\mathbb{R}^4$. According to [DKZ15b, Theorem II] no DKZ-manifold can holomorphically be embedded into a compact complex manifold. Since it is crucial that $\mathbb{C}^2$ is embedded in the compact complex manifold $\mathbb{P}^2_{\mathbb{C}}$ it follows that E can not be biholomorphically equivalent to $\mathbb{C}^2$.

4. The Cohomology of DKZ-Manifolds

According to [DKZ15b, Theorem III] the Picard group of each DKZ-manifold $E = E(\varrho_1, \varrho_2)$ is uncountable. In this section we will slightly improve this result by proving that it is of infinite dimension, when considered as a complex vector space. Note that is not clear à-priori that the Picard group carries the structure of a complex vector space.

4.1. **Review on line bundles and Čech cohomology.**

4.1.1. *Picard Group.* Let X be a complex manifold of dimension $n \in \mathbb{Z}_{>0}$. Recall that every line bundle $\pi : L \to X$ over X is uniquely determined by its family of transition functions. It is possible to interpret isomorphisms of line bundles in terms of this family (see [FG02, p.175]). We quote this as a lemma.

Lemma 4.1. *Two line bundles $\pi^{'} : L^{'} \to X$ and $\pi^{''} : L^{''} \to X$ on X given by transitions functions $(g^{'}_{\alpha\beta})_{(\alpha,\beta)\in\mathcal{A}_1^2}$ and $(g^{''}_{\alpha\beta})_{(\alpha,\beta)\in\mathcal{A}_2^2}$ over open covers $\mathcal{U}_1 = \{U^1_\alpha \mid \alpha \in \mathcal{A}_1\}$ and $\mathcal{U}_2 = \{U^2_\alpha \mid \alpha \in \mathcal{A}_2\}$ of X are isomorphic as line bundles on X if and only if there exists a common refinement $\mathcal{U}_3 = \{U^3_\alpha \mid \alpha \in \mathcal{A}_2\}$ together with refinement maps τ_{31},τ_{32} and a family $(h_\alpha)_{\alpha\in\mathcal{A}_3}$ of nowhere vanishing holomorphic functions over $\mathcal{U}_3$ such that*

$$h_\alpha g^{(1)}_{\tau_{31}(\alpha)\tau_{31}(\beta)} = g^{(2)}_{\tau_{32}(\alpha)\tau_{32}(\beta)} h_\beta \text{ on } U^{(3)}_{\alpha\beta} \text{ for all } \alpha, \beta \in \mathcal{A}_3.$$

This gives an interpretation of the Picard group $\mathrm{Pic}(X)$ of X as the set of all families of transition functions over open covers of X modulo the equivalence relation defined by lemma 4.1. We will denote the class of a line bundle $\pi : L \to X$ over X in $\mathrm{Pic}(X)$ by $[L]$. Moreover we denote the line bundle given by a family $(g_{\alpha\beta})_{\alpha\beta}$ by $L(g_{\alpha\beta})$. The group structure is obtained by multiplying the defining transition functions pointwise on a common refinement.

4.1.2. *Sheaves.* Let X be a complex manifold of dimension $n \in \mathbb{Z}_{>0}$. We denote the topology of X by τ_X and furthermore for every $x \in X$ the set of open subsets of X containing x by $\tau_{X,x}$. Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups (respectively complex vector spaces) on X. Then for every open subset $U \subset X$ we have a group homomorphism

$$\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$$

which is compatible with the restriction maps. The stalk $\mathcal{F}_x$ of $\mathcal{F}$ at a point $x \in X$ is defined to be the inductive limit of all systems of local sections at x ordered by inclusion; that is

$$\mathcal{F}_x := \operatorname*{colim}_{U\in\tau_{X,x}} \mathcal{F}(U) := \bigsqcup_{U\in\tau_{X,x}} \mathcal{F}(U)\big/\!\sim,$$

where $\sqcup$ denotes the disjoint union $\sim$ denotes the equivalence relation induced by the formula

$$(f, U) \sim (f_{|V}, V)$$

where $f_{|V}$ denotes the restriction of f to V. The elements in this limit are usually denoted by $f_x := [(f, U)]_x$. Note that $\mathcal{F}_x$ is equipped with the structure of an abelian group in the obvious way and that the morphism φ induces a group homomorphism (homomorphism of complex vector spaces, respectively)

$$\begin{aligned} \varphi_x : \mathcal{F}_x &\to \mathcal{G}_x \\ [(f, U)]_x &\mapsto [(\varphi(U)(f), U)]_x \end{aligned}$$

(see [Har06, p.63]).

