CANONICAL EXTENSIONS AND POSITIVITY OF CURVATURE

Master's thesis

submitted by

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in partial fulfillment of the requirements for the degree of Master of Science in the Department of Mathematics at University Duisburg-Essen

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August 18, 2022

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Niklas Müller

Acknowledgements

I want to express my gratitude towards my parents for always supporting me and allowing me to fully concentrate on my studies. Without them I would not stand where I stand today. Along with them I want to thank all the people who always believed in me and encouraged me to embrace my love for mathematics. Special thanks to Mr. Mirko Stappert for many helpful conversations and suggestions during my studies and for pointing me towards some of the beauties of mathematics.

Furthermore, I am truly indebted towards Mr. Vladimir Lazić for his immense support and for always encouraging me to 'choose the hard way'. I was fortunate to learn many things from him (not all of them mathematical) and hope for this to continue for a long time. The same goes for his entire research group which has warmly welcomed me times and times again in Saarbrücken, specifically Mr. Nikolaos Tsakanikas who always gave his best advice to me. Every stay was a great one and I am very much looking forward to the next visit!

Of course, I also want to thank the many people who were involved in the creation of this work through various means of giving advise, providing references or helpful comments.

Last but certainly not least I am very grateful towards Mr. Daniel Greb, my supervisor, for the many thorough thoughts he spent on me during the last year. His advice always turned out to be a great one; for example I could not be happier with the topic of this thesis which emerged from his suggestion. I am glad that he will continue to supervise me also in the future and I am very much looking forward to this!

Contents

Conventions and Notation

Our notation is for the most part standard and certainly in any case widely used. Modulo minor modifications it agrees with the typical references [Har77], [Vak17], [Laz04a], [Huy05] and [Dem12]. Regarding the more ambitious terminology concerning projective bundles and positivity of forms we refer the reader to the detailed discussion in the appendix.

Throughout this work, we will use Serre's GAGA theorem to identify smooth complex projective varieties with their analytification. To stay consistent, we usually speak of projective manifolds although we may sometimes use the term projective variety if we want to stress that we think of it algebraically (this is useful, for example, when speaking of non-compact sub varieties). To avoid all ambiguities concerning the term *smooth* in this context, a holomorphic map whose differential is surjective will be called a submersion and we usually simply speak of differentiable functions/sections/maps when in fact we really mean that they are infinitely often differentiable in the real sense (or C^∞ for short). We are confident, that the latter convention should not cause too much confusion.

By a *holomorphic vector bundle* $\mathcal{E}$ we will always mean a locally free sheaf of $\mathcal{O}$-modules. The underlying variety will be denoted $|\mathcal{E}|$ and we call this the *total space* of the bundle. We will denote by $\mathcal{A}^0(\mathcal{E}) := \mathcal{E} \otimes_{\mathcal{O}} C^\infty$ the associated sheaf of differentiable sections and, accordingly, we denote by $\mathcal{A}_X^{p,q}$ the sheaf of differentiable (p,q)-forms on a complex manifold X. As per usual, we denote by $\Gamma(X,-)$ the vector space of global sections of a bundle. Of course, when talking specifically about holomorphic bundles we mostly prefer the notation $\Gamma(X,-) = H^0(X,-)$. Moreover, we often use the abbreviation $\sigma \in \mathcal{E}$ to mean that σ is a section over *some* (unspecified) open subset of X.

Finally, a manifold for us will typically be connected. This should, however, most often also be clear from the context.

Introduction and Overview

Structure Theory and Positivity of Curvature - Some History

Classification is one of the most central aspirations of modern mathematics. Specifically in geometry possibly the most fruitful approach to classifying and subsequently studying geometric objects is to distinguish them by curvature or - more algebraically - by the positivity of their tangent bundle. The guiding principle is that manifolds whose curvature is bounded below should be special and, in the best case scenario, even completely classifiable. A classical, celebrated example of this philosophy is Moris resolution [Mor79] of *Hartshornes conjecture*: Any smooth projective variety whose tangent bundle is ample is isomorphic to some $\mathbb{P}^n$. Even more classical is the Borel-Remmert theorem [BR62] stating that any smooth projective variety whose tangent bundle is generated by global sections is a direct product of a torus and a variety which is homogeneous for the action of a semi simple group. Naturally, one wonders what may be said if one further relaxes the positivity assumptions; one possible answer is the following result obtained by Demailly-Peternell-Schneider:

Theorem 0.1. (Main theorem of [DPS94])
Let X be a smooth complex projective variety with a nef tangent bundle. Then, there exists a finite étale covering $\widetilde{X} \to X$ of X which admits a submersion $\alpha\colon \widetilde{X} \to T$ onto a complex torus T. In fact, α is a flat analytic fibre bundle and its fibres are Fano varieties with nef tangent bundle.

The above results have all been stated in the algebraic language. However, largely in parallel a similar theory was developed for complex analytic manifolds using the language of differential geometry. The natural setting in this case is the one of compact Kähler manifolds. Indeed, Moris result implies that any compact Kähler manifold of positive (holomorphic bi-) sectional curvature is biholomorphic to a complex projective space. The Borel-Remmert theorem is not only true in the Kähler case, it also naturally admits a generalisation [Mok88] due to Mok to manifolds of non-negative (holomorphic bi-) sectional curvature. Unfortunately, however, the notion of holomorphic bisectional curvature is rather a heritage of the differential geometric language and does not directly correspond to some algebraic positivity notion. This impeded cooperation between the subjects.

In contrast, in [DPS94] an analytic notion of nef vector bundles was introduced with a strong focus on generalising the algebraic one. Roughly speaking, from the analytic point of view the tangent bundle is nef if and only if the negative component of the sectional curvature may be chosen arbitrarily small. The authors then proceeded to extend part of their theorem to the Kähler setting. Their paper may thus be seen as part of a change of philosophy.

The following years saw a huge drive towards an understanding of algebraic notions of positivity from the analytic view points. Both complex and algebraic geometry have benefited immensely from this development. As just one sample evidence of this, let us mention the impressive generalisations of Theorem 0.1 achieved in very recent years by the work of [Cao13], [CH17], [MW21], [HIM21] and many others. This is still a very active area of research which composes many of the recent achievements in algebraic and complex geometry.

A classical Example: Manifolds with Nef Tangent Bundle

One of my main motivations while writing this thesis was to understand this circle of ideas and the main techniques needed to prove them. Surprisingly, many of these ideas were already contained in [DPS94] although the proofs of course nowadays are much more technical and involved. With this in mind, the first goal of this thesis is to explain in detail the proof of Theorem 0.1. As such, I hope that this work might serve as an introduction if one is further interested in the above mentioned recent developments.

To this end, we of course need to first introduce said analytic definitions of the algebraic positivity notions. This is the role of **Chapter I**: We start out by quickly recalling the basic notions of differential geometry needed; most prominently the notions of connections and curvature. We then define analytically what it means for a line bundle to be nef (or ample) before extending this theory to the higher rank case. To get accustomed to these definitions we use them to reprove well-known results from the algebraic setting. We also spend some words on further positivity notions used throughout the main text. Finally, we conclude with some words on flat bundles. In particular, we will state the *Non-Abelian Hodge Theorem* and see how to deduce from it a useful numerical criterion for flatness.

Chapter II is devoted to the structure theory of compact Kähler manifolds with nef tangent bundle. To settle our expectations we begin by quickly considering the case of surfaces. Then, following [DPS94] and [Cao13] we develop the complete structure theory and prove Theorem 0.1 with a focus on conveying all the ingredients necessary for the proof. A particular emphasis will be laid on the special case of Fano varieties with nef tangent bundle. According to a well-known conjecture of Campana and Peternell these should be homogeneous:

Conjecture 0.2. (Campana-Peternell)
Let F be a Fano manifold with nef tangent bundle. Then, F is homogeneous for the action of some complex Lie group.

Following up on this proof, we also consider the converse question of when a fibre bundle of the form in Theorem 0.1 has a nef tangent bundle. In particular, we prove the following characterisation which is (as of my knowledge) completely new:

Theorem 0.3. *Let $\alpha\colon X \to T$ be a holomorphic fibre bundle with typical fibre F. Assume, that T is a complex torus and that F is a homogeneous Fano manifold. Then, the tangent bundle $\mathcal{T}_X$ of X is nef if and only if α is a flat fibre bundle.*

Here, an analytic fibre bundle is said to be flat if its defining Čech cocycle admits a representative with locally constant transition functions. In particular, Theorem 0.3 allows us to construct many explicit but non-trivial examples of manifolds with a nef tangent bundle.

We conclude the second chapter with an outlook towards more recent structure theory. We also collect some open follow-up problems.

A new Approach: Canonical Extensions

The second principal goal of this thesis and the problem which started it off is to explain a new - possibly more geometric - approach to studying manifolds with nef tangent bundle. To this end we have to introduce a construction which at first glance seems totally unrelated: For any Kähler manifold (X, ω) one can show the existence of a universal complex manifold $Z_X \xrightarrow{p} X$ on which the cohomology class $[p^*\omega] = 0$ vanishes. This manifold is called the *canonical extension* of (X, ω). Canonical extensions were introduced in [Don02] to prove smoothness properties of geodesics in the space of Kähler metrics - so called *solutions to the Monge-Ampère flow.* To this end, Donaldson translated the problem into a question about certain submanifolds of Z_X. The Monge-Ampère equation has been heavily investigated over the past years as it is related to the existence of Kähler-Einstein metrics and, thus, to the Calabi-conjecture and (conjecturally) K-stability of Fanos. Subsequently, also canonical extensions have been used a number of times in this context, for example in [Tia92] and [GKP22].

Now, in [GW20] a new point of view on this topic was suggested: More concretely, the authors investigated the question whether the global geometry of Z_X is related to the posititvity of the tangent bundle of X:

Conjecture 0.4. (Greb-Wong, Höring-Peternell)
Let (X, ω) be a compact Kähler manifold. Then, $\mathcal{T}_X$ is nef if and only if Z_X is a Stein space.

Note that both of these uses of canonical extensions are in some sense connected through Conjecture 0.2: If X is a compact Kähler manifold whose tangent bundle is nef, then by Theorem 0.1 X fibres over a complex torus with fibre F a Fano manifold with nef tangent bundle. Hence, according to Conjecture 0.2 F would be a homogeneous Fano and, in particular, it would admit a Kähler-Einstein metric and be K-poly stable.

In any case, **Chapter III.** will be devoted to the study of canonical extensions of complex manifolds. We begin by discussing several possible constructions all of which will be important later on. The heart of this chapter is the second section in which we give the following new partial answer to Conjecture 0.4:

Theorem 0.5. *Let (X, ω) be a compact Kähler manifold with a nef tangent bundle. If the Campana-Peternell conjecture holds true, then the canonical extension Z_X is a Stein manifold.*

In fact, assuming a weakened version of Conjecture 0.2 suffices.

The converse question of whether manifolds whose canonical extension is Stein have a nef tangent bundle seems to be very difficult however. We end this chapter by surveying some partial results obtained by [HP21]. We also complement one of their results regarding surfaces. Nevertheless, many (even basic) questions remain unsettled.

We conclude this thesis with a small **Appendix**: First, it contains a short encyclopaedia of sorts in which we state some well-known results from algebraic and complex geometry used within the main text. I feel like this may turn out to be rather convenient if one feels a bit uncertain about the precise prerequisites for a theorem or if the attribution of the result is slightly ambitious.

The rest of the appendix is devoted to some more in depth discussions regarding our notational conventions. This includes a detailed explanation of our convention regarding projective bundles and wedge products and a short exposition regarding positivity of forms. I hope that these sections may help to avoid possible confusions stemming from the multitude of conventions used in the literature.

Chapter I

Positivity in Complex Geometry

Positivity of vector bundles is one of the most fundamental concepts in contemporary algebraic geometry. When one tries to extend these notions to the setting of not necessarily projective complex manifolds, however, many of the classical definitions seem to break down. This chapter is devoted to explaining the perhaps simplest approach of adjusting various notions of positivity to this case. As such, it will lay the foundation for all our further work.

We start off by reviewing the basics concepts at play in the differential calculus on holomorphic vector bundles such as (Chern) connections, curvature, characteristic classes and Dolbeaut cohomology. This section may be seen as a four-page summary of the main results in [Huy05, Chapter 4.]. Its primary objectives are to bring us up to pace, to set up the notation we are going to use later on and to serve as reference for later chapters.

In the second section we are going to discuss how to characterise positivity of line bundles from the differential geometric point of view. A specific focus will be laid on ample and nef bundles and we will see how to re-prove their basic properties employing analytic methods. Finally, we are going to spend a few words on big line bundles and positivity relative to a holomorphic map.

The third section will be devoted to the extension of these concepts to the higher rank case. There are two natural notions of positivity available and we will study their relationship before generalising the classical results about positivity of bundles to our setting. These ideas were pioneered by [DPS94].

Finally, in a fourth section we inspect two different concepts of flatness and their interplay. This is going to be crucial in the later chapters. Moreover, we quickly sketch how to extend the differential calculus of holomorphic vector bundles to the setting of principal bundles.

1 Differential Calculus on Holomorphic Bundles

As explained in the introduction, the following should be seen as a quick summary of the most essential results from [Huy05, Chapter 4.].

Notation I.1.1. Throughout this section, let X denote a complex manifold and let $\mathcal{E}$ denote a holomorphic vector bundle on X, i.e. a locally free sheaf of $\mathcal{O}_X$-modules (of finite rank). We write $\mathcal{A}^0(\mathcal{E})$ for the sheaf of all differentiable (not necessarily holomorphic) sections to $\mathcal{E}$. More generally, we let

$$\mathcal{A}^k(\mathcal{E}) := \mathcal{A}^0(\mathcal{E}) \otimes_{C^\infty_X} \mathcal{A}^k_X, \quad \mathcal{A}^{p,q}(\mathcal{E}) := \mathcal{A}^0(\mathcal{E}) \otimes_{C^\infty_X} \mathcal{A}^{p,q}_X$$

denote the sheaves of differential k-forms (respectively (p,q)-forms) with values in $\mathcal{E}$.

In this situation, according to [Huy05, Lemma 2.6.23.] there exist $\mathbb{C}$-linear maps of sheaves

$$\bar\partial_{\mathcal{E}} \colon \mathcal{A}^{p,q}(\mathcal{E}) \to \mathcal{A}^{p,q+1}(\mathcal{E})$$

uniquely determined by the rule $\bar\partial_{\mathcal{E}}(\sigma \otimes \eta) = \bar\partial_{\mathcal{E}}(\sigma) \wedge \eta + \sigma \otimes \bar\partial(\eta)$ for all $\sigma \in \mathcal{E}$ and all differential forms η. Then, $\bar\partial^2_{\mathcal{E}} = 0$ and one can show that for any $p \geq 0$ the complex

$$0 \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^p_X \to \mathcal{A}^{p,0}(\mathcal{E}) \xrightarrow{\bar\partial} \mathcal{A}^{p,1}(\mathcal{E}) \xrightarrow{\bar\partial} \mathcal{A}^{p,2}(\mathcal{E}) \xrightarrow{\bar\partial} \dots$$

is an acyclic resolution (see [Huy05, Corollary 2.6.25]). In particular, there exist natural identifications

$$H^q(X, \mathcal{E} \otimes \Omega^p_X) = \left\{ \eta \in \Gamma(X, \mathcal{A}^{p,q}(\mathcal{E})) \,\middle|\, \bar\partial\eta = 0 \right\} \big/ \left\{ \bar\partial\zeta \,\middle|\, \zeta \in \Gamma(X, \mathcal{A}^{p,q+1}(\mathcal{E})) \right\}.$$

Given a cohomology class $a \in H^q(X, \mathcal{E} \otimes \Omega^p_X)$ a closed form $\eta \in \Gamma(X, \mathcal{A}^{p,q}(\mathcal{E}))$ in the cohomology class $[\eta] = a$ is called a *Dolbeaut representative* of a.

Reminder I.1.2. Recall, that a *connection* ∇ *in* $\mathcal{E}$ is a $\mathbb{C}$-linear map of sheaves

$$\mathcal{A}^0(\mathcal{E}) \to \mathcal{A}^1(\mathcal{E}), \quad \sigma \mapsto (V \mapsto \nabla_V \sigma)$$

satisfying the Leibniz rule $\nabla(f\sigma) = f \cdot \nabla\sigma + \sigma \otimes df$ for any (local) differentiable function $f \in C^\infty_X$ and any (local) differentiable section σ to $\mathcal{E}$. In this case, the *curvature* of ∇ is defined by the rule

$$F_\nabla(\sigma)(V,W) = \nabla_V\nabla_W\sigma - \nabla_W\nabla_V\sigma - \nabla_{[V,W]}\sigma, \quad \forall\, V,W \in T^{\mathbb{C}}X, \forall\, \sigma \in \mathcal{A}^0(\mathcal{E}).$$

It is not hard to show that F_∇ is not only C^∞_X linear in V, W but also in σ. In other words, F_∇ is a section to $\mathcal{A}^2(\mathrm{End}(\mathcal{E}))$.

A connection ∇ is said to be *compatible with the holomorphic structure on $\mathcal{E}$* if

$$\nabla_W \sigma = \left(\bar{\partial}_{\mathcal{E}} \sigma\right)(W), \quad \forall\, W \in T^{0,1}X, \forall\, \sigma \in \mathcal{A}^0(\mathcal{E}).$$

Moreover, given any hermitean metric h on $\mathcal{E}$ the connection ∇ is called *metric* (with respect to h) if the following product rule is satisfied:

$$V(h(\sigma_1, \sigma_2)) = h(\nabla_V \sigma_1, \sigma_2) + h(\sigma_1, \nabla_{\overline{V}} \sigma_2), \quad \forall\, V \in T^{\mathbb{C}}X, \forall\, \sigma_1, \sigma_2 \in \mathcal{A}^0(\mathcal{E}).$$

Proposition I.1.3. (Chern connection, [Huy05, Proposition 4.2.14.])
For any hermitean metric h on $\mathcal{E}$ there exists a unique connection ∇ on $\mathcal{E}$ which is both metric w.r.t. h and compatible with the holomorphic structure of $\mathcal{E}$. This connection is called the Chern connection *of $(\mathcal{E}, h)$.*

Definition I.1.4. *Given a hermitean metric h on $\mathcal{E}$ with Chern connection ∇ we denote $\Theta_h := \Theta_h(\mathcal{E}) := \frac{i}{2\pi} F_\nabla$ and call this the* Chern curvature (tensor) *of $(\mathcal{E}, h)$.*

A priori, the Chern curvature $\Theta_h \in \mathcal{A}^2(\mathrm{End}(\mathcal{E}))$ is just some endomorphism valued 2-form. One may check however that $\Theta_h \in \mathcal{A}^{1,1}(\mathrm{End}(\mathcal{E}))$ is of type $(1,1)$ and that it is real and in fact even self-adjoint with respect to h in the sense that

$$h\Big(\Theta_h(V, W)\, \sigma, \sigma\Big) = h\Big(\sigma, \Theta_h\left(\overline{V}, \overline{W}\right) \sigma\Big), \quad \forall\, V, W \in T^{\mathbb{C}}X, \forall\, \sigma \in \mathcal{A}^0(\mathcal{E}).$$

The following formulae are well-known:

Example I.1.5. (see [Huy05, Proposition 4.3.7])
Let $(\mathcal{E}, h)$, $(\mathcal{E}', h')$ be hermitean vector bundles on X and let $f\colon Y \to X$ be a holomorphic map. Then, the Chern curvature of the induced hermitean metric on

(1) $f^*\mathcal{E}$ is given by $\Theta_{f^*h}(f^*\mathcal{E}) = f^*\Theta_h(\mathcal{E})$.
(2) $\mathcal{E}^*$ is given by $\Theta(\mathcal{E}^*) = -\Theta(\mathcal{E})^T$.
(3) $\mathcal{E} \oplus \mathcal{E}'$ is given by $\Theta_{h\oplus h'}(\mathcal{E} \oplus \mathcal{E}') = \Theta_h(\mathcal{E}) \oplus \Theta_{h'}(\mathcal{E}')$.
(4) $\mathcal{E} \otimes \mathcal{E}'$ is given by $\Theta_{h\otimes h'}(\mathcal{E} \otimes \mathcal{E}') = \Theta_h(\mathcal{E}) \otimes \mathrm{id}_{\mathcal{E}'} + \mathrm{id}_{\mathcal{E}} \otimes \Theta_{h'}(\mathcal{E}')$.

Example I.1.6. Let $\mathscr{L}$ be a holomorphic line bundle over X equipped with a hermitean metric h. In this case, the curvature tensor $\Theta_h \in \mathcal{A}^{1,1}(\mathrm{End}\mathscr{L}) = \mathcal{A}_X^{1,1}$ is a real $(1,1)$-form. In fact, using a direct calculation (which may be found in [Huy05, Example 4.3.9.]) one can prove that for any open subset $U \subseteq X$ and any non-vanishing holomorphic section $\sigma \in H^0(U, \mathscr{L})$ it holds that

$$\Theta_h|_U = -\frac{i}{2\pi}\, \partial\bar{\partial} \log(h(\sigma, \sigma))\Big|_U. \tag{I.1}$$

In particular, Θ_h is d-closed. Moreover, it follows from Eq. (I.1) that for any integer m the induced metric h^m on the line bundle $\mathscr{L}^{\otimes m}$ satisfies $\Theta_{h^m}(\mathscr{L}^{\otimes m}) = m\,\Theta_h(\mathscr{L})$. Conversely, given a hermitean metric h on $\mathscr{L}^{\otimes m}$ the expression $\sqrt[m]{h}$ makes sense as a smooth hermitean metric on $\mathscr{L}$ and using the same formula one readily computes $\Theta_{h^{1/m}}(\mathscr{L}) = \frac{1}{m}\Theta_h(\mathscr{L}^{\otimes m})$.

Example I.1.7. Let $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{Q} \to 0$ be a short exact sequence of holomorphic vector bundles on X and let h be a hermitean metric on $\mathcal{E}$. By abuse of notation, we continue to denote by h the metrics on $\mathcal{F}, \mathcal{Q}$ induced by the C^∞ h-orthogonal splitting $\mathcal{E} \cong_{C^\infty} \mathcal{F} \oplus \mathcal{Q}$. Then, decomposing the Chern connection $\nabla_{\mathcal{E}}$ on $(\mathcal{E}, h)$ according to this splitting one finds that

$$\nabla_{\mathcal{E}} = \begin{pmatrix} \nabla_{\mathcal{F}} & -A^* \\ A & \nabla_{\mathcal{Q}} \end{pmatrix},$$

where by $\nabla_{\mathcal{F}}, \nabla_{\mathcal{Q}}$ we denote the respective Chern connections on $\mathcal{F}, \mathcal{Q}$. Moreover, $A \in \mathcal{A}^1(\mathscr{H}om(\mathcal{F}, \mathcal{Q}))$ is called the *second fundamental form.* It follows that

$$\begin{aligned} \Theta_h(\mathcal{E})|_{\mathcal{F}} &= \Theta_h(\mathcal{F}) + \frac{i}{2\pi} A \wedge A^*, \\ \Theta_h(\mathcal{E})|_{\mathcal{Q}} &= \Theta_h(\mathcal{Q}) - \frac{i}{2\pi} A \wedge A^*. \end{aligned} \tag{I.2}$$

Here, $\Theta_h(\mathcal{E})|_{\mathcal{F}}$ denotes the component of $\Theta_h(\mathcal{E})$ in $\mathcal{A}^2(\mathrm{End}(\mathcal{F}))$.

Definition I.1.8. *A* holomorphic connection D *on* $\mathcal{E}$ *is a a* $\mathbb{C}$*-linear map of vector bundles* $D\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X$ *satisfying the Leibniz rule* $D(f\sigma) = f \cdot D\sigma + \sigma \otimes \partial f$*. In this case,* $\nabla := D + \bar\partial_{\mathcal{E}}$ *is a connection on* $\mathcal{E}$ *in the ordinary sense which is compatible with the holomorphic structure. By abuse of notation, one often conflates* D, ∇*.*

Reminder I.1.9. Let us now recall the *Chern-Weil construction* of characteristic classes as described in [Huy05, Section 4.4.]: Fix a connection ∇ on $\mathcal{E}$ and denote $r := \mathrm{rk}\mathcal{E}$. Let $P\colon \mathrm{M}_r(\mathbb{C})^k \to \mathbb{C}$ be any multilinear symmetric form on the algebra $\mathrm{M}_r(\mathbb{C})$ of complex $r \times r$ matrices and assume that P is invariant under the natural action of $\mathrm{GL}_r(\mathbb{C})$ by conjugation. Then, the expression $P(\frac{i}{2\pi}F_\nabla, \dots, \frac{i}{2\pi}F_\nabla)$ makes sense as a $2k$-form on X. Using a direct calculation one may prove that

$$d\left(P\left(\frac{i}{2\pi}F_\nabla\right)\right) = \sum P\left(\frac{i}{2\pi}F_\nabla, \dots, \frac{i}{2\pi}\nabla F_\nabla, \dots, \frac{i}{2\pi}F_\nabla\right) = 0,$$

where in the last step we used the well-known *Bianchi identity* $\nabla F_\nabla = 0$ which is valid for any connection. In particular, $P(\frac{i}{2\pi}F_\nabla, \dots, \frac{i}{2\pi}F_\nabla) \in H^{2k}(X, \mathbb{C})$ determines a well-defined cohomology class. One may verify that this class does not depend on the choice of connection ∇ and that it is functorial in the vector bundle $\mathcal{E}$. Thus, the association $\mathcal{E} \mapsto P(\mathcal{E}) := P(\frac{i}{2\pi}F_\nabla, \dots, \frac{i}{2\pi}F_\nabla) \in H^{2k}(X, \mathbb{C})$ defines a *characteristic class*. Conversely, it is a fact that any characteristic class (including Chern classes, the Chern character, Todd classes, ...) is of this form.

Example I.1.10.

(1) By the above, the cohomology class $P(\mathcal{E}) = P(\frac{i}{2\pi}F_\nabla, \dots, \frac{i}{2\pi}F_\nabla)$ does not depend on the connection we use. In particular, choosing ∇ to be the Chern connection of a hermitean metric h on $\mathcal{E}$, (in which case $\frac{i}{2\pi}F_\nabla = \Theta_h$ is real of type $(1,1)$) we see that $P(\mathcal{E}) = P(\Theta_h, \dots, \Theta_h) \in H^{k,k}(X, \mathbb{C})$ is of type (k,k). Moreover, $P(\mathcal{E}) \in H^{k,k}(X, \mathbb{R})$ is real if P is so.

(2) Suppose that $\mathcal{E}$ admits a holomorphic connection $\nabla = D + \bar\partial$. Then, a direct computation shows that $F_\nabla = F_D \in \operatorname{End}(\mathcal{E}) \otimes \Omega_X^2$ is holomorphic and, in particular, of type $(2,0)$. Consequently, any characteristic class $P(\mathcal{E})$ is of type $(2k,0)$. In combination with item (1) this shows that any characteristic class of $\mathcal{E}$ must vanish.

Example I.1.11. The first Chern class c_1 corresponds to the linear functional $\mathrm{c}_1 \colon \mathrm{M}_r(\mathbb{C}) \to \mathbb{C}, A \mapsto \operatorname{tr}(A)$. In other words, for any holomrphic vector bundle $\mathcal{E}$ and any connection ∇ on $\mathcal{E}$ it holds that $\mathrm{c}_1(\mathcal{E}) = [\frac{i}{2\pi} \operatorname{tr} F_\nabla]$. In particular, in case $\mathcal{E} = \mathscr{L}$ is a holomorphic line bundle and $\frac{i}{2\pi}F_\nabla = \Theta_h$ is the Chern curvature of some hermitean metric h on $\mathscr{L}$ we simply have that $[\Theta_h] = \mathrm{c}_1(\mathscr{L}) \in H^{1,1}(X, \mathbb{R})$. Here, we consider $\Theta_h \in \mathcal{A}^{1,1}(\operatorname{End}\mathscr{L}) = \mathcal{A}_X^{1,1}$ as a real $(1,1)$-form. In view of Eq. (I.1) we see how $\mathrm{c}_1(\mathscr{L})$ may be computed explicitly from h.

Conversely, (assuming that X is compact Kähler) it follows from Eq. (I.1) that for any closed real $(1,1)$-form η representing $\mathrm{c}_1(\mathscr{L})$ there exists a metric h' on $\mathscr{L}$ such that $\Theta_{h'} = \eta$: Indeed, any other hermitean metric on $\mathscr{L}$ is of the form $h' = e^\varphi \cdot h$ for some differentiable real function φ on X. Using Eq. (I.1) locally we compute

$$\begin{aligned}\Theta_{h'} &= -\frac{i}{2\pi}\,\partial\bar\partial \log(h'(\sigma,\sigma)) = -\frac{i}{2\pi}\,\partial\bar\partial \log(e^\varphi h(\sigma,\sigma)) \\ &= -\frac{i}{2\pi}\,\Big(\partial\bar\partial\varphi + \partial\bar\partial\log(h(\sigma,\sigma))\Big) = \Theta_h - \frac{i}{2\pi}\,\partial\bar\partial\varphi.\end{aligned}$$

Now, by assumption $\eta - \Theta_h$ is an d-exact real $(1,1)$-form and so according to the *$\partial\bar\partial$-lemma* (which may be found e.g. in [Huy05, Corollary 3.2.10]) there exists a real function φ such that $\partial\bar\partial\varphi = \Theta_h - \eta$ so that $\Theta_{h'} = \eta$.

Example I.1.12. The k-th Chern Character ch_k belongs to the linear functional $\mathrm{ch}_k \colon M_r(\mathbb{C})^k \to \mathbb{C}, (A_1, \dots, A_k) \mapsto \frac{1}{k!}\operatorname{tr}(A_1A_2\dots A_k)$. In particular, $\mathrm{ch}_1(-) = \mathrm{c}_1(-)$.

Remark I.1.13. Let us end this section by briefly explaining the relation between the positivity notions for the Ricci curvature in Riemannian geometry and the positivity of the canonical divisor. Indeed, it is not hard to show (compare [Huy05, Exercise 4.A.3.]) that if g is a Kähler metric on X, then the Chern curvature form of the induced hermitean metric on $\mathcal{O}_X(-K_X)$ is

$$\Theta_g(v, Iv) = \frac{1}{2\pi}\mathrm{Ric}_g(v, v), \quad \forall\, v \in T_x^{\mathbb{R}}X.$$

Conversely, by *Yau's* celebrated resolution of the *Calabi conjecture* for any hermitean metric h on $\mathcal{O}_X(-K_X)$ there exists a Kähler metric g on X for which it holds that $\frac{1}{2\pi}\mathrm{Ric}_g = \Theta_g(-, I-) = \Theta_h(-, I-)$. In conclusion, X admits a Kähler metric of positive/ negative/ vanishing Ricci curvature if and only if $\mathcal{O}_X(-K_X)$ admits a hermitean metric of (strictly) positive/ negative/ vanishing Chern curvature (cf. Proposition IV.3.7). This will be extended upon in Example I.2.7

2 Positivity of Line Bundles

2.1 Ample and Nef Line Bundles

Reminder I.2.1. A line bundle $\mathscr{L}$ on a compact complex manifold X is called *very ample* if it is generated by global sections and if the associated holomorphic map

$$X \to \mathbb{P}(H^0(X, \mathscr{L}))$$

is an embedding. More generally, $\mathscr{L}$ is called *ample* if some multiple $\mathscr{L}^{\otimes m}$ of $\mathscr{L}$ is very ample for some $m > 0$.

Definition I.2.2. *Let X be a compact complex manifold. A holomorphic line bundle $\mathscr{L}$ on X is called* positive *if there exists a smooth hermitean metric h on $\mathscr{L}$ whose Chern curvature form Θ_h (which automatically is a closed real $(1,1)$-form by Eq.* (I.1)*) is strictly positive in the sense of Proposition IV.3.7. In this case, X is automatically Kähler and Θ_h is a Kähler form on X.*

Example I.2.3. Consider the tautological bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ on $\mathbb{P}^n$. We define a smooth hermitean metric on $\mathcal{O}_{\mathbb{P}^n}(1)$ by the formula

$$h_{\mathrm{FS}}(f_1, f_2)|_{(z^0:\,\dots:\,z^n)} := \frac{f_1(z^0, \dots, z^n) \cdot \overline{f_2(z^0, \dots, z^n)}}{|z^0|^2 + \dots + |z^n|^2}, \quad \forall\, f_1, f_2 \in \mathcal{A}^0(\mathcal{O}_{\mathbb{P}^n}(1)).$$

Note that h_{FS} is well-defined. Using Eq. (I.1) we compute on $\{z^0 \neq 0\} \subset \mathbb{P}^n$

$$\begin{aligned}
\Theta_{h_{\mathrm{FS}}}|_{(1:\, z^1 :\, \dots :\, z^n)} &= -\frac{i}{2\pi}\partial\bar{\partial}\log h(z^0, z^0)\Big|_{(1:\, z^1 :\, \dots :\, z^n)} \\
&= -\frac{i}{2\pi}\partial\bar{\partial}\log\left(\frac{1}{1+|z^1|^2+\dots|z^n|^2}\right) \\
&= \frac{i}{2\pi}\partial\bar{\partial}\log\left(1+|z^1|^2+\dots|z^n|^2\right) \\
&= \frac{i}{2\pi}\partial\left(\sum_k \frac{z^k}{1+|z^1|^2+\dots|z^n|^2}\, d\bar{z}^k\right) \\
&= \frac{i}{2\pi}\sum_{j,k}\frac{\delta_{j,k}\,(1+|z^1|^2+\dots|z^n|^2) - \bar{z}^j z^k}{\left(1+|z^1|^2+\dots|z^n|^2\right)^2}\, dz^j \wedge d\bar{z}^k.
\end{aligned}$$

It is straightforward to verify that this is point-wise a strictly positive form. A similar computation yields the strict positivity of $\Theta_{h_{\mathrm{FS}}}$ over $\{z^i \neq 0\}$ for $i \neq 0$. It follows that $\mathcal{O}_{\mathbb{P}^n}(1)$ is a positive bundle.

Example I.2.4. Let $\mathscr{L}$ be a holomorphic line bundle on a compact complex manifold X and fix an integer $m > 0$. Then, $\mathscr{L}$ is positive if and only if $\mathscr{L}^{\otimes m}$ is so as follows from the discussion in Example I.1.6. In particular, $\mathcal{O}_{\mathbb{P}^n}(m)$ is positive for any $m > 0$.

Corollary I.2.5. *Let X be a projective manifold and let $\mathscr{L}$ be an ample line bundle on X. Then, $\mathscr{L}$ is positive.*

Proof. Say $\mathscr{L}^{\otimes m}$ is very ample. Let $\phi_m \colon X \hookrightarrow \mathbb{P}H^0(X, \mathscr{L}^{\otimes m})$ be the associated embedding so that $\mathscr{L}^{\otimes m} = \phi_m^* \mathcal{O}(1)$. Let h_{FS} be the metric on $\mathcal{O}(1)$ determined in Example I.2.3 so that $\Theta_{h_{\mathrm{FS}}}$ is strictly positive. Then, $\phi^* h_{\mathrm{FS}} = h_{\mathrm{FS}}|_X$ is a smooth hermitean metric on $\mathscr{L}^{\otimes m} = \phi_m^* \mathcal{O}(1)$ with curvature form $\Theta_{h_{\mathrm{FS}}}|_X$ - which is a strictly positive form. Thus, $\mathscr{L}^{\otimes m}$ and, hence, by Example I.2.4 also $\mathscr{L}$ is positive. □

The following result is fundamental. We will often use it without mention.

Theorem I.2.6. (Kodaira's embedding theorem)
A holomorphic line bundle $\mathscr{L}$ on a compact complex manifold X is positive if and only if it is ample. In particular, in this case X is projective.