Definition 4.2 (Short exact sequence of sheaves)**.** A sequence

$$0 \to \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \to 0$$

of morphisms of sheaves of abelian groups (respectively complex vector spaces) on X is said to be short exact if the induced sequence

$$0 \to \mathcal{F}_x \to \mathcal{G}_x \to \mathcal{H}_x \to 0$$

is short exact for every $x \in X$.

Example 4.3. Let $\underline{\mathbb{Z}}_X$ be the sheaf of locally constant holomorphic functions on X with values in $\mathbb{Z}$. Furthermore let $\mathcal{O}_X$ be the sheaf of holomorphic functions and $\mathcal{O}_X^*$ the sheaf of nowhere vanishing holomorphic functions on X, respectively. For $U \in \tau_X$ define

$$\begin{aligned} \iota(U) : \underline{\mathbb{Z}}_X(U) &\to \mathcal{O}_X(U) \\ f &\mapsto f \end{aligned}$$

and

$$\begin{aligned} \exp(U) : \mathcal{O}_X(U) &\to \mathcal{O}_X^*(U) \\ f &\mapsto e^{2\pi i f}. \end{aligned}$$

Then the sequence

$$0 \to \underline{\mathbb{Z}} \xrightarrow{\iota} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

is a short exact sequence of sheaves. Obviously ι_x is injective for all $x \in X$. Now let $[(f, U)]_x \in \mathcal{O}_{X,x}^*$. Without loss of generality we assume that U admits a holomorphic logarithm, say log. Then $[(\log(f), U)]_x$ is a preimage of $[(f, U)]_x$ in $\mathcal{O}_{X,x}$.

4.1.3. *Čech Cohomology.* In this subsection we want to achieve a representation of the Picard group of a complex manifold as a cohomology group. Concerning [Gun15, Chapter 3] we assume the notion of Čech cohomology. In particular let X be a complex manifold with open cover $\mathcal{U} = \{U_i \mid i \in I\}$, $\mathcal{F}$ a sheaf of abelian groups (respectively complex vector spaces) on X and $q \in \mathbb{Z}_{\geq 0}$ a non-negative integer. Then we denote the q-th coboundary operator with respect to $\mathcal{U}$ and coefficients in $\mathcal{F}$ by $\delta^q = \delta^q_{\mathcal{U},\mathcal{F}}$. Therefore the q-th Čech cohomology group with respect to $\mathcal{U}$ and coefficients in $\mathcal{F}$ is given by

$$\check{\mathrm{H}}^q(\mathcal{U}, \mathcal{F}) = \frac{\ker(\delta^q)}{\mathrm{Im}(\delta^{q-1})}.$$

Hence the absolute q-th Čech cohomology group of X with coefficients in $\mathcal{F}$ is given by the inductive limit

$$\check{\mathrm{H}}^q(X, \mathcal{F}) = \operatorname{colim} \check{\mathrm{H}}^q(\mathcal{U}, \mathcal{F}).$$

Recall that the inductive system is given by all open coverings of X ordered by refinements. Moreover for any morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ on X we denote the induced homomorphism by $\check{\mathrm{H}}^q(\varphi)$.

Lemma 4.4. *The assignment*

$$\begin{aligned} \check{\mathrm{H}}^1(X, \mathcal{O}^*_X) &\to \mathrm{Pic}(X) \\ \left[(s_{\alpha\beta})_{\alpha,\beta\in I^2} \cdot \mathrm{Im}(\delta^0_{\mathcal{U}_i,\mathcal{O}^*_X})\right] &\mapsto [L(s_{\alpha\beta})] \end{aligned}$$

defines an ismorphism of abelian groups.

Proof. Essentially the proof follows from construction and the representation of the Picard group provided in 4.1.1. For a more explicit proof see [Deb05, 5.9.]. □

Theorem 4.5 (A long exact sequence)**.** *Let*

$$0 \to \mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H} \to 0$$

be a short exact sequence of sheaves of abelian groups (respectively, complex vector spaces) on X. Then for every $q \in \mathbb{Z}_{\geq 0}$ there exists a group homomorphism (respectively complex vector space homomorphism)

$$\delta^q : \check{\mathrm{H}}^q(X, \mathcal{H}) \to \check{\mathrm{H}}^{q+1}(X, \mathcal{F}),$$

such that the sequence

$$\begin{aligned} 0 \to \check{\mathrm{H}}^0(X, \mathcal{F}) &\xrightarrow{\check{\mathrm{H}}^0(f)} \check{\mathrm{H}}^0(X, \mathcal{G}) \xrightarrow{\check{\mathrm{H}}^0(g)} \check{\mathrm{H}}^0(X, \mathcal{H}) \xrightarrow{\delta^0} \dots \\ \dots \check{\mathrm{H}}^1(X, \mathcal{F}) &\xrightarrow{\check{\mathrm{H}}^1(f)} \check{\mathrm{H}}^1(X, \mathcal{G}) \xrightarrow{\check{\mathrm{H}}^1(g)} \check{\mathrm{H}}^1(X, \mathcal{H}) \xrightarrow{\delta^1} \dots \end{aligned}$$

is a long exact sequence of cohomology groups respectively complex vector spaces).

Proof. See [Gun15, Chapter 3 Theorem 1]. □

Theorem 4.6. *(Mayer-Vietoris Sequence) Let $X = U \cup V$ be an open covering of X by two open subsets. Then there exists a long exact sequence of cohomology groups*

$$0 \to \check{\mathrm{H}}^0(X, \mathcal{F}) \to \check{\mathrm{H}}^0(U, \mathcal{F}_{|U}) \oplus \check{\mathrm{H}}^0(V, \mathcal{F}_{|V}) \to \check{\mathrm{H}}^0(U \cap V, \mathcal{F}_{|U\cap V}) \dots$$
$$\dots \to \check{\mathrm{H}}^1(X, \mathcal{F}) \to \check{\mathrm{H}}^1(U, \mathcal{F}_{|U}) \oplus \check{\mathrm{H}}^1(V, \mathcal{F}_{|V}) \to \dots$$

Here the sheaf $\mathcal{F}_{|U}$ is simply the sheaf that assigns to $W \subset U$ open the group $\mathcal{F}(W)$ (see [Har06, p. 65]).

Sketch of proof. One easily convinces oneself that the set of all bases of the topology of X together with the ordering defined on the set of open covers form a cofinal system of open coverings. This means that we can also form the direct limit over this family of open coverings instead of considering all open covers. Now let $\mathcal{U} = \{U_\alpha \mid \alpha \in \mathcal{A}\}$ be a basis of the topology of X. For any $W \in \tau_X$ we denote

$$\mathcal{A}_W := \{\alpha \in \mathcal{A} \mid U_\alpha \subset W\}.$$

It is easy to see that $(\mathcal{U})_{|W} := \{U_\alpha \mid \alpha \in \mathcal{A}_W\}$ is a basis of W and conversely that every basis of W is of this form. Now the sequence

$$0 \to \mathcal{C}^q(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^q((\mathcal{U})_{|U}, \mathcal{F}) \oplus \mathcal{C}^q((\mathcal{U})_{|V}, \mathcal{F}) \to \mathcal{C}^q((\mathcal{U})_{|U\cap V}, \mathcal{F}) \to 0$$

defined by

$$(s_\alpha)_{\alpha \in \mathcal{A}^{q+1}} \mapsto \left((s_\alpha)_{\alpha \in \mathcal{A}_U^{q+1}}, (s_\alpha)_{\alpha \in \mathcal{A}_V^{q+1}} \right)$$

and

$$\left((s_\alpha)_{\alpha \in \mathcal{A}_U^{q+1}}, (t_\alpha)_{\alpha \in \mathcal{A}_V^{q+1}} \right) \mapsto (s_\alpha)_{\alpha \in \mathcal{A}_{U\cap V}^{q+1}} - (t_\alpha)_{\alpha \in \mathcal{A}_{U\cap V}^{q+1}}$$

is a short exact sequence of abelian groups. One can check that these sequences are compatible with the coboundary operators, so they induce a short exact sequence of complexes of abelian groups

$$0 \to \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^\bullet((\mathcal{U})_{|U}, \mathcal{F}) \oplus \mathcal{C}^\bullet((\mathcal{U})_{|V}, \mathcal{F}) \to \mathcal{C}^\bullet((\mathcal{U})_{|U\cap V}, \mathcal{F}) \to 0.$$

Using the snake lemma one can construct transition homomorphisms

$$\eta : H^q((\mathcal{U})_{|U\cap V}, \mathcal{F}) \to H^{q+1}(\mathcal{U}, \mathcal{F}).$$

Obviously everything is compatible with refinements so that all of the latter homomorphism induce the sequence.