Proof. A detailed proof may be found in [Huy05, Proposition 5.3.1.]. □

Example I.2.7. According to Remark I.1.13 X admits a Kähler metric of positive Ricci curvature if and only if $\mathcal{O}_X(-K_X)$ admits a metric of strictly positive curvature, i.e. if and only if $\mathcal{O}_X(-K_X)$ is positive or - equivalently - ample. In this case we call X a *Fano manifold*. The most famous example of a Fano manifold is of course $\mathbb{P}^n$.

According to the next result, being ample is a *numerical property*:

Proposition I.2.8. *Let X be a compact Kähler manifold. A holomorphic line bundle $\mathscr{L}$ on X is positive if and only if $c_1(\mathscr{L})$ is a Kähler class (i.e. there exists a Kähler metric ω on X such that $[\omega] = c_1(\mathscr{L}) \in H^{1,1}(X, \mathbb{R})$ as cohomology classes).*

Proof. First, if $\mathscr{L}$ is positive then there exists a metric h on $\mathscr{L}$ whose curvature form $\Theta_h \in c_1(\mathscr{L})$ is a Kähler form. Conversely, if there exists a Kähler form ω representing $c_1(\mathscr{L})$ then according to Example I.1.11 there exists a hermitean metric h on $\mathscr{L}$ so that $\Theta_h = \omega$ as differential forms. In other words, Θ_h, i.e. $\mathscr{L}$ is positive. □

According to Proposition I.2.8 above $\mathscr{L}$ is positive (i.e. ample) if and only if the class $c_1(\mathscr{L}) \in H^{1,1}(X, \mathbb{R})$ is a Kähler class. Note that the set of Kähler classes form a real, open, convex cone in the finite dimensional real vector space $H^{1,1}(X, \mathbb{R})$. We call this cone the *Kähler cone* of X.

Proposition I.2.9. *Let X be a compact Kähler manifold and let $\mathscr{L}$ be a holomorphic line bundle on X. The following assertions are equivalent:*

(1) The class $c_1(\mathscr{L}) \in H^{1,1}(X, \mathbb{R})$ is contained in the closure of the Kähler cone.

(2) For some (respectively any) Kähler form ω on X and any $\varepsilon > 0$ the cohomology class $c_1(\mathscr{L}) + \varepsilon[\omega] \in H^{1,1}(X, \mathbb{R})$ is Kähler.

(3) For some (respectively any) Kähler form ω on X and any $\varepsilon > 0$ there exists a smooth hermitean metric h on $\mathscr{L}$ such that $\Theta_h \geq -\varepsilon\omega$ as smooth $(1,1)$-forms.

Proof. Clearly (3) ⇒ (2) ⇒ (1). Conversely, (1) ⇒ (2) is an immediate consequence of the fact that the Kähler cone is open. Moreover, (3) ⇒ (2) may be deduced from Example I.1.11 using the same arguments as in the proof of Proposition I.2.8. □

Definition I.2.10. *Let X be a compact Kähler manifold.*

(1) A holomorphic line bundle $\mathscr{L}$ on X satisfying any of the equivalent conditions in Proposition I.2.9 above is called a nef *line bundle.*

(2) A cohomology class in $H^{1,1}(X, \mathbb{R})$ is called nef *if it is contained in the closure of the Kähler cone of X.*

Example I.2.11. Let $\mathscr{L}$ be a line bundle on a compact Kähler manifold X.

- Clearly, the set of nef cohomology classes form a (closed, convex) cone as well. In particular, $\mathscr{L}$ is nef if and only if $\mathscr{L}^{\otimes m}$ is so (for any $m > 0$).
- If $c_1(\mathscr{L}) = 0$, then $\mathscr{L}$ is nef.

- Conversely, if both $\mathscr{L}$ and $\mathscr{L}^*$ are nef, then $c_1(\mathscr{L}) = 0$. Indeed, it is shown in [DPS94, Corollary 1.5.] that in this case $\mathscr{L}$ must admit a smooth hermitean metric h with $\Theta_h = 0$. The proof is not hard but uses the theory of distributions and so we avoid it.

Remark I.2.12. Let $NS(X)_{\mathbb{R}} \subseteq H^{1,1}(X, \mathbb{R})$ denote the real sub vector space generated by the first Chern classes of holomorphic line bundles on X. In case X is projective, the real convex cone generated by the Chern classes of ample line bundles is clearly contained in the intersection of the Kähler cone with $NS(X)_{\mathbb{R}}$. It is not entirely trivial but true (by the work of [DP04, Theorem 4.7.]) that these two cones in fact coincide. It follows, that in the projective case the classical definition of nefness agrees with Definition I.2.10; compare also [Laz04a, Theorem 1.4.23.].

Often the ampleness (nefness) of a line bundle is employed in order to obtain estimates for intersection numbers:

Theorem I.2.13. (Nakai-Moishezon-Kleiman criterion)
Let X be a compact Kähler manifold and let $\mathscr{L}$ be a holomorphic line bundle on X.

(1) Suppose that $\mathscr{L}$ is positive (equivalently ample). Then, for all closed subvarieties $Y \subseteq X$ of positive dimension $\dim Y =: k > 0$ *it holds that*

$$\int_Y c_1(\mathscr{L})^k := [Y] \cap c_1(\mathscr{L})^k > 0.$$

(2) Conversely, suppose that X is projective (!) and that for all closed subvarieties $Y \subseteq X$ of positive dimension $\dim Y =: k > 0$ *it holds that*

$$\int_Y c_1(\mathscr{L})^k := [Y] \cap c_1(\mathscr{L})^k > 0.$$

Then, $\mathscr{L}$ is ample.

Proof. We are only going to prove (1) in case Y is smooth. Indeed, in this case for any hermitean metric h on $\mathscr{L}$ one may literally compute the intersection number $\int_Y c_1(\mathscr{L})^k$ as the integral

$$\int_Y \Theta_h \wedge \cdots \wedge \Theta_h.$$

Since Θ_h is Kähler for a suitable choice of h ($\mathscr{L}$ being positive), this integral is clearly positive. Here, we use Proposition IV.3.6. In fact, as described in [Dem12, Chapter III.] using the theory of currents the case of singular Y may be dealt with in exactly the same way.

(2) is classical; see for example [Laz04a, Theorem 1.2.23.] for a proof. □

Theorem I.2.14. (Kleiman)
Let X be a compact Kähler manifold and let $\mathscr{L}$ be a holomorphic line bundle on X.

(1) Suppose that $\mathscr{L}$ is nef. Then, for all closed subvarieties $Y \subseteq X$ of positive dimension $\dim Y =: k > 0$ *it holds that*

$$\int_Y \mathrm{c}_1(\mathscr{L})^k \geq 0.$$

(2) Conversely, suppose that X is projective (!) and that for any closed curve $C \subset X$ it holds that

$$\int_C \mathrm{c}_1(\mathscr{L}) \geq 0.$$

Then, $\mathscr{L}$ is nef.

Proof. Regarding (1), for any fixed Kähler form ω on X and any $\varepsilon > 0$ the class $\mathrm{c}_1(\mathscr{L}) + \varepsilon\omega$ is Kähler (by the very definition of nefness). Thus, as in Theorem I.2.13

$$0 < \int_Y (\mathrm{c}_1(\mathscr{L}) + \varepsilon\omega)^k$$

for any $\varepsilon > 0$, i.e. $\int_Y \mathrm{c}_1(\mathscr{L})^k \geq 0$. The proof of (2) is again classical. It may be found for example in [Laz04b, Theorem 1.4.9.]. □

Proposition I.2.15. *Let $f\colon Y \to X$ be a holomorphic map between compact Kähler manifolds. If $\mathscr{L}$ is a nef line bundle on X then $f^*\mathscr{L}$ is a nef line bundle on Y. Conversely, if f is a submersion and if $f^*\mathscr{L}$ is nef on Y, then $\mathscr{L}$ is nef on X.*

Proof. First, suppose that $\mathscr{L}$ is nef. Fix a Kähler form ω_X on X, a Kähler form ω_Y on Y and $C > 0$ such that $C \cdot \omega_Y \geq f^*\omega_X$ (this is possible because ω_Y is strictly positive and Y is compact). Now, for any $\varepsilon > 0$ there exists a metric h on $\mathscr{L}$ such that $\Theta_h \geq -\varepsilon\omega_X$. Then, f^*h is a smooth hermitean metric on $f^*\mathscr{L}$ with curvature $f^*\Theta_h \geq -\varepsilon f^*\omega_X \geq -\varepsilon \cdot C \cdot \omega_Y$. Since this construction is possible for any $\varepsilon > 0$ it follows that $f^*\mathscr{L}$ is nef.

In the projective case the second assertion is easily verified using Kleiman's criterion Theorem I.2.14. A proof in the general case is provided in [DPS94, Proposition 1.8.]. □

Corollary I.2.16. *Semi ample line bundles are nef.*

Proof. Let $\mathscr{L}$ be a semi ample bundle on the compact Kähler manifold X. Fix an integer $m > 0$ such that the map $\phi_m\colon X \to \mathbb{P}H^0(X, \mathscr{L}^{\otimes m})$ is holomorphic. Let h_{FS} be the metric on $\mathcal{O}(1)$ determined in Example I.2.3 so that $\Theta_{h_{\mathrm{FS}}}$ is strictly positive. Then, $\phi_m^* h_{\mathrm{FS}}$ is a smooth hermitean metric on $\mathscr{L}^{\otimes m} = \phi_m^*\mathcal{O}(1)$ with curvature form $\phi_m^*\Theta_{h_{\mathrm{FS}}} \geq 0$. Thus, $\mathscr{L}^{\otimes m}$ and, hence, by Example I.2.11 $\mathscr{L}$ is nef. □

Corollary I.2.17. *Let X be a smooth projective manifold and let $D \subset X$ be a smooth divisor. Then, $\mathcal{O}_X(D)$ is nef if and only if the normal bundles $\mathcal{N}_{D/X} = \mathcal{O}_X(D)|_D$ is nef.*

Proof. According to Proposition I.2.15 if $\mathcal{O}_X(D)$ is nef then so is $\mathcal{O}_X(D)|_D$. Conversely, if $\mathcal{O}_X(D)|_D$ is nef then so is $\mathcal{O}_X(D)$ as follows from Kleiman's criterion Theorem I.2.14: Let $C \subset X$ be a curve. If $C \subset D$, then $D \cdot C \geq 0$ as $\mathcal{O}_X(D)|_D$ is nef. But if C is not contained in D, then it is clear anyway that $D \cdot C \geq 0$. □

Remark I.2.18. As is customary, we often conflate divisors $D \subset X$ with their corresponding line bundle $\mathcal{O}_X(D)$. In this sense, we sometimes say that D is an ample/nef/…divisor if $\mathcal{O}_X(D)$ satisfies the corresponding property.

2.2 Positivity relative to a Holomorphic Map

As in the classical setting, there also exists a notion of positivity relative to a map in complex geometry:

Definition I.2.19. *Let $f\colon X \to Y$ be a proper submersion between complex manifolds. A holomorphic line bundle $\mathscr{L}$ on X is called f-*relatively positive *if the restriction of $\mathscr{L}$ to any fibre $F_y := f^{-1}(y)$ is positive in the sense of Definition I.2.2.*

Theorem I.2.20. *Let $f\colon X \to Y$ be a proper submersion between complex manifolds. A holomorphic line bundle $\mathscr{L}$ on X is f-relatively positive if and only if $\mathscr{L}|_{F_y}$ is ample for any $y \in Y$.*

Moreover, if Y is compact then there exists $m_0 > 0$ such that $\mathscr{L}^{\otimes m}|_{F_y}$ is very ample for any integer $m > m_0$ and any $y \in Y$. In particular, the natural rational map $X \to \mathbb{P}(f_*\mathscr{L}^{\otimes m})$ is holomorphic and a closed embedding for all $m > m_0$.

Proof. The first assertion is clear by Kodaira's Theorem I.2.6. The second assertion is essentially just a consequence of the compactness of Y and the fact that being very ample is an open condition. A more detailed reference for the algebraic case is provided in [Laz04a, Theorem 1.7.6.] and the general complex case may be dealt with analogously. □

Lemma I.2.21. *Let $f\colon X \to Y$ be a submersion between compact complex manifolds with fibres $F_y := f^{-1}(y)$. Suppose that there exists an f-relatively ample holomorphic line bundle $\mathscr{L}$ on X. Then, for any Kähler form ω_Y on Y there exists a real number $0 < \varepsilon \ll 1$ such that the class*

$$f^*[\omega_Y] + \varepsilon\, \mathrm{c}_1(\mathscr{L}) \in H^{1,1}(X, \mathbb{R})$$

is a Kähler class on X. In particular, if Y is Kähler then so is X.

Proof. Choose a base point $y_0 \in Y$ and denote $F_0 := f^{-1}(y_0)$. Since $\mathscr{L}|_{F_0}$ was assumed to be positive, there exists a smooth hermitean metric h_0 on $\mathscr{L}|_{F_0}$ with strictly positive curvature form. Now, by the classical theorem of Ehresmann, as $f\colon X \to Y$ is a submersion there exists a neighbourhood $y_0 \in U_0$ such that $f^{-1}(U_0) \cong U_0 \times F_0$ as differentiable manifolds. Shrinking U_0 if necessary we may clearly assume that $\mathscr{L}|_{U_0 \times F_0} \cong_{C^\infty} pr_2^*\mathscr{L}|_{F_0}$ as differentiable complex line bundles (for example by [BT82, Theorem 6.8.]). Let us denote by $\widetilde{h_0} := pr_2^* h_0$ the hermitean metric on $\mathscr{L}|_{f^{-1}(U_0)}$ induced by any such identification. Since Y was assumed to be compact, there exists a finite cover of Y by such neighbourhoods U_i. Fix a partition of unity ρ_i subordinate to this cover and put $h := \sum f^*\rho_i \cdot \widetilde{h_i}$. Then, h is a smooth hermitean metric on $\mathscr{L}$ and for any $y \in Y$, $h|_{F_y} = \sum \rho_i(y) h_i$ is a metric with curvature form $\Theta_h|_{F_y} = \sum \rho_i(y)\Theta_{h_i}$ on $\mathscr{L}|_{F_y}$. Note that the later is a strictly positive form.

In total, we see that $\mathrm{c}_1(\mathscr{L})$ may be represented by the differentiable $(1,1)$-form Θ_h which is strictly positive along any fibre. Since $f^*\omega_Y$ is trivial along the fibres, we see that also the form

$$f^*\omega_Y + \varepsilon\Theta_h$$

is strictly positive along the fibres for any $\varepsilon > 0$. Since $f^*\omega_Y$ is strictly positive in the horizontal directions and since Y is compact, we see that $f^*\omega_Y + \varepsilon\Theta_h$ is also strictly positive in the horizontal directions for sufficiently small $\varepsilon > 0$. Altogether, we conclude that $f^*\omega_Y + \varepsilon\Theta_h$ is Kähler for sufficiently small $0 < \varepsilon \ll 1$. □

Corollary I.2.22. *Let $f\colon X \to Y$ be a submersion between compact Kähler manifolds. Suppose that there exists an f-relatively ample line bundle $\mathscr{L}$ on X which is at the same time nef on X. Then, for any Kähler form ω_Y on Y the class*

$$\mathrm{c}_1(\mathscr{L}) + f^*[\omega_Y] \in H^{1,1}(X, \mathbb{R})$$

is a Kähler class on X.

Proof. According to the preceding result Lemma I.2.21 there exists a real number $0 < \varepsilon \ll 1$ such that $f^*[\omega_Y] + \varepsilon\, \mathrm{c}_1(\mathscr{L})$ is a Kähler class on X. But then,

$$\mathrm{c}_1(\mathscr{L}) + f^*[\omega_Y] = (1-\varepsilon)\, \mathrm{c}_1(\mathscr{L}) + (f^*[\omega_Y] + \varepsilon\, \mathrm{c}_1(\mathscr{L}))$$

is a Kähler class as well as the sum of a nef class and a Kähler class (see item (2) in Proposition I.2.9). □

2.3 Big Line Bundles

Let us end this section by collecting some basic results concerning big line bundles that we are going to require later on.

Definition I.2.23. *Let X be a compact complex manifold and let $\mathscr{L}$ be a holomorphic line bundle on X. We call $\mathscr{L}$* big *if there exists an integer $m > 0$ such that $H^0(X, \mathscr{L}^{\otimes m}) \neq 0$ and such that the rational map*

$$\phi_m \colon X \dashrightarrow \mathbb{P}H^0(X, \mathscr{L}^{\otimes m})$$

is generically finite (i.e. generically has finite fibres).

Example I.2.24. Let $\mathscr{L}$ be a holomorphic line bundle on a compact complex manifold X and let $m > 0$ be an integer. Then, clearly $\mathscr{L}$ is big if and only if $\mathscr{L}^{\otimes m}$ is so.

Remark I.2.25. Essentially by definition, a compact complex manifold X carries a big line bundle if and only if it is a *Moishezon* manifold (i.e. the transcendence degree of the field of meromorphic functions of X over $\mathbb{C}$ is equal to the dimension of X). In particular, due to a famous theorem of Moishezon any compact Kähler manifold carrying a big line bundle must necessarily be projective.

Proposition I.2.26. *Let $f \colon Y \to X$ be a surjective, generically finite holomorphic map between projective manifolds. If $\mathscr{L}$ is a big line bundle on X, then $f^*\mathscr{L}$ is a big line bundle on Y.*

Proof. Indeed, if the map $\phi_m \colon X \to \mathbb{P}H^0(X, \mathscr{L}^{\otimes m})$ is generically finite onto its image, then so is the map $\phi_m \circ f \colon Y \to \mathbb{P}H^0(X, \mathscr{L}^{\otimes m})$. But the latter is just the rational map associated to the linear series $H^0(X, \mathscr{L}^{\otimes m}) \subseteq H^0(Y, f^*\mathscr{L}^{\otimes m})$. In particular, the full linear series of $f^*\mathscr{L}$ also defines a generically finite map. □

Lemma I.2.27. (Kodaira's trick, [Laz04a, Corollary 2.2.7.])
A holomorphic line bundle $\mathscr{L}$ on a projective manifold X is big if and only if there exists an integer $m > 0$, an ample divisor A on X and an effective divisor D on X such that

$$\mathscr{L}^{\otimes m} \cong \mathcal{O}_X(A + D).$$

Theorem I.2.28. *Let $\mathscr{L}$ be a nef line bundle on a projective manifold X of dimension $\dim X = n$. Then, $\mathscr{L}$ is a big bundle if and only if $\mathrm{c}_1(\mathscr{L})^n > 0$.*

Note that according to Theorem I.2.14 $\mathrm{c}_1(\mathscr{L})^n \geq 0$ holds in any case as soon as $\mathscr{L}$ is nef.

Proof. This is essentially a consequence of the asymptotic Riemann-Roch theorem. A detailed exposition is contained in [Laz04a, Theorem 2.2.16.]. □

3 Positivity of Vector Bundles

Throughout this section, let (X, ω) denote a compact Kähler manifold and let $\mathcal{E}$ be a holomorphic vector bundle over X. Recall, that for any hermitean metric h on $\mathcal{E}$ the Chern curvature form $\Theta_h \in \mathcal{A}^{1,1}(\mathrm{End}(\mathcal{E}))$ is a real endomorphism valued $(1,1)$-form on X. Moreover, it is self-adjoint with respect to h (see the remark after Definition I.1.4). Perhaps the most straightforward attempt at defining positivity of higher rank vector bundles is the following:

Definition I.3.1. *We call $\mathcal{E}$* positive in the sense of Griffiths *if there exists a hermitean metric h on $\mathcal{E}$ such that $\Theta_h > 0$ in the sense of real endomorphism valued $(1,1)$-forms, i.e. $h(\Theta_h \sigma, \sigma) > 0$ as real $(1,1)$-forms for all non-vanishing $\sigma \in \mathcal{A}^0(\mathcal{E})$. Similarly, $\mathcal{E}$ is said to be* nef in the sense of Griffiths *if for any $\varepsilon > 0$ there exists a hermitean metric h on $\mathcal{E}$ such that $\Theta_h \geq -\varepsilon Id_{\mathcal{E}} \cdot \omega$.*

Proposition I.3.2. *Tensor products, direct sums and quotients of holomorphic vector bundles which are positive (resp. nef) in the sense of Griffiths are positive (resp. nef) in the sense of Griffiths.*

Proof. The assertion about tensor products and direct sums follows immediately from Example I.1.5. Moreover, if $\mathcal{E} \twoheadrightarrow \mathcal{Q}$ is a holomorphic quotient of a hermitean bundle $(\mathcal{E}, h)$ then Eq. (I.2) yields

$$h\left(\Theta_h(\mathcal{E})\sigma, \sigma\right) = h\left(\Theta_h(\mathcal{Q})\sigma, \sigma\right) + \frac{i}{2\pi} h\left(A \wedge A^* \sigma, \sigma\right) \leq h\left(\Theta_h(\mathcal{Q})\sigma, \sigma\right), \tag{I.3}$$

for all $\sigma \in \mathcal{A}^0(\mathcal{E})$. Here, we use that $\frac{i}{2\pi} A \wedge A^* \geq 0$. In other words, the curvature of the quotient can at most increase and it immediately follows that $\mathcal{Q}$ is Griffiths nef (positive) as soon as $\mathcal{E}$ is. □

Proposition I.3.3. *Let $f\colon X \to Y$ be a holomorphic map between compact Kähler manifolds. If $\mathcal{E}$ is nef in the sense of Griffiths, then so is $f^*\mathcal{E}$.*

Proof. This may be proved completely analogously to Proposition I.2.15. □

Alternatively, we can also introduce positivity notions for vector bundles of higher rank by just imitating the approach usually taken in algebraic geometry:

Definition I.3.4. *The bundle $\mathcal{E}$ is said to be ample (resp. nef) if the tautological bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ on the projectivisation $\mathbb{P}(\mathcal{E}) \xrightarrow{\pi} X$ of $\mathcal{E}$ is ample (resp. nef). Our convections regarding projective bundles are recalled in Section 2 of the appendix.*

Remark I.3.5. Note that together with (X, ω) also $\mathbb{P}(\mathcal{E})$ is compact Kähler so that it makes sense to ask whether $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is nef. In fact, $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is π-relatively ample by construction and so $\mathrm{c}_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) + C \cdot [\pi^*\omega]$ is a Kähler class on $\mathbb{P}(\mathcal{E})$ for any sufficiently large $C \gg 0$ by Lemma I.2.21.

Let us clarify below how these two notions of positivity are related to each other:

Proposition I.3.6. *If $\mathcal{E}$ is nef in the sense of Griffiths, then it is nef.*

Proof. According to Proposition I.3.3 also $\pi^*\mathcal{E}$ on $\mathbb{P}(\mathcal{E}) \xrightarrow{\pi} X$ is nef in the sense of Griffiths. But then, also $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is nef as a quotient bundle of the nef bundle $\pi^*\mathcal{E}$.□

Lemma I.3.7. *The bundle $\mathcal{E}$ is nef if and only if for any $\varepsilon > 0$ there exists a sequence of metrics h_m on $\operatorname{Sym}^m \mathcal{E}$ and $m_0 > 0$ such that*

$$\Theta_{h_m}(\operatorname{Sym}^m \mathcal{E}) \geq -\varepsilon\, m \operatorname{Id}_{\operatorname{Sym}^m \mathcal{E}} \cdot \omega, \quad \forall\, m > m_0. \tag{I.4}$$

Proof. Suppose that there exists $m > 0$ and a metric h_m on $\operatorname{Sym}^m \mathcal{E}$ satisfying Eq. (I.4). Then, also $\Theta_{\pi^* h_m}(\pi^* \operatorname{Sym}^m \mathcal{E}) \geq -\varepsilon\, m\, Id \cdot \pi^*\omega$. Since curvature can only increase in quotients by Eq. (I.3) it follows that the induced hermitean metric h'_m on $\pi^* \operatorname{Sym}^m \mathcal{E} \twoheadrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(m)$ satisfies $\Theta_{h'_m} \geq -\varepsilon\, m\, \pi^*\omega \geq -\varepsilon\, m\, \omega_{\mathbb{P}(\mathcal{E})}$. Here, $\omega_{\mathbb{P}(\mathcal{E})}$ is any fixed background metric on $\mathbb{P}(\mathcal{E})$ such that $\omega_{\mathbb{P}(\mathcal{E})} \geq \pi^*\omega$. Then, according to Example I.1.6 $h := \sqrt[m]{h'_m}$ is a smooth hermitean metric on $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ of curvature $\Theta_h \geq -\varepsilon\omega_{\mathbb{P}(\mathcal{E})}$. Since ε was arbitrary, this proves that $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is nef. Thus, $\mathcal{E}$ is nef by definition.

The proof of the converse is not hard but rather technical and so we avoid it. It may be found in [DPS94, Theorem 1.12.]. □

A similar statement is true of positive bundles. In fact, Griffiths conjectured that a holomorphic vector bundle is ample if and only if it is Griffiths positive but this conjecture is still wide open.

Lemma I.3.7 is very useful as it allows to prove properties of nef bundles via reduction to the case of Griffiths nef bundles where we already know them:

Corollary I.3.8. *Quotients of nef vector bundles are nef.*

Proof. Suppose that $\mathcal{E}$ is a nef vector bundle on X and let $\mathcal{E} \twoheadrightarrow \mathcal{Q}$ be a holomorphic quotient bundle. Fix $\varepsilon > 0$ and let h_m be a sequence of hermitean metrics on $\operatorname{Sym}^m \mathcal{E}$ as in Lemma I.3.7 above. Note that we may consider $\operatorname{Sym}^m \mathcal{Q}$ as a holomorphic quotient bundle of $\operatorname{Sym}^m \mathcal{E}$. In particular, an application of Eq. (I.3) yields that

$$h_m\Big(\Theta_{h_m}(\operatorname{Sym}^m \mathcal{Q})\sigma, \sigma\Big) \geq h_m\Big(\Theta_{h_m}(\operatorname{Sym}^m \mathcal{E})\sigma, \sigma\Big) \geq -\varepsilon m\, h_m\Big(\operatorname{Id}(\sigma), \sigma\Big) \cdot \omega,$$

for all sections $\sigma \in \mathcal{A}^0(\operatorname{Sym}^m \mathcal{Q})$ and for any $m > m_0$. In other words, we have the inequality $\Theta_{h_m}(\operatorname{Sym}^m \mathcal{Q}) \geq -\varepsilon m\, Id \cdot \omega$. Invoking Lemma I.3.7 once more, we see that also $\mathcal{Q}$ is nef. □

Corollary I.3.9. *Symmetric tensor powers of nef vector bundles are nef.*

Proof. Suppose that $\mathcal{E}$ is a nef vector bundle on X and fix $\ell > 0$ and $\varepsilon > 0$. Then, Lemma I.3.7, yields the existence of a sequence of metrics $h_{m\cdot\ell}$ on the vector bundle $\operatorname{Sym}^{m\ell}\mathcal{E} = \operatorname{Sym}^m(\operatorname{Sym}^\ell \mathcal{E})$ and an integer $m_0 > 0$ such that

$$\Theta_{h_{m\ell}}(\operatorname{Sym}^{m\ell}\mathcal{E}) \geq -\varepsilon\, \ell \cdot m \operatorname{Id}_{\operatorname{Sym}^m(\operatorname{Sym}^\ell \mathcal{E})} \cdot \omega, \quad \forall\, m > m_0.$$

Thus, by the same token also $\operatorname{Sym}^\ell \mathcal{E}$ is nef. □

Lemma I.3.10. *Extensions of nef vector bundles are nef.*

Proof. The proof is similar in spirit to the other proofs above. However, as it is rather technical we avoid it. For details see [DPS94, Proposition 1.15.]. □

Corollary I.3.11. *If $\mathcal{E}_1, \mathcal{E}_2$ are nef vector bundles on X, then also $\mathcal{E}_1 \otimes \mathcal{E}_2$ is nef. In particular, the determinant (or more generally any exterior power) of a nef vector bundle is nef.*

Proof. Since $\mathcal{E}_1, \mathcal{E}_2$ are nef also the trivial extension $\mathcal{E}_1 \oplus \mathcal{E}_2$ is nef (see Lemma I.3.10). Now, an application of Corollary I.3.9 shows that also

$$\operatorname{Sym}^2(\mathcal{E}_1 \oplus \mathcal{E}_2) = \operatorname{Sym}^2(\mathcal{E}_1) \oplus \mathcal{E}_1 \otimes \mathcal{E}_2 \oplus \operatorname{Sym}^2(\mathcal{E}_2)$$

is nef and, hence, so is $\mathcal{E}_1 \otimes \mathcal{E}_2$ as a quotient of this bundle (see Corollary I.3.8). In particular, if $\mathcal{E}$ is nef then so is $\mathcal{E}^{\otimes m}$ for any $m > 0$. Consequently, also all $\bigwedge^m \mathcal{E}$ are nef as they are quotients of $\mathcal{E}^{\otimes m}$. □

For the sake of later reference, let us collect below the most important hereditary properties of nef bundles in short exact sequences:

Theorem I.3.12. *Consider an exact sequence of holomorphic vector bundles on a compact Kähler manifold*

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{Q} \to 0.$$

(1) If $\mathcal{E}$ is nef, then so is $\mathcal{Q}$.

(2) If $\mathcal{F}$ and $\mathcal{Q}$ are nef, then so is $\mathcal{E}$.

(3) If $\mathcal{E}$ is nef and if $\det(\mathcal{Q})^*$ is nef, then also $\mathcal{F}$ is nef.

Proof. Part (1) and (2) have already been proved above as Corollary I.3.8 and Lemma I.3.10. Regarding (3), we consider the natural bilinear pairing

$$\mathcal{F} \oplus \bigwedge^{s-1} \mathcal{F} \to \det(\mathcal{F}), \quad (\sigma, \tau) \mapsto \sigma \wedge \tau.$$

Here, we denote $s := \mathrm{rk}(\mathcal{F})$. This pairing is non-degenerate, hence gives rise to an identification $\mathcal{F} \cong \bigwedge^{s-1} \mathcal{F}^* \otimes \det(\mathcal{F})$. Now, the later is naturally a quotient of $\bigwedge^{s-1} \mathcal{E}^* \otimes \det(\mathcal{F})$ which in turn may be identified in a similar manner with

$$\bigwedge^{s-1} \mathcal{E}^* \otimes \det(\mathcal{F}) = \bigwedge^{r-(s-1)} \mathcal{E} \otimes \det(\mathcal{E})^* \otimes \det(\mathcal{F}) = \bigwedge^{r-(s-1)} \mathcal{E} \otimes \det(\mathcal{Q})^*.$$

Here, $r := \mathrm{rk}(\mathcal{E})$. Finally, since $\mathcal{E}$ is nef so is $\bigwedge^{r-(s-1)} \mathcal{E}$. Since tensor products of nef vector bundles and quotients thereof remain nef, it follows that also the bundle $\bigwedge^{s-1} \mathcal{E}^* \otimes \det(\mathcal{F})$ and, hence, $\bigwedge^{s-1} \mathcal{F}^* \otimes \det(\mathcal{F}) \cong \mathcal{F}$ are nef and so we are done. □

Remark I.3.13. By ad verbatim the same argumentation, all of the above results remain valid for ample vector bundles.

Proposition I.3.14. *Let $f\colon X \to Y$ be a holomorphic map between compact Kähler manifolds and let $\mathcal{E}$ be a vector bundle on X. If $\mathcal{E}$ is nef on Y, then $f^*\mathcal{E}$ is nef on X. Conversely, if f is a surjective submersion and if $f^*\mathcal{E}$ is nef on X, then also $\mathcal{E}$ is nef on Y.*

Proof. Consider the induced map $\widetilde{f}\colon \mathbb{P}(f^*\mathcal{E}) \to \mathbb{P}(\mathcal{E})$. Then, both $\mathbb{P}(f^*\mathcal{E}), \mathbb{P}(\mathcal{E})$ are compact Kähler and $\widetilde{f}^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{O}_{\mathbb{P}(f^*\mathcal{E})}(1)$. Moreover, if f is a submersion, then so is $\widetilde{f}$. Thus, the result is reduced to the case of line bundles which was already treated in Proposition I.2.15. □

We end our discussion of nef bundles by stating without proof the following two technical results which we are going to need later on:

Theorem I.3.15. *Let $\mathcal{E}$ be a nef vector bundle. Then, the global sections of $\mathcal{E}^*$ do not admit any zeros.*

Proof. The proof of this fact may be found in [DPS94, Proposition 1.16.]. The idea is rather straightforward but to implement it one heavily relies on the theory of distributions. □

Lemma I.3.16. *Let X be a compact Kähler manifold of $\dim X = n$ and let $\mathcal{E}$ be a nef vector bundle on X. If there exists an integer k such that $\mathrm{c}_1(\mathcal{E})^k = 0$, then for any homogeneous polynomial $\zeta \in H^{k,k}(X, \mathbb{C})$ of (cohomological) degree $2k$ in the Chern classes of $\mathcal{E}$ and for any Kähler form ω on X it holds that*

$$\int_X \zeta \wedge \omega^{n-k} = 0.$$

Note that passing to the limit, the same conclusion holds true if one replaces ω by any nef cohomology class. We will later apply the theorem for ζ a combination of Chern characters and Todd classes of $\mathcal{E}$.

Proof. This is essentially just a consequence of the classical *Fulton-Lazarsfeld inequalities* which remain valid in the Kähler setting by the work of [DPS94, Corollary 2.6.]. □

Finally, let us conclude this chapter by quickly reviewing other positivity notions which are going to be important in later chapters.

Definition I.3.17. *A holomorphic vector bundle $\mathcal{E}$ on X is called* strongly semi ample *if some multiple* $\operatorname{Sym}^m \mathcal{E}$ *is globally generated. It is called* semi ample *if* $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(m)$ *is globally generated for some $m > 0$.*

Reminder I.3.18. Recall for later reference the following elementary facts:

- A holomorphic vector bundle $\mathcal{E}$ on X is globally generated if and only if the bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is so.
- Strongly semi ample bundles are always semi ample. The converse is not true in general (see [MU19, Example 3.2.]).
- Any quotient of a globally generated vector bundle is globally generated itself.

Corollary I.3.19. *Semi ample vector bundles are nef.*

Proof. By definition, a vector bundle $\mathcal{E}$ is semi ample (respectively nef) if and only if $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is semi ample (nef). Since semi ample line bundles are nef according to Corollary I.2.16 we conclude. □

Definition I.3.20. *A holomorphic vector bundle $\mathcal{E}$ on X is said to be* big *if* $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ *is a big line bundle.*

4 Stability and Flatness

In this section we want to study flat vector bundles, i.e. such with vanishing curvature. It turns out that there exists an especially well behaved subclass of flat bundles called *numerically flat bundles* and we will have a closer look at these. Finally, we indicate how the theory may be extended to the setting of more general principal bundles.

4.1 Non-Abelian Hodge Theory

Reminder I.4.1. Let X be a complex manifold. Recall, that a connection ∇ on a holomorphic vector bundle $\mathcal{E}$ over X is said to be *flat* if its curvature tensor $F_\nabla = 0$ vanishes. Note that in this case all Chern classes of $\mathcal{E}$ vanish (because any Chern class of $\mathcal{E}$ is just a polynomial in F_∇). Recall moreover, that given any representation $\rho\colon \pi_1(X) \to \mathrm{GL}_r(\mathbb{C})$ we may construct a flat vector bundle $\mathcal{E}_\rho$ on X (which we call the vector bundle *constructed form* ρ) as follows: Denoting by $\widetilde{X} \to X$ the universal cover of X we put

$$\mathcal{E}_\rho := (\widetilde{X} \times \mathbb{C}^r)/\pi_1(X).$$

Here, $\pi_1(X)$ acts on $\widetilde{X}$ in the natural way and on $\mathbb{C}^r$ through ρ. Then, $\mathcal{E}_\rho$ is naturally a holomorphic vector bundle over X and the ordinary component-wise exterior derivative gives a well-defined flat connection ∇ on X. In fact, it is well-known that all flat vector bundles are of this form (compare also Lemma I.4.12 below). In summary, we see that the association $\rho \mapsto \mathcal{E}_\rho$ defines a functor

$$\{\text{rep's } \pi_1(X) \to GL_r\} \to \{\text{holomorphic bundles of rank } r \text{ over } X\}.$$

with essential image the bundles admitting a flat connection. It is a fundamental observation that this functor is *not* full:

Example I.4.2. Consider the elliptic curve $E := \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ and the flat line bundles $\mathscr{L}_1, \mathscr{L}_2$ over E defined by the representations

$$\rho_1\colon \pi_1(E) = \mathbb{Z} + i\mathbb{Z} \to \mathbb{C}^\times, \quad n + im \mapsto e^m$$
$$\rho_2\colon \pi_1(E) = \mathbb{Z} + i\mathbb{Z} \to \mathbb{C}^\times, \quad n + im \mapsto e^{in}.$$

Then, the map

$$f\colon \mathscr{L}_1 \to \mathscr{L}_2, \quad [(z,w)] \mapsto \left[\left(z, e^{iz}w\right)\right]$$

is a well defined holomorphic isomorphism $\mathscr{L}_1 \cong \mathscr{L}_2$. However, the representations ρ_1, ρ_2 are not isomorphic (two one-dimensional representations are isomorphic if and only if they are identical). In fact, ρ_2 is unitary while ρ_1 is not.

The *Non-Abelian Hodge theorem* may be seen as a remedy of this deficiency. To formulate it, we need some more terminology: First, recall that a representation $\rho\colon \pi_1(X) \to \mathrm{GL}_r(\mathbb{C})$ is called *unitary* if there exists a hermitean inner product on $\mathbb{C}^r$ such that $\rho(\gamma) \in U(r)$ for all $\gamma \in \pi_1(X)$. Second, we need the concept of *stability*:

Definition I.4.3. *Let (X, ω) be a compact Kähler manifold of dimension n and let $\mathcal{F}$ be a torsion-free coherent $\mathcal{O}_X$-module on X of rank $r > 0$.*

(i) The determinant bundle *of $\mathcal{F}$ is* $\det(F) := (\bigwedge^r \mathcal{F})^{**}$*. It is a coherent reflexive $\mathcal{O}_X$-module of rank one and, hence, a line bundle on X. In particular, the expression* $c_1(\mathcal{F}) := c_1(\det(\mathcal{F}))$ *makes sense.*

(ii) The degree *of $\mathcal{F}$ (with respect to ω) is the real number* $\deg(\mathcal{F}) := c_1(\mathcal{F}) \cap [\omega]^{n-1}$. *The* slope $\mu(\mathcal{F})$ *of $\mathcal{F}$ (with respect to ω) is the number* $\mu(\mathcal{F}) = \deg(\mathcal{F})/\mathrm{rk}(\mathcal{F})$.

(iii) We call $\mathcal{F}$ stable *(respectively* semi stable*) if for any torsion-free coherent quotient $\mathcal{F} \twoheadrightarrow \mathcal{Q}$ it holds that $\mu(\mathcal{Q}) > \mu(\mathcal{F})$ (respectively $\mu(\mathcal{Q}) \geq \mu(\mathcal{F})$). Moreover, we call $\mathcal{F}$* poly stable *if it is a direct sum of stable sub sheaves of equal slope.*

The following celebrated result emerged as a combination of works of Donaldson, Kobayashi, Hitchin, Uhlenbeck, Yau and many others:

Theorem I.4.4. (Non-Abelian Hodge Theorem, Unitary version)
Let (X, ω) be a compact Kähler manifold. There exist equivalences of categories:

$$\{\textit{irreducible unitary rep's of } \pi_1(X)\} \xrightarrow{\sim} \left\{\begin{matrix}\textit{stable holomorphic vector bundles with} \\ c_1(-) \cap [\omega]^{n-1} = \mathrm{ch}_2(-) \cap [\omega]^{n-2} = 0\end{matrix}\right\}$$

$$\{\textit{unitary representations of } \pi_1(X)\} \xrightarrow{\sim} \left\{\begin{matrix}\textit{poly stable holomorphic bundles with} \\ c_1(-) \cap [\omega]^{n-1} = \mathrm{ch}_2(-) \cap [\omega]^{n-2} = 0\end{matrix}\right\}$$

Holomorphic bundles contained in the categories on the RHS are called (irreducible) hermitean flat.

Remark I.4.5. Let X be a compact Kähler manifold and let $\rho\colon \pi_1(X) \to \mathrm{GL}_r$ be a unitary representation (w.r.t. some inner product h) with corresponding hermitean flat bundle $\mathcal{E}_\rho$. Then, h extends to a hermitean metric h on $\mathcal{E}_\rho$ compatible with the flat connection ∇ induced by ρ. Thus, $\Theta_h = \frac{i}{2\pi} F_\nabla = 0$. Conversely, if a bundle $\mathcal{E}$ admits a hermitean metric h with vanishing Chern curvature $\Theta_h = 0$ then $\mathcal{E}$ is hermitean flat but we will not need this. In any case, we certainly see that hermitean flat bundles are always nef (even in the sense of Griffiths).

The case of semi stable vector bundles is slightly more complicated: Let us call a representation $\rho\colon \pi_1(X) \to \mathrm{GL}_r(\mathbb{C})$ *graded unitary* if there exists a ρ-invariant filtration $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_s = \mathbb{C}^r$ such that the induced representations $\pi_1(X) \to V_i/V_{i-1}$ are all unitary. Accordingly, the associated holomorphic vector bundle $\mathcal{E}_\rho$ is called *graded hermitean flat*.

Now, it follows from the work of Simpson and Nie-Zhang (see [Den21, Theorem 1.2.] for more details) that any semi stable vector bundle $\mathcal{E}$ satisfying the conditions $c_1(\mathcal{E}) \cap [\omega]^{n-1} = \mathrm{ch}_2(\mathcal{E}) \cap [\omega]^{n-2} = 0$ is indeed graded hermitean flat. Conversely, we have the following:

Lemma I.4.6. *Let (X, ω) be a compact Kähler manifold and let $\mathcal{E}$ be a holomorphic vector bundle over X. Then, the following assertions are equivalent:*

(1) The bundle $\mathcal{E}$ is graded hermitean flat.

(2) Both $\mathcal{E}$ and $\mathcal{E}^*$ are nef.

(3) Both $\mathcal{E}$ and $\det(\mathcal{E}^*)$ are nef.

(4) The bundle $\mathcal{E}$ is nef and $\mathrm{c}_1(\mathcal{E}) = 0$.

(5) The bundle $\mathcal{E}$ is semi stable and $\mathrm{c}_1(\mathcal{E}) \cap [\omega]^{n-1} = \mathrm{ch}_2(\mathcal{E}) \cap [\omega]^{n-2} = 0$.

Alternatively, we call a bundle satisfying one of the above conditions numerically flat*.*

Proof. Let us start by proving (1) ⇒ (2): Let $\rho\colon \pi_1(X) \to \mathrm{GL}_r$ be a graded unitary representation. Then, by definition the bundle $\mathcal{E}_\rho$ is an extension of hermitean flat bundles which are always nef by Remark I.4.5. As extensions of nef bundles are nef by Theorem I.3.12, it follows that also $\mathcal{E}_\rho$ is nef. But together with ρ, also $\bar{\rho}^*$ is graded unitary and so the same argument also shows that $(\mathcal{E}_\rho)^* = \mathcal{E}_{\bar{\rho}^*}$ is nef.

The implication (2) ⇒ (3) is tautologous. Moreover, (3) ⇒ (4) was contained in Example I.2.11.

Now, assuming (4) let us prove that $\mathcal{E}$ is semi stable. Fix any torsion-free coherent quotient sheaf $\mathcal{E} \twoheadrightarrow \mathcal{Q}$ of rank s say. Since $\mathcal{Q}$ is torsion-free it is free away from an analytic subset Z of codimension at least 2. In particular, $\bigwedge^s \mathcal{E} \twoheadrightarrow \det(\mathcal{Q})$ is surjective away from Z and, hence, everywhere (by reflexivity of $\bigwedge^s \mathcal{E}, \det(\mathcal{Q})$). Consequently, $\det(\mathcal{Q})$ is necessarily nef as a quotient of the nef bundle $\bigwedge^s \mathcal{E}$ and so

$$\deg(\mathcal{Q}) = \mathrm{c}_1(\mathcal{Q}) \cap [\omega]^{n-1} = \lim_{\varepsilon \to 0} (\mathrm{c}_1(\mathcal{Q}) + \varepsilon[\omega]) \cap [\omega]^{n-1} \geq 0.$$

Here we use that $\mathrm{c}_1(\mathcal{Q}) + \varepsilon[\omega]$ is Kähler for any $\varepsilon > 0$. On the other hand, clearly $\mu(\mathcal{E}) = 0$ as $\deg(\mathcal{E}) = \mathrm{c}_1(\mathcal{E}) \cap [\omega]^{n-1} = 0$. Thus, $\mu(\mathcal{Q}) \geq 0 = \mu(\mathcal{E})$ and so $\mathcal{E}$ is semi stable. Moreover, since $\mathcal{E}$ is nef and since $\mathrm{c}_1(\mathcal{E})^2 = 0$ an application of Lemma I.3.16 yields that also $\mathrm{ch}_2(\mathcal{E}) \cap [\omega]^{n-2} = 0$.

Finally, as stated above (5) ⇒ (1) follows from [Den21, Theorem 1.2.]. □

Remark I.4.7. Lemma I.4.6 shows that the functor

$$\{\text{graded unitary rep's of } \pi_1(X)\} \twoheadrightarrow \left\{\begin{matrix}\text{semi stable holomorphic bundles with} \\ \mathrm{c}_1(-) \cap [\omega]^{n-1} = \mathrm{ch}_2(-) \cap [\omega]^{n-2} = 0\end{matrix}\right\}.$$

is essentially surjective. In contrast to Theorem I.4.4 it is however *not* faithful (compare [Den21, Remark 3.4.]). Nevertheless, the construction of Simpson and Nie-Zhang actually provides a fully faithful section

$$\left\{\begin{matrix}\text{semi stable holomorphic bundles with} \\ \mathrm{c}_1(-) \cap [\omega]^{n-1} = \mathrm{ch}_2(-) \cap [\omega]^{n-2} = 0\end{matrix}\right\} \xhookrightarrow{\text{SNZ}} \{\text{graded unitary rep's of } \pi_1(X)\}.$$

With this in mind, from now on, if $\mathcal{E}$ is any numerically flat vector bundle on X, then we always consider it as a graded hermitean flat bundle using the uniquely determined graded unitary structure given by the functor SNZ. The fully faithfulness then translates to the fact, that any vector bundle homomorphism $\phi\colon \mathcal{E}_1 \to \mathcal{E}_2$ between numerically flat bundles comes from a morphism between the underlying representations of $\pi_1(X)$ specified by SNZ. In other words, if we denote by ∇_1, ∇_2 the corresponding flat connections on $\mathcal{E}_1, \mathcal{E}_2$ and if σ is any flat local section to $\mathcal{E}_1$ (i.e. if $\nabla_1\sigma = 0$), then also $\phi(\sigma)$ is a flat section.

Note that this is certainly not true with respect to arbitrary flat structures on $\mathcal{E}_1, \mathcal{E}_2$ (Example I.4.2 gives an explicit counter example).

Let us conclude this subsection by noting that numerically flat bundles enjoy excellent hereditary properties.

Proposition I.4.8. *Let $\mathcal{E}$ be a numerically flat vector bundles on a compact Kähler manifold X. Then, all symmetric tensor powers $\operatorname{Sym}^m \mathcal{E}$ are numerically flat as well.*

Proof. Let $\rho\colon \pi_1(X) \to \mathrm{GL}(\mathbb{C}^r)$ be the distinguished underlying graded unitary representation from Remark I.4.7 so that $\mathcal{E} \cong \mathcal{E}_\rho$. Then, clearly $\operatorname{Sym}^m \mathcal{E} \cong \mathcal{E}_{\operatorname{Sym}^m \rho}$, where $\operatorname{Sym}^m \rho\colon \pi_1(X) \to \mathrm{GL}(\operatorname{Sym}^m \mathbb{C}^r)$ is the induced representation which is of course graded unitary itself. Alternatively, one may also use that according to Lemma I.4.6 $\mathcal{E}$ is numerically flat if and only if $\mathcal{E}$ and $\mathcal{E}^*$ are nef and then apply Corollary I.3.11. □

Lemma I.4.9. *Let $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{Q} \to 0$ be a short exact sequence of holomorphic vector bundles on a compact Kähler manifold X. If any two of the three bundles are numerically flat, then so is the third.*

Proof. Using the characterisations of numerically flat bundles provided in Lemma I.4.6, this is a direct consequence of our result Theorem I.3.12 on nef bundles in short exact sequences. □

4.2 Notions of Flatness for general Fibre Bundles

In the last subsection we discussed flatness properties of vector bundles. Recall, that any vector bundle is equivalently determined by its underlying frame bundle which is a principal GL-bundle. With some more technical effort, most of the results discussed in the previous subsection remain valid in the more general setting of arbitrary principal bundles. This is what we want to explain in the following. To this end, throughout this section we fix a complex manifold X, a complex Lie group G and a holomorphic principal G-bundle $\mathcal{G} \xrightarrow{\pi} X$.

Reminder I.4.10. A (holomorphic) connection in $\mathcal{G}$ is nothing but a (holomorphic) splitting of the short exact sequence

$$0 \to \mathfrak{ad}(\mathcal{G}) \to (\pi_* \mathcal{T}_{\mathcal{G}})^G \to \mathcal{T}_X \to 0.$$

Here, $\mathfrak{ad}(\mathcal{G})$ is the vector bundle on X associated to $\mathcal{G}$ via the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$, where as per usual $\mathfrak{g} = \mathcal{T}_G|_1$. Moreover, $(\pi_* \mathcal{T}_{\mathcal{G}})^G \subset \pi_* \mathcal{T}_{\mathcal{G}}$ denotes the sub sheaf of sections which are invariant under the natural action of G. Equivalently, a (holomorphic) connection is a (holomorphic) G-invariant sub bundle $\mathcal{H} \subset \mathcal{T}_{\mathcal{G}}$ which is point wise complementary to $\mathcal{T}_{\mathcal{G}/X}$. The equivalence of both points of view follows immediately from the natural identification $\mathfrak{ad}(\mathcal{G}) = (\pi_* \mathcal{T}_{\mathcal{G}/X})^G$.

Given a connection $\mathcal{H}$ in $\mathcal{G}$ and a differentiable path $\gamma\colon [0,1] \to X$ one may define the *parallel transport* P_γ *along* γ by the following rule:

$$P_\gamma\colon G \cong \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(1)) \cong G, \quad y \mapsto P_\gamma(y) := \widetilde{\gamma}_y(1).$$

Here, $\widetilde{\gamma}_y\colon [0,1] \to \mathcal{G}$ is the unique curve lifting γ, starting at y and which is tangent to $\mathcal{H}$ at all times. It follows immediately from the uniqueness of the parallel transport that $P_\gamma\colon G \to G$ is just multiplication by some element in G; in particular it is a biholomorphism. Note that the identification $P_\gamma \in G$ is independent of choices of identifications $\pi^{-1}(\gamma(0)) \cong \pi^{-1}(\gamma(1)) \cong G$.

Example I.4.11. Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation, let $\mathcal{E}$ denote the associated vector bundle and suppose we are given a (holomorphic) connection $\mathcal{H}$ in $\mathcal{G}$. Fix a point $x \in X$, a section σ to $\mathcal{E}$ and a tangent vector $v = \frac{d}{dt}\gamma(t)|_{t=0} \in \mathcal{T}_X|_x$. Then, the rule

$$\nabla_v(\sigma) := \frac{d}{dt}\left(\rho \circ P_{\gamma|_{[0,t]}}\right)^{-1}\left(\sigma\left(\gamma(t)\right)\right)\Big|_{t=0}$$

defines a (holomorphic) connection in the usual sense in the holomorphic vector bundle $\mathcal{E}$. More concretely, $\mathcal{G}$ might be the frame bundle of a holomorphic vector bundle $\mathcal{E}$ and ρ might be the natural representation. In this case on can show that connections in $\mathcal{E}$ are in fact in one-to-one correspondence with connections in $\mathcal{G}$.

Now, as in the case of vector bundles (compare with [Lee18, Theorem 7.11.] or [Dem12, Section V.6.]) one may define the *curvature* of a connection to be the derivative of the parallel transport map along small loops as these loops shrink to points. We say that the connection is *flat* if its curvature vanishes or - equivalently - if the parallel transport map P_γ only depends of the homotopy class of the path γ. Then, following [Ati57, Proposition 14] one may show:

Lemma I.4.12. *Let $\widetilde{X} \to X$ denote the universal cover of X. The following assertions are equivalent:*

1. *The principal bundle $\mathcal{G}$ admits a flat (automatically holomorphic) connection.*
2. *There exists a representation $\rho\colon \pi_1(X) \to G$ and a biholomorphism of fibre bundles*
$$(\widetilde{X} \times G)/\pi_1(X) \cong \mathcal{G}.$$
Here, $\pi_1(X)$ acts on $\widetilde{X}$ in the natural way and on G through ρ.
3. *The short exact sequence*
$$0 \to \mathcal{T}_{\mathcal{G}/X} \to \mathcal{T}_{\mathcal{G}} \to \pi^*\mathcal{T}_X \to 0$$
admits a global holomorphic splitting establishing $\pi^*\mathcal{T}_X$ as an integrable sub bundle of $\mathcal{T}_{\mathcal{G}}$.
4. *The transition functions of $\mathcal{G}$ may be chosen to be locally constant.*

Example I.4.13. In the situation of Example I.4.11 above, if the connection on $\mathcal{G}$ is flat, then so is the induced connection on $\mathcal{E}$.

We will also need the following terminology:

Definition I.4.14. *Suppose that X is compact Kähler and that G is a connected reductive complex Lie group. We say that the principal G-bundle $\mathcal{G}$ is* semi stable *(resp.* numerically flat*), if the adjoint vector bundle $\mathfrak{ad}(\mathcal{G})$ is so.*

Lemma I.4.15. (Biswas-Subramanian, [BS05])
Let X be a compact Kähler manifold, let G be a (connected) semi simple complex Lie group and let $\mathcal{G} \to X$ be a holomorphic principal G-bundle. Let $\rho\colon G \to \mathrm{GL}(V)$ be a complex representation of G and let us denote the associated vector bundle by $\mathcal{E}$.

1. *If $\mathcal{G}$ is semi stable (resp. numerically flat) as a principal bundle, then so is $\mathcal{E}$.*
2. *If $\mathcal{E}$ is semi stable (resp. numerically flat) as a holomorphic bundle and if $\ker\rho \subseteq G$ is a* finite *group, then also $\mathcal{G}$ is semi stable (resp. numerically flat).*

Remark I.4.16. Lemma I.4.15 remains true also when G is a complex Lie group with possibly finitely many connected components as long as the connected component of the identity G^0 is semi simple. Indeed, in this case $\pi\colon \widetilde{X} := \mathcal{G}/G^0 \to X$ is a finite étale cover of X and by construction the structure group of $\pi^*\mathcal{G}$ may be reduced to the group G^0. Thus, we may apply Lemma I.4.15 on $\widetilde{X}$. The assertion now follows as being semi stable (numerically flat) is invariant under taking finite étale covers.

Chapter II

Manifolds with Nef Tangent Bundle

In this chapter we want to discuss the structure theory of compact Kähler manifolds with nef tangent bundle. In particular, we will prove that (up to finite étale covers) these manifolds are flat fibre bundles over a complex torus via their Albanese map. Moreover, conjecturally the fibres are homogeneous Fano manifolds. This explicit structure theory will be crucial in the following chapter for our work on canonical extensions of such manifolds.

We begin this chapter by surveying which projective surfaces may posses a nef tangent bundle. This already gives a good sense for the structure theory to be developed in the successive sections.

In the second section, we discuss the fundamental result by Demailly, Peternell and Schneider that the Albanese of a compact Kähler manifold with nef tangent bundle is a holomorphic submersion and that (possibly after finite étale cover) the fibres are Fano manifolds with nef tangent bundle.

The third section is devoted to the study of Fano manifolds with nef tangent bundle. In particular, the conjecture of Campana and Peternell predicts that these should be homogeneous. We will quickly discuss what is known about the conjecture. Then, we will discuss what is known (conjecturally) about the automorphism groups of such manifolds. This knowledge will be important later on.

A proof that the Albanese of compact Kähler manifolds with nef tangent bundle is (up to finite étale cover) even a flat fibre bundle is contained in the fourth paragraph. We give quite some details, since this result was not yet contained in [DPS94].

In the fifth section we answer the converse question of precisely which fibre bundles over complex tori with fibres homogeneous Fano manifolds have a nef tangent bundle. The case of projective bundles is particularly enlightening.

Finally, we end this chapter by stating some more recent generalisations of the result of Demailly, Peternell and Schneider and we explain which of the natural follow-up questions remain unanswered.

1 Surfaces with Nef Tangent Bundle

To obtain a feeling for what kind of compact Kähler manifolds we may expect to have a nef tangent bundle we will use this section to survey the case of smooth projective surfaces. This situation is already interesting enough to obtain a sense of the general theory. To this end, recall that one classically distinguishes surfaces according to whether or not they contain a -1-*curve* (i.e. a smooth rational curve $C \cong \mathbb{P}^1$ of self intersection $C^2 = -1$). Our first observation is that projective surfaces with nef tangent bundle can not support such curves:

Proposition II.1.1. *Let X be a projective manifold with nef tangent bundle and let D be a smooth divisor in X. Then, D is a nef divisor.*

Proof. Indeed, if $\mathcal{T}_X$ is nef then so is $\mathcal{T}_X|_D$ by Proposition I.3.14. Since $\mathcal{N}_{D/X}$ is naturally a quotient of $\mathcal{T}_X|_D$ it is nef as well and it follows by Corollary I.2.17 that D itself is nef. □

In particular, if X is any smooth projective surface with a nef tangent bundle and if $C \subset X$ is a smooth curve, then $\mathrm{c}_1(\mathcal{O}(C))^2 = C^2 \geq 0$ and so C can not be a -1-curve. Consequently, according to the Kodaira classification of surfaces X must fit into precisely one of the following mutually exclusive categories:

(1) X is isomorphic to $\mathbb{P}^2$ or to $\mathbb{P}^1 \times \mathbb{P}^1$.

(2) X is a *ruled surface*: There exists a smooth projective curve C of genus $g(C) \geq 1$ and a rank 2 vector bundle $\mathcal{E}$ on C such that $X \cong \mathbb{P}(\mathcal{E})$. Moreover, X does not contain a -1-curve.

(3) The canonical line bundle $\mathcal{O}_X(K_X)$ is nef. In this case X is called a *minimal* surface.

In the following we will have a look at which of these surfaces (may) exhibit a nef tangent bundle:

(1) The tangent bundle of both $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ is globally generated and, hence, (invoking Corollary I.2.16) nef: In general, the *Euler sequence* establishes the tangent bundle of any projective space $\mathbb{P}^n$ as a quotient of the ample and globally generated bundle $\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)}$. Thus, the tangent bundle of any projective space is ample and globally generated. Moreover, since

$$\mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^1} = pr_1^*\mathcal{T}_{\mathbb{P}^1} \oplus pr_2^*\mathcal{T}_{\mathbb{P}^1} = pr_1^*\mathcal{O}_{\mathbb{P}^1}(2) \oplus pr_2^*\mathcal{O}_{\mathbb{P}^1}(2)$$

we see that also $\mathcal{T}_{\mathbb{P}^1\times\mathbb{P}^1}$ is certainly globally generated. Note that both $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are homogeneous Fano manifolds. In fact, we will prove in Section 3 that all homogeneous manifolds have a nef tangent bundle.

(2) If $X \xrightarrow{\pi} C$ is a ruled surface over a curve of genus $g(C) \geq 2$, then X never has a nef tangent bundle. The reason is the short exact sequence

$$0 \to \mathcal{T}_{X/C} \to \mathcal{T}_X \xrightarrow{d\pi} \pi^*\mathcal{T}_C \to 0.$$

If $\mathcal{T}_X$ is nef then so is its quotient $\pi^*\mathcal{T}_C$ and, hence, $\mathcal{T}_C = \mathcal{O}_C(-K_C)$ (the latter implication was proved in Proposition I.2.15). In particular, the genus of C can not be greater or equal to two because in this case $\mathcal{O}_C(K_C)$ is ample.

The case of elliptic curves however is very interesting: Indeed, on the one hand as for the case of $\mathbb{P}^1 \times \mathbb{P}^1$ one proves that $\mathbb{P}^1 \times C = \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C)$ has a globally generated hence nef tangent bundle if C is an elliptic curve. On the other hand, some ruled surfaces over elliptic curves do not exhibit a nef tangent bundle: For any $n > 0$ and any point $p \in C$ the ruled surface $X := \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(-np))$ does not contain a -1-curve, yet it contains the curve $C' := \mathbb{P}(\mathcal{O}_C) \subset X$ which is of self-intersection $(C')^2 = -n < 0$. Thus, it follows from Proposition II.1.1 that X can *not* posses a nef tangent bundle. In Section 5 we will investigate this situation (and its higher dimensional analogues) in detail. In particular, we will prove that the tangent bundle of a ruled surface $X = \mathbb{P}(\mathcal{E})$ is nef if and only if the bundle $\mathcal{E}$ defining X is semi stable (see Corollary II.5.10 below).

(3) If X is a minimal surface with nef tangent bundle then both $\mathcal{O}_X(K_X)$ and $\mathcal{O}_X(-K_X) = \det(\mathcal{T}_X)$ are nef. Thus, $\mathcal{O}_X(K_X)$ is numerically flat. It follows from *abundance* (which is proved for surfaces) that in this case $\mathcal{O}_X(K_X)$ is a torsion line bundle. In other words, some finite étale cover $\widetilde{X}$ of X has trivial canonical bundle. Appealing again to the classification of surfaces, $\widetilde{X}$ must be a torus or a *K3-surface*. One can show (using e.g. the results obtained in Section 2) that K3 surfaces never exhibit a nef tangent bundle. The tangent bundle of a torus on the other hand is always nef. Indeed, the tangent bundle of any complex torus $X = \mathbb{C}^q/\Gamma$ may of course be identified with $\mathcal{T}_X = \mathcal{O}_X^{\oplus q}$, i.e. it is the trivial bundle which is nef as an extension of nef bundles.

Altogether, we see that any smooth projective surface with a nef tangent bundle is either a homogeneous Fano manifold, a quotient of a complex torus or admits a holomorphic map onto an elliptic curve with fibres $\mathbb{P}^1$ (a homogeneous Fano as well). We may summarise this by saying that all such surfaces admit a holomorphic map onto a complex torus such that the fibres are homogeneous Fano manifolds. In the following sections we will see that similar results remain true also in higher dimensions.

2 The Albanese Map: Local Structure

This chapter is devoted to studying compact Kähler manifolds with nef tangent bundle. As already pointed out in Section 1 from the case of surfaces we expect that such manifolds always admit a holomorphic map onto a torus with very special properties. Indeed, any complex manifold admits a universal map onto a torus, called its *Albanese* map, and this is the map we will study. Concretely, within this section we want to explain the important structural results on Albanese maps of compact Kähler manifolds with nef tangent bundle discovered in [DPS94]. For the proof we will follow the original publication.

Let us start by defining the Albanese map. To this end, let for the moment X be any compact Kähler manifold. Recall, that in this case $H^0(X, \Omega_X^1)$ is a finite dimensional vector space over $\mathbb{C}$. Recall also the following basic result:

Proposition II.2.1. *Let X be a compact Kähler manifold. If $\eta \in H^0(X, \Omega_X^p)$ is a global holomorphic p-form for some p, then $d\eta = 0$.*

Proof. Since η is holomorphic by assumption,

$$d\eta = \partial\eta + \bar{\partial}\eta = \partial\eta$$

and the form $\partial\eta$ is clearly ∂-exact. Consequently, according to the important $\partial\bar{\partial}$-*lemma* [Huy05, Corollary 3.2.10] it is also $\bar{\partial}$-exact, i.e. there exists some (differentiable) differential form ζ on X such that $\bar{\partial}\zeta = \partial\eta$. But then ζ must be a form of bidegree $(-1, p+1)$ and, hence, $\zeta = 0$. □

In particular, for any fixed base point $x_0 \in X$ and any path $\gamma\colon [0,1] \to X$ in X with starting point $\gamma(0) = x_0$ the linear functional

$$\alpha_{x_0}(\gamma)\colon H^0(X, \Omega_X^1) \to \mathbb{C}, \quad \eta \mapsto \int_\gamma \eta$$

only depends on the end-point $\gamma(1) = x$ and the homotopy type of γ. Thus, if we denote

$$\Gamma := \left\{\alpha_{x_0}(\delta) \,\middle|\, \delta \in H_1(X, \mathbb{Z})\right\} = \operatorname{Im}\left(H_1(X, \mathbb{Z}) \to H^0\left(X, \Omega_X^1\right)^*\right), \tag{II.1}$$

then the class $\alpha_{x_0}(x) := [\alpha_{x_0}(\gamma)] \in H^0(X, \Omega_X^1)^*/\Gamma$ of the linear functional

$$\alpha_{x_0}(\gamma)\colon H^0(X, \Omega_X^1) \to \mathbb{C}, \quad \eta \mapsto \int_\gamma \eta$$

only depends on the end point $\gamma(1) = x$ of γ. We call $\mathrm{Alb}(X) := H^0(X, \Omega_X^1)^*/\Gamma$ the *Albanese variety* of X and the map $\alpha_{x_0}\colon X \to \mathrm{Alb}(X), x \mapsto \alpha_{x_0}(x)$ described above is called the *Albanese morphism* (with respect to the base point x_0) of X. Note that according to Hodge theory the group Γ is a lattice in $H^0(X,\Omega^1_X)^*$ and so $\mathrm{Alb}(X) := H^0(X,\Omega^1_X)^*/\Gamma$ is indeed a complex torus.

Clearly, α_{x_0} is a differentiable map. Moreover it is easily seen that there exists a natural identification

$$d\alpha_{x_0}|_x \colon T_xX \to T_{\alpha(x)}\,\mathrm{Alb}(X) = H^0(X,\Omega^1_X)^*, \quad v \mapsto ([v]\colon \eta \mapsto \eta_x(v)) \tag{II.2}$$

for every $x \in X$. In particular, observing that the differential of α_{x_0} is $\mathbb{C}$-linear we conclude that α_{x_0} is holomorphic.

Remark II.2.2. If $x_1, x_2 \in X$ are two base points and if γ is a path connecting them then by construction the Albanese maps $\alpha_{x_1}, \alpha_{x_2}\colon X \to \mathrm{Alb}(X)$ agree up to a translation of $\mathrm{Alb}(X)$ by the vector $\alpha_{x_1}(\gamma) \in H^0(X,\Omega^1_X)^*$. Ergo, we often drop the subscripts and speak of *the* Albanese map $\alpha\colon X \to \mathrm{Alb}(X)$.

Example II.2.3. If $X = T$ is a complex torus itself, then the Albanese map $\alpha\colon T \to \mathrm{Alb}(T)$ is an isomorphism. Indeed, write $T = V/\Gamma$ for some finite dimensional $\mathbb{C}$ vector space V and a lattice Γ in V, fix the base point $[0] \in T$ and choose any point $[v] \in T$. Then, $\gamma_v\colon [0,1] \to T, t \mapsto tv$ is a path in T from $[0]$ to $[v]$. We may identify $H^0(T,\Omega^1_T) = V^*$ and under this identification the Albanese corresponds to

$$\alpha_{[0]}\colon T \to \mathrm{Alb}(T), \quad [v] \mapsto \left(\eta \in V^* \mapsto \int_{\gamma_v} \eta = \eta(v) - \eta(0) = \eta(v)\right)$$

which is the identity up to the identification $V = (V^*)^*$.

Note that the Albanese is functorial: Given a holomorphic map $f\colon X \to Y$ between compact Kähler manifolds, the map $\mathrm{Alb}(f)\colon \mathrm{Alb}(X) \to \mathrm{Alb}(Y)$ induced by the pull back $f^*H^0(X,\Omega^1_Y) \to H^0(X,\Omega_X)$ fits by construction into the commutative diagram:

$$\begin{CD} X @>{f}>> Y \\ @V{\alpha_x}VV @VV{\alpha_{f(x)}}V \\ \mathrm{Alb}(X) @>{\mathrm{Alb}(f)}>> \mathrm{Alb}(Y) \end{CD}$$

In particular, in case $Y = T'$ is any other complex torus it is not hard to deduce from this and Example II.2.3 the following universal property:

Proposition II.2.4. (Universal property of the Albanese)
Let X be a compact Kähler manifold. If $f\colon X \to T'$ is any holomorphic map onto a complex torus T', then f factors uniquely through the Albanese $\alpha\colon X \to \mathrm{Alb}(X)$.

We are now ready to start exploring the structure theory of compact Kähler manifolds with nef tangent bundle:

Proposition II.2.5. *The Albanese morphism $\alpha\colon X \to T := \operatorname{Alb}(X)$ of any compact Kähler manifold X with a nef tangent bundle is surjective and a submersion.*

Proof. If α were no submersion there would exist a point $x \in X$ for which $d\alpha|_x$ would not be surjective. In other words, there would exist a non zero linear functional $\eta \in (T_{\alpha(x)} \operatorname{Alb}(X))^* = H^0(X, \Omega_X^1)$ vanishing on $d\alpha(T_x X)$:

$$\eta\Big(d\alpha|_x(v)\Big) \overset{\text{Eq. (II.2)}}{=\!=\!=} \eta(v) =\!= 0, \quad \forall\, v \in T_x X.$$

This is to say the form $\eta \in H^0(X, \Omega_X^1)$ would vanish at x. But in view of the assumed nefness of $\mathcal{T}_X$ this would contradict Theorem I.3.15. We conclude that α is a submersion and so in particular the image of α is open. As X is compact we deduce that also $\alpha(X)$ is compact, hence closed. Since $\operatorname{Alb}(X)$ is connected we conclude that α is indeed surjective. □

Lemma II.2.6. *Let X be a compact Kähler manifold with nef tangent bundle so that according to the preceding result Proposition II.2.5 the Albanese morphism $\alpha\colon X \to T := \operatorname{Alb}(X)$ is a surjective submersion. We will denote its fibres by $F_t := \alpha^{-1}(t)$. Then, for any $t \in T$ the fibre F_t is a compact connected Kähler manifold with nef tangent bundle.*

Proof. Fix some $t \in T$. First of all, since α is a submersion F_t is a complex manifold. Moreover, it is compact Kähler as a closed submanifold of the compact Kähler manifold X.

Next, let us prove that F_t has nef tangent bundle as well. Since α is a submersion the relative tangent bundle sequence

$$0 \to \mathcal{T}_{X/T} \to \mathcal{T}_X \to \alpha^* \mathcal{T}_T \to 0 \tag{II.3}$$

is exact. Note that $\mathcal{T}_T = \mathcal{O}_T^{\oplus q}$ and, hence, also $\alpha^* \mathcal{T}_T = \mathcal{O}_X^{\oplus q}$ are trivial because T is a torus. Since $\mathcal{T}_X$ is nef, Theorem I.3.12 on the inheritance of nefness in short exact sequences yields also the nefness of the relative tangent bundle $\mathcal{T}_{X/T}$. As restrictions of nef bundles to subspaces remain nef by Proposition I.3.14 we conclude that also

$$\mathcal{T}_{X/T}|_{F_t} = \mathcal{T}_{F_t}$$

is nef.

Finally, let us prove that the fibres of α are connected. To this end, we are going to require some basic algebraic topology: First, we claim that the natural map

$$\alpha_*\colon H_1(X, \mathbb{Z}) \to H_1(T, \mathbb{Z}) \tag{II.4}$$

is surjective by construction of the Albanese variety T. Indeed, recall that by definition $T = H^0(X, \Omega^1_X)^*/\Gamma$. Now, $H_1(T, \mathbb{Z})$ may be identified with Γ as follows: The class of the line from $0 \in H^0(X, \Omega^1_X)^*$ to some element of Γ corresponds to the respective element of Γ. But the natural map $H_1(X, \mathbb{Z}) \to \Gamma$ is surjective by Eq. (II.1) which is the very definition of Γ. Explicitly, given $\alpha(\delta) \in \Gamma \cong H_1(T, \mathbb{Z})$ the closed 1-cycle $\delta \in H_1(X, \mathbb{Z})$ maps under α to the line from 0 to $\alpha(\delta)$ in T. This proves the surjectivity of α_* in Eq. (II.4).