For an explicit proof for singular cohomology see [Hat01, p.203]. This proves the theorem for Čech cohomology, too (see [PS08, B.15]).

Remark 4.7. Note that the Mayer-Vietoris sequence starts with

$$\mathcal{F}(X) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \qquad f \mapsto (f_{|U}, f_{|U})$$

and

$$\mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V) \qquad (f, g) \mapsto f_{|U\cap V} - g_{|U\cap V}.$$

4.2. **The Cohomology of DKZ-manifolds.** In this subsection we will prove our main result; that is that the Picard group of every DKZ-manifold is of infinite dimension, when considered as a complex vector space. Note that is not clear jet that the Picard group carries the structure of a complex vector space. It is remarkable that the cohomology of sheafs that do not depend on the complex structure of the DKZ-manifolds are well known to be the cohomology groups of $\mathbb{R}^4$. We start with some technical lammas.

Lemma 4.8. *Let $G, G' \subset \mathbb{C}$ be open domains in the complex plane, $f \in \mathcal{O}(G \times G')$ and $\varrho > 0$. Assume that $\{\zeta \in \mathbb{C} \mid |\zeta| = \varrho\} \subset G'$. Then for every $n \in \mathbb{Z}$ the function*

$$c_n : G \to \mathbb{C} \qquad z \mapsto \frac{1}{2\pi i} \oint\limits_{\gamma} \frac{f(z,\zeta)}{\zeta^{n+1}} d\zeta$$

with

$$\gamma : [0,1] \to \mathbb{C} \qquad t \mapsto e^{2\pi i}$$

is holomorphic.

Proof. Let $z_0 \in G$ and $(a_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{C} \setminus \{0\}$ with $a_k \to 0$ as $k \to \infty$. Since G is open we find an $\varepsilon > 0$ such that $\overline{\mathrm{D}_\varepsilon(z_0)} \subset G$. Since the sequence $(a_k)_k$ is convergent, we find a positive integer $K \in \mathbb{Z}_{>0}$ such that $a_k + z_0 \in \mathrm{D}_\varepsilon(z_0)$ for $k \geq K$. Now we fix $k_0 \geq K$ and $t \in [0,1]$. We define

$$\varphi_t : \mathrm{D}_\varepsilon(z_0) \to \mathbb{C} \qquad z \mapsto f(z, \gamma(t))$$

and

$$\eta : [0,1] \to \mathrm{D}_\varepsilon(z_0) \qquad s \mapsto z_0 + s a_k.$$

Then we have

$$\begin{aligned} | f(z_0 + a_k, \gamma(t)) - f(z_0, \gamma(t))| &= | \varphi_t(z_0 + a_k) - \varphi_k(z_0) | \\ &= \left| \oint\limits_{\eta} \varphi_t'(\zeta) d\zeta \right| \\ &\leq \oint\limits_{\eta} \left| \varphi_t'(\zeta) \right| d\zeta \\ &\leq \sup_{\zeta \in \overline{\mathrm{D}_\varepsilon(z_0)}} | \varphi_t'(\zeta) \big| |a_k|. \end{aligned}$$

Now define

$$L(t) := \sup_{\zeta \in \overline{\mathrm{D}_\varepsilon(z_0)}} | \varphi_t'(\zeta) \big| |a_k|.$$

This is a continuous function on the bounded set $[0,1]$. Therefore it is integrable. Now the theorem of Lebesgue (see [For17, 5.3.]) implies

$$\lim_{k\to\infty}\int_0^1\left|\frac{f(z_0+a_k,\gamma(z))}{a_k}-\frac{d}{dz}f(z_0,\gamma(t))\right|dt$$

$$=\int_0^1\lim_{k\to\infty}\left|\frac{f(z_0+a_k,\gamma(z))}{a_k}-\frac{d}{dz}f(z_0,\gamma(t))\right|dt=0.$$

Finally we conclude

$$\left|\frac{c_n(z_0+a_k)-c_n(z_0)}{a_k}-\frac{1}{2\pi i}\oint_{|\zeta|=\varrho}\frac{1}{\zeta^{n+1}}\frac{d}{dz}f(z_0,\zeta)d\zeta\right|$$

$$\leq\int_0^1\left|\frac{f(z_0+a_k,\gamma(t))-f(z_0,\gamma(t)}{a_k}-\frac{d}{dz}f(z_0,\gamma(t))\right|dt$$

which vanishes for $k\to\infty$. □

Corollary 4.9. *Let $G\subset\mathbb{C}$ be an open domain in the complex plane and $0<r<R$. Then*

(1) For all $f\in\mathcal{O}(G\times\mathrm{D}(r,R))$ there exists a sequence $(c_n)_{n\in\mathbb{Z}}$ in $\mathcal{O}(G)$ such that

$$f(z,q)=\sum_{n=-\infty}^{\infty}c_n(z)q^n\quad\textit{for all}\quad(z,q)\in G\times\mathrm{D}(r,R).$$

Furthermore for every $z_0\in G$ the sequences

$$\sum_{n=0}^{\infty}c_{-n}(z_0)q^{-n}\quad\textit{and}\quad\sum_{n=}^{\infty}c_n(z_0)q^n$$

have radii of convergence r^{-1} and R, respectively.

(2) For all $g\in\mathcal{O}(G\times\mathrm{D}(R))$ there exists a sequence $(b_n)_{n\in\mathbb{Z}_{\geq 0}}$ in $\mathcal{O}(G)$ such that

$$g(z,q)=\sum_{n=0}^{\infty}b_n(z)q^n\quad\textit{for all}\quad(z,q)\in G\times\mathrm{D}(R).$$

Furthermore for every $z_0\in G$ the sequence

$$\sum_{n=0}^{\infty}b_n(z_0)q^n$$

has radius of convergence R.

Proof. This is an immediate consequence of lemma 4.8 together with the Laurent expansion theorem [Fre06, III 5.2.] and the power series expansion theorem for holomorphic functions[Fre06, III 2.2.], respectively. $\square$

Lemma 4.10. *Let ϱ_1 be a positive number with $0 < \varrho_1 \leq 1$ and $g \in \mathcal{O}(W_1)$. Then there exists a holomorphic map $h \in \mathcal{O}(\mathrm{D}(\varrho_1))$, such that the diagram*

$$\begin{tikzcd} W_1 \arrow[rr, "f_1"] \arrow[dr, "g"'] & & \mathrm{D}(\varrho_1) \arrow[dl, "h"] \\ & \mathbb{C} & \end{tikzcd}$$

commutes.

Proof. Every fiber of f_1 is essentialy a cubic curve in $\mathbb{P}^2_{\mathbb{C}}$ and therefore compact as a closed set in the compact space $\mathbb{P}^2_{\mathbb{C}}$. Since g restricted to such a fiber is still holomorphic it is constant by the maximum principle (see [FG02, I 4.11.]). Therefore we can define h to be

$$h(q) := g([x, y, z], q),$$

where $[x : y : z]$ is any solution of $P_q(x, y, z) = 0$. In order to conclude that h is holomorphic note that proposition 3.10 together with the fundamental theorem of algebra implies that every fiber $f_1^{-1}(q_0)$ of f_1 contains a point $([x_0 : y_0 : z_0], q_0)$ at which f_1 is a submersion. Therefore [FG02, IV 1.16.] yields the existence of a local holmorphic section s_0 for f_1 on an open neighborhood $U = U(q_0) \subset \mathrm{D}(\varrho)$ of the base point q_0. Now we can conclude that $h_{|U} = g_{|s_0(U)} \circ s$ is holomorphic on U. Since q_0 has been chosen arbitrarily this proves the Lemma. $\square$

Lemma 4.11. *Let ϱ_1 and ϱ_2 be positive numbers so that $1 < \varrho_2 < \varrho_1^{-1} < \infty$. Furthermore let $\underline{\mathbb{Z}}_X$ be the sheaf of locally constant functions with values in $\mathbb{Z}$ on $E = E(\varrho_1, \varrho_2)$. Then*

$$\check{\mathrm{H}}^q(E, \underline{\mathbb{Z}}) = \{0\}$$

for every $q > 0$.