Note that the proof shows that also the natural map

$$\alpha_* \colon \pi_1(X) \to \pi_1(T) = H_1(T, \mathbb{Z})$$

is surjective. Alternatively, this also immediately follows from the surjectivity of Eq. (II.4) and the theorem of *Hurewicz*. But then, the fibres of α must be connected for if $x_1, x_2 \in F_t$ are two points in the same fibre and if γ is a path in X from x_1 to x_2 then $\alpha \circ \gamma$ is a closed path in T. Since $\alpha_* \colon \pi_1(X) \to \pi_1(T) = H_1(T, \mathbb{Z})$ is surjective by what we just proved there exists a loop $\delta \in \pi_1(X, x_1)$ such that $\alpha \circ \delta$ is homotopic to $\alpha \circ \gamma$. Thus, replacing γ by $\gamma \cdot \delta^{-1}$ we may assume that $\alpha \circ \gamma$ is null homotopic in T. Since α is a submersion we may lift this homotopy between $\alpha \circ \gamma$ and the constant path in T to a homotopy in X between γ and a path γ' contained completely in the fibre F_t. In other words, γ' is a path in F_t connecting x_1 and x_2. Since $x_1, x_2 \in F_t$ were chosen arbitrarily the fibre F_t is connected and we are done. □

Definition II.2.7. *Let X be any compact Kähler manifold. The integer*

$$q(X) := \dim_{\mathbb{C}} H^0(X, \Omega^1_X) = \dim \operatorname{Alb}(X)$$

is called the irregularity *of X. We will also be interested in the* augmented irregularity

$$\widetilde{q}(X) := \sup \left\{ q\left(\widetilde{X}\right) \,\middle|\, \widetilde{X} \to X \text{ is a finite étale cover} \right\}$$

of X. In case $q(X) = \widetilde{q}(X)$ we say that X is of maximal irregularity.

Example II.2.8. (1) Let X be a compact Kähler manifold with nef tangent bundle. Then, the Albanese $\alpha \colon X \to \operatorname{Alb}(X)$ is a submersion according to Lemma II.2.6 and, thus, $q(X) \leq \dim X$. It follows that also $\widetilde{q}(X) \leq \dim X$. In particular, $\widetilde{q}(X)$ is finite and so there always exists a finite étale cover of maximal irregularity in this case.

(2) Let F be a Fano manifold. Then, $q(F) = 0$. Indeed, from Hodge theory it follows that $\dim H^0(F, \Omega^1_F) = \dim H^1(F, \mathcal{O}_F)$. However,

$$H^1(F, \mathcal{O}_F) = H^1(F, \mathcal{O}_F(-K_F + K_F)) = 0$$

is equal to zero according to Kodaira vanishing, see Theorem IV.1.5.

Since étale covers of Fanos are clearly Fano as well we see that also $\widetilde{q}(F) = 0$. In fact, it is well-known that Fanos are always simply connected hence do not admit any étale covers after all.

(3) Let $T = V/\Gamma$ be a complex torus of dimension q. Then, clearly

$$q(T) = \dim H^0(T, \Omega_T^1) = \dim(V^*) = q$$

Since we already know from (1) that $\widetilde{q}(T) \leq \dim T$, it follows that $\widetilde{q}(T) = q$. Alternatively, this may be concluded from the fact that any finite étale cover of a torus is clearly a torus itself.

Here now follows the main result of this section which is also the main theorem of [DPS94]:

Theorem II.2.9. (Demailly-Perternell-Schneider)
Let X be a compact Kähler manifold with nef tangent bundle and suppose that X is of maximal irregularity. Then, the Albanese $\alpha\colon X \to \mathrm{Alb}(X) =: T$ is a surjective submersion. Its fibres are connected Fano manifolds with nef tangent bundle.

Explaining the entire proof in detail would be too much to ask for the purpose of this thesis. Let us however at least sketch the main steps of the proof:

As we already saw in Example II.2.8 above, if X is Fano, then $\widetilde{q}(X) = 0$. The first step of the proof (and in fact the crucial one according to [DPS94]) is to show the converse statement:

Proposition II.2.10. (see [DPS94])
Let X be a compact Kähler manifold of dimension $n := \dim X$ with nef tangent bundle. Then, the augmented irregularity $\widetilde{q}(X)$ vanishes if and only if X is Fano.

To prove the proposition in turn, one distinguishes the two cases $\mathrm{c}_1(X)^n > 0$ and $\mathrm{c}_1(X)^n = 0$. Note that $\mathrm{c}_1(X)^n := \mathrm{c}_1(\mathcal{T}_X)^n = \mathrm{c}_1(-K_X)^n \geq 0$ holds true in any case since $\mathcal{T}_X$ and, hence, $\mathcal{O}_X(-K_X) = \det(\mathcal{T}_X)$ were assumed to be nef (this was part of Theorem I.2.14).

In case $\mathrm{c}_1(X)^n > 0$ Theorem I.2.28 yields that $\mathcal{O}_X(-K_X)$ is big. According to Kodaira's trick Lemma I.2.27 we may write $\mathcal{O}_X(-mK_X) = \mathcal{O}_X(A + D)$ with A an ample and D an effective divisor. Then, by Proposition II.1.1 D is nef (the argument there was only given in case D is smooth but this is no restriction: Indeed, in our situation X is necessarily projective according to Remark I.2.25 and so the general case may be dealt with in ad verbatim the same fashion using the classical algebraic definitions which work also in the singular setting). Hence, $\mathcal{O}_X(-mK_X) = \mathcal{O}_X(A+D)$ is ample as the sum of an ample and a nef divisor. We conclude that X must be Fano and so $\widetilde{q}(X) = 0$ by Example II.2.8.

Conversely, in case $\mathrm{c}_1(X)^n = 0$ it follows from Theorem I.2.13 that X can not be Fano and so we need to prove that $\widetilde{q}(X) \neq 0$: First, using the Hirzebruch-Riemann-Roch theorem we compute

$$\sum_p h^0\left(X, \Omega_X^p\right) = \chi(X, \mathcal{O}_X) = \int_X \mathrm{td}_n(X) \xlongequal{\text{Lemma I.3.16}} 0.$$

Since X is assumed to be compact it holds that $h^0(X, \mathcal{O}_X) = 1$ and one concludes that there must exist a non trivial global holomorphic p-form η for some odd integer p. We may consider η as a morphism of sheaves

$$\bigwedge^{p-1} \mathcal{T}_X \to \Omega^1_X.$$

Let us denote the image of this map by $\mathcal{E} \subseteq \Omega^1_X$. One can prove that $\mathcal{E}$ is in fact a vector bundle. Then, $\mathcal{E}$ is nef as a quotient of the nef bundle $\bigwedge^{p-1} \mathcal{T}_X$. On the other hand, $\mathcal{E}$ is a sub bundle of Ω^1_X. Thus, $\mathcal{E}^*$ is a quotient of $(\Omega^1_X)^* = \mathcal{T}_X$ and, hence, nef as well. Altogether, $\mathcal{E}$ is a numerically flat sub bundle of Ω^1_X, i.e. it is defined by a graded unitary representation ρ of the fundamental group of X. In particular, some sub bundle $\mathcal{E}' \subseteq \mathcal{E}$ arises from a unitary sub representation $\rho' \leq \rho$. Since the tangent bundle of X was assumed to be nef one has some control over the size of the fundamental group of X (similar in spirit to the well known *theorem of Myer's* [Mye41]). This suffices to conclude that the image of the representation ρ' is finite. In particular, we see that after some finite étale cover $\pi\colon \widetilde{X} \to X$ the pull back $\pi^*\mathcal{E}' \subset \Omega^1_{\widetilde{X}}$ is a trivial vector bundle. Thus, $q(\widetilde{X}) \geq h^0(\widetilde{X}, \pi^*\mathcal{E}') = \mathrm{rk}(\pi^*\mathcal{E}') \neq 0$. This concludes the proof of the proposition.

Let us turn back to explaining the proof of Theorem II.2.9. To this end, let X be a compact Kähler manifold of maximal irregularity and with nef tangent bundle and consider the Albanese $\alpha\colon X \to \mathrm{Alb}(X)$. According to what we have proved before, α is a smooth submersion with connected fibres. The idea is now of course to use induction on the dimension and the augmented irregularity of X and apply the induction hypothesis to the fibres of α.

In view of Proposition II.2.10 there are two cases: If the augmented irregularity of the fibres vanishes, then the fibres are Fano and we are done. In the second case, heading for a contradiction we suppose that X is of maximal irregularity and that the augmented irregularity of the fibres does *not* vanish. Then, one considers the *relative Albanese*

$$\alpha'\colon X \to \mathrm{Alb}(X/\mathrm{Alb}(X)).$$

Here, $\mathrm{Alb}(X/\mathrm{Alb}(X))$ is a manifold admitting a submersion p to $\mathrm{Alb}(X)$ such that for any $t \in \mathrm{Alb}(X)$, the restriction

$$\alpha'|_{F_t}\colon F_t \to p^{-1}(t) = \mathrm{Alb}(F_t) \subset \mathrm{Alb}(X/\mathrm{Alb}(X))$$

is the usual Albanese map of F_t. We now claim that p is a fibre bundle (so that in particular all the tori $\mathrm{Alb}(F_t)$ are isomorphic) with finite monodromy. Grant this for the moment. Then, after a finite étale cover $\widetilde{X} \to X$ the monodromy is trivial and so the relative Albanese splits:

$$\alpha'\colon \widetilde{X} \to \mathrm{Alb}(\widetilde{X}/\mathrm{Alb}(\widetilde{X})) = \mathrm{Alb}(X) \times \mathrm{Alb}(F_t) =: T'$$

In other words, there exists a surjective holomorphic map from $\widetilde{X}$ to a torus T' of dimension $q(X) + \widetilde{q}(F) > q(X) = \dim \mathrm{Alb}(X)$. But this is a contradiction, because in this case the pull back map

$$\alpha^* \colon H^0(T', \Omega^1_{T'}) \hookrightarrow H^0(\widetilde{X}, \Omega^1_{\widetilde{X}})$$

would be injective. Hence,

$$q(\widetilde{X}) = \dim H^0(\widetilde{X}, \Omega^1_{\widetilde{X}}) \geq H^0(T', \Omega^1_{T'}) = \dim T' > \dim \mathrm{Alb}(X) = q(X)$$

and this contradicts our assumption that X is of maximal irregularity.

To conclude the proof it thus remains to establish that p is a fibre bundle with finite monodromy. The argument for this is rather technical: One may consider p as a variation of the complex structure on $\mathrm{Alb}(F_t)$ over the torus $\mathrm{Alb}(X)$. But there exist no non-trivial variations of the complex structure of a torus over another torus (a more or less elementary argument for this may be found in [DPS94, Proposition 3.12.]). That the monodromy of p is finite may be proved by another induction argument and we will not comment further on this. The detailed proof is given in [DPS94, Proposition 3.12.].

3 The special Case of Homogeneous Fanos

In this section we want to take a closer look at Fano manifolds with nef tangent bundle. These naturally appeared in the important Theorem II.2.9. In 1991, Campana and Peternell conjectured that such manifold are always homogeneous. We will start off this section by making more explicit the (partly conjectural) connections between the homogeneouity of a manifold and the positivity of its tangent bundle. Afterwards, we will discuss what is already known about the holomorphic automorphism group of such manifolds. During this section we do not follow any particular source but rather collect some more or less well known facts.

3.1 Homogeneouity and Positivity of the Tangent Bundle

Let us start by recalling the situation for curves and surfaces which was already discussed in Section 1:

Example II.3.1. The only one-dimensional Fano manifold is of course $\mathbb{P}^1$. Note that $\mathcal{T}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-K_{\mathbb{P}^1}) = \mathcal{O}_{\mathbb{P}^1}(2)$ is even ample. Note also that $\mathbb{P}^1$ is homogeneous for the natural action of $\mathrm{GL}_2(\mathbb{C})$.

More generally, any projective space $\mathbb{P}^n$ has ample tangent bundle. The reason is that the *Euler sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \to \mathcal{T}_{\mathbb{P}^n} \to 0$$

establishes $\mathcal{T}_{\mathbb{P}^n}$ as a quotient of the ample vector bundle $\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)}$.

Example II.3.2. By the classification of smooth projective surfaces we know that the only Fano surfaces are $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$ and blow ups of $\mathbb{P}^2$ in at most 8 general points.

- The tangent bundle of $\mathbb{P}^1 \times \mathbb{P}^1$ is $pr_1^*\mathcal{O}_{\mathbb{P}^1}(2) \oplus pr_2^*\mathcal{O}_{\mathbb{P}^1}(2)$. Here, by $pr_i \colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ we denote the projections. In particular, $\mathcal{T}_{\mathbb{P}^1\times\mathbb{P}^1}$ is nef. In fact, it is globally generated.
- As discussed in the previous example, $\mathbb{P}^2$ has ample tangent bundle.
- No blow up of $\mathbb{P}^2$ in a finite set of points has a nef tangent bundle. Indeed, the exceptional curves would be -1-curves and we already proved in Section 1 that such curves can not exist on a compact Kähler surface with nef tangent bundle.

In summary, the only Fano manifolds with nef tangent bundle of dimension at most two are $\mathbb{P}^1, \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$. Note that all of these are homogeneous for the action of some complex Lie group, namely $\mathrm{GL}_2(\mathbb{C}), \mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$ and $\mathrm{GL}_3(\mathbb{C})$. More generally, the tangent bundle of any compact homogeneous Kähler manifold is nef as follows from the following proposition:

Proposition II.3.3. *Let F be a (connected (!)) complex manifold. If F is homogeneous for the action of some complex Lie group, then $\mathcal{T}_F$ is generated by global sections. Conversely, if F is compact and if $\mathcal{T}_F$ is globally generated, then for any two points $x, y \in F$ there exists a holomorphic automorphism φ of F such that $\varphi(x) = y$.*

Proof. Let us start by assuming that F is homogeneous. Let G be a complex Lie group acting holomorphically and transitively on F via $\sigma \colon G \times F \to F$ so that for any $x \in F$ the map $\sigma_x \colon G \to F$, $g \mapsto g \cdot x$ is holomorphic. Then, for any vector $A \in \mathfrak{g} := T_1G$ the assignment $x \mapsto d\sigma_x|_1(A)$ defines a holomorphic vector field on F which we are going to denote $d\sigma(A)$.

We want to prove that $\mathcal{T}_F$ is generated by global sections. To this end, fix any point $x \in F$ and any vector $v \in \mathcal{T}_{F,x}$. We need to construct a global holomorphic vector field V on F such that $V|_x = v$. Indeed, since the action of G on F was assumed to be transitive, the map $\sigma_x \colon G \to F$, $g \mapsto g \cdot x$ is surjective. In particular, according to *Sard's theorem* [Lee13, Theorem 6.10.] (which is basically the C^∞-analogue of the theorem on generic smoothness), σ_x is a submersion over almost all points of F. On the other hand,

$$d\sigma_x|_g = d(\sigma_x \circ \ell_g \circ \ell_{g^{-1}})|_g = d(g \circ \sigma_x \circ \ell_{g^{-1}})|_g = dg|_x \circ d\sigma_x|_1 \circ d\ell_{g^{-1}}|_g \qquad \text{(II.5)}$$

so that the rank of $d\sigma_x$ is constant. Here, $\ell_g \colon G \to G$ denotes (as per usual) the multiplication by g from the left map and we conflate the elements of G with the automorphism of F they define.

It follows, that σ_x is a submersion. In particular, there exists a vector $A \in \mathfrak{g} = \mathcal{T}_{G,1}$ such that $d\sigma_x|_1(A) = v$. But then, $d\sigma(A)$ is a global holomorphic vector field on F with $d\sigma(A)|_x = d\sigma_x|_1(A) = v$ as required.

Conversely, suppose that F is compact and that $\mathcal{T}_F$ is generated by global sections.

Claim: For any $x \in F$ there exists a neighbourhood $x \in U = U_x \subseteq F$ such that for any point $y \in U$ there exists an automorphism φ of F satisfying $\varphi(x) = y$

Grant this for a moment. Fix two points $x, y \in F$ and a path $\gamma \colon [0,1] \to F$ from x to y. Then, since $[0,1]$ is compact there exists a partition $0 = t_0 < t_1 < \cdots < t_s = 1$ such that for any i the path $\gamma|_{[t_i, t_{i+1}]}$ is completely contained in a neighbourhood $U_{\gamma(t_i)}$ as above. According to the claim, there then exist automorphisms $\varphi_1, \ldots \varphi_r$ of F such that $\varphi_i(\gamma(t_{i-1})) = \gamma(t_i)$. It follows, that $\varphi := \varphi_r \circ \ldots \varphi_1$ is an automorphism of F taking x to y and so we are done (assuming the claim).

Finally, let us turn to the proof of the above claim. Fix $x \in F$. Since $\mathcal{T}_F$ is globally generated by assumption, there exist global holomorphic vector fields $V_1, \ldots, V_m \in H^0(F, \mathcal{T}_F)$ such that $V_1|_x, \ldots, V_m|_x$ is a basis for the holomorphic tangent space at x. Since F was assumed to be compact, the *flows*

$$\phi^i \colon F \times \mathbb{C} \to F, \quad (y, z) \mapsto \phi^i_z(y)$$

to the vector fields V_i exist. Recall, that by definition the flows ϕ^i_z are characterised by the fact that for any $z \in \mathbb{C}$ the map ϕ^i_z is a holomorphic automorphism of F and

$$\frac{d}{dz}\phi^i_z(y)\Big|_{z=z_0} = V_i\Big|_{\phi^i_{z_0}(y)}. \tag{II.6}$$

Consider the holomorphic map

$$\psi \colon \mathbb{C}^m \to F, \quad (z_1, \ldots, z_m) \mapsto \phi^m_{z_m}\left(\ldots\left(\phi^1_{z_1}(x)\right)\right).$$

It follows immediately from Eq. (II.6) that $d\psi|_0(\frac{\partial}{\partial z_i}) = V_i|_x$. Since the vector fields $V_1, \ldots, V_m$ were chosen to form a basis of $\mathcal{T}_F|_x$ this implies that $d\psi|_0$ is non degenerate. Hence, there exists a neighbourhood $x \in U \subseteq F$ such that $\psi^{-1}|_U$ is biholomorphic onto its image. But this implies, that for any $y \in U$ there exists a point $(z_1, \ldots, z_m) \in \mathbb{C}^m$ such that

$$y = \psi(z_1, \ldots, z_m) = \phi^m_{z_m}\left(\ldots\left(\phi^1_{z_1}(x)\right)\right).$$

Thus, the holomorphic automorphism $\varphi := \phi^m_{z_m} \circ \cdots \circ \phi^1_{z_1}$ of F takes x to y. This concludes the proof of our claim and we have already seen above that this suffices to prove the statement. □

Example II.3.4. Fix a finite dimensional complex vector space V and a sequence of natural numbers $0 < k_1 < \cdots < k_s \leq \dim V$. Then, the set

$$F := \{0 \subset V_1 \subset \ldots V_s \subset V | \quad V_i \subset V \text{ is a sub vector space of } \dim V_i = k_i\} \\ \subseteq \prod_{i=1}^{s} \mathbb{G}(k_i, V)$$

is naturally a closed subvariety and in fact a submanifold of $\prod \mathbb{G}(k_i, V)$. Here, (as per usual) $\mathbb{G}(k, V)$ denotes the *Grassmanian* of complex subspaces of dimension k in V. Such a manifold F is called a *flag manifold.* Especially important examples of flag manifolds are projective spaces ($s = 1$, $k_1 = \dim V - 1$) and Grassmanians manifolds ($s = 1$).

Since all Grassmanians are projective, so is F. Moreover, F is clearly homogeneous for the action of the complex Lie group $\mathrm{GL}(V)$ and so the tangent bundle of F is globally generated (in particular nef) according to Proposition II.3.3 above. Furthermore, it is a classical fact that all Flag manifolds are Fano.

Since globally generated vector bundles are always nef, Proposition II.3.3 implies that homogeneous Fano manifolds always posses a nef tangent bundle. In fact, Example II.3.1 and Example II.3.2 show that a Fano manifold of dimension at most two has a nef tangent bundle if and only if it is homogeneous and all other examples that we saw thus far were of this form as well. It is conjectured that this is true in general:

Conjecture II.3.5. (Campana-Peternell, [CP91])
Every Fano manifold with nef tangent bundle is homogeneous.

In view of Proposition II.3.3 the conjecture of Campana and Peternell predicts that as soon as the tangent bundle of a Fano manifold is nef, it is in fact even globally generated. Conjecture II.3.5 has been verified for manifolds of dimension at most five by the work of [Kan15] and many others before him and also in some other special cases. In full generality however it has not even been proved yet that the tangent bundle must be semi ample. For now, we only have the following characterisation:

Lemma II.3.6. *Let F be a Fano manifold with nef tangent bundle.*

(1) If the tangent bundle $\mathcal{T}_F$ is generated by global sections, then $\mathcal{T}_F$ is also big.
(2) If the tangent bundle $\mathcal{T}_F$ is big, then it is also semi ample.

Proof. Let us start by proving the second statement.To this end, suppose that $\mathcal{T}_F$ is big and nef. Denote $m := \dim F$ and consider the manifold $\mathbb{P}(\mathcal{T}_F) \xrightarrow{\pi} F$. Recall, that

$$\begin{aligned}\mathcal{O}_{\mathbb{P}(\mathcal{T}_F)}\Big(K_{\mathbb{P}(\mathcal{T}_F)}\Big) &= \mathcal{O}_{\mathbb{P}(\mathcal{T}_F)}(-(m-1)) \otimes \pi^*\left(\mathcal{O}_F(K_F) \otimes \det(\mathcal{T}_F)\right) \\ &= \mathcal{O}_{\mathbb{P}(\mathcal{T}_F)}(-(m-1)). \end{aligned} \tag{II.7}$$

In other words, by the very definition of nefness (bigness) for the vector bundle $\mathcal{T}_F$ the line bundle $-K_{\mathbb{P}(\mathcal{T}_F)}$ is big and nef. Of course, then also $-K_{\mathbb{P}(\mathcal{T}_F)} + (-K_{\mathbb{P}(\mathcal{T}_F)})$ is big so that the *base point free theorem* Theorem IV.1.7 applies and yields that $-K_{\mathbb{P}(\mathcal{T}_F)}$ is even semi ample. In view of Eq. (II.7) this means that $\mathcal{O}_{\mathbb{P}(\mathcal{T}_F)}(1)$ (and, hence, also $\mathcal{T}_F$) is semi ample. This concludes the proof of (2).

We could in principle give the proof of (1) right now but it feels more natural to me to present it in the context of Chapter III. and this is what we will do (see Corollary III.2.8). □

In this sense, we will record the following:

Conjecture II.3.7. (weak Campana-Peternell conjecture)
If the tangent bundle of a Fano manifold is nef then it is also big.

In later sections we will often rely on the additional positivity provided by this weak version of the Campana-Peternell conjecture. As a sample case, let us portray how Conjecture II.3.7 may be used to refine Theorem II.2.9. We need the following auxiliary result:

Lemma II.3.8. *Let F be a Fano manifold with big and nef tangent bundle. Then,*

$$H^1(F, \mathcal{T}_F) = 0. \tag{II.8}$$

In particular, Fano manifolds with big and nef tangent bundle admit no deformations: If $f\colon X \to T$ is a holomorphic submersion and if every fibre of f is a (connected) Fano manifold with big and nef tangent bundle, then f is in fact a holomorphic fibre bundle.

Proof. Since $H^1(F, \mathcal{T}_F) = H^1(F, \mathcal{T}_F \otimes \det(\mathcal{T}_F) \otimes \mathcal{O}_F(K_F))$, the required vanishing Eq. (II.8) is just a direct consequence of Griffith's-Kawamata-Viehweg vanishing Theorem IV.1.5. In fact, Theorem IV.1.5 proves that

$$H^j(F, \mathcal{T}_F) = 0, \quad \forall\, j > 0. \tag{II.9}$$

That the second assertion follows from the first one is a standard fact; we will prove it nevertheless below: Let $f\colon X \to T$ be a holomorphic submersion as required. Consider the natural sequence of vector bundles

$$0 \to \mathcal{T}_{X/T} \to \mathcal{T}_X \to f^*\mathcal{T}_T \to 0$$

which is exact since f is a submersion. Pushing this sequence down, we have an exact sequence of sheaves on T

$$0 \to f_*\mathcal{T}_{X/T} \to f_*\mathcal{T}_X \to f_*f^*\mathcal{T}_T \to R^1 f_*\mathcal{T}_{X/T}. \tag{II.10}$$

Now, the projection formula yields $f_*f^*\mathcal{T}_T = \mathcal{T}_T \otimes f_*\mathcal{O}_X$. Since the fibres of f are connected by assumption, the standard variant Lemma IV.1.1 of Zariskis main theorem shows that $f_*\mathcal{O}_X = \mathcal{O}_T$. Moreover, it follows directly from Grauert's theorem on higher direct images Theorem IV.1.2 and Eq. (II.9) that $R^1 f_*\mathcal{T}_{X/T} = 0$. In effect, we may rewrite Eq. (II.10) as the following short exact sequence:

$$0 \to f_*\mathcal{T}_{X/T} \to f_*\mathcal{T}_X \to \mathcal{T}_T \to 0. \tag{II.11}$$

Now, fix a point $t_0 \in T$ and a Stein open neighbourhood $t_0 \in U \subseteq T$ which is also a coordinate neighbourhood with coordinates $(z^1, \dots, z^q)$. For example one may take U to be a polydisc. We will construct an explicit biholomorphism $f^{-1}(U) \cong U \times F_{t_0}$: Since U is Stein it follows from the exactness of Eq. (II.11) that also the sequence

$$0 \to H^0(U, f_*\mathcal{T}_{X/T}) \to H^0(U, f_*\mathcal{T}_X) \to H^0(U, \mathcal{T}_T) \to 0$$

is exact. In particular, due to the surjectivity of the last map in this sequence there exists holomorphic vector fields $V_1, \dots, V_q$ on $f^{-1}(U)$ such that $df(V_j) = \frac{\partial}{\partial z^j}$ for all $j = 1, \dots, q$. Since the fibres of F are in particular assumed to be compact, the holomorphic flows $\phi^1, \dots, \phi^q$ to the vector fields $V_1, \dots, V_q$ exist on $f^{-1}(U)$. But then, the map

$$U \times F_{t_0} \to f^{-1}(U), \quad (z^1, \dots, z^q, x) \mapsto \phi^1_{z_1}\left(\dots\left(\phi^q_{z_q}(x)\right)\right)$$

is easily seen to be a biholomorphism $U \times F_{t_0} \cong f^{-1}(U)$. This shows that f is a holomorphic fibre bundle and so we are done. □

Corollary II.3.9. *Let X be a compact Kähler manifold with nef tangent bundle and suppose that X is of maximal irregularity so that the Albanese $\alpha\colon X \to \mathrm{Alb}(X) =: T$ is a surjective submersion. Assume that the weak Campana-Peternell conjecture holds true so that its fibres are Fano manifolds with big and nef tangent bundle. Then, α is even a holomorphic fibre bundle.*

In the next subsection this result will be improved significantly. Indeed, we will show (without even assuming the weak Campana-Peternell conjecture) that α must be a *flat* fibre bundle (see Definition II.4.1 below). The proof of this strengthening of Theorem II.2.9 is however much harder.

3.2 Automorphisms of Fano Manifolds

As discussed above, we will soon prove that any compact Kähler manifold with nef tangent bundle is naturally (up to finite étale cover) a fibre bundle with fibre a Fano manifold with nef tangent bundle over a torus. It will be important for us later on to have some control over the structure groups that may occur. It is for this reason that during the rest of this section we will be studying the group of holomorphic automorphisms of Fano manifolds with nef tangent bundle. After all, the structure group of α will be a closed subgroup of this.

We start by recalling without proof the following important result:

Theorem II.3.10. *Let Y be any compact complex variety. Then, the group* $\mathrm{Aut}(Y)$ *of holomorphic automorphisms of Y is in a natural way a (finite-dimensional (!)) complex Lie group. In fact, its Lie algebra is naturally identified with* $H^0(Y, \mathcal{T}_Y)$.

Moreover, if Y is projective, then the connected component of the identity $\mathrm{Aut}^0(Y) \subseteq \mathrm{Aut}(Y)$ *is even an algebraic group.*

Proof. An essentially complete proof of the complex case may be found in [Akh95, Section 2.3.] (only the proof of some auxiliary results concerning locally compact groups are omitted). Regarding the projective algebraic case, note first of all that according to one of the famous GAGA theorems, the group of all holomorphic automorphisms of (the analytification of) a projective algebraic variety agrees with the group of algebraic automorphisms of this variety. In the latter setting, a proof is sketched in [Bri18, Theorem 2.3.]. □

Example II.3.11. It is well-known, that for all $n \in \mathbb{N}$ it holds that

$$\mathrm{Aut}(\mathbb{P}^n) = PGL_n(\mathbb{C}).$$

Corollary II.3.12. *Let F be a compact complex manifold. Then, the following assertions are equivalent:*

(1) F is homogeneous (for the action of some complex Lie group),

(2) the action of $\mathrm{Aut}^0(F)$ *on F is transitive,*

(3) the bundle $\mathcal{T}_F$ is generated by global sections.

Proof. The implication (1) ⇒ (3) was already proved in Proposition II.3.3.

Regarding (3) ⇒ (2), the second statement of Proposition II.3.3 yields for any points $x, y \in F$ the existence of an automorphism φ of F mapping x to y. In fact, the proof of Proposition II.3.3 shows that we may take φ to be a composition of flows to holomorphic vector fields. However, any flow ϕ_z to a holomorphic vector field V is clearly contained in $\mathrm{Aut}^0(F)$ as the map $[0,1] \to \mathrm{Aut}(F), t \mapsto \phi_{tz}$ is a (differentiable) path in $\mathrm{Aut}(F)$ from 1 to ϕ_z. It follows that also $\varphi \in \mathrm{Aut}^0(F)$ as a product of elements in $\mathrm{Aut}^0(F)$. Altogether, this proves that $\mathrm{Aut}^0(F)$ acts transitively on F.

Finally, the implication (2) ⇒ (1) is tautologous. □

Remark II.3.13. In general, it is *not* true that the automorphism group of a projective variety is an algebraic group because it may very well admit infinitely many connected components. For example, if $Y = E \times E$ is a product of an elliptic curve with itself, then $\mathrm{Aut}(E)/\mathrm{Aut}^0(E)$ contains $GL_2(\mathbb{Z})$ in the following natural way: A matrix $A \in GL_2(\mathbb{Z})$ acts on $E \times E$ via

$$A\colon E \times E \to E \times E, \quad (x_1, x_2) \mapsto ([a_{11}](x_1) + [a_{12}](x_2), [a_{21}](x_1) + [a_{22}](x_2)).$$

Here, as per usual, $[n]\colon E \to E$ is the multiplication-by-n map.

If the manifold is Fano however, this phenomenon can not occur:

Lemma II.3.14. *Let F be a Fano manifold. Then,* $\mathrm{Aut}(F)$ *is a* linear algebraic group*, i.e. a Zariski closed subgroup of some general linear group. In particular, the group*

$$\mathrm{Aut}(F)/\mathrm{Aut}^0(F)$$

is finite.

Proof. Certainly, by Theorem II.3.10 it holds that $\mathrm{Aut}(F)$ is a scheme, locally of finite type over $\mathbb{C}$. The following is an extension of the argument given in [Bri18, Corollary 2.17.]:

By definition, $\mathcal{O}_F(-K_F)$ is ample. Fix a natural number m for which the natural rational map

$$\phi_m\colon F \to \mathbb{P}H^0(F, \mathcal{O}_F(-mK_F))$$

is an embedding. Now, note that $\mathrm{Aut}(F)$ acts naturally on $H^0(F, \mathcal{O}_F(-mK_F))$ and, hence, also on $\mathbb{P}H^0(F, \mathcal{O}_F(-mK_F))$. In other words, there exists a natural group homomorphism

$$\rho\colon \mathrm{Aut}(F) \to \mathrm{Aut}(\mathbb{P}H^0(F, \mathcal{O}_F(-mK_F))) \cong \mathrm{PGL}(H^0(F, \mathcal{O}_F(-mK_F))).$$

Of course this is a morphism of algebraic schemes and in particular a Lie group homomorphism. In the following, we will prove that ρ defines a closed embedding of $\mathrm{Aut}(F)$ into the linear algebraic group $\mathrm{PGL}(H^0(F, \mathcal{O}_F(-mK_F))$ so that $\mathrm{Aut}(F)$ is itself linear algebraic.

First, let us prove that ρ is injective and that its image is closed: By construction ϕ_m is equivariant with respect to ρ. In particular, ρ is injective for if $\rho(g) = \mathrm{id}$ then

$$\phi_m \circ g \xlongequal{\text{equivariance}} \rho(g) \circ \phi_m \xlongequal{\rho(g)=\mathrm{id}} \phi_m.$$

Since ϕ_m is an embedding by construction this in turn implies $g = \mathrm{id}$. Moreover, the equivariance of ϕ_m directly implies that $\mathrm{Aut}(F)$ may be identified via ρ with the subgroup of elements of $\mathrm{Aut}(\mathbb{P}H^0(F, \mathcal{O}_F(-mK_F)))$ which map F isomorphically onto itself. But this subset is clearly Zariski closed: If the image of ϕ_m is cut out by the homogeneous polynomials $f_1, ..., f_r$, then the group of automorphisms mapping $\mathrm{Im}(\phi_m)$ isomorphically onto itself is the closed subset

$$\bigcap_{x \in \mathrm{Im}(\phi)} \{g \in \mathrm{PGL}| \quad f_1(g(x)) = f_r(g^{-1}(x)) = \ldots f_1(g(x)) = f_r(g^{-1}(x)) = 0\}$$

of $\mathrm{PGL}(H^0(F, \mathcal{O}_F(-mK_F))$.