Sketch of proof. It is well known that Čech cohomology and singular cohomology coincide for paracompact Hausdorff spaces (see [PS08, B.15]) and hence for complex manifolds. Furthermore the singular cohomology of E only depends on the homotopy class of the E (see [Hat01, p. 201]). We have already seen that E is diffeomorphically equivalent to $\mathbb{R}^4$ which is homotopy equivalent to the one point space. Obviously the cohomology groups of the one point space are trivial.

Corollary 4.12. *Let ϱ_1 and ϱ_2 be positive numbers so that $1 < \varrho_2 < \varrho_1^{-1} < \infty$ and $E := E(\varrho_1, \varrho_2)$. Then there exists an isomorphism of abelian groups*

$$\check{\mathrm{H}}^1(E, \mathcal{O}_E) \simeq \check{\mathrm{H}}^1(E, \mathcal{O}_E^*).$$

Proof. Consider the short exact exponential sequence on E from example 4.3

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_E \to \mathcal{O}_E^* \to 0.$$

By theorem 4.5 we get an exact sequence

$$\check{\mathrm{H}}^1(E, \underline{\mathbb{Z}}) \to \check{\mathrm{H}}^1(E, \mathcal{O}_E) \to \check{\mathrm{H}}^1(E, \mathcal{O}_E^*) \to \check{\mathrm{H}}^2(E, \underline{\mathbb{Z}}). \tag{4}$$

Therefore Lemma 4.11 implies the desired result. □

Note that $\check{\mathrm{H}}^1(E, \mathcal{O}_E)$ carries the structure of a complex vector space. Furthermore the abelian group of this vector space is isomorphic to $\check{\mathrm{H}}^1(E, \mathcal{O}_E^*)$. Therefore there is a unique structure of a complex vector space on $\check{\mathrm{H}}^1(E, \mathcal{O}_E^*)$ such that the isomorphism from equation 4 is an isomorphism of complex vector spaces.

Theorem 4.13. *Let ϱ_1 and ϱ_2 be positive numbers so that $1 < \varrho_2 < \varrho_1^{-1} < \infty$. Furthermore let $E := E(\varrho_1, \varrho_2)$. Then the Picard group of E is of infinite dimension, when considered as a complex vector space.*

Proof. Fix a non-negative real number ϱ_0 satisfying

$$1 < \varrho_2 < \varrho_1^{-1} < \varrho_0^{-1} \leq \infty$$

and let

$$V := \mathrm{D}(1, \varrho_2) \times \mathrm{D}(\varrho_0, \varrho_1).$$

Furthermore let ι_1 and ι_2 be the inclusions embedding V into W_1 and W_2, respectively. According to the Mayer-Vietoris sequence (theorem 4.6) we get an exact sequence

$$\mathcal{O}(W_1) \oplus \mathcal{O}(W_2) \xrightarrow{\iota_1^* - \iota_2^*} \mathcal{O}(V) \longrightarrow \check{\mathrm{H}}^1(E, \mathcal{O}_E).$$

of complex vector spaces. Therefore there exists an injective vector space homomorphism

$$\frac{\mathcal{O}(V)}{(\iota_1^* - \iota_2^*)(\mathcal{O}(W_1) \oplus \mathcal{O}(W_2))} \to \check{\mathrm{H}}^1(E, \mathcal{O}_E).$$

Hence it is enough to show that the quotient is of infinite dimension. Let $\phi \in \mathcal{O}(V)$. Then corollary 4.9 implies the existence of a sequence $(c_n)_{n \in \mathbb{Z}}$ in $\mathcal{O}(\mathrm{D}(1, \varrho_2))$ such that

$$\phi(z, q) = \sum_{n=-\infty}^{\infty} c_n(z) q^n \quad \text{for all} \quad (z, q) \in \mathrm{D}(1, \varrho_2) \times \mathrm{D}(\varrho_0, \varrho_1).$$