In summary, $\rho\colon \mathrm{Aut}(F) \to \mathrm{PGL}(H^0(F, \mathcal{O}_F(-mK_F)))$ is an injective Lie group homomorphism and its image is a (Zariski) closed subgroup of the latter group. To prove that ρ is a diffeomorphism onto its image, it thus remains to show that the differential of ρ is injective, i.e. that it is an immersion. This is a consequence of standard results from the theory of smooth manifolds: Since ρ is a Lie group homomorphism,

$$d\rho|_g = d\ell_{\rho(g)}|_1 \circ d\rho|_1 \circ d\ell_{g^{-1}}|_g \quad \forall\, g \in \mathrm{Aut}(F) \tag{II.12}$$

so that the rank of the differential of ρ is constant. Here, again, ℓ_g denotes the map of multiplication by g from the left. Invoking the standard theorem [Lee13, Theorem 4.12.] and using that the rank of $d\rho$ is constant we may choose local coordinates on $\mathrm{Aut}(F)$ and on $\mathrm{PGL}(H^0(F, \mathcal{O}_F(-mK_F)))$ such that in these local coordinates, ρ is of the form

$$(z_1, \ldots, z_{\dim \mathrm{Aut}(F)}) \mapsto (z_1, \ldots, z_k, 0, \ldots, 0).$$

Here, k denotes the rank of ρ. In particular, since ρ is injective we see that we must have $k = \dim \mathrm{Aut}(F)$, i.e. that $d\rho$ is injective. This concludes the proof that $\mathrm{Aut}(F)$ is linear algebraic. The second statement is clear, because $\mathrm{Aut}(F)$ is in particular of finite type over $\mathbb{C}$ as a closed sub scheme of the finite type scheme GL. In particular, it has only finitely many connected components. □

Finally, assuming the conjecture of Campana and Peternell and using some classical theorem from Lie theory one can say even more:

Theorem II.3.15. *Let F be a homogeneous Fano manifold. Then,* $\mathrm{Aut}^0(F)$ *is a semi simple algebraic group.*

Proof. First of all, we already know from Lemma II.3.14 that $\mathrm{Aut}^0(F)$ is linear algebraic. Now, recall that by the classical *Levi decomposition* theorem (see [HN12, Section 5.6.] for a discussion of this result) there exists a normal, connected and solvable Lie subgroup R of $\mathrm{Aut}^0(F)$ such that $\mathrm{Aut}^0(F)/R$ is a semi simple algebraic group. We need to show that R is the trivial group. Indeed, by *Borel's fixed point theorem* (the statement of which may be found e.g. in [Akh95, Section 3.1.]) the action of R on F has a fixed point. In other words, $h \cdot x_0 = x_0$ for all $h \in R$. Now, fix any other point $x \in F$. Since $\mathrm{Aut}^0(F)$ acts transitively on F by Corollary II.3.12 there exists an element $g \in \mathrm{Aut}^0(F)$ such that $x_0 = g \cdot x$. Note that

$$h \cdot x_0 = x_0 \quad \Leftrightarrow \quad h \cdot g \cdot x = g \cdot x \quad \Leftrightarrow \quad (g^{-1}hg) \cdot x = x.$$

In particular, $g^{-1}Rg$ fixes x since R fixes x_0. But R is a normal subgroup. Thus, $R = g^{-1}Rg$ fixes x as well. Since x was arbitrary we see that $R \subseteq \mathrm{Aut}^0(F)$ acts trivially on F and so $R = \{1\}$. This implies that $\mathrm{Aut}^0(F) = \mathrm{Aut}(F)^0/R$ is semi simple. □

Remark II.3.16. Let X be a compact Kähler manifold which is homogeneous for the action of a complex Lie group. According to a classical result of Borel and Remmert (see [Akh95, Section 3.9.]) it holds that X is isomorphic to a direct product of a complex torus and a homogeneous Fano manifold. The latter are called *generalised flag manifolds*.

4 The Albanese Map: Global Structure

The goal of this section is to prove that in the situation of Theorem II.2.9 the Albanese map α is even a (flat) fibre bundle. This result was already contained in [DPS94] for the special case of projective manifolds and we will more or less follow their presentation. The modification to the Kähler case (specifically Theorem II.4.11) is originally due to Cao.

To begin with, let us define what we mean by a *flat* fibre bundle.

Definition II.4.1. *Let $f\colon X \to T$ be a holomorphic fibre bundle with fibre F and structure group $G \subseteq \mathrm{Aut}(F)$. Denote by $\widetilde{T} \to T$ the universal covering of T. Then, f is called a* flat *(or also* locally constant*)* fibre bundle *if there exists a group homomorphism $\rho\colon \pi_1(T) \to G$ and an isomorphism of fibre bundles*

$$X \cong (\widetilde{T} \times F)/\pi_1(T).$$

Here, $\pi_1(T)$ acts diagonally via the natural action on $\widetilde{T}$ and via ρ on F.

In analogy with the case of principal bundles (compare with Lemma I.4.12) one may equivalently characterise flat bundles as those bundles which may be chosen to have locally constant transition functions or equivalently as those bundles for which $f^*\mathcal{T}_T$ may be viewed as an integrable sub bundle of $\mathcal{T}_X$.

Our main technical tool for proving that a holomorphic submersion is a flat fibre bundle is the following result. Its proof is an extension of the argument in [CH17, Proposition 4.1.].

Theorem II.4.2. *Let $f\colon X \to T$ be a holomorphic submersion with fibres F_t. Suppose that both X, T are compact Kähler and that there exists an f-relatively positive line bundle $\mathscr{L}$ on X (i.e. $\mathscr{L}|_{F_t}$ is ample for all points $t \in T$).*

(1) The sheaves $f_*\mathscr{L}^{\otimes m}$ are locally free for all sufficiently large $m \gg 0$.

(2) If $f_*\mathscr{L}^{\otimes m}$ is a numerically flat vector bundle for all sufficiently large $m \gg 0$, then f is a flat holomorphic fibre bundle.

Proof. Regarding the first assertion, for any fixed $t_0 \in T$ the line bundle $\mathscr{L}|_{F_{t_0}}$ is ample by assumption. Thus, by Serre's vanishing theorem there exists an integer $m_{t_0} > 0$ such that

$$H^j\left(F_t, \mathscr{L}^{\otimes m}|_{F_{t_0}}\right) = 0 \quad \forall\, j > 0, \quad \forall\, m > m_{t_0}.$$

Then, by Grauert's semi continuity theorem in flat families in fact

$$H^j\left(F_t, \mathscr{L}^{\otimes m}|_{F_t}\right) = 0 \quad \forall\, j > 0, \quad \forall\, m > m_{t_0}$$

for all t in an open neighbourhood of t_0 in T. Since T was assumed to be compact, we see that

$$H^j\left(F_t, \mathscr{L}^{\otimes m}|_{F_t}\right) = 0 \quad \forall\, j > 0, \quad \forall\, t \in T, \quad m \gg 0.$$

Thus, item (1) immediately follows by another theorem of Grauert, namely Theorem IV.1.2.

Let us now turn to the proof of (2): According to what we just proved there exists an integer $m_0 > 0$ such that $f_*\mathscr{L}^{\otimes m}$ is a vector bundle for all $m \geq m_0$. Moreover, by Theorem I.2.20 we may assume that the natural rational map

$$\phi_m\colon X \hookrightarrow \mathbb{P}(f_*\mathscr{L}^{\otimes m})$$

is holomorphic and a closed embedding with $\phi_m^*\mathcal{O}_{\mathbb{P}(f_*\mathscr{L}^{\otimes m})}(1) = \mathscr{L}^{\otimes m}$ for all $m \geq m_0$. Replacing $\mathscr{L}$ by $\mathscr{L}^{\otimes m_0}$ we may of course assume that $m_0 = 1$. Denote by $\mathscr{I}$ the ideal sheaf defining the closed subvariety $X \hookrightarrow \mathbb{P}(f_*\mathscr{L})$. Then, there exists a short exact sequence

$$0 \to \mathscr{I} \to \mathcal{O}_{\mathbb{P}(f_*\mathscr{L})} \to \mathcal{O}_X \to 0.$$

Twisting this sequence by $\mathcal{O}_{\mathbb{P}(f_*\mathscr{L})}(m)$ and pushing it down by the projection map $\pi\colon \mathbb{P}(f_*\mathscr{L}) \to T$ we find the exact sequence

$$0 \to \pi_*\Big(\mathscr{I} \otimes \mathcal{O}_{\mathbb{P}(f_*\mathscr{L})}(m)\Big) \to \pi_*\Big(\mathcal{O}_{\mathbb{P}(f_*\mathscr{L})}(m)\Big) \to f_*\Big(\mathscr{L}^{\otimes m}\Big) \to R^1\pi_*\Big(\mathscr{I} \otimes \mathcal{O}_{\mathbb{P}(f_*\mathscr{L})}(m)\Big). \tag{II.13}$$

Here, we use that $\mathcal{O}(1)|_X = \mathscr{L}$ and that $\pi \circ \phi_1 = f$ by construction. Since the line bundle $\mathcal{O}_{\mathbb{P}(f_*\mathscr{L})}(1)$ is π-ample it follows from the relative Serre vanishing theorem and a theorem of Grauert (namely Theorem IV.1.2) that

$$\begin{aligned} R^j\pi_*\Big(\mathscr{I} \otimes \mathcal{O}_{\mathbb{P}(f_*\mathscr{L})}(m)\Big) &= 0, \quad \forall\, j > 0, \forall\, m \gg 0, \\ \pi_*\Big(\mathscr{I} \otimes \mathcal{O}_{\mathbb{P}(f_*\mathscr{L})}(m)\Big)|_t &= H^0(F_t, \mathscr{I}_{F_t}(m)), \quad \forall\, t \in T, \forall\, m \gg 0. \end{aligned} \tag{II.14}$$

Moreover, we may identify $\pi_*\mathcal{O}_{\mathbb{P}(f_*\mathscr{L})}(m) = \operatorname{Sym}^m(f_*\mathscr{L})$. Altogether, the short exact sequence Eq. (II.13) takes the shape

$$0 \to \pi_*\Big(\mathscr{I}(m)\Big) \to \operatorname{Sym}^m(f_*\mathscr{L}) = \pi_*\Big(\mathcal{O}_{\mathbb{P}(f_*\mathscr{L})}(m)\Big) \to f_*(\mathscr{L}^{\otimes m}) \to 0. \tag{II.15}$$

For any point $t \in T$, let us appreciate $V_t := f_*\mathscr{L}|_t = H^0(\mathbb{P}(V_t), \mathcal{O}(1))$. Note that via the inclusion $\pi_*(\mathscr{I} \otimes \mathcal{O}(m))|_t \subseteq \pi_*\mathcal{O}(m)|_t$ given by Eq. (II.15) we may identify the elements of $\pi_*(\mathscr{I} \otimes \mathcal{O}(m))|_t = H^0(F_t, \mathscr{I}_{F_t}(m))$ (here we use Eq. (II.14)) with the set of homogeneous polynomials in $H^0(\mathbb{P}(V_t), \mathcal{O}(m))$ cutting out $F_t \subseteq \mathbb{P}(V_t)$.

Now, the bundles $f_*(\mathscr{L}^{\otimes m})$ are all numerically flat by our very assumptions. Fix a base point $t_0 \in T$ and let $\rho\colon \pi_1(T, t_0) \to \operatorname{GL}(V_{t_0})$ be the unique underlying graded unitary structure on $f_*\mathscr{L}$ (see Remark I.4.7). In particular, denoting by $\widetilde{T} \to T$ the universal cover of T there exists a natural identification $f_*\mathscr{L} = (\widetilde{T} \times V_{t_0})/\pi_1(T, t_0)$ as holomorphic vector bundles. Consequently,

$$\mathbb{P}(f_*\mathscr{L}) = (\widetilde{T} \times \mathbb{P}(V_{t_0}))/\pi_1(T, t_0)$$

Moreover, along with $f_*\mathscr{L}$ also the bundles $\operatorname{Sym}^m(f_*\mathscr{L})$ are numerically flat for any $m > 0$ (see Proposition I.4.8). Since numerically flat bundles satisfy the 2-out-of-3-property in short exact sequences by Lemma I.4.9 we deduce from Eq. (II.15) that also $\pi_*\Big(\mathscr{I} \otimes \mathcal{O}(m)\Big)$ is numerically flat. In conclusion, both $\pi_*\mathcal{O}(m)$ and $\pi_*(\mathscr{I} \otimes \mathcal{O}(m))$ are numerically flat. In particular, according to Remark I.4.7 the inclusion $\pi_*(\mathscr{I}(m)) \subseteq \pi_*\mathcal{O}(m)$ is compatible with the flat structures. In other words, $\pi_*(\mathscr{I} \otimes \mathcal{O}(m))|_t = H^0(F_t, \mathscr{I}_{F_t}(m))$ is invariant under the parallel transport in $\pi_*\mathcal{O}(m)$ and, hence, so is the zero locus of the homogeneous polynomials in

$$\pi_*(\mathscr{I} \otimes \mathcal{O}(m))|_t = H^0(F_t, \mathscr{I}_{F_t}(m)) \subseteq H^0(\mathbb{P}(V), \mathcal{O}(m)) = \pi_*\mathcal{O}(m)|_t$$

in $\mathbb{P}(V_t)$. But recall that this zero locus is just F_t. In other words, we may identify

$$X = (\widetilde{T} \times F_{T_0})/\pi_1(T) \hookrightarrow (\widetilde{T} \times \mathbb{P}(V_{t_0}))/\pi_1(T) = \mathbb{P}(f_*\mathscr{L}).$$

We conclude that $f\colon X \to T$ is a flat bundle which was to prove. □

Let now X be a compact Kähler manifold of maximal irregularity and with nef tangent bundle so that according to Theorem II.2.9 the Albanese $\alpha\colon X \to T := \mathrm{Alb}(X)$ is a holomorphic submersion and the fibres are Fano manifolds. We want to show that α is flat and to this end we aim to apply our criterion Theorem II.4.2 to the line bundle $\mathscr{L} = \mathcal{O}_X(-K_{X/T})$ on X. Indeed, $\mathcal{O}_X(-K_{X/T})|_{F_t} = \mathcal{O}_{F_t}(-K_{F_t})$ is ample for all $t \in T$ as F_t is Fano. It thus remains to prove that $\alpha_*\mathcal{O}_X(-mK_{X/T})$ is numerically flat for all sufficiently large $m > 0$. According to Lemma I.4.6 this amounts to showing that $\alpha_*\mathcal{O}_X(-mK_{X/T})$ is nef and that $\mathrm{c}_1(\alpha_*\mathcal{O}_X(-mK_{X/T})) = 0$ for all $m \gg 0$. This is our goal for the rest of this section.

4.1 Nefness of the Bundles $\alpha_*\mathcal{O}_X(-mK_{X/T})$ and Positivity of Direct Image Sheaves

The aim of this subsection is to prove that in the situation of Theorem II.2.9 it holds that $\alpha_*\mathcal{O}_X(-mK_{X/T})$ is nef for all $m \geq 0$. To this end, note that taking determinants in the short exact sequence

$$0 \to \mathcal{T}_{X/T} \to \mathcal{T}_X \to \alpha^*\mathcal{T}_T \to 0$$

yields the identification

$$\begin{aligned}\mathcal{O}_X(-K_X) &= \mathcal{O}_X(\,-K_{X/T}) \otimes \alpha^*\mathcal{O}_T(-K_T) \\ &= \mathcal{O}_X(-K_{X/T}) \otimes \alpha^*\mathcal{O}_T = \mathcal{O}_X(-K_{X/T}) \qquad \text{(II.16)}\end{aligned}$$

so that together with $\mathcal{O}_X(-K_X)$ also $\mathcal{O}_X(-K_{X/T})$ is nef. The main idea is now to chose metrics h on $\mathcal{O}_X(-mK_{X/T})$ with small negative curvature component and *integrate* these *along the fibres* to obtain metrics on $\alpha_*\mathcal{O}_X(-mK_{X/T})$ with similarly small negative curvature component. This technique was pioneered in [Ber09]; the generalisation to our setting was developed in [Cao13] and we follow this proof. However, we add a lot more details.

Let us start by recalling the definition of *integration along the fibres*:

Definition II.4.3. *Let $f\colon X \to T$ be a holomorphic submersion with fibres F_t. Assume that the fibres F_t are compact and denote $m := \dim F$. Given any integer $k \geq 2m$ and a differentiable k-form $\eta \in \mathcal{A}^k_X$ on X, we define the $(k-2m)$-form $f_*\eta$ on T by the rule*

$$\Big(f_*\eta\Big)(V_1, ..., V_{k-2m})|_t := \int_{F_t} \eta\left(\widetilde{V}_1, ..., \widetilde{V}_{k-2m}, -\right), \quad \forall\, V_1, ..., V_{k-2m} \in T^{\mathbb{C}}T.$$

Here, $\widetilde{V}_1, ..., \widetilde{V}_{k-2m}$ are any locally defined vector fields on X satisfying

$$df(\widetilde{V}_j)|_t = V_j|_t, \quad \forall\, j = 1, ..., k-2m.$$

That such vector fields $\widetilde{V}_1, ..., \widetilde{V}_{k-2m}$ always exist may be seen using a partition of unity argument. Moreover, if η is a differentiable k-form on X and $k < 2m$, then we put $f_*\eta = 0$.

We call $f_*\eta$ the form obtained by integrating η along the fibres *or the* push forward *of η by f.*

The following properties of the push-forward are straightforward to verify:

Proposition II.4.4. *Integration along the fibres induces well-defined $\mathbb{C}$-linear maps*

$$f_*\colon \mathcal{A}^k_X \to \mathcal{A}^{k-2m}_T.$$

Moreover, it satisfies the following formulae:

(1) Push forward preserves type: If $\eta \in \mathcal{A}^{p,q}_X$, then $f_*\eta \in \mathcal{A}^{p-m,q-m}_X$.
(2) Push forward commutes with the exterior derivative: $d \circ f_* = f_* \circ d$. In particular, f_* induces morphisms

$$f_*\colon H^k(X, \mathbb{C}) \to H^{k-m}(T, \mathbb{C}).$$

Similarly, f_* commutes also with $\partial, \bar{\partial}$.

(3) Push forward satisfies the projection formula*: For all differential forms ζ on T and η on X it holds that*

$$f_*(f^*\zeta \wedge \eta) = \zeta \wedge f_*\eta.$$

(4) The push forward of a (strictly) positive form on X is a (strictly) positive form on T.

In particular, if ω_X is a Kähler form on X, then $f_*(\omega_X^{m+1})$ is a strictly positive closed $(1,1)$-form on T, i.e. a Kähler form.

Here now follows the main technical result of this section:

Theorem II.4.5. (Push forward of metrics)
Let $f\colon X \to T$ be a holomorphic submersion where both X, T are compact Kähler and fix a Kähler form ω_T on T. We denote the fibres of f by F_t and put $m := \dim F_t$.

Assume that there exists an f-relatively ample line bundle $\mathscr{L}$ on X carrying a smooth hermitean metric h whose Chern curvature satisfies

$$\Theta_h(\mathscr{L}) \geq \lambda\, f^*\omega_T \tag{II.17}$$

for some constant $\lambda \in \mathbb{R}$. Then, $f_*(\mathscr{L} \otimes \mathcal{O}_X(K_{X/T}))$ naturally carries a smooth hermitean metric h^* satisfying $\Theta_{h^*}(f_*(\mathscr{L} \otimes \mathcal{O}_X(K_{X/T}))) \geq \lambda \, \mathrm{Id} \cdot \omega_T$ in the sense of endomorphism-valued $(1,1)$-forms, i.e.

$$h^*(\Theta_{h^*}\sigma, \sigma) \geq \lambda \, h^*(\sigma, \sigma) \, \omega_T, \quad \forall \, \sigma \in \mathcal{A}^0 \left(f_*(\mathscr{L} \otimes \mathcal{O}_X(K_{X/T})) \right).$$

Remark II.4.6. Note that if $\mathscr{L}$ is f-relatively ample on X, then $f_*(\mathscr{L} \otimes \mathcal{O}_X(K_{X/T}))$ is indeed a vector bundle on T so that the assertion in Theorem II.4.5 makes sense. Indeed, for any $t \in T$ an application of the Kodaira vanishing theorem (see Theorem IV.1.4) yields

$$H^j \left(F_t, (\mathscr{L} \otimes \mathcal{O}_X(K_X)) \Big|_{F_t} \right) = H^j \Big(F_t, \mathscr{L}|_{F_t} \otimes \mathcal{O}_{F_t}(K_{F_t}) \Big) = 0, \quad \forall \, j > 0.$$

Thus, $f_*(\mathscr{L} \otimes \mathcal{O}_X(K_{X/T}))$ is a vector bundle by Grauert's Theorem IV.1.2. In fact, the same token also shows that

$$f_*(\mathscr{L} \otimes \mathcal{O}_X(K_{X/T}))|_t = H^0 \left(F_t, (\mathscr{L} \otimes \mathcal{O}_X(K_X)) \Big|_{F_t} \right), \quad \forall \, t \in T. \tag{II.18}$$

Proof. (of Theorem II.4.5)
Let us start by recalling that a (locally defined) differentiable section σ of the bundle $f_*(\mathscr{L} \otimes \mathcal{O}_X(K_{X/T}))$ is nothing but a smooth section $\sigma \in \mathcal{A}^0(f_*(\mathscr{L} \otimes \mathcal{O}_X(K_{X/T})))$ such that the restriction $\sigma|_{F_t}$ to any fibre is a holomorphic $\mathscr{L}$-valued m-form. Here, we use Eq. (II.18).

Now, note that the metric h on $\mathscr{L}$ induces a non-degenerate sesquilinear pairing

$$h \colon \mathcal{A}^{m,0}(f_*\mathscr{L}|_{F_t}) \times \mathcal{A}^{m,0}(f_*\mathscr{L}|_{F_t}) \to \mathcal{A}^{m,m}_{F_t}$$

which by abuse of notation we continue to denote by h and which is determined by

$$h(s_1 \otimes \eta_1, s_2 \otimes \eta_2) := i^{m^2} \cdot h(s_1, s_2) \, \eta_1 \wedge \overline{\eta_2}, \qquad \forall \, s_1, s_2 \in \mathcal{A}^0(\mathscr{L}|_{F_t}), \, \forall \, \eta_1, \eta_2 \in \mathcal{A}^{m,0}_{F_t}.$$

Here, the factor i^{m^2} ensures that $h(\sigma, \sigma) \geq 0$ for all $\sigma \in \mathcal{A}^{m,0}(\mathscr{L}|_{F_t})$. With this notation in place we define the metric h^* on $f_*(\mathscr{L} \otimes \mathcal{O}_X(K_{X/T}))$ via the rule

$$h^*(\sigma_1, \sigma_2)|_t := f_*(h(\sigma_1, \sigma_2))|_t = \int_{F_t} h(\sigma_1, \sigma_2), \quad \forall \, \sigma_1, \sigma_2 \in \mathcal{A}^0(f_*(\mathscr{L} \otimes \mathcal{O}_X(K_{X/T}))).$$

Then, clearly h^* is a smooth hermitean metric on $f_*(\mathscr{L} \otimes \mathcal{O}_X(K_{X/T}))$. We need to verify the inequality

$$h^*(\Theta_{h^*}\sigma, \sigma) \geq \lambda\, h^*(\sigma, \sigma)\, \omega_T \tag{II.19}$$

for any fixed section $\sigma \in \mathcal{A}^0\left(f_*(\mathscr{L} \otimes \mathcal{O}_X(K_{X/T})\right)$. One can do so point wise and in effect (using that all quantities in question are tensorial) one may replace σ by any other section attaining the same value at a given point $t \in T$. The idea of the proof is now to choose this section satisfying suitable favourable properties (for example being holomorphic) and then verify Eq. (II.19) explicitly by a direct calculation. This leads to the following formula, first discovered in [Ber09]:

$$h^*(\Theta_{h^*}\sigma, \sigma)|_t = f_*(h(\Theta_h\sigma, \sigma))|_t + \int_{F_t} |\eta|^2. \tag{II.20}$$

Here $|\eta|^2$ is some non-negative term which roughly speaking measures the infinitesimal variation of the pair $(\mathscr{L}|_{F_t}, h|_{F_t})_{t \in T}$. Once the appropriate choice of σ has been made, the proof of Eq. (II.20) is not hard but the calculations are somewhat tedious. Moreover, the choice of σ involves some non-trivial analysis and so we will avoid the proof. In any case, granting Eq. (II.20) we obtain the estimates

$$\begin{aligned} h^*(\Theta_{h^*}\sigma, \sigma)|_t &\overset{\text{Eq. (II.20)}}{\geq} f_*(h(\Theta_h\sigma, \sigma))|_t \\ &\overset{\text{Eq. (II.17)}}{\geq} f_*(\lambda\, h(\sigma, \sigma) \wedge f^*\omega_T) \\ &= \lambda \cdot f_*(h(\sigma, \sigma))\, \omega_T \\ &=: \lambda \cdot h^*(\sigma, \sigma)\, \omega_T, \end{aligned}$$

where in the second to last step we used Proposition II.4.4. This concludes the proof. □

Lemma II.4.7. (Positivity of direct image sheaves)
Let $f\colon X \to T$ be a holomorphic submersion between compact Kähler manifolds. We denote the fibres of f by F_t and put $m := \dim F_t$. Let $\mathscr{L}$ be an f-relatively ample line bundle on X. If $\mathscr{L}$ is nef, then so is $f_*(\mathscr{L} \otimes \mathcal{O}_X(K_{X/T}))$.

Proof. Fix a Kähler form ω_T on T. First of all, since by our assumption $\mathscr{L}$ is nef and f-relatively positive an application of Corollary I.2.22 yields that

$$\mathrm{c}_1(\mathscr{L}) + f^*[\varepsilon\omega_T] \in H^{1,1}(X, \mathbb{R})$$

is a Kähler class on X for any fixed $\varepsilon > 0$. In other words, there exists a Kähler form ω_X on X such that

$$[\omega_X] = \mathrm{c}_1(\mathscr{L}) + f^*[\varepsilon\omega_T],$$

so that $c_1(\mathscr{L})$ is represented by the closed $(1,1)$-form $\omega_X - \varepsilon f^*\omega_T$. Recall, that according to Example I.1.11 we can then find a smooth hermitean metric h on $\mathscr{L}$ whose curvature form Θ_h is actually given by $\Theta_h = \omega_X - \varepsilon f^*\omega_T$. In particular, it follows that $\Theta_h \geq -\varepsilon f^*\omega_T$. But in this situation, Theorem II.4.5 applies and yields the existence of a smooth hermitean metric h^* on the bundle $f_*(\mathscr{L} \otimes \mathcal{O}_X(K_{X/T}))$ such that

$$\Theta_{h^*} \geq -\varepsilon \operatorname{Id} \cdot \omega_T.$$

Since ε was arbitrary, we conclude that $f_*(\mathscr{L} \otimes \mathcal{O}_X(K_{X/T}))$ is nef in the sense of Griffiths. In particular, according to Proposition I.3.6 it is nef. □

Remark II.4.8. It turns out that the general line of argument used in Theorem II.4.5 is well suited for generalisations (to for example the case where f is no longer assumed to be a submersion) and it is indeed an active ongoing area of research to push these methods to there limit. Fittingly, this theory is referred to as *positivity of direct image bundles.*

Corollary II.4.9. (Cao)
Let X be a compact Kähler manifold with nef tangent bundle. Assume that X is of maximal irregularity so that the Albanese $\alpha\colon X \to \operatorname{Alb}(X) =: T$ is a holomorphic submersion whose fibres F_t are Fano manifolds with nef tangent bundle.

Then, the sheaves

$$\alpha_*(\mathcal{O}_X(-mK_{X/T}))$$

are in fact nef vector bundles on T for all $m \geq 1$.

Proof. The line bundle $\mathcal{O}_X(-mK_{X/T})$ satisfies

$$\mathcal{O}_X(-mK_{X/T})|_{F_t} = \mathcal{O}_{F_t}(-mK_{F_t}), \quad \forall\, t \in T.$$

As F_t is Fano, it follows that $\mathcal{O}_X(-mK_{X/T})$ is α-ample. In particular, according to Remark II.4.6 $\alpha_*(\mathcal{O}_X(-mK_{X/T}))$ is a vector bundle on T. Since we also know that $\mathcal{O}_X(-mK_{X/T})$ is nef (compare with the discussion around Eq. (II.17)), Lemma II.4.7 applies and yields the nefness of $\alpha_*(\mathcal{O}_X(-mK_{X/T}))$ for all $m \geq 1$. □

4.2 Numerical Flatness of $\alpha_*\mathcal{O}_X(-mK_{X/T})$

In the preceding subsection we proved that in the situation of Theorem II.2.9 $\alpha_*\mathcal{O}_X(-mK_{X/T})$ is a nef vector bundle on T for all $m \geq 1$. Following up on this, this subsection is devoted to proving that

$$c_1(\alpha_*\mathcal{O}_X(-mK_{X/T})) = 0$$

so that the bundles $\alpha_*\mathcal{O}_X(-mK_{X/T})$ are in fact numerically flat. This will immediately imply that the Albanese map α is a flat fibre bundle.

Throughout this subsection we follow the approach of [DPS94] but we also require some results to deal with the general Kähler case which are due to Cao. Let us start by introducing the following concept:

Definition II.4.10. *Let X be a compact Kähler manifold and let $\mathscr{L}$ be a nef line bundle on X. The integer*

$$\nu(\mathscr{L}) = \max\left\{k \in \mathbb{N} \;\middle|\; \mathrm{c}_1(\mathscr{L})^k \neq 0\right\} \in \{0, ..., \dim X\}$$

is called the numerical dimension *of $\mathscr{L}$.*

Theorem II.4.11. (Cao)
Let X be a compact Kähler manifold with nef tangent bundle. Assume that X is of maximal irregularity so that the Albanese map $\alpha\colon X \to \mathrm{Alb}(X) =: T$ is a holomorphic submersion and so that its fibres F_t are Fano manifolds.

Then, the numerical dimension of $\mathcal{O}_X(-K_{X/T})$ is equal to

$$\nu(\mathcal{O}_X(-K_{X/T})) = \dim X - \dim T = \dim F_t.$$

Proof. We will prove the theorem only in case X is projective. The proof in the general case is essentially the same. However, it is complicated by the fact that (as of my knowledge) the variant of Kawamata-Viehweg vanishing (Theorem IV.1.4) that we apply below is still unknown in the general Kähler setting. Indeed, it was Cao's contribution to prove an ad hoc variant of Theorem IV.1.4 in this specific situation. More details on the proof in the Kähler case may be found in [Cao13, Theorem 3.1.4].

Step 1: It holds that $\nu(\mathcal{O}_X(-K_{X/T})) \geq \dim X - \dim T = \dim F_t$.

Indeed, since all fibres F_t are Fano $-K_{X/T}|_{F_t} = -K_{F_t}$ is ample. According to the Nakai-Moishezon criterion Theorem I.2.13 applied to the subvariety $F_t \subseteq F_t$

$$\left(\mathrm{c}_1(\mathcal{O}_X(-K_{X/T}))|_{F_t}\right)^{\dim F_t} = \mathrm{c}_1(\mathcal{O}_{F_t}(-K_{F_t}))^{\dim F_t} > 0$$

is in particular non-zero. It follows, that $\mathrm{c}_1(\mathcal{O}_X(-K_{X/T}))^{\dim F_t} \neq 0$ since its restriction to any fibre F_t does not vanish.

Step 2: It holds that $\nu(\mathcal{O}_X(-K_{X/T})) \leq \dim X - \dim T = \dim F_t$.

Let us abbreviate $q := \dim T$ and $\nu := \nu(\mathcal{O}_X(-K_{X/T}))$. Since $\mathcal{O}_X(-K_{X/T})$ is nef due to Eq. (II.16) and our assumption, the Kawamata-Viehweg vanishing Theorem IV.1.4 applies in this situation and yields

$$H^j(X, \mathcal{O}_X) = H^j\Big(X, \mathcal{O}_X(-K_X) \otimes \mathcal{O}_X(K_X)\Big) = 0, \quad \forall\, j > \dim X - \nu. \tag{II.21}$$

On the other hand, since T is a torus of dimension q it holds that $H^0(T, \Omega_T^q) \cong \mathbb{C}$ is non-zero. Let $\eta \in H^0(T, \Omega_T^q)$ be a non trivial form. Then, also $\alpha^*\eta \in H^0(X, \Omega_X^q)$ is non-zero. But according to Hodge theory it holds that

$$H^q(X, \mathcal{O}_X) \cong H^0(X, \Omega_X^q) \ni \alpha^*\eta \neq 0.$$

Thus, it follows from Eq. (II.21) that $q \leq \dim X - \nu$ which we wanted to prove. □

Corollary II.4.12. *Let X be a compact Kähler manifold of maximal irregularity and with nef tangent bundle so that the Albanese morphism $\alpha\colon X \to \mathrm{Alb}(X) =: T$ is a submersion and its fibres F_t are Fano manifolds. Then,*

$$\mathrm{c}_1(\alpha_*\mathcal{O}_X(-mK_{X/T})) = 0, \quad \forall\, m > 0.$$

Proof. Consider the short exact sequence

$$0 \to \mathcal{T}_{X/T} \to \mathcal{T}_X \to \alpha^*\mathcal{T}_T. \to 0,$$

Since $\mathcal{T}_X$ is nef by assumption and since $\mathcal{T}_T$ is trivial it follows from Theorem I.3.12 that also $\mathcal{T}_{X/T}$ is a nef vector bundle on X. On the other hand, according to the preceding result Theorem II.4.11 we know that

$$\mathrm{c}_1(\mathcal{O}_X(-K_{X/T}))^k = \mathrm{c}_1(\mathcal{T}_{X/T})^k = 0, \quad \forall\, k > \dim F_t.$$

In particular, Lemma I.3.16 implies that any homogeneous polynomial $\zeta \in H^{k,k}(X, \mathbb{C})$ in the Chern classes of $\mathcal{T}_{X/T}$ of (cohomological) degree $2k \geq 2(\dim F + 1)$ satisfies

$$\int_X \zeta \wedge (\alpha^*\omega)^{n-k} = 0 \tag{II.22}$$

for *any* Kähler metric ω on T. Here, we abbreviate $n := \dim X$ and we use that $\alpha^*\omega$ certainly defines a nef cohomology class.

Now, recall from Remark II.4.6 that

$$H^j\left(\mathcal{O}_{F_t}(-mK_{F_t})\right) = 0, \quad \forall\, j > 0, \forall\, t \in T \text{ and } \forall\, m > 0.$$

In view of Grauert's result on direct image sheaves Theorem IV.1.2 this implies that

$$R^j\alpha_*\mathcal{O}_X(-mK_{X/T}) = 0, \quad \forall\, j>0, \quad \forall\, m>0. \tag{II.23}$$

Thus, using the Grothendieck-Riemann-Roch formula Theorem IV.1.8 we compute

$$\begin{aligned}\sum_{j=0}(-1)^j \operatorname{ch}(R^j\alpha_*\mathcal{O}_X(-mK_{X/T})) &\overset{\text{Eq. (II.23)}}{=\!=\!=} \operatorname{ch}(\alpha_*\mathcal{O}_X(-mK_{X/T})) \\ &\overset{\text{Riem.-Roch}}{=\!=\!=} \alpha_*\left(\operatorname{ch}\left(\mathcal{O}_X(-mK_{X/T})\right)\wedge \operatorname{td}(\mathcal{T}_{X/T})\right).\end{aligned} \tag{II.24}$$

In particular, we see that

$$\begin{aligned}&\int_T c_1(\alpha_*\mathcal{O}_X(-mK_{X/T}))\wedge\omega^{q-1}\\ &=\!=\!= \int_T \operatorname{ch}_1(\alpha_*\mathcal{O}_X(-mK_{X/T}))\wedge\omega^{q-1}\\ &\overset{\text{Eq. (II.24)}}{=\!=\!=} \int_T \alpha_*\left(\operatorname{ch}(\mathcal{O}_X(-mK_{X/T})\wedge\operatorname{td}(\mathcal{T}_{X/T})\right)\wedge\omega^{q-1}\\ &\overset{\text{Proposition II.4.4}}{=\!=\!=} \int_T \alpha_*\left(\operatorname{ch}(\mathcal{O}_X(-mK_{X/T})\wedge\operatorname{td}(\mathcal{T}_{X/T})\wedge\alpha^*\left(\omega^{q-1}\right)\right)\\ &\overset{\text{Fubini}}{=\!=\!=} \int_X \operatorname{ch}(\mathcal{O}_X(-mK_{X/T})\wedge\operatorname{td}(\mathcal{T}_{X/T})\wedge\alpha^*\left(\omega^{q-1}\right)\\ &\overset{\operatorname{ch}(V\otimes W)=\operatorname{ch}(V)\operatorname{ch}(W)}{=\!=\!=} \int_X \operatorname{ch}(\mathcal{O}_X(-K_{X/T})^m\wedge\operatorname{td}(\mathcal{T}_{X/T})\wedge\alpha^*\omega^{n-(\dim F+1)}\\ &\overset{\text{Eq. (II.22)}}{=\!=\!=} 0\end{aligned}$$

vanishes for *any* Kähler form ω on T. Here, we abbreviate $q := \dim T$. As we already know from Corollary II.4.9 that $\alpha_*\mathcal{O}_X(-mK_{X/T})$ is nef the required result $c_1(\alpha_*\mathcal{O}_X(-mK_{X/T})) = 0$ thus directly follows from the following fact: ☐

Proposition II.4.13. *Let X be a compact Kähler manifold of dimension n and let $a \in H^{1,1}(X,\mathbb{R})$ be a nef cohomology class on X. If $a\cap[\omega]^{n-1} = 0$ for any Kähler form ω on X, then $a = 0$.*

Proof. Fix a Kähler form ω. Then, also $[\omega'] := [\omega] + a$ is a Kähler class and so $a \cap [\omega']^{n-1} = 0$. In other words, a is ω'-primitive. On the other hand we compute

$$0 = a\cap([\omega]+a)^{n-1} = a^2\cap([\omega]+a)^{n-2} + a\cap\omega\cap([\omega]+a)^{n-2} \geq 0.$$

Here, we used in the last step that both intersection numbers are non-negative thanks to all classes in question being nef (this is similar to Theorem I.2.14). It follows that $a^2 \cap [\omega']^{n-2} = 0$. Since a is also a real ω'-primitive $(1,1)$ class, the *Hodge-Riemann bilinear relations* (see [Huy05, Proposition 3.3.15.] for details) yield $a = 0$. ☐

Theorem II.4.14. (Cao, Demailly-Peternell-Schneider)
Let X be a compact Kähler manifold with nef tangent bundle. There exists a finite étale cover $\widetilde{X} \to X$ such that the Albanese map $\alpha\colon \widetilde{X} \to \mathrm{Alb}(\widetilde{X})$ is a flat fibre bundle. The typical fibre is a Fano manifold with nef tangent bundle.