Note that for every $z_0 \in \mathrm{D}(1, \varrho_2)$ the series

$$\sum_{m=0}^{\infty} c_n(z_0)(q^{-1})^m \quad \text{and} \quad \sum_{n=1}^{\infty} c_n(z_0) q^n \tag{5}$$

have radii of convergence ϱ_0^{-1} and ϱ_1 respectively. Vice versa every sequence $(c_n)_{n \in \mathbb{Z}}$ in $\mathcal{O}(\mathrm{D}(1, \varrho_2))$ so that the sequences in 5 have radii of convergence greater than ϱ_0^{-1} and ϱ_1, respectively, defines an element in $\mathcal{O}(V_1)$. Now let $\alpha \in \mathcal{O}(W_1)$. Then lemma 4.10 implies the existence of a holomorphic function $\overline{\alpha} \in \mathcal{O}(\mathrm{D}(\varrho_1))$ such that the diagram

$$\begin{tikzcd} W_1 \arrow[rr, "f_1"] \arrow[dr, "\alpha"'] & & \mathrm{D}(\varrho_1) \arrow[dl, "\overline{\alpha}"] \\ & \mathbb{C} & \end{tikzcd}$$

commutes. Note that $\overline{\alpha}$ has a representation as a power series

$$\overline{\alpha}(q) = \sum_{n=0}^{\infty} a_n q^n \quad \text{for all} \quad q \in \mathrm{D}(\varrho_1)$$

converging on $\mathrm{D}(\varrho_1)$. Therefore for all $(z, q) \in V$ we have

$$\iota_1^*(\alpha)(z,q) = \alpha \circ \iota_1(z,q) = \overline{\alpha} \circ f_1 \circ \iota_1(z,q) = \overline{\alpha}(q) = \sum_{n=0}^{\infty} a_n q^n.$$

Vice versa every sequence $(a_n)_{n\in\mathbb{N}}$ with corresponding radius of convergence grater than ϱ_1 defines a holmorphic function on $\mathrm{D}(\varrho_1)$ and hence on W_1. For $\beta \in \mathcal{O}(W_2)$ corollary 4.9 yields a sequence $(b_n)_{n\in\mathbb{Z}_{\geq 0}}$ in $\mathcal{O}(\mathrm{D}(1,\varrho_2))$ with

$$\beta(z,q) = \sum_{n=0}^{\infty} b_n(z) q^n \quad \text{for all} \quad (z,q) \in W_2.$$

Note that for every $z_0 \in \mathrm{D}(1,\varrho_2)$ the series $\beta(z_0, q)$ has radius of convergence ϱ_0^{-1}. Hence for every $(z,q) \in V$ we have

$$\iota_2^*(\beta)(z,q) = \beta \circ \iota_2(z,q) = \beta(z, q^{-1}) = \sum_{n=0}^{\infty} b_n(z)(q^{-1})^n.$$

Vice versa let $(b_n)_{n\in\mathbb{Z}_{\geq 0}}$ be a sequence in $\mathcal{O}(\mathrm{D}(1,\varrho_2))$ such that for every $z_0 \in \mathrm{D}(1,\varrho_2)$ the series

$$\sum_{n=0}^{\infty} b_n(z_0) q^n$$

has radius of convergence grater than ϱ_o^{-1}. Then

$$\sum_{n=0}^{\infty} b_n(z_0)(q^{-1})^n$$

defines an element in $\iota_2^*(\mathcal{O}(W_2))$. In conclusion every element in $(\iota_1^* - \iota_2^*)(\mathcal{O}(W_1) \oplus \mathcal{O}(W_2))$ is of the form

$$\sum_{n=0}^{\infty} a_n q^n - \sum_{n=0}^{\infty} b_n(z)(q^{-1})^n, \tag{6}$$

with $(a_n)_n$ and $(b_n)_n$ as above. Now define

$$\phi_k(z,q) := z^k q \quad \text{for} \quad k \in \mathbb{Z}_{\neq 0}.$$

Then ϕ_k is of the form 5 for all $k \in \mathbb{Z}_{\neq 0}$. Moreover no linear combination

$$\sum_{k \neq 0} \lambda_k \phi_k(z,q),$$

over $\mathbb{C}$ is of the form 6. In other words the family $(\phi_k)_k$ is linearly independent in the quotient

$$\frac{\mathcal{O}(V_1)}{(\iota_1^* - \iota_2^*)(\mathcal{O}(W_1) \oplus \mathcal{O}(W_2))}.$$

□

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I hereby declare that this thesis is based on my own unaided work, unless stated otherwise. All references and verbatim extracts have been marked as such and I assure that no other sources have been used. Moreover, this thesis has not been part of any other exam in this or any other form.

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