Proof. Let $\widetilde{X} \to X$ be any finite étale cover of maximal irregularity (such a cover always exists by Example II.2.8). It follows from Theorem II.2.9 that α is a holomorphic submersion and that its typical fibre is a Fano manifold with nef tangent bundle. Thus, it only remains to prove the flatness of α. We want to apply our criterion Theorem II.4.2 to the line bundle $\mathcal{O}_X(-K_{X/T})$ on X and to this end it remains to check that $\alpha_*\mathcal{O}_X(-mK_{X/T})$ is numerically flat on T for all $m > 1$. But indeed, on the one hand we know from Theorem II.4.5 that $\alpha_*\mathcal{O}_X(-mK_{X/T})$ is a nef vector bundle on T for all $m \geq 1$. On the other hand, by Corollary II.4.12 it also holds that

$$\mathrm{c}_1(\alpha_*\mathcal{O}_X(-mK_{X/T})) = 0.$$

Thus, an application of Lemma I.4.6 yields the numerical flatness of $\alpha_*\mathcal{O}_X(-mK_{X/T})$ for all $m \geq 1$. In particular, we may apply Theorem II.4.2 to the α-relatively ample bundle $\mathcal{O}_X(-K_{X/T})$, proving that α is flat. □

5 A Characterisation of Manifolds with Nef Tangent Bundle

In the preceding sections we proved that (up to finite étale covers) Kähler manifolds with nef tangent bundle are flat fibre bundles over tori. Moreover, we saw that (conjecturally) the fibres are homogeneous Fano manifolds. In this section we will prove that this statement is optimal by proving the converse implication: Any flat fibre bundle over a torus with fibre a homogeneous Fano manifold exhibits a nef tangent bundle. This gives many non-trivial examples of manifolds with nef tangent bundle. To my knowledge (apart from the special case of ruled surfaces which has been known for quite a while, see [GR85], and some scattered results, see [DPP15, Theorem 5.2.]) these results have not been obtained before. In any case, the presentation in this section is original.

Let us begin by proving that all fibre bundles of this form are necessarily Kähler:

Theorem II.5.1. *Let $f\colon X \to T$ be a holomorphic fibre bundle with fibre F over a complex torus T. Assume that F is a Fano manifold. Then, X is Kähler. Moreover, X is of maximum irregularity and f is the Albanese of X.*

Proof. As F is Fano, the relative anti-canonical bundle $\mathcal{O}_X(-K_{X/T})$ is f-relatively ample. In particular, since T is Kähler Lemma I.2.21 yields that also X is Kähler.

Regarding the second assertion, let us abbreviate $q := \dim T$. Using some arguments from algebraic topology it is proved in [DPS94, Proposition 3.12.] that

$$\widetilde{q}(X) \leq \widetilde{q}(T) + \widetilde{q}(F) = q + 0. \tag{II.25}$$

Here, in the second equality we used that we know from Example II.2.8 that complex tori are always of maximal irregularity and that the (augmented) irregularity of Fanos vanishes. On the other hand, f clearly induces an injection

$$f^*\colon H^0(T, \Omega_T^1) \to H^0(X, \Omega_X^1).$$

In particular, $\widetilde{q}(X) \geq q(X) \geq \dim H^0(T, \Omega_T^1) = q$. Combining this with Eq. (II.25) we see that $\widetilde{q}(X) = q(X) = q$. Hence, X is of maximal irregularity. Finally, according to the universal property of the Albanese Proposition II.2.4 f factors uniquely through the Albanese $\alpha\colon X \to \mathrm{Alb}(X)$ of X; denote by $g\colon \mathrm{Alb}(X) \to T$ this unique map such that $f = g \circ \alpha$. It remains to show, that g is an ismomorphism. But indeed, since f is a surjective submersion with connected fibres so is g. Since $\dim T = q = q(X) = \dim \mathrm{Alb}(X)$ by the above and since g is surjective, g must also be generically finite. Finally, since g is a submersion all fibres are of the same dimension. It follows, that g is an étale cover with connected fibres, i.e. an ismomorphism. Thus, $f = \alpha$ is the Albanese of X and we are done. □

The goal of this section is to prove the following result characterising precisely which flat fibre bundles admit a nef tangent bundle:

Theorem II.5.2. (Characterisation of manifolds with nef tangent bundle)
Let $\alpha\colon X \to T$ be a holomorphic fibre bundle with typical fibre F over a complex torus T. Suppose that F is a homogeneous Fano manifold. The following assertions are equivalent:

1. *The tangent bundle $\mathcal{T}_X$ of X is nef,*
2. *the anti-canonical divisor $-K_X$ of X is nef,*
3. *the bundle $\alpha_*\mathcal{T}_X$ is a numerically flat vector bundle on T,*
4. *the bundle $\alpha_*\mathcal{O}_X(-K_X)$ is a numerically flat vector bundle on T,*
5. *the fibre bundle α is a flat fibre bundle,*
6. *the natural short exact sequence $0 \to \mathcal{T}_{X/T} \to \mathcal{T}_X \to \alpha^*\mathcal{T}_T \to 0$ admits a (global) holomorphic splitting.*

In the following we will prove Theorem II.5.2 in a series of steps. Let us start by observing that $\alpha_*\mathcal{T}_X$ and $\alpha_*\mathcal{O}_X(-K_X)$ *are* actually vector bundles (and not just coherent sheaves):

Proposition II.5.3. *Let $\alpha\colon X \to T$ be a holomorphic fibre bundle over a complex torus T with typical fibre F. Suppose that F is a Fano manifold with big and nef tangent bundle (according to Lemma II.3.6 F may e.g. be a homogeneous Fano).*

(i) For all integers $j > 0$ it holds that

$$R^j\alpha_*\mathcal{T}_{X/T} = R^j\alpha_*\mathcal{O}_X(-K_{X/T}) = 0.$$

In particular, $\alpha_*\mathcal{T}_{X/T}$ and $\alpha_*\mathcal{O}_X(-K_{X/T})$ are vector bundles over T with fibres $H^0(F,\mathcal{T}_F)$ and $H^0(F,\mathcal{O}_F(-K_F))$ respectively.

(ii) There exist natural short exact sequences

$$0 \to \mathcal{T}_{X/T} \to \mathcal{T}_X \to \alpha^*\mathcal{T}_T \to 0, \tag{II.26}$$

$$0 \to \alpha_*\mathcal{T}_{X/T} \to \alpha_*\mathcal{T}_X \to \mathcal{T}_T \to 0. \tag{II.27}$$

In particular, also $\alpha_*\mathcal{T}_X$ is a vector bundle on T.

Proof. Regarding item (i), note that

$$H^j(F,\mathcal{T}_F) = H^j(F,\mathcal{O}_F(-K_F)) = 0, \qquad \forall\, j > 0. \tag{II.28}$$

by the Griffiths-Kawamata-Viehweg vanishing Theorem IV.1.5. Indeed, we may write

$$\mathcal{T}_F = \mathcal{O}_F(K_F) \otimes \mathcal{T}_F \otimes \det(\mathcal{T}_T),$$
$$\mathcal{O}_F(-K_F) = \mathcal{O}_F(K_F) \otimes \mathcal{O}_F(-K_F) \otimes \det(\mathcal{O}_F(-K_F)).$$

Here, we used that by assumption the bundle $\mathcal{T}_T$ is big and nef and that $\mathcal{O}_F(-K_F)$ is ample. Thus, (i) immediately follows from Grauert's theorem on direct image sheaves Theorem IV.1.2.

As to item (ii), the exactness of the first sequence Eq. (II.26) is clear. For the second sequence, start by pushing down the first one to obtain the short exact sequence

$$0 \to \alpha_*\mathcal{T}_{X/T} \to \alpha_*\mathcal{T}_X \to \alpha_*\alpha^*\mathcal{T}_T \to 0. \tag{II.29}$$

Here, we used that $R^1\alpha_*\mathcal{T}_{X/T} = 0$ by part (i) of the proposition. We claim that $\alpha_*\alpha^*\mathcal{T}_T$ may be naturally identified with $\mathcal{T}_T$. Indeed, notice that by the projection formula for sheaves

$$\alpha_*\alpha^*\mathcal{T}_T = \alpha_*\left(\alpha^*\mathcal{T}_T \otimes \mathcal{O}_X\right) = \mathcal{T}_T \otimes \alpha_*\mathcal{O}_X.$$

Finally, by Zariski's main theorem (see Lemma IV.1.1 in the appendix) $\alpha_*\mathcal{O}_X$ may be naturally identified with $\mathcal{O}_T$ via the structure morphism. Thus, Eq. (II.27) follows from Eq. (II.29). Now, a simple dimension count yields that all fibres of $\alpha_*\mathcal{T}_X$ have the same dimension. Hence, $\alpha_*\mathcal{T}_X$ is a vector bundle and we are done. ☐

Our strategy for the proof of Theorem II.5.2 is to first prove the equivalence of the first four tokens. To do this, it will be favourable to work with the relative tangent bundles at times. This is no problem as the next lemma (which we also have used implicitly before) shows. In the following, we will often use it without mention:

Lemma II.5.4. *Let $\alpha\colon X \to T$ be a holomorphic fibre bundle with typical fibre F over a complex torus T. Suppose that F is a Fano manifold with big and nef tangent bundle (e.g. F may be a homogeneous Fano manifold). The following statements are pairwise equivalent:*

(1) $\mathcal{T}_X$ *is nef and*

(1') $\mathcal{T}_{X/T}$ *is nef.*

(2) $\mathcal{O}_X(-K_X)$ *is nef and*

(2') $\mathcal{O}_X(-K_{X/T})$ *is nef.*

(3) $\alpha_*\mathcal{T}_X$ *is a numerically flat vector bundle on* T *and*

(3') $\alpha_*\mathcal{T}_{X/T}$ *is a numerically flat vector bundle on* T*.*

(4) $\alpha_*\mathcal{O}_X(-K_X)$ *is a numerically flat vector bundle on* T *and*

(4') $\alpha_*\mathcal{O}_X(-K_{X/T})$ *is a numerically flat vector bundle on* T*.*

Proof. First of all, by Proposition II.5.3 $\alpha_*\mathcal{T}_X$, $\alpha_*\mathcal{T}_{X/T}$ and $\alpha_*\mathcal{O}_X(-K_{X/T})$ are all in fact vector bundles on T. We consider the short exact sequence Eq. (II.26):

$$0 \to \mathcal{T}_{X/T} \to \mathcal{T}_X \to \alpha^*\mathcal{T}_T \to 0.$$

Since T is a torus $\mathcal{T}_T = \mathcal{O}_T^{\oplus q}$ is trivial, where $q := \dim T$. In particular, it immediately follows from Theorem I.3.12 which discussed hereditary properties of nefness in short exact sequences that $\mathcal{T}_{X/T}$ is nef if and only if $\mathcal{T}_X$ is nef. We conclude, that items (1) and (1′) are equivalent.

Note that ad verbatim the same argument with Eq. (II.27) in place of Eq. (II.26) shows that also (3) and (3′) are pairwise equivalent.

Moreover, taking determinants in the short exact sequence Eq. (II.26) above we see that

$$\begin{aligned}\mathcal{O}_X(-K_X) &= \det(\mathcal{T}_X) = \det(\mathcal{T}_{X/T}) \otimes \det(\alpha^*\mathcal{T}_T) \\ &= \mathcal{O}_X(-K_{X/T}) \otimes \mathcal{O}_X = \mathcal{O}_X(-K_{X/T}).\end{aligned}$$

Hence, also (2), (2′) and (4), (4′) are pairwise equivalent. □

We are now ready to start proving the equivalence of the first four items in Theorem II.5.2. The equivalence will be proved in the following order:

$$(1) \Rightarrow (2) \Rightarrow (4) \Rightarrow (3) \Rightarrow (1)$$

Note that the first implication (1) ⇒ (2) is obvious as $\mathcal{O}_X(-K_X) = \det(\mathcal{T}_X)$ and determinants of nef bundles are nef by Corollary I.3.11.

Moreover, recall that we certainly already proved in Section 4 that (1) ⇒ (4): Indeed, using Theorem II.5.1 this is just Theorem II.4.14. But going through the proof of Theorem II.4.14 we see that we did not use the nefness of $\mathcal{T}_X$ all that much. Instead, the nefness of $\mathcal{O}_X(-K_{X/T})$ (and the fibres of our bundle being Fano) usually was sufficient. In fact, the only place where we used directly that $\mathcal{T}_X$ is nef was in the proof of Corollary II.4.12 to deduce the vanishing of the relative Todd classes in high degree. In [Cao13], Cao avoided the use of the Grothendieck-Riemann-Roch formula (at the cost of a more complicated argument). Using his [Cao13, Theorem 5.6.1.] instead, we deduce that the implication (2) ⇒ (4) does in fact hold true.

Also the implication (3) ⇒ (1) is not too hard:

Proposition II.5.5. *Let $\alpha\colon X \to T$ be a holomorphic fibre bundle over a complex torus T with typical fibre F. Suppose that F is a homogeneous Fano manifold. If $\alpha_*\mathcal{T}_{X/T}$ is nef (e.g. numerically flat) then so is $\mathcal{T}_{X/T}$.*

Proof. We claim that the natural morphism

$$e\colon \alpha^*\alpha_*\left(\mathcal{T}_{X/T}\right) \to \mathcal{T}_{X/T} \tag{II.30}$$

is surjective. Indeed, according to Proposition II.5.3 the fibres of $\alpha_*\mathcal{T}_{X/T}$ may be naturally identified with

$$\left(\alpha_*\mathcal{T}_{X/T}\right)\Big|_t = H^0\left(F_t, \mathcal{T}_{F_t}\right).$$

Hence, on every fibre F_t the map e is just the natural morphism

$$H^0(F, \mathcal{T}_F) \otimes_{\mathbb{C}} \mathcal{O}_F \to \mathcal{T}_F,$$

which is surjective due to the global generation assumption. In other words, the cokernel of $\alpha^*\alpha_*\mathcal{T}_{X/T} \to \mathcal{T}_{X/T}$ is supported on no fibre, i.e. it is trivial. This proves the surjectivity of e in Eq. (II.30).

But now, if $\alpha_*\mathcal{T}_{X/T}$ is nef then so is $\alpha^*\alpha_*\mathcal{T}_{X/T}$ by Proposition I.2.15. Since nefness is preserved under taking quotients as well, the surjectivity of $\alpha^*\alpha_*\mathcal{T}_{X/T} \to \mathcal{T}_{X/T}$ proves that also $\mathcal{T}_{X/T}$ is nef and we are done. □

To establish the equivalence of the first four items in Theorem II.5.2 it thus remains to prove the implication (4) $\Rightarrow$ (3), i.e. that the numerical flatness of $\alpha_*\mathcal{O}_X(-K_{X/T})$ implies the one of $\alpha_*\mathcal{T}_{X/T}$. We begin with some observations: Denote by $G := \mathrm{Aut}(F)$. Since the action of G on F is effective by definition there exists a unique holomorphic principal G-bundle $\mathcal{G} \xrightarrow{\pi} T$ so that $\alpha\colon X \to T$ is the associated bundle with typical fibre F. Note that also $\alpha_*\mathcal{T}_{X/T}$ and $\alpha_*\mathcal{O}_X(-K_{X/T})$ are associated to $\mathcal{G}$: Indeed,

$$\alpha_*\mathcal{T}_{X/T} = \mathcal{G} \times_G H^0(F, \mathcal{T}_F), \quad \alpha_*\mathcal{O}_X(-K_{X/T}) = \mathcal{G} \times_G H^0(F, \mathcal{O}_F(-K_F)).$$

Here, $G = \mathrm{Aut}(F)$ acts on $H^0(F, \mathcal{T}_F)$ and $H^0(F, \mathcal{O}_F(-K_F))$ in the natural way and we used Proposition II.5.3 to see that the fibres of $\alpha_*\mathcal{T}_{X/T}$ may be identified with $H^0(F, \mathcal{T}_F)$ (and similarly for $\alpha_*\mathcal{O}_X(-K_{X/T})$).

The idea is now to invoke Lemma I.4.15 to prove that the numerical flatness of $\alpha_*\mathcal{O}_X(-K_{X/T})$ implies the numerical flatness of $\mathcal{G}$ which will yield the numerical flatness of $\alpha_*\mathcal{T}_{X/T}$. We start with the following auxiliary result:

Proposition II.5.6. *Let F be a homogeneous Fano manifold and let $G := \mathrm{Aut}(F)$. Then, the natural action of G on $H^0(F, \mathcal{O}_F(-K_F))$ has finite kernel.*

Proof. We will prove that the kernel of the composition

$$\rho\colon G \to \mathrm{GL}\left(H^0(F, \mathcal{O}_F(-K_F))\right) \twoheadrightarrow \mathrm{PGL}\left(H^0(F, \mathcal{O}_F(-K_F))\right)$$

is finite. Clearly this suffices to prove the assertion.

> *Step 1: The natural rational map $F \dashrightarrow \mathbb{P}H^0(F, \mathcal{O}_F(-K_F))$ is everywhere defined and in fact a finite map.*

Indeed, since F is homogeneous $\mathcal{T}_F$ is globally generated. But then also its determinant $\mathcal{O}_F(-K_F)) = \det(\mathcal{T}_F)$ is globally generated. In particular, the rational map

$$\psi\colon F \dashrightarrow \mathbb{P}H^0(F, \mathcal{O}_F(-K_F))$$

is everywhere defined. Let F' be any fibre. On the one hand, $\mathcal{O}_F(-K_F)|_{F'}$ is the trivial line bundle since F' is a fibre of ψ. On the other hand, $\mathcal{O}_F(-K_F)|_{F'}$ is also ample because $\mathcal{O}_F(-K_F)$ is so. It follows, that F' is a discrete (hence finite) set of points. This concludes the proof of *Step 1*.

In the following, let us abbreviate $H := H^0(F, \mathcal{O}_F(-K_F))$

> *Step 2: The kernel K of the natural map $\rho\colon G \to PGL(H) = \mathrm{Aut}(\mathbb{P}H)$ is a finite group.*

Exactly as in the proof of Lemma II.3.14 the map $\psi\colon F \to \mathbb{P}H$ is equivariant with respect to ρ. Now, K acts on F and by definition this action leaves $\operatorname{Im}\psi$ invariant. In other words, the map $\psi\colon F \to \operatorname{Im}\psi$ is K-invariant. But since ψ is finite according to *Step 1*, K must be finite as well: Let us denote by $U \subseteq \operatorname{Im}\psi$ the (dense) Zariski open subset over which ψ is smooth. Then, $\psi\colon \psi^{-1}(U) \to U$ is a holomorphic covering (say of degree d) and K acts on $\psi^{-1}(U)$ by Deck transformations. In particular, the action of K on $\psi^{-1}(U)$ (and hence by density the action of K on F) is determined by the action of K on any fibre of $\psi\colon \psi^{-1}(U) \to U$. But then, $K \hookrightarrow S_d$ has at most $d!$ elements. We conclude that also the kernel of $G \to \mathrm{GL}(H)$ is finite. □

Corollary II.5.7. *Let $\alpha\colon X \to T$ be a holomorphic fibre bundle over T, a complex torus. Suppose that the typical fibre F of α is a homogeneous Fano manifold. If $\alpha_*\mathcal{O}_X(-K_{X/T})$ is numerically flat then so is $\alpha_*\mathcal{T}_{X/T}$.*

Proof. As above we denote by $G := \operatorname{Aut}(F)$ and we let $\mathcal{G}$ denote the underlying principal G-bundle of X so that $X = \mathcal{G} \times_G F$. Note that as F is a homogeneous Fano G has only at most finitely many connected components (see Lemma II.3.14) and G^0 is a semi simple group (see Theorem II.3.15). Consider the two associated bundles

$$\alpha_*\mathcal{T}_{X/T} = \mathcal{G} \times_G H^0(F, \mathcal{T}_F), \quad \alpha_*\mathcal{O}_X(-K_{X/T}) = \mathcal{G} \times_G H^0(F, \mathcal{O}_F(-K_F)).$$

According to Proposition II.5.6, the kernel of the action of G on $H^0(F, \mathcal{O}_F(-K_F))$ is finite. In view of part (ii) of Lemma I.4.15 the numerical flatness of $\alpha_*\mathcal{O}_X(-K_{X/T})$ thus yields the numerical flatness of $\mathcal{G}$ which in turn by part (i) of Lemma I.4.15 gives the numerical flatness of $\alpha_*\mathcal{T}_{X/T}$. See also Remark I.4.16. □

An alternative strategy for proving Corollary II.5.7 may be found in [DPP15, Theorem 5.2.]. In any case, our proof of the equivalence of statements $(1), (2), (3), (4)$ in Theorem II.5.2 is completed.

Let us now turn towards the other statements; the implication $(4) \Rightarrow (5)$ has essentially been proved already: Indeed, if $\alpha_*\mathcal{O}_X(-K_{X/T})$ is numerically flat then by the same method of proof as in $(4) \Rightarrow (3)$ so is $\alpha_*\mathcal{O}_X(-mK_{X/T})$ for any $m > 0$ because both are associated to the same principal bundle $\mathcal{G}$. But then α is flat by Theorem II.4.2 applied to the line bundle $\alpha_*\mathcal{O}_X(-K_{X/T})$. Moreover, the implication $(5) \Rightarrow (6)$ is also essentially clear as by the discussion after Definition II.4.1 α is flat if and only if the sequence $0 \to \mathcal{T}_{X/T} \to \mathcal{T}_X \to \alpha^*\mathcal{T}_T \to 0$ admits a holomorphic splitting establishing $\alpha^*\mathcal{T}_T$ as an integrable (!) sub bundle of $\mathcal{T}_X$.

To complete the proof of Theorem II.5.2 it finally remains to prove that the splitting of the relative tangent sequence implies any of the first four items, e.g. the numerical flatness of $\alpha_*\mathcal{T}_{X/T}$.

Theorem II.5.8. *Let $\alpha\colon X \to T$ be a holomorphic fibre bundle over T, a complex torus. Assume that the typical fibre F of α is a homogeneous Fano manifold. If there exists a (global) holomorphic splitting of the relative tangent sequence*

$$0 \to \mathcal{T}_{X/T} \to \mathcal{T}_X \to \alpha^*\mathcal{T}_T \to 0, \tag{II.31}$$

then $\alpha_*\mathcal{T}_{X/T}$ is numerically flat.

Proof. First of all, recall that according to Proposition II.5.3 also the sequence

$$0 \to \alpha_*\mathcal{T}_{X/T} \to \alpha_*\mathcal{T}_X \to \mathcal{T}_T \to 0 \tag{II.32}$$

is exact. Moreover, the proof of the same token shows that this is just the sequence obtained by pushing down Eq. (II.30) by α. Thus, also the sequence Eq. (II.32) admits a holomorphic splitting.

As before, let us abbreviate $G := \mathrm{Aut}(F)$ and let us denote by $\pi\colon \mathcal{G} \to T$ the principal G-bundle underlying $X \to T$.

Step 1: The above splitting of Eq. (II.32) *induces naturally a holomorphic connection on the principal G-bundle $\pi\colon \mathcal{G} \to T$ underlying $\alpha\colon X \to T$.*

Recall, that by our definition a holomorphic connection in $\mathcal{G}$ is nothing but a holomorphic splitting of the short exact sequence of vector bundles

$$0 \to \mathfrak{ad}(\mathcal{G}) \to (\pi_*\mathcal{T}_{\mathcal{G}})^G \to \mathcal{T}_T \to 0. \tag{II.33}$$

Now, as F is homogeneous for the action of G we may identify the Lie algebra $\mathfrak{g}$ of G with $H^0(F, \mathcal{T}_F)$ and the natural action of G on $H^0(F, \mathcal{T}_F)$ may be identified with the adjoint action (the first statement was contained in Theorem II.3.10 and the latter one is straightforward to verify). In particular, both $\alpha_*\mathcal{T}_{X/T}$ and $\mathfrak{ad}(\mathcal{G})$ are associated to the adjoint representation of G, i.e. there exists a natural identification

$$\mathfrak{ad}(\mathcal{G}) = \alpha_*\mathcal{T}_{X/T}. \tag{II.34}$$

We claim that there also exists a natural morphism

$$(\pi_*\mathcal{T}_{\mathcal{G}})^G \to \alpha_*\mathcal{T}_X.$$

To see this, write $F = G/P$ and consider the natural submersion $p\colon \mathcal{G} \to \mathcal{G}/P = X$. This is a principal P-bundle, so in analogy with Eq. (II.33) there exists a natural short exact sequence

$$0 \to \mathfrak{ad}(P) \to (p_*\mathcal{T}_{\mathcal{G}})^P \to \mathcal{T}_X \to 0.$$

Pushing down the map $(p_*\mathcal{T}_\mathcal{G})^P \to \mathcal{T}_X$ by α, this gives a natural morphism

$$(\pi_*\mathcal{T}_\mathcal{G})^P = \alpha_*\left((p_*\mathcal{T}_\mathcal{G})^P\right) \to \alpha_*\mathcal{T}_X.$$

Here, we used that push forward commutes with taking invariants. Then, the composition

$$\iota\colon\ (\pi_*\mathcal{T}_\mathcal{G})^G \hookrightarrow (\pi_*\mathcal{T}_\mathcal{G})^P \to \alpha_*\mathcal{T}_X$$

is the map we are looking for.

Altogether, we have a diagram of natural maps

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{ad}(\mathcal{G}) & \longrightarrow & (\pi_*\mathcal{T}_\mathcal{G})^G & \longrightarrow & \mathcal{T}_T & \longrightarrow & 0 \\
\Vert & & \Vert & & \downarrow \iota & & \Vert & & \Vert \\
0 & \longrightarrow & \alpha_*\mathcal{T}_{X/T} & \longrightarrow & \alpha_*\mathcal{T}_X & \longrightarrow & \mathcal{T}_T & \longrightarrow & 0
\end{array}$$

with exact rows which is immediately verified to commute. In particular, by the 5-Lemma ι is an isomorphism. Thus, the splitting of the sequence

$$0 \to \alpha_*\mathcal{T}_{X/T} \to \alpha_*\mathcal{T}_X \to \mathcal{T}_T \to 0$$

which exists by Eq. (II.32) induces a splitting of the sequence

$$0 \to \mathfrak{ad}(\mathcal{G}) \to (\pi_*\mathcal{T}_\mathcal{G})^G \to \mathcal{T}_T \to 0.$$

But again, this is the same thing as a holomorphic connection in $\mathcal{G}$.

Step 2: The vector bundle $\alpha_*\mathcal{T}_{X/T}$ admits a holomorphic connection.

Indeed, we already saw in *Step 1* that $\mathcal{G}$ admits a holomorphic connection. Of course, then also all bundles which are associated to $\mathcal{G}$ admit a holomorphic connection and, in particular, this is true of $\alpha_*\mathcal{T}_{X/T}$.

Step 3: The vector bundle $\alpha_*\mathcal{T}_{X/T}$ is semi stable.

By a theorem of Biswas [Bis95, Remark 3.7.(ii)], if more generally $\mathcal{E}$ is any vector bundle admitting a holomorphic connection on a compact Kähler manifold T whose tangent bundle $\mathcal{T}_T$ is semi stable and of non-negative slope $\mu(\mathcal{T}_T) \geq 0$, then $\mathcal{E}$ is automatically semi stable. In fact, [Bis95, Remark 3.7.(ii)] applies in even greater generality and granting some basic properties of semi stable bundles its proof is not hard. We avoid it however, as it would lead us to far astray.

In any case, we may apply this result to $\mathcal{E} := \alpha_*\mathcal{T}_{X/T}$ because as T is a torus $\mathcal{T}_T$ is trivial. In particular it is numerically flat and, hence, semi stable (here we apply Lemma I.4.6 twice). Moreover, clearly $\mu(\mathcal{T}_T) = \deg(\mathcal{T}_T) = 0$.

Step 4: The vector bundle $\alpha_*\mathcal{T}_{X/T}$ is numerically flat.

Recall, that we already proved that $\alpha_*\mathcal{T}_{X/T}$ is semi stable (*Step 3*) and that it admits a holomorphic connection (*Step 2*). In particular, by Example I.1.10 all of its Chern classes vanish. Thus, it follows from our characterisation result Lemma I.4.6 that $\alpha_*\mathcal{T}_{X/T}$ is numerically flat and we are finally done. □

Let us conclude this section by making precise what Theorem II.5.2 states in the case where $X := \mathbb{P}(\mathcal{E})$ is the projectivisation of a vector bundle:

Proposition II.5.9. *Let T be a complex torus and let $\mathcal{E}$ be a holomorphic vector bundle of rank r on T. Denote $X := \mathbb{P}(\mathcal{E}) \xrightarrow{\alpha} T$. Then, there exist natural identifications*

$$\alpha_*\mathcal{T}_{X/T} = (\mathcal{E} \otimes \mathcal{E}^*)/\mathcal{O}_T, \quad \alpha_*\mathcal{O}_X(-K_X) = \operatorname{Sym}^r \mathcal{E} \otimes \det(\mathcal{E}^*).$$

Proof. Consider the relative Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{E})} \to \alpha^*\mathcal{E}^* \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \to \mathcal{T}_{\mathbb{P}(\mathcal{E})/T} \to 0.$$

Pushing down by α we find the exact sequence

$$0 \to \alpha_*\mathcal{O}_X \to \mathcal{E}^* \otimes \alpha_*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \to \alpha_*\mathcal{T}_{X/T} \to R^1\alpha_*\mathcal{O}_X.$$

Now, $\alpha_*\mathcal{O}_X = \mathcal{O}_T$ by Zariski's Lemma IV.1.1, $\alpha_*\mathcal{O}_X(1) = \mathcal{E}$ essentially by construction and $R^1\alpha_*\mathcal{O}_X = R^1\alpha_*\mathcal{O}_X(-K_{X/T}+K_{X/T}) = 0$ by the Kodaira vanishing theorem. In total, we find the short exact sequence $0 \to \mathcal{O}_T \to \mathcal{E}^* \otimes \mathcal{E} \to \alpha_*\mathcal{T}_{X/T} \to 0$ which proves the first formula. Regarding the second formula, we simply compute

$$\alpha_*\mathcal{O}_X(-K_X) = \alpha_*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(r) \otimes \alpha^* \det(\mathcal{E}^*)) = \operatorname{Sym}^r \mathcal{E} \otimes \det(\mathcal{E}^*).$$ □

Corollary II.5.10. (compare [DPS94, Proposition 3.18.])
Let (T, ω) be a complex torus and let $\mathcal{E}$ be a holomorphic vector bundle of rank r on T. Denote $X := \mathbb{P}(\mathcal{E}) \xrightarrow{\alpha} T$. Then, the following assertions are equivalent:

(1) *The tangent bundle $\mathcal{T}_X$ of X is nef.*
(2) *The holomorphic fibre bundle $X := \mathbb{P}(\mathcal{E}) \xrightarrow{\alpha} T$ is flat.*
(3) *The vector bundle $\mathcal{E} \otimes \mathcal{E}^* = \operatorname{End}(\mathcal{E})$ is numerically flat.*
(4) *The vector bundle $\operatorname{Sym}^r \mathcal{E} \otimes \det(\mathcal{E}^*)$ is numerically flat.*
(5) *$\mathcal{E}$ is semi stable w.r.t. ω and satisfies $2r\,\mathrm{ch}_2(\mathcal{E}) \cap [\omega]^{n-2} = \mathrm{c}_1(\mathcal{E})^2 \cap [\omega]^{n-2}$.*

Proof. Modulo the identifications in Proposition II.5.9 the equivalence of (1), (2), (3) and (4) is literally just Theorem II.5.2 (note that the bundle $\mathcal{E} \otimes \mathcal{E}^*$ is numerically flat if and only if $\mathcal{E} \otimes \mathcal{E}^*/\mathcal{O}_T$ is so by Theorem I.3.12). Finally, the equivalence of (3) and (5) is a standard fact: Let us denote by $\mathcal{G}$ the frame bundle of $\mathcal{E}$ which is a principal GL_r-bundle. Then, according to [AB01, Remark 1.2.] $\mathcal{E}$ is semi stable if and only if so is $\mathcal{G}$. But by definition, the latter is equivalent to the semi stability of $\mathfrak{ad}(\mathcal{G}) = \mathrm{End}(\mathcal{E}) = \mathcal{E} \otimes \mathcal{E}^*$ (compare also [AB01, Proposition 2.10.]). Since $\mathrm{c}_1(\mathcal{E} \otimes \mathcal{E}^*) = \mathrm{c}_1(\mathcal{E}) + \mathrm{c}_1(\mathcal{E}^*) = 0$ always vanishes and

$$\mathrm{ch}_2(\mathcal{E} \otimes \mathcal{E}^*) = r\,\mathrm{ch}_2(\mathcal{E}) + \mathrm{c}_1(\mathcal{E})\,\mathrm{c}_1(\mathcal{E}^*) + r\,\mathrm{ch}_2(\mathcal{E}^*) = 2r\,\mathrm{ch}_2(\mathcal{E}) - \mathrm{c}_1(\mathcal{E})^2$$

we conclude by applying Lemma I.4.6. □

6 Outlook: Some more general Structure Theory

Throughout the preceding sections we have come to a rather complete understanding of compact Kähler manifolds exhibiting a nef tangent bundle. Starting with [Cao13], the last 10 years have seen quite some work aimed towards extending this understanding to more general set-ups. We want to conclude this chapter by surveying some of the most far reaching generalisations obtained. Additionally, we present some questions which remain unsettled.

Roughly speaking, mainly two natural generalisations of the condition of having a nef tangent bundle have been considered. First, one may try to impose weaker positivity assumptions on the tangent bundle. Alternatively, one can also ask for results in case one only assumes the anti-canonical line bundle $\mathcal{O}_X(-K_X) = \det(\mathcal{T}_X)$ to be nef. The latter case has seen quite some attention (see for example [CH17], [Wan21] and [MW21]) and by now is nearly completely settled:

Theorem II.6.1. (Matsumura-Wang, [MW21, Theorem 1.1.])
Let X be a projective manifold with nef anti-canonical line bundle $\mathcal{O}_X(-K_X)$. Then, there exists a finite étale cover $\widetilde{X} \to X$ and a holomorphic map $f\colon \widetilde{X} \to Y$ such that

(1) Y is a projective manifold with trivial canonical bundle $\mathcal{O}_Y(K_Y) = \mathcal{O}_Y$,

(2) f is a flat fibre bundle,

(3) the typical fibre F of f is a rationally connected projective manifold with nef anti-canonical bundle.

In fact, one may even allow for mild singularities of X (singularities of *klt*-pairs to be precise). The proof of Theorem II.6.1 is very similar to the one of Theorem II.4.14: The ingredients are all the same, however it is of course far more technical. Having in mind the (weak) Campana-Peternell conjecture one would be tempted to conjecture the following:

Question II.6.2. *Let F be a rationally connected projective manifold with nef anti-canonical bundle $\mathcal{O}_F(-K_F)$. Is $\mathcal{O}_F(-K_F)$ big? Is it semi ample? Or maybe even globally generated?*

Note that completely analogously to Lemma II.3.6 one may show that

$$\mathcal{O}_F(-K_F) \text{ globally generated } \Rightarrow \mathcal{O}_F(-K_F) \text{ big } \Rightarrow \mathcal{O}_F(-K_F) \text{ semi ample.}$$

It turns out however, that Question II.6.2 is (much) too optimistic: An explicit counter example is provided already by blow ups of $\mathbb{P}^2$ in certain configurations of 9 points. In fact, [Koi17] even proved that there exist such configurations of 9 points in $\mathbb{P}^2$ for which $\mathcal{O}_F(-K_F)$ is nef but not semi positive (i.e. does not admit a smooth hermitean metric of $\Theta_h \geq 0$; by the proof of Corollary I.2.16 this is even weaker then being semi ample). This had been an open question for about 20 years.

One may also wonder about the converse statement, similar to our Theorem II.5.2:

Question II.6.3. *Let $f\colon X \to Y$ be a flat holomorphic fibre bundle with fibre F. Assume, that Y is a projective Calabi-Yau manifold and that F is a rationally connected manifold with nef anti-canonical bundle. Is it true, that also $\mathcal{O}_X(-K_X)$ is nef?*

It seems plausible to me that one can extend the arguments given in the proof of Theorem II.5.2 to answer Question II.6.3 affirmatively at least in case $\mathcal{O}_F(-K_F)$ is semi ample (the semi ampleness would be needed for an analogue of Proposition II.5.5). In the general case however, our methods break down entirely. Nevertheless, we see that manifolds with nef anti canonical bundle are now fairly well understood.

In contrast, the idea of relaxing the positivity assumptions on the tangent bundle has only recently been picked up in [HIM21] and the precise structure to be expected is still not entirely clear. In said paper, several weakened positivity conditions on the tangent bundle have been considered which may be summarised under the title *pseudo-effectivity*. As a sample result, let us consider the case of projective manifolds X with (strongly) pseudo-effective tangent bundle. This means, that for any ample line bundle $\mathscr{L}$ on X and any integer $N > 0$ there exists an integer $m > 0$ such that the bundle

$$\operatorname{Sym}^{mN} \mathcal{E} \otimes \mathscr{L}^{\otimes m}$$

is generically generated by global sections.

Theorem II.6.4. (Hosono, Iwai, Matsumura)
Let X be a projective manifold with strongly pseudo-effective tangent bundle. Then, there exists a finite étale cover $\widetilde{X} \to X$ and a holomorphic map $f\colon \widetilde{X} \to T$ such that

(1) T is an abelian variety (i.e. a projective torus),

(2) f is a submersion,

(3) a general fibre of f is rationally connected and has a strongly pseudo-effective tangent bundle.

In this case however, it is all but clear whether the result is optimal. There is some evidence that f should be a holomorphic fibre bundle (in [HIM21] this is proved assuming some slightly stronger positivity of the tangent bundle) but this is not entirely clear. As of now, we only know that one can not expect f to be a flat fibre bundle by an example of [HIM21, Theorem 1.5.]: They prove that any ruled surface of the form $\mathbb{P}(\mathcal{E}) \xrightarrow{\pi} C$ over an elliptic curve C has a strongly pseudo-effective tangent bundle. However, π can not be flat if $\mathcal{E}$ is unstable as follows from Theorem II.5.2.

Again, one may ask:

Question II.6.5. (posed in [HIM21, Problem 3.12.])
Let F be a rationally connected projective manifold with strongly pseudo-effective tangent bundle. Is $\mathcal{O}_F(-K_F)$ big?

Finally, in view of all of these results it seems natural to expect that one should be able to extend the structure theory in similar fashion to compact Kähler manifold with pseudo-effective anti-canonical bundle. At the moment however, basically nothing is known in such generality.

Chapter III

Canonical Extensions

To any compact Kähler manifold (X, ω_X) one may associate in a canonical way a bundle of affine spaces $Z_X \xrightarrow{p} X$ called a canonical extension of X. It may be characterised as the universal complex manifold on which the cohomology class $[p^*\omega_X] = 0$ vanishes. Canonical extensions are interesting because they have featured in some surprising contexts as for example in the verification of smoothness of solutions to the Monge-Ampère equation in [Don02].

More recently, in [GW20] it was asked, whether the positivity of the tangent bundle of X is related to the geometry of Z_X. Concretely, it was proposed that Z_X could be Stein if and only if the tangent bundle of X is nef. The principal result of this chapter will give a partial answer to this question.

We begin by discussing several possible constructions of Z_X and more general affine bundles over X. Then, we properly introduce the problem posed by Greb and Wong. We will quickly discuss what is know thus far, including some basic properties of the manifold Z_X.

The second section is the heart of this chapter. It is devoted to proving that, assuming the weak Campana-Peternell conjecture to hold true, any canonical extension of a compact Kähler manifold with nef tangent bundle is a Stein manifold, thereby answering affirmatively the first half of the problem proposed in [GW20]. The special cases of tori and homogeneous Fano manifolds have already been settled by [GW20] and [HP21] respectively and we will start by considering these results. Appealing to the structure theory proved in Chapter II the general case is then solved by putting these together.

In the converse direction only partial results are known due to the work of [HP21]. In the third section we will survey their most important discoveries. In particular, we discuss the case of surfaces and complement one of their results. Nevertheless, perhaps the most interesting case remains open.

1 Canonical Extensions of Complex Manifolds

Canonical extensions may be seen as a special case of a more general construction of affine bundles over a Kähler manifold which we will consider below. We give a plethora of equivalent approaches to this topic to serve the tastes of different readers. Then, in a second subsection we introduce canonical extensions and explain their (conjectural) connection to the positivity of the tangent bundle of X. Finally, we prove some basic properties of the construction for later reference.

1.1 Constructing Extensions from Complex Vector Bundles

In the following we will introduce a general approach to constructing bundles of affine spaces over complex manifolds. This theory will eventually lead to the definition of canonical extensions in the next subsection. What follows is essentially an expansion of the results in [GW20, Section 1.3.].

Reminder III.1.1. Let X be a complex manifold and let $\mathcal{F}$, $\mathcal{Q}$ be coherent sheaves on X. An *extension* of $\mathcal{Q}$ by $\mathcal{F}$ is a (necessarily coherent) sheaf $\mathcal{G}$ along with a short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{Q} \to 0.$$

A *morphism* between an extension $0 \to \mathcal{F} \to \mathcal{G}_1 \to \mathcal{Q} \to 0$ and another extension $0 \to \mathcal{F} \to \mathcal{G}_2 \to \mathcal{Q} \to 0$ is a morphism of sheaves $\phi\colon \mathcal{G}_1 \to \mathcal{G}_2$ for which the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G}_1 & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \\ \| & & \| & & \downarrow{\scriptstyle\phi} & & \| & & \| \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{G}_2 & \longrightarrow & \mathcal{Q} & \longrightarrow & 0 \end{array}$$

commutes. Note that in this case ϕ must be an isomorphism by the 5-Lemma.

It is a well-known fact from category theory that there exists a natural one-to-one correspondence between isomorphism classes of extensions $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{Q} \to 0$ and the cohomology group $\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{Q}, \mathcal{F})$, where $\mathrm{Ext}_{\mathcal{O}}(\mathcal{Q}, -)$ is defined to be the right-derived functor $\mathrm{Ext}_{\mathcal{O}}(\mathcal{Q}, -) = R\mathrm{Hom}_{\mathcal{O}}(\mathcal{Q}, -)$.

Example III.1.2. Let us consider the special case $\mathcal{Q} = \mathcal{O}_X$. Note that

$$\mathrm{Ext}_{\mathcal{O}}(\mathcal{O}_X, -) := R\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}_X, -) = R\Gamma(X, -).$$

In particular, we may identify $\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}_X, \mathcal{F}) = H^1(X, \mathcal{F})$ so that by the discussion above isomorphism classes of extensions $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{O}_X \to 0$ are in one-to-one correspondence with cohomology classes in $H^1(X, \mathcal{F})$.

Now, fix a class $a \in H^1(X, \mathcal{F})$ and consider the corresponding extension

$$0 \to \mathcal{F} \to \mathcal{V}_a \to \mathcal{O}_X \to 0.$$

In [GW20, Section 1.3.] assuming $\mathcal{F} =: \mathcal{E}$ is a holomorphic vector bundle the authors describe explicitly a possible construction of $\mathcal{V}_a$. Below, we want to sketch this approach: Indeed, in this case $H^1(X, \mathcal{E})$ may be computed using Dolbeaut-cohomology, i.e. there exists a $\bar{\partial}$-closed differentiable form $\omega \in \mathcal{A}^{0,1}(\mathcal{E})$ representing the class a. Now, as a differentiable vector bundle one just defines $\mathcal{V}_a$ to be the direct sum $\mathcal{F} \oplus \mathcal{O}_X$. Moreover, a section $s = (\sigma, f)$ to $\mathcal{V}_a =_{C^\infty} \mathcal{F} \oplus \mathcal{O}_X$ is defined to be holomorphic if and only if

$$\bar{\partial}_{\mathcal{V}_a}(s) := \begin{pmatrix} \bar{\partial}_{\mathcal{E}} & \omega \\ 0 & \bar{\partial}_{\mathcal{O}_X} \end{pmatrix} \begin{pmatrix} \sigma \\ f \end{pmatrix} = 0. \tag{III.1}$$

Construction III.1.3. Let X be a complex manifold, let $\mathcal{E}$ be a holomorphic vector bundle on X and fix any cohomology class $a \in H^1(X, \mathcal{E})$. As we saw above, to a we can associate an extension

$$0 \to \mathcal{E} \to \mathcal{V}_a \xrightarrow{p} \mathcal{O}_X \to 0 \tag{III.2}$$

of holomorphic vector bundles on X. Consider the sub sheaf $\mathcal{Z}_a := p^{-1}(1) \subset \mathcal{V}_a$ of sections of $\mathcal{V}_a$ mapping under p to the constant function 1. Note that $\mathcal{Z}_a$ is *not* a sheaf of $\mathcal{O}_X$-modules. However, it comes with a natural action of $\mathcal{E}$ by translations making $\mathcal{Z}_a$ into an $\mathcal{E}$-torsor. In this sense, $\mathcal{Z}_a$ is an *affine bundle*: Its underlying total space $Z_a := |\mathcal{Z}_a| \to X$ is a fibre bundle over X and the fibre $Z_a|_x$ over any point x is in a natural way an affine vector space with group of translations $\mathcal{E}|_x$. In the following, we will call the total space $Z_a := |\mathcal{Z}_a| \to X$ (an) *extension* of X. Sometimes we may also denote it by $Z_{\mathcal{E},a}$ if we want to make explicit the dependence on the bundle $\mathcal{E}$. Below we want to describe three more constructions of Z_a:

First, we may consider p as a holomorphic map between the underlying total spaces of the bundles $\mathcal{E}$ and $\mathcal{O}_X$:

$$|p|: |\mathcal{E}| \to |\mathcal{O}_X| = X \times \mathbb{C}.$$

Then, Z_a may be naturally identified with the pre-image

$$Z_a = |p|^{-1}(X \times \{1\}).$$

Since p is a surjective morphism of vector bundles, $|p|$ is a submersion. In particular, we see from this that Z_a is indeed a manifold and we also see that we may view the affine space structure on the fibres $Z_a|_x$ as arising from the embedding $Z_a|_x \subset \mathcal{V}_a|_x$.

A second, possibly more geometric construction of Z_a is as follows: Dualising the short exact sequence Eq. (III.2) we find the short exact sequence

$$0 \to \mathcal{O}_X \to (\mathcal{V}_a)^* \to \mathcal{E}^* \to 0$$

which defines an embedding

$$\mathbb{P}(\mathcal{E}^*) \hookrightarrow \mathbb{P}(\mathcal{V}_a^*).$$

We claim that there exists a natural identification of the affine bundle Z_a with the complement $\mathbb{P}(\mathcal{V}_a^*) \setminus \mathbb{P}(\mathcal{E}^*)$. Indeed, for any $x \in X$ the fibre $\mathbb{P}(\mathcal{V}_a^*)|_x$ is just the space of lines in $\mathcal{V}_a|_x$ passing through the origin. Now, of course any point in the affine space $Z_a|_x \subset \mathcal{V}_a|_x$ defines a unique line passing trough itself and the origin (here, we use that that $0 \notin Z_a|_x = p_x^{-1}(1)$) and so $Z_a|_x \subset \mathbb{P}(\mathcal{V}_a^*)|_x$ in a natural way. Moreover, the set $\mathbb{P}(\mathcal{V}_a^*)|_x \setminus Z_a|_x$ consists precisely in those lines which are parallel to $Z_a|_x$, i.e. contained in $\mathbb{P}(\mathcal{E}^*)|_x$. This concludes the proof of the claim.

Finally, Z_a also satisfies a universal property (which of course determines it uniquely). In short, this may be summarised by saying that $Z_a \to X$ is the universal holomorphic map making the class a trivial upon pull back. We will provide more details on this characterisation in Lemma III.1.8 below.

Remark III.1.4. In the situation of Construction III.1.3 we have seen that Z_a may alternatively be defined as the complement $\mathbb{P}(\mathcal{V}_a^*) \setminus \mathbb{P}(\mathcal{E}^*)$. For later reference, let us record that the inclusion $\mathbb{P}(\mathcal{E}^*) \hookrightarrow \mathbb{P}(\mathcal{V}_a^*)$ is defined by the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{V}_a^* \to \mathcal{E}^* \to 0.$$

In particular, $\mathbb{P}(\mathcal{E}^*)$ is a smooth hypersurface in the linear series of $\mathcal{O}_{\mathbb{P}(\mathcal{V}_a^*)}(1)$. Note that in this case according to Proposition IV.2.3 in the appendix its normal bundle is given by

$$\mathcal{N}_{\mathbb{P}(\mathcal{E}^*)/\mathbb{P}(\mathcal{V}_a^*)} = \mathcal{O}_{\mathbb{P}(\mathcal{E}^*)}(1).$$

This interpretation will be crucial in the next section.

Example III.1.5. Let X be a complex manifold and let $\mathcal{E}$ be a holomorphic vector bundle on X. Consider the trivial cohomology class $0 \in H^1(X, \mathcal{E}) = \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}_X, \mathcal{E})$. Then, the extensiona

$$0 \to \mathcal{E} \to \mathcal{V}_0 \xrightarrow{p} \mathcal{O}_X \to 0$$

is (holomorphically) isomorphic to the trivial extension

$$0 \to \mathcal{E} \to \mathcal{E} \oplus \mathcal{O}_X \to \mathcal{O}_X \to 0.$$

It follows that $|p|^{-1}(X \times \{1\}) = |\mathcal{E}| \times \{1\}$. In other words, Z_0 is isomorphic as an affine bundle to the affine bundle obtained from $\mathcal{E}$ by forgetting the zero-section.

Proposition III.1.6. *Let X be a complex manifold, let $\mathcal{E}$ be a holomorphic vector bundle on X and fix any cohomology class $a \in H^1(X, \mathcal{E}) = \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}_X, \mathcal{E})$. Then, the bundle $Z_a \to X$ admits a global holomorphic section if and only if $a = 0$.*

Proof. First, by Example III.1.5 if $a = 0$, then $Z_a \cong |\mathcal{E}|$ as affine bundles over X. In particular, Z_a admits holomorphic sections (i.e. the zero-section).

Conversely, assume that $Z_a \to X$ admits a global holomorphic section. Equivalently, the underlying sheaf $\mathcal{Z}_a$ admits a global section s. Fix a Dolbeaut representative $\omega \in \mathcal{A}^{0,1}(\mathcal{E})$ of a. Recall, that at least in the differentiable category we have $\mathcal{Z}_a = pr_2^{-1}(1) \subset \mathcal{V}_a =_{C^\infty} \mathcal{E} \oplus \mathcal{O}_X$, i.e. s is of the form $s = (\sigma, 1)$. Since s was moreover assumed to be holomorphic, it satisfies Eq. (III.1):

$$\bar\partial_{\mathcal{V}_a}(s) := \begin{pmatrix} \bar\partial_{\mathcal{E}} & \omega \\ 0 & \bar\partial_{\mathcal{O}_X} \end{pmatrix} \begin{pmatrix} \sigma \\ 1 \end{pmatrix} = 0.$$

In other words, σ is a differentiable section to $\mathcal{E}$ such that $\bar\partial_{\mathcal{E}}(-\sigma) = \omega$. This means that ω is $\bar\partial$-exact, i.e. that the class $a = [\omega] = 0 \in H^1(X, \mathcal{E})$ vanishes. □

The construction of extensions is functorial:

Proposition III.1.7. *Let $f \colon X \to T$ be a holomorphic map between complex manifolds. Let $\mathcal{E}$ be a holomorphic vector bundle on T and fix any cohomology class $a \in H^1(T, \mathcal{E})$.*

There exists a natural isomorphism of affine bundles

$$Z_{f^*\mathcal{E}, f^*a} \cong f^* Z_{\mathcal{E},a} = Z_{\mathcal{E},a} \times_T X.$$

We will denote the induced map $Z_{f^*\mathcal{E}, f^*a} \to Z_{\mathcal{E},a}$ by Z_f.

Proof. Recall, that the class $a \in H^1(T, \mathcal{E}) = \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}_T, \mathcal{E})$ defines an extension

$$0 \to \mathcal{E} \to \mathcal{V}_a \xrightarrow{p} \mathcal{O}_T \to 0$$

and by definition, $Z_a = |p|^{-1}(T \times \{1\})$. Now, by the functoriality of the Ext-functor the class $f^*a \in H^1(X, f^*\mathcal{E}) = \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}_X, f^*\mathcal{E})$ defines the extension

$$0 \to f^*\mathcal{E} \to f^*\mathcal{V}_a \xrightarrow{f^*p} f^*\mathcal{O}_T = \mathcal{O}_X \to 0.$$

Here, the total spaces $|f^*\mathcal{E}|, |f^*\mathcal{V}_a|, |\mathcal{O}_X|$ are of course just given by

$$|f^*\mathcal{E}| = |\mathcal{E}| \times_T X, \quad |f^*\mathcal{V}_a| = |\mathcal{V}_a| \times_T X, \quad |f^*\mathcal{O}_T| = |\mathcal{O}_T| \times_T X$$

and $|f^*p|$ is just the map $|p| \times_T \mathrm{id}_X$. It follows that

$$\begin{aligned} Z_{f^*a} &:= |f^*p|^{-1}(X \times \{1\}) = |f^*p|^{-1}\Big((T \times \{1\}) \times_T X\Big) \\ &= \Big(|p|^{-1}(T \times \{1\})\Big) \times_T (\mathrm{id}_X^{-1}(X)) = Z_a \times_T X. \end{aligned}$$

□

Combining Proposition III.1.6 and Proposition III.1.7 we find another possible definition of the canonical extension:

Lemma III.1.8. (Universal property of extensions)
Let X be a complex manifold, let $\mathcal{E}$ be a holomorphic vector bundle on X and fix a cohomology class $a \in H^1(X, \mathcal{E})$. Then, $Z_a \xrightarrow{p} X$ enjoys the following universal property: Let Y be any complex manifold and let $h\colon Y \to X$ be any holomorphic map such that the cohomology class $h^*a = 0 \in H^1(Y, f^*\mathcal{E})$ vanishes. Then, h factors uniquely through $Z_a \xrightarrow{p} X$.

In this sense, $Z_a \to X$ is the universal manifold on which the cohomology class a vanishes.

Proof. By Proposition III.1.7 there exists a natural identification $Z_{f^*a} = Z_a \times_X Y$; we denote by $Z_f\colon Z_{f^*a} \to Z_a$ the natural induced holomorphic map. Now, since the class $f^*a = 0 \in H^1(Y, f^*\mathcal{E})$ vanishes we know by Proposition III.1.6 that the map $Z_{f^*a} \to Y$ admits a section s. Then, the composition $Z_f \circ s$ of s with $Z_f\colon Z_{f^*a} \to Z_a$ is a factorisation of $Y \to X$ via $Z_a \to X$ as required. It is unique because if conversely $h'\colon Y \to Z_a$ is any factorisation, then $s := (h', \mathrm{id}_Y)\colon Y \to Z_a \times_X Y = Z_{f^*a}$ is a section of the bundle $Z_{f^*a} \to Y$ such that $h' = Z_f \circ s$.

Note also that the final assertion is justified because $p^*a \in H^1(Z_a, p^*\mathcal{E})$ indeed vanishes: According to Proposition III.1.6 this is equivalent to the existence of a section to the bundle $Z_{p*a} = Z_a \times_X Z_a \to Z_a$. But the latter condition is satisfied as there exists for example the tautological section $(\mathrm{id}_{Z_a} \times_X \mathrm{id}_{Z_a})$. ☐

Corollary III.1.9. *Let X be a complex manifold, let $\mathcal{E}$ be a holomorphic vector bundle on X and fix a cohomology class $a \in H^1(X, \mathcal{E})$. Then, for any $\lambda \in \mathbb{C}^\times$ there exists a canonical isomorphism of affine bundles*

$$Z_a = Z_{\lambda \cdot a}.$$

Proof. Both bundles share the same universal property Lemma III.1.8, hence are canonically isomorphic. ☐

1.2 Canonical Extensions of Kähler Manifolds

Definition III.1.10. *Let (X, ω_X) be a complex Kähler manifold. Then, ω_X is a $\bar{\partial}$-closed form hence defines a cohomology class $[\omega_X] \in H^1(X, \Omega^1_X)$. The associated extension $Z_{[\omega_X]}$ is called (a)* canonical extension *of X. Alternatively, we also write $Z_{X,[\omega_X]}$ if we want to stress the dependence on X or simply Z_X if the dependence on $[\omega_X] \in H^1(X, \Omega^1_X)$ is not important in that situation.*

In the preceding subsection we have seen many equivalent constructions for $Z_{[\omega_X]}$:

(1) As a bundle of affine spaces over X (more precisely: as a Ω^1_X-torsor). The holomorphic structure may be described explicitly in terms of ω_X, see Eq. (III.1).
(2) As the complement $Z_{[\omega_X]} = \mathbb{P}(\mathcal{V}^*_{[\omega_X]}) \setminus \mathbb{P}(\mathcal{T}_X)$ of the smooth hypersurface $\mathbb{P}(\mathcal{T}_X)$ which is an element in the linear series of $\mathcal{O}_{\mathbb{P}(\mathcal{V}^*)}(1)$ and has normal bundle $\mathcal{N}_{\mathbb{P}(\mathcal{T}_X)/\mathbb{P}(\mathcal{V}^*)} = \mathcal{O}_{\mathbb{P}(\mathcal{T}_X)}(1)$ (this was part of Remark III.1.4).
(3) As the universal manifold on which the cohomology class $[\omega_X]$ vanishes.

In the following we are going to use all of these constructions interchangeably without mention.

In the paper [GW20] the authors posed the question whether the positivity of the tangent bundle of a compact Kähler manifold X is related to the Steiness/Affiness of its canonical extensions Z_X. A slightly naive but instructive motivation for this question is as follows: By item (2) above we may consider Z_X as the complement of the divisor $D := \mathbb{P}(\mathcal{T}_X)$ in the compact Kähler manifold $Y := \mathbb{P}(\mathcal{V}^*)$ with normal bundle $\mathcal{N}_{D/Y} = \mathcal{O}_{\mathbb{P}(\mathcal{T}_X)}(1)$. In particular, by definition $\mathcal{N}_{D/Y}$ is ample/big/nef if and only if so is $\mathcal{T}_X$.

Now, if D is an ample divisor in Y (i.e. if $\mathcal{O}_X(D)$ is ample), then $Y \setminus D$ is of course affine. Conversely, if $Y \setminus D$ does not contain any compact curve (e.g. if it is Stein), then certainly $D \cdot C > 0$ for any curve $C \not\subseteq D$ and so one might expect that D is positive in some sense (cf. Theorem I.2.13).

The relation between the positivity of a divisor and the geometry of its complement is a classical problem. For example it was conjectured by Goodman that if $Y \setminus D$ is Stein then D should be nef (see [Har70, Conjecture II.5.2.]). The subject is subtle however, and in fact Goodmans conjecture was disproved in [HP21]. Nevertheless, the authors of said paper supported the question posed by [GW20] and in fact posed it as a conjecture:

Conjecture III.1.11. (Greb-Wong, Höring-Peternell)
Let X be a compact Kähler manifold. Then, the tangent bundle $\mathcal{T}_X$ is nef (respectively big and nef) if and only if some canonical extension Z_X of X is Stein (respectively affine).

Remark III.1.12. According to a famous example of Serre (which may be found e.g. in [Har77, Example B.2.0.1]), *being affine* is not really a well-defined condition for complex manifolds because there exists a non-affine scheme whose analytification is biholomorphic to the analytification of an affine scheme. Thus, the question whether Z_X is affine in Conjecture III.1.11 above only makes sense when we additionally assume that X is projective. In this case X and, hence, $Z_X \subset \mathbb{P}(\mathcal{V}^*)$ may be defined in purely algebraic terms as smooth varieties over $\mathbb{C}$ and so it makes sense to ask if Z_X is affine (as an algebraic variety). In fact, conversely if $\mathcal{T}_X$ is big then X is necessarily projective (this is essentially a consequence of Remark I.2.25) so that the statement is consistent.

Example III.1.13. Let us provide some evidence for Conjecture III.1.11 by proving it in the case of curves. To this end, let $X = C$ be a compact curve and fix a Kähler form ω on C. We consider $Z_{[\omega]} := \mathbb{P}(\mathcal{V}^*_{[\omega]}) \setminus \mathbb{P}(\mathcal{T}_C)$, where

$$\mathcal{V}_{[\omega]} \in \operatorname{Ext}^1_{\mathcal{O}}(\mathcal{O}_C, \Omega^1_C) = H^1(C, \Omega^1_C) = H^0(C, \mathcal{O}_C) = \mathbb{C} \tag{III.3}$$

is the unique (up to scaling) non-trivial extension. Moreover, $\mathbb{P}(\mathcal{T}_C) \cong C$ is a section to $\mathbb{P}(\mathcal{V}^*) \to C$ of self-intersection

$$\Big(\mathbb{P}(\mathcal{T}_C)\Big)^2 = \deg\Big(\mathcal{N}_{\mathbb{P}(\mathcal{T}_C)/\mathbb{P}(\mathcal{V}^*)}\Big) = \deg\Big(\mathcal{O}_{\mathbb{P}(\mathcal{T}_C)}(1)\Big) = \deg(\mathcal{T}_C). \tag{III.4}$$

Let us start by dealing with the case $C = \mathbb{P}^1$: The tangent bundle of $\mathbb{P}^1$ is ample as follows from the Euler sequence $0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(2)} \to \mathcal{T}_{\mathbb{P}^1} \to 0$ and so we expect $Z_{\mathbb{P}^1}$ to be affine. In fact, the Euler sequence is the (up to scaling unique) extension class in $\operatorname{Ext}^1_{\mathcal{O}}(\mathcal{O}_C, \Omega^1_C)$ and so $Z_{\mathbb{P}^1}$ is the complement of a divisor in the linear series of

$$\mathcal{O}_{\mathbb{P}(\mathcal{O}(1)^{\oplus(2)})}(1) = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1) \otimes p^*\mathcal{O}(2)$$

in $\mathbb{P}^1 \times \mathbb{P}^1$. Clearly, this divisor is ample and so $Z_{\mathbb{P}^1}$ is affine. Explicitly, one may show that $Z_{\mathbb{P}^1}$ is the affine surface $\{x^2 + y^2 + z^2 = 1\}$ in $\mathbb{C}^3$.

If $C = E$ is an elliptic curve, then $\mathcal{V}_{[\omega]}$ is the (unique) non-trivial extension of $\mathcal{O}_E$ by itself and $\mathbb{P}(\mathcal{T}_E)$ is the (unique) section of self-intersection 0. Thus, Z_E is just Serre's example of a non-affine scheme whose analytification is Stein (see [Har70, Example VI.3.2] for more details). Alternatively, we will see in Corollary III.2.3 below, that Z_E is biholomorphic to $(\mathbb{C}^\times)^2$. Note that the observation that Z_E is Stein but not affine is compatible with the fact that $\mathcal{T}_E = \mathcal{O}_E$ is nef but not big.

Finally, if $g(C) \geq 2$ then Z_C is never Stein (as we expect since $\mathcal{T}_C$ is not nef). The reason is that in this case $\mathbb{P}(\mathcal{T}_C)$ is a curve of negative self-intersection by Eq. (III.4). In particular, by Grauert's contraction Theorem IV.1.3 there exists a bimeromorphic map $\pi\colon \mathbb{P}(\mathcal{V}^*) \to X'$ onto a normal surface X' which is an isomorphism over Z_C and contracts $\mathbb{P}(\mathcal{T}_C)$ to a single point $p' \in X'$. Thus, $Z_C \cong X' \setminus \{p'\}$ can not be Stein as by Hartogs theorem, any holomorphic function on Z_C would extend to a holomorphic function on the compact space X'. But then, any function on Z_C is constant.

Altogether, we record that Conjecture III.1.11 holds true at least for curves.

Below, let us quickly summarise what is known thus far about Conjecture III.1.11 in general:

- The conjecture is known to hold for (most) projective surfaces by [HP21, Theorem 1.13.]. More details on the cases left open will be provided in Section 3.
- If X is projective and any one canonical extension of X is affine, then the tangent bundle of X is big by [GW20, Corollary 4.4.]. Conversely, if $\mathcal{T}_X$ is nef and big, then all canonical extensions of X are affine. The latter result is due to [HP21, Theorem 1.2.] but we will also recall the argument in Corollary III.2.6. Thus, (at least modulo the nef case) the big case is settled.
- Building on the work of [GW20] and [HP21], in the following section we are going to prove that if the tangent bundle of X is nef (and if the weak form of the conjecture of Campana and Peternell Conjecture II.3.7 holds true), then the canonical extensions of X are always Stein.
- The remaining case is to prove that if a canonical extension of X is Stein, then the tangent bundle of X is nef. This problem is still almost completely open. In fact, arguably it is not even entirely clear if it is correct in this form. We will spent a few more words on this in Section 3.

Let us end this section by observing some basic consequences of the construction of Z_X. The first one may be interpreted as saying that canonical extensions can never be too far away from being Stein; it is due to [GW20, Proposition 2.7.].

Proposition III.1.14. *Let (X, ω_X) be a complex Kähler manifold. Then, the canonical extension $Z_{[\omega_X]}$ does not contain any compact complex subvarieties of positive dimension.*

Proof. Assume to the contrary that there exists a compact analytic subvariety $Y \subseteq Z_X$ of dimension $k > 0$. To keep the argument more elementary we will in the following assume that Y is a manifold. At the end of the proof we will indicate how to modify the argument to deal with the general case.

Denote by $p\colon Z_X \to X$ the projection. Then, for any $x \in X$ the set $Y \cap p^{-1}(x)$ is a compact subvariety of the Stein manifold $p^{-1}(x) = Z_X|_x \cong \mathbb{C}^n$ and, hence, a finite set of points. In other words, the map $p|_Y : Y \to p(Y)$ is finite, of degree d say. Fix a dense, Zariski open subset $U \subseteq Y$ such that $p|_Y$ is étale over U. Then, $\omega_X|_U$ is a Kähler form on and so $(\omega_X|_U)^k = k! \operatorname{vol}_U$ is a volume form on U according to Example IV.3.8. In particular, the number

$$\begin{aligned}\int_Y p^*\omega_X^k = \int_{p^{-1}(U)} p^*\omega_X^k = d \cdot \int_U \omega_X^k = d \cdot k! \cdot \operatorname{vol}(U) \\ = k! \cdot \operatorname{vol}(p^{-1}(U)) = k! \cdot \operatorname{vol}(Y) > 0 \qquad \text{(III.5)}\end{aligned}$$

is positive. Here, we use twice that $p^{-1}(U) \subseteq Y$ is dense and that p is étale over U. On the other hand, the cohomology class $[p^*\omega_X] = 0$ is trivial according to the universal property of Z_X Lemma III.1.8. Consequently, so is the cohomology class $[p^*\omega_X^k] = 0$ and so

$$\int_Y p^*\omega_X^k = 0,$$

by Stokes theorem, in contradiction to Eq. (III.5). This proves that Z_X can not contain compact submanifolds of positive dimension. In fact, *Lelong* introduced the concept of integration over analytic subvarieties and he showed that Stokes theorem is still valid in this setting. Thus, the general case may be treated ad verbatim in the same way. The concept of integration over singular varieties is related to the theory of currents, see [Dem12, Theorem III.2.7.] for an introduction to these ideas and Lelongs theorem. □

Finally, In view of Theorem II.4.14 the following result will be useful:

Proposition III.1.15. *Let $\pi\colon \widetilde{X} \to X$ be an étale cover between Kähler manifolds. Then, for any Kähler form ω_X on X there exists a natural isomorphism of affine bundles*

$$Z_{\widetilde{X},[\pi^*\omega_X]} \cong \pi^* Z_{X,[\omega_X]} := Z_{X,[\omega_X]} \times_X \widetilde{X}. \tag{III.6}$$

Moreover, if π is finite then $Z_{X,[\omega_X]}$ is Stein if and only if $Z_{\widetilde{X},[\pi^*\omega_X]}$ is so.

Proof. First of all, we have already proved in Proposition III.1.7 that

$$Z_{X,[\omega_X]} \times_X \widetilde{X} = \pi^* Z_{X,[\omega_X]} \cong Z_{\pi^*\mathcal{T}_X,[\pi^*\omega_X]}.$$

Since π is étale the natural morphism $d\pi\colon \mathcal{T}_{\widetilde{X}} \to \pi^*\mathcal{T}_X$ is an isomorphism. Thus, Eq. (III.6) is proved.

Regarding the second assertion, the identification

$$Z_{\widetilde{X},[\pi^*\omega_X]} \cong Z_{X,[\omega_X]} \times_X \widetilde{X}$$

which we just proved shows that together with $\pi\colon \widetilde{X} \to X$ also the holomorphic map $Z_\pi\colon Z_{\widetilde{X}} \to Z_X$ is a finite étale cover. But in general, if $\widetilde{Z} \to Z$ is a any finite map between complex manifolds, then Z is Stein if and only if $\widetilde{Z}$ is Stein (see for example [Nar62, Lemma 2.] for a rather short proof). □

2 Extensions of Kähler Manifolds with Nef Tangent Bundle

This section contains the main result of this chapter: (assuming Conjecture II.3.5) all canonical extensions of a compact Kähler manifold with nef tangent bundle are necessarily Stein manifolds. This result is new and partially answers Conjecture III.1.11. In view of our structure result Theorem II.4.14 for such manifolds, the proof will proceed by first dealing with the special cases of complex tori and Fano manifolds. Afterwards, we are going to deduce from this the result in the general case.

2.1 The Case of Complex Tori

Let $T = V/\Gamma$ be a complex torus and fix any Kähler form ω_T on T. In this subsection we want to prove that $Z_{[\omega_T]}$ is Stein. Following the approach in [GW20, Section 2.3.] this will be achieved by a direct calculation:

Fix a hermitean inner product h on a finite dimensional complex vector space $V \cong \mathbb{C}^q$. We may consider h as a (constant) smooth hermitean metric on the complex manifold V. Writing $h = (h_{k\ell})_{k\ell}$ in terms of some linear complex coordinates (z^k), the associated fundamental form is given by

$$\omega_h = \frac{i}{2} \sum_{k,\ell=1}^{q} h_{k\ell} \, dz^k \wedge d\bar{z}^\ell.$$

Then, clearly $\partial\omega_h = \bar{\partial}\omega_h = d\omega_h = 0$, i.e. h is a Kähler metric on the complex manifold V which is translation invariant by construction. In particular, h also induces a (constant) Kähler metric on any complex torus V/Γ. By abuse of notation we continue to denote the corresponding Kähler form by ω_h. We call ω_h a *constant* Kähler metric. The next result advises us that it suffices to compute canonical extensions of such constant metrics.

Proposition III.2.1. *Let V be a $\mathbb{C}$-vector space of dimension $\dim V = q$ and let $T = V/\Gamma$ be a complex torus. Then, any cohomology class of a Kähler metric on T contains the class of a (unique) constant Kähler metric.*

Proof. Fix linear coordinates $(z^k = x^k + iy^k)$ on V, choose any Kähler metric ω_T on T and consider its cohomology class $[\omega_T] \in H^1(T, \Omega_T^1)$. Below, we will find a hermitean inner product h on V such that $[\omega_T] = [\omega_h]$. Let us begin by recalling that the second cohomology group of a torus may be computed explicitly to be

$$\bigwedge\nolimits_{\mathbb{R}}^2 V^* \cong H^2(T, \mathbb{R}), \qquad \begin{aligned} dx^k \wedge dx^\ell &\mapsto [dx^k \wedge dx^\ell], \\ dx^k \wedge dy^\ell &\mapsto [dx^k \wedge dy^\ell], \\ dy^k \wedge dy^\ell &\mapsto [dy^k \wedge dy^\ell]. \end{aligned} \tag{III.7}$$

In particular, complexifying and decomposing according to type we find that there exists a canonical identification

$$\bigwedge^{(1,0)} V^* \otimes \bigwedge^{(0,1)} V^* \cong H^1(T, \Omega^1_T), \quad \frac{i}{2} \sum_{k,\ell} a_{k\ell}\, dz^k \otimes d\bar{z}^\ell \mapsto \left[\frac{i}{2} \sum_{k,\ell} a_{k\ell}\, dz^k \wedge d\bar{z}^\ell\right].$$

In other words, any cohomology class contains a (unique) constant differential form representing it. In particular, there exists a matrix $H = (h_{k\ell})$ such that

$$[\omega_T] = \left[\frac{i}{2} \sum_{k,\ell} h_{k\ell}\, dz^k \wedge d\bar{z}^\ell\right] \in H^1(T, \Omega^1_T)$$

and it remains to prove that $\omega_h := \frac{i}{2} \sum h_{k\ell}\, dz^k \wedge d\bar{z}^\ell$ is in fact a Kähler form or, equivalently, that H is a hermitean positive definite matrix. To this end, note that H is at least hermitean symmetric since ω_h is real of type $(1,1)$ by construction. In the following we will use an ad hoc argument to deduce that H is positive definite.

Step 1: H is positive semi definite, i.e. $(-i) \cdot \omega_h(v, \bar{v}) \geq 0$ for all $v \in V$.

Fix a non-zero vector $v \in V$. We need to prove that $(-i)\omega_h(v, \bar{v}) \geq 0$. Since both this assertion and the isomorphism Eq. (III.7) are coordinate-independent, we may adapt our choice of (z^k) in such a way that $v = e_1$ is the first basis vector. Then, to prove that H is positive semi definite we only need to verify that

$$(-i) \cdot \omega_h(e_1, e_1) = h_{11} \geq 0. \tag{III.8}$$

To this end, for any $\varepsilon > 0$ we consider the constant Kähler metrics

$$\omega_j := \omega_j(\varepsilon) := \frac{i}{2} \left(dz^j \wedge d\bar{z}^j + \varepsilon \sum_{k \neq j} dz^k \wedge d\bar{z}^k\right), \quad j = 1, \ldots, q = \dim V.$$

on T. Since products of positive form are positive by Proposition IV.3.6 also the form

$$\omega_T \wedge \omega_2 \wedge \cdots \wedge \omega_q \in \mathcal{A}^{q,q}(T)$$

is positive for any $\varepsilon > 0$, i.e. it is a volume form on T. In follows that

$$\begin{aligned}
0 &\leq \int_T \omega_T \wedge \omega_2 \wedge \cdots \wedge \omega_q \\
&\overset{\text{Stokes}}{=\!=\!=} \int_T \omega_h \wedge \omega_2 \wedge \cdots \wedge \omega_q \\
&=\!=\!= \left(\frac{i}{2}\right)^q \left(\int_T h_{11}\, dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^q \wedge d\bar{z}^q + \varepsilon \int_T \ldots\right) \\
&\longrightarrow \left(\frac{i}{2}\right)^q \int_T h_{11}\, dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^q \wedge d\bar{z}^q \\
&= h_{11} \int_T dx^1 \wedge dy^1 \wedge \cdots \wedge dx^q \wedge dy^q = h_{11} \cdot \mathrm{vol}_{\mathrm{std}}(T), \qquad \text{as } \varepsilon \to 0.
\end{aligned}$$

This proves Eq. (III.8) above (i.e. that $h_{11} \geq 0$) and concludes our proof that H is positive semi definite.

Step 2: H is positive definite, i.e. ω_h is a Kähler metric.

Indeed, we have already seen that H is positive semi definite. To prove that H is positive definite it thus suffices to show that $\det(H) > 0$ and indeed we compute

$$\begin{aligned}
0 < q! \cdot \mathrm{vol}_{\omega_T}(T) &\overset{\text{Example IV.3.8}}{=\!=\!=\!=\!=} \int_T \omega_T^q \\
&\overset{\text{Stokes}}{=\!=\!=\!=\!=} \int_T \omega_h^q \\
&=\!=\!=\!=\!= \int_T \left(\frac{i}{2} \sum_{k,\ell} h_{k\ell}\, dz^k \wedge d\bar{z}^\ell\right)^q \\
&=\!=\!=\!=\!= \det(H) \int_T \left(\frac{i}{2} \sum_{k,\ell} dz^k \wedge d\bar{z}^\ell\right)^q \\
&=\!=\!=\!=\!= \det(H) \cdot q! \cdot \mathrm{vol}_{\mathrm{std}}(T).
\end{aligned}$$

Thus, $\det(H) > 0$ and we are done. □

Fix an inner product h on $\mathbb{C}^q$. In the following, we will compute explicitly the canonical extension

$$Z_{[\omega_h]} \to \mathbb{C}^q$$

of $\mathbb{C}^q$ with respect to ω_h. Indeed, $Z_{\mathbb{C}^q}$ is abstractly isomorphic as an affine bundle to $|\Omega^1_{\mathbb{C}^q}| \cong \mathbb{C}^{2q}$ by Proposition III.1.6 since all higher cohomology groups on $\mathbb{C}^q$ vanish but for our work on tori later on we are going to require a more explicit description of this isomorphism. To this end, recall that the class $[\omega_h]$ defines an extension of vector bundles

$$0 \to \Omega^1_{\mathbb{C}^q} \to \mathcal{V}_{[\omega_h]} \xrightarrow{p} \mathcal{O}_{\mathbb{C}^q} \to 0.$$

and a holomorphic section w to $\mathcal{V}_{\omega_h}$ is a tuple (η, f), where f is a holomorphic function on $\mathbb{C}^q$ and η is a differentiable 1-form satisfying $\bar{\partial}\eta = f \cdot \omega_h$ (cf. Eq. (III.1)). In particular, the $(q+1)$ tuples

$$(dz^k, 0), \quad \forall\, k = 1, \dots q, \qquad \left(\frac{i}{2}\sum_{k=1}^{q} \bar{z}^k\, dz^k, 1\right)$$

define global holomorphic sections to $\mathcal{V}_{[\omega_h]}$. Note that clearly they in fact form a global holomorphic frame. Thus, the map

$$\mathbb{C}^{2q+1} = \mathbb{C}^q \times \mathbb{C}^q \times \mathbb{C} \to \left|\mathcal{V}_{[\omega_h]}\right|, \quad (z, w, y) \mapsto \left(z,\ \sum_{k=1}^{q}\left(w_k + \frac{i}{2}\, y\bar{z}^k\right) dz^k, y\right)$$

define global holomorphic coordinates on (the total space of) $V_{[\omega_h]}$. Now, by definition $\mathcal{Z}_{[\omega_h]} \subset \mathcal{V}_{[\omega_h]}$ consists in those sections mapped under p to the constant function 1. In particular, we see that global holomorphic coordinates on $Z_{[\omega_h]}$ are given by

$$\psi\colon \mathbb{C}^{2q} \to Z_{[\omega_h]}, \quad (z, w) \mapsto \left(z,\ \sum_{k=1}^{q}\left(w_k + \frac{i}{2}\, \bar{z}^k\right) dz^k,\ 1\right). \tag{III.9}$$

Proposition III.2.2. (Greb-Wong, [GW20, Proposition 2.13.])
Let $T = \mathbb{C}^q/\Gamma$ be a complex torus and let ω_h be a constant Kähler metric on T. Then, the canonical extension $Z_{[\omega_h]}$ is a Stein manifold.

Proof. Below, we will explicitly compute $Z_{T,[\omega_h]}$ in terms of the coordinates ψ defined by Eq. (III.9): We start by considering the universal cover $\pi\colon \mathbb{C}^q \to T$ of T. According to Proposition III.1.15 there exists a natural isomorphism of affine bundles

$$Z_{\mathbb{C}^q,[\omega_h]} \cong Z_{T,[\omega_h]} \times_T \mathbb{C}^q.$$

In particular, $\pi_1(T) = \Gamma$ acts on the extension $Z_{\mathbb{C}^q,[\omega_h]}$ and the quotient may be identified with $Z_{T,[\omega_h]}$.

Step 1: Determination of the action of $\pi_1(T)$ on $Z_{\mathbb{C}^q,[\omega_h]}$ in terms of the coordinates Eq. (III.9).

By construction, the action of an element $\gamma \in \pi_1(T) = \Gamma$ on sections s to $Z_{\mathbb{C}^q,[\omega_h]}$ is given by the formula $(\gamma \cdot s)(z) := s(z+\gamma)$. In other words,

$$\gamma \cdot \left(z,\ \sum_{k=1}^{q} \left(w_k + \frac{i}{2}\,\bar{z}^k\right) dz^k,\ 1\right) = \left(z+\gamma,\ \sum_{k=1}^{q} \left(w_k + \frac{i}{2}\,\bar{z}^k\right) dz^k,\ 1\right) \qquad \forall\, \gamma \in \Gamma.$$

Spelling this out in terms of ψ we find that

$$\gamma \cdot \psi(z,w) = \psi\left(z+\gamma, w - \frac{i}{2}\,\bar{\gamma}\right).$$

Altogether, the canonical extension $Z_{T,[\omega_h]}$ of T is biholomorphic via ψ to the quotient of $\mathbb{C}^{2q}$ by the group

$$\widehat{\Gamma} := \left\{ \left(\gamma, -\frac{i}{2}\,\bar{\gamma}\right) \,\Big|\, \gamma \in \Gamma \right\}$$

which acts on $\mathbb{C}^{2q}$ via translations. In the following, we will prove that this quotient is Stein. To this end, fix a $\mathbb{Z}$-basis $\gamma_1, \ldots, \gamma_{2q}$ for Γ. We will denote

$$\widehat{\gamma}_j = \left(\gamma_j, -\frac{i}{2}\,\overline{(\gamma_j)}\right), \quad j = 1, \ldots 2q.$$

Then, clearly $\widehat{\gamma}_1, \ldots, \widehat{\gamma}_{2q}$ generate $\widehat{\Gamma}$.

Step 2: The vectors $\widehat{\gamma}_1, \ldots, \widehat{\gamma}_{2q}$ form a $\mathbb{C}$-basis for $\mathbb{C}^{2q}$.

Consider the matrix

$$J := \begin{pmatrix} \mathrm{id}_{\mathbb{C}^q} & 0 \\ 0 & \frac{i}{2}\,\mathrm{id}_{\mathbb{C}^q} \end{pmatrix} \in \mathrm{GL}_{2q}(\mathbb{C}).$$

Denote $\tilde{\gamma}_j := (\gamma_j, \overline{\gamma_j})$. Then, $J\tilde{\gamma}_j = \widehat{\gamma}_j$. In particular, it suffices to show that $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{2q}$ is a $\mathbb{C}$-basis for $\mathbb{C}^{2q}$ (or equivalently that they are $\mathbb{C}$-linearly independent). But indeed, this immediately follows from the fact that by construction the elements $\gamma_1, \ldots, \gamma_{2q}$ are an $\mathbb{R}$-basis of $\mathbb{C}^q$ (since they form a $\mathbb{Z}$-basis of the lattice $\Gamma \subset \mathbb{C}^q$): Suppose that there exist complex numbers a_j such that

$$\sum a_j \tilde{\gamma}_j = \sum a_j \left(\gamma_j, \overline{(\gamma_j)}\right) = 0. \tag{III.10}$$

Then, $\sum a_j\gamma_j = \sum a_j\overline{\gamma_j} = 0$. The latter equation implies that also $\sum \overline{a_j}\gamma_j = 0$. But then, also

$$\sum a_j\gamma_j + \sum \overline{a_j}\gamma_j = 2\sum \mathrm{Re}(a_j)\gamma_j = 0.$$

Since the γ_j are $\mathbb{R}$-linearly independent, it follows that $\mathrm{Re}(a_j) = 0$ for all j. Using Eq. (III.10) again we find

$$0 = \sum a_j \gamma_j = i \left(\sum \mathrm{Im}(a_j)\gamma_j \right).$$

It follows that also $\mathrm{Im}(a_j) = 0$ for all j and we conclude that $\widehat{\gamma}_1, \dots, \widehat{\gamma}_{2q}$ is a $\mathbb{C}$-basis for $\mathbb{C}^{2q}$.

Step 3: The complex manifold $Z_{T,[\omega_h]}$ is Stein.

From *Step 1* we know that $Z_{T,[\omega_h]}$ is biholomorphic to the quotient $\mathbb{C}^{2q}/\widehat{\Gamma}$, where $\widehat{\Gamma} \cong \mathbb{Z}^{2q}$ acts on $\mathbb{C}^{2q}$ by translations. According to *Step 2* we know that the $\mathbb{Z}$-basis $\widehat{\gamma}_1, \dots, \widehat{\gamma}_{2q}$ for $\widehat{\Gamma}$ is also a $\mathbb{C}$-basis for $\mathbb{C}^{2q}$. Thus, in terms of the complex coordinates defined by the basis $\widehat{\gamma}_1, \dots, \widehat{\gamma}_{2q}$ the manifold $Z_{T,[\omega_h]}$ is just the quotient

$$Z_{T,[\omega_h]} \cong (\mathbb{C}/\mathbb{Z})^{2q}.$$

where in each factor $\mathbb{Z} \subset \mathbb{C}$ acts by translations. But in this case, the exponential function

$$e^{2\pi i -} \colon \mathbb{C}/\mathbb{Z} \to \mathbb{C}^{\times}, \quad z \mapsto e^{2\pi i z}$$

defines a biholomorphism $\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^{\times}$. Altogether, we see that $Z_{T,[\omega_h]} \cong (\mathbb{C}^{\times})^{2q}$ is Stein. ☐

Corollary III.2.3. *Let $T = \mathbb{C}^q/\Gamma$ be a complex torus. Fix any Kähler form ω_T on T. Then the canonical extension of T with respect to ω_T is a Stein manifold. In fact,*

$$Z_{T,[\omega_T]} \cong (\mathbb{C}^{\times})^{2q}.$$

Proof. According to Proposition III.2.1 there exists a constant Kähler metric ω_h in the same cohomology class as ω_T. In particular,

$$Z_{[\omega_T]} = Z_{[\omega_h]} \cong (\mathbb{C}^{\times})^{2q}$$

is Stein according to the preceding result Proposition III.2.2. ☐

2.2 The Case of Homogeneous Fanos

In this subsection we consider canonical extensions of Fano manifolds with nef tangent bundle. In particular, we will prove that (assuming the weak Campana-Peternell conjecture Conjecture II.3.7) these are always affine. We also complete the proof of Lemma II.3.6. The main results in this subsection are due to [HP21] and we will follow them for all proofs.

We will need the following two auxiliary results:

Proposition III.2.4. *Let (F, ω_F) be a smooth projective manifold. Fix a Kähler form ω_F on F and let*

$$Z_{[\omega_F]} = \mathbb{P}\left(\mathcal{V}^*_{[\omega_F]}\right) \setminus \mathbb{P}(\mathcal{T}_F)$$

be the corresponding canonical extension. Then,

(1) the tangent bundle $\mathcal{T}_F$ of F is a nef bundle if and only if $\mathbb{P}(\mathcal{T}_F)$ is a nef divisor in $\mathbb{P}(\mathcal{V}^*_{[\omega_F]})$ (equivalently, the bundle $\mathcal{O}_{\mathbb{P}(\mathcal{V}^*)}(1)$ is nef).

(2) The tangent bundle $\mathcal{T}_F$ of X is big and nef if and only if $\mathbb{P}(\mathcal{T}_F)$ is a big and nef divisor in $\mathbb{P}(\mathcal{V}^*_{[\omega_F]})$ (equivalently, the bundle $\mathcal{O}_{\mathbb{P}(\mathcal{V}^*)}(1)$ is big and nef).

If moreover it holds that $H^1(F, \mathbb{C}) = 0$, then

(3) The tangent bundle $\mathcal{T}_F$ of X is generated by global sections if and only if the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{V}^*)}(\mathbb{P}(\mathcal{T}_F))$ is so.

Proof. Let us denote $Y := \mathbb{P}(\mathcal{V}^*_{[\omega_F]})$, $D := \mathbb{P}(\mathcal{T}_F)$ and $n := \dim Y$. Recall, that by the discussion at the beginning of Section 1.2 D is a smooth hypersurface in Y and

$$\mathcal{O}_Y(D)|_D =: \mathcal{O}_D(D) = \mathcal{N}_{D/X} = \mathcal{O}_{\mathbb{P}(\mathcal{T}_F)}(1).$$

In particular, $\mathcal{N}_{D/Y}$ is nef/ big/ globally generated if and only if $\mathcal{T}_F$ is so.

Thus, item (1) immediately follows from the fact that a divisor is nef if and only if its normal bundle is so (this was proved in Corollary I.2.17).

Regarding item (2) we note that $D^n = (D|_D)^{n-1}$. Since a nef divisor is big if and only if its top self-intersection number is positive according to Theorem I.2.28 it follows that D is nef and big if and only if $D|_D$ is nef and big.

Finally, let us prove item (3). First, if $\mathcal{O}_Y(D)$ is generated by global sections then clearly so is $\mathcal{O}_Y(D)|_D$. Conversely, assume that $\mathcal{O}_Y(D)|_D$ is globally generated and choose a point $x \in Y$. We need to show, that there exists a global section of $\mathcal{O}_Y(D)$ which does not vanish at x. Note that by definition of $\mathcal{O}_Y(D)$ there exists a global section which vanishes precisely on D. This settles the case $x \in Y \setminus D$ and it remains to consider the case $x \in D$. To this end, consider the short exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(D) \to \mathcal{O}_D(D) \to 0. \tag{III.11}$$

We claim that together with $H^1(F, \mathbb{C})$ also $H^1(Y, \mathbb{C}) = 0$ vanishes. Grant this for the moment. Then, according to Hodge theory also $H^1(Y, \mathcal{O}_Y) = 0$. In particular, from the long exact sequence in cohomology associated to the short exact sequence Eq. (III.11) it follows that the natural map

$$H^0(Y, \mathcal{O}_Y(D)) \twoheadrightarrow H^0(D, \mathcal{O}_D(D)) \tag{III.12}$$

is surjective. Since $\mathcal{O}_D(D)$ is generated by global sections by assumption, it follows that there exists a global section of $\mathcal{O}_D(D)$ which does not vanish at x. By surjectivity of Eq. (III.12), this section is the restriction of a global section of $\mathcal{O}_Y(D)$. It follows that the global sections of the sheaf $\mathcal{O}_Y(D)$ generate its stalks for all points $x \in D$ as well.

It remains to prove that $H^1(Y, \mathbb{C}) = 0$. There are several ways to see this; e.g. one may use that $Y \to F$ is a fibre bundle with fibre $\mathbb{P}^n$ so that $H^1(F, \mathbb{C}) = H^1(\mathbb{P}^n, \mathbb{C}) = 0$. Then, the vanishing $H^1(Y, \mathbb{C}) = 0$ immediately follows from the *Serre spectral sequence.* □

Lemma III.2.5. *Let Y be a smooth projective manifold, let $D \subset Y$ be a divisor and assume that the line bundle $\mathcal{O}_Y(D)$ is semi ample. If $Y \setminus D$ does not contain any compact sub varieties of positive dimension, then $Y \setminus D$ is affine. Moreover, in this case $\mathcal{O}_Y(D)$ is a big line bundle.*

Proof. Choose a positive integer $m > 0$ such that the natural rational map

$$\phi_m \colon Y \to \mathbb{P}H^0(Y, \mathcal{O}_Y(mD)) =: \mathbb{P}$$

is holomorphic. Put $Y' := \operatorname{Im} \phi_m$ and let $\tau \in H^0(Y, \mathcal{O}_Y(mD))$ be the section defining the divisor mD. By construction of ϕ_m, the pull back map

$$\phi_m^* \colon H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \to H^0(Y, \mathcal{O}_Y(mD))$$

is an isomorphism. Consequently, we may consider τ as a section of $\mathcal{O}_{\mathbb{P}}(1)$ defining a hypersurface $H \subset \mathbb{P}$. Then, $mD = \phi_m^{-1}(H)$ as complex analytic spaces. It follows, that $D = \phi_m^{-1}(Y' \cap H)$ in the set-theoretic sense so that in particular $H \cap Y' \subset Y'$ is a strict sub variety.

Now, fix any point $y \in Y' \setminus (Y' \cap H)$. Then, the fibre $\phi_m^{-1}(y)$ is a closed, hence compact sub variety of Y which is contained in $Y \setminus D = Y \setminus \phi_m^{-1}(H)$. Since $Y \setminus D$ does not contain any compact sub varieties of positive dimension by assumption, it follows that $\phi_m^{-1}(y)$ is a finite set. Thus, $\phi_m\colon Y\setminus D \to Y'\setminus(Y'\cap H)$ is a finite map (i.e. ϕ_m is generically finite). Since $Y'\setminus(Y'\cap H)\subset \mathbb{P}\setminus H$ is affine as a closed subset of an affine variety, it follows that also $Y\setminus D$ is affine. Moreover, $\mathcal{O}_Y(D)=\phi_m^*\mathcal{O}_{\mathbb{P}}(1)$ is big as the pull back of an ample bundle under a generically finite map by Proposition I.2.26.□

Corollary III.2.6. (Höring-Peternell, [HP21, Theorem 1.2.])
Let F be a Fano manifold with big and nef tangent bundle. Then, any canonical extension of F is affine and, hence, Stein.

Proof. Fix any Kähler form ω on F and let $Z_F=\mathbb{P}(\mathcal{V}_\omega^*)\setminus\mathbb{P}(\mathcal{T}_F)$ be the corresponding canonical extension. Denote $m:=\dim F$ and let $\mathbb{P}(\mathcal{V}_\omega^*)\xrightarrow{\pi} X$ be the projection. Since $\mathcal{T}_F$ is big and nef by assumption it follows from Proposition III.2.4 that the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{V}_\omega^*)}(1)$ is big and nef. Note that

$$\begin{aligned}\mathcal{O}_{\mathbb{P}(\mathcal{V}_\omega^*)}(K_{\mathbb{P}(\mathcal{V}_\omega^*)}) &= \mathcal{O}_{\mathbb{P}(\mathcal{V}_\omega)}(-(m+1))\otimes\pi^*\Big(\mathcal{O}_F(K_F)\otimes\det\Big(\big(\mathcal{V}_{[\omega]}\big)^*\Big)\Big)\\ &= \mathcal{O}_{\mathbb{P}(\mathcal{V}_\omega)}(-(m+1))\otimes\pi^*\Big(\mathcal{O}_F(K_F)\otimes\det(\mathcal{T}_F)\otimes\det(\mathcal{O}_F)\Big)\\ &= \mathcal{O}_{\mathbb{P}(\mathcal{V}_\omega)}(-(m+1)).\end{aligned}$$

In summary, we see that both the line bundles $\mathcal{O}_{\mathbb{P}(\mathcal{V}^*)}(1)$ and also

$$\mathcal{O}(-K_{\mathbb{P}(\mathcal{V}^*)})\otimes\mathcal{O}_{\mathbb{P}(\mathcal{V}^*)}(1)=\mathcal{O}_{\mathbb{P}(\mathcal{V}^*)}(m+2)$$

are big and nef. Hence, the base-point free theorem Theorem IV.1.7 applies to show that

$$\mathcal{O}_{\mathbb{P}(\mathcal{V}^*)}(1)=\mathcal{O}_{\mathbb{P}(\mathcal{V}^*)}(\mathbb{P}(\mathcal{T}_F))$$

is semi ample. Now, since by Proposition III.1.14 we know that the canonical extension $Z_F=\mathbb{P}((\mathcal{V}_{\omega_F})^*)\setminus\mathbb{P}(\mathcal{T}_F)$ does not contain any compact subvarieties of positive dimension it follows from the preceding result Lemma III.2.5 that Z_F is an affine manifold. This is what we wanted to prove. □

Remark III.2.7. Assuming the Campana-Peternell conjecture another proof of Corollary III.2.6 is given in [GW20, Proposition 2.23.]: Therein, the authors provide an explicit description of the canonical extension of a homogeneous Fano manifold F from which it follows that Z_F is affine. For concreteness, let us only make this explicit in case $F=\mathbb{P}^n$. To this end, let us abbreviate $G:=\mathrm{PGL}_n=\mathrm{Aut}(\mathbb{P}^n)$. Then, we may identify $\mathbb{P}^n=G/P$, where $P:=\{A\in G|Ae_1=\lambda e_1\}$. Let $L=(\mathbb{C}^\times\times\mathrm{GL}_n)/\mathbb{C}^\times\subset P$ be the subgroup of block diagonal matrices (note that L is a Levi subgroup of P). Then, the bundle $Z_{\mathbb{P}^n}\to\mathbb{P}^n$ may naturally be identified with $G/L\to G/P$.

Let us complete this sections by completing the proof of Lemma II.3.6 which was left open in Chapter II. Section 3:

Corollary III.2.8. *Let F be a Fano manifold with nef tangent bundle. If the tangent bundle $\mathcal{T}_F$ of F is generated by global sections, then it is also big.*

Proof. Fix any Kähler form ω on F and let $Z_F = \mathbb{P}(\mathcal{V}_\omega^*) \setminus \mathbb{P}(\mathcal{T}_F)$ be the corresponding canonical extension. Recall from Example II.2.8 that $H^1(F, \mathcal{O}_F) = 0$ for any Fano manifold. In particular, also $H^1(F, \mathbb{C}) = 0$ by Hodge theory. Since $\mathcal{T}_F$ was assumed to be globally generated we see from Proposition III.2.4 that the line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{V}_\omega^*)}(\mathbb{P}(\mathcal{T}_F))$ is generated by global sections as well. Since $Z_F = \mathbb{P}(\mathcal{V}_\omega^*) \setminus \mathbb{P}(\mathcal{T}_F)$ does not contain compact subvarieties of positive dimension as is true for any canonical extension (see Proposition III.1.14) we again see that Lemma III.2.5 above applies. In particular, Z_F is affine and $\mathcal{O}_{\mathbb{P}(\mathcal{V}_\omega)^*}(\mathbb{P}(\mathcal{T}_F))$ is big. Since this bundle is also generated by global sections (and hence in particular is nef according to Corollary I.2.16) it follows immediately from Proposition III.2.4 that $\mathcal{T}_F$ is big. ☐

2.3 The general Case

Let X be a compact Kähler manifold X with nef tangent bundle. According to Theorem II.4.14 there exists a finite étale cover $\widetilde{X}$ of X whose Albanese map $\alpha\colon \widetilde{X} \to \mathrm{Alb}(\widetilde{X}) =: T$ is a flat fibre bundle. Moreover, the typical fibre F of α is (assuming the weak Campana-Peternell conjecture) a Fano manifold with big and nef tangent bundle. In this subsection we are going to prove that any canonical extension Z_X of X may be viewed in a natural way as a fibre bundle over a canonical extension Z_T of T and with fibre Z_F a canonical extension of F. This will immediately imply that all canonical extensions of X are Stein, thus partially confirming Conjecture III.1.11.

To explain the existence of the fibre bundle structure on Z_X we need the following technical result which may be found in [HP21, Lemma 5.5]:

Proposition III.2.9. *Let (X, ω_X) be a Kähler manifold. Suppose that there exists a decomposition $\mathcal{T}_X = \mathcal{E} \oplus \mathcal{F}$ into holomorphic sub bundles. Let $[\omega_X] = [\omega_\mathcal{E}] + [\omega_\mathcal{F}]$ be the induced decomposition in*

$$\mathrm{Ext}^1_\mathcal{O}(\mathcal{O}_X, \Omega^1_X) = \mathrm{Ext}^1_\mathcal{O}(\mathcal{O}_X, \mathcal{E}^*) \oplus \mathrm{Ext}^1_\mathcal{O}(\mathcal{O}_X, \mathcal{F}^*).$$

Then, there exists a natural isomorphism of affine bundles over X

$$Z_{[\omega_X]} \cong Z_{[\omega_\mathcal{E}]} \times_X Z_{[\omega_\mathcal{F}]}.$$

Proof. The lemma is essentially just a consequence of the definition of the additive structure on the Ext-functor which we will presuppose in the following: Let us denote by $\mathcal{V}_\mathcal{E}, \mathcal{V}_\mathcal{F}$ the extensions defined by the classes $[\omega_\mathcal{E}] \in \mathrm{Ext}^1_\mathcal{O}(\mathcal{O}_X, \mathcal{E}^*)$, respectively $[\omega_\mathcal{F}] \in \mathrm{Ext}^1_\mathcal{O}(\mathcal{O}_X, \mathcal{F}^*)$. Then, by the construction of the identification $\mathrm{Ext}^1_\mathcal{O}(\mathcal{O}_X, \Omega^1_X) = \mathrm{Ext}^1_\mathcal{O}(\mathcal{O}_X, \mathcal{E}^*) \oplus \mathrm{Ext}^1_\mathcal{O}(\mathcal{O}_X, \mathcal{F}^*)$ one can show that $\mathcal{V}_\mathcal{E}, \mathcal{V}_\mathcal{F}$ are related with $\mathcal{V}_X$ by means of the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & \Omega^1_X & = & \mathcal{E}^* \oplus \mathcal{F}^* & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{V}_X & \longrightarrow & \mathcal{V}_{\mathcal{E}} \oplus \mathcal{V}_{\mathcal{F}} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
\| & & \downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle p_{\mathcal{E}} \oplus p_{\mathcal{E}}} & & \| & & \| \\
0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{(1,1)} & \mathcal{O}_X \oplus \mathcal{O}_X & \xrightarrow{(1,-1)} & \mathcal{O}_X & \longrightarrow & 0
\end{array}$$

In particular, chasing through the diagram we find that the image of

$$\mathcal{Z}_X = p_X^{-1}(1) \subset \mathcal{V}_X \hookrightarrow \mathcal{V}_{\mathcal{E}} \oplus \mathcal{V}_{\mathcal{F}}$$

may be identified with

$$\mathcal{Z}_{\mathcal{E}} \oplus \mathcal{Z}_{\mathcal{F}} = p_{\mathcal{E}}^{-1}(1) \oplus p_{\mathcal{F}}^{-1}(1)$$

In other words, $Z_X = Z_{\mathcal{E}} \times_X Z_{\mathcal{F}}$. □

Corollary III.2.10. *Let (X, ω_X) be a compact Kähler manifold with nef tangent bundle. Assume that X is of maximal irregularity so that the Albanese morphism $\alpha \colon X \to \mathrm{Alb}(X) =: T$ is a flat holomorphic fibre bundle. Denote by F the typical fibre. Then, there exists a natural isomorphism of affine bundles*

$$Z_{\mathcal{T}_X, [\omega_X]} \cong Z_{\mathcal{T}_{X/T}, [\omega_{X/T}]} \times_X Z_{\alpha^* \mathcal{T}_T}.$$

Here, by $[\omega_{X/T}]$ we denote the image of $[\omega_X]$ under the natural homomorphism

$$H^1(X, \Omega^1_X) \to H^1(X, \Omega^1_{X/T}).$$

Remark III.2.11. Within the statement of Corollary III.2.10 above, we leave the extension class that $Z_{\alpha^* \mathcal{T}_T}$ is build from ambiguous on purpose. Indeed, the proof below will implicitly determined this class but the given description is not all that useful for us. Our next order of buissnes will thus be to have a closer look at this class and also give a more explicit description of it.

Proof. (of Corollary III.2.10)
Since α is a flat bundle the natural short exact sequence

$$0 \to \mathcal{T}_{X/T} \to \mathcal{T}_X \to \alpha^* \mathcal{T}_T \to 0$$

admits a global holomorphic splitting (we may even assume that $\alpha^*\mathcal{T}_T \subseteq \mathcal{T}_X$ is integrable; compare the discussion after Definition II.4.1). Hence,

$$Z_{\mathcal{T}_X,[\omega_X]} \cong Z_{\mathcal{T}_{X/T}} \times_X Z_{\alpha^*\mathcal{T}_T}.$$

according to Proposition III.2.9 above. Here, the class defining the affine bundle $Z_{\mathcal{T}_{X/T}}$ is the image of $[\omega_X]$ under the induced map

$$\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}_X, \Omega^1_X) \to \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}_X, \Omega^1_{X/T}).$$

Modulo the identification

$$\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}_X, -) = H^1(X, -)$$

this is the proclaimed class. □

As explained in Remark III.2.11 our next goal is to give an explicit description of the cohomology class defining the extension $Z_{\alpha^*\mathcal{T}_T}$ in Corollary III.2.10 above. To this end, we will require some auxiliary results.

Proposition III.2.12. *Let $f\colon X \to T$ be a holomorphic submersion of relative dimension m between compact Kähler manifolds. Let us denote by F_t the fibres of f and fix a Kähler form ω_X on X. Then, the function*

$$\mathrm{vol}(F_t, \omega_X|_{F_t}) := \frac{1}{m!}\int_{F_t} (\omega_X|_{F_t})^m$$

is constant (i.e. does not depend on t).

Proof. Note that by definition

$$\mathrm{vol}(F_t, \omega_X|_{F_t}) = \frac{1}{m!}\, f_*\left(\omega_X^m\right)\Big|_t.$$

In particular, according to the basic properties of f_* Proposition II.4.4 since ω_X is d-closed (X being Kähler) also the function $t \mapsto \mathrm{vol}(F_t)$ is d-closed, i.e. constant. □

Corollary III.2.13. *Let $f\colon X \to T$ be a holomorphic submersion between compact Kähler manifolds. Suppose that every fibre F_t of f is Fano and denote $m := \dim F_t$. Fix a Kähler form ω_X on X and recall that by Proposition III.2.12 the volume $\mathrm{vol}(F_t)$ of any fibre is the same. Then, the composition*

$$P\colon H^q(X, f^*\Omega^p_T) \xrightarrow{i_*} H^q(X, \Omega^p_X) \xrightarrow{\wedge\frac{\omega^m}{m!}} H^{q+m}(X, \Omega^{p+m}_X) \xrightarrow{f_*} H^q(T, \Omega^p_T)$$

is an isomorphism for all p, q. In fact, the inverse is given (up to a factor of $\frac{1}{\mathrm{vol}(F)}$) by the natural map

$$f^*\colon H^q(T, \Omega^p_T) \to H^q(X, f^*\Omega^p_T).$$

Proof. First, let us prove that $P \circ f^* = \mathrm{vol}(F) \cdot \mathrm{id}$ using Dolbeaut representatives: Fix any integers p, q and any closed differentiable (p,q)-form η on T. Using the properties of the push forward we compute

$$\begin{aligned} P(f^*([\eta])) &=\!=\!=: \frac{1}{m!}\left[f_*\Big(f^*\eta \wedge \omega_X^m\Big)\right] \\ &\overset{\text{Proposition II.4.4}}{=\!=\!=} \frac{1}{m!}\,[\eta \wedge f_*(\omega_X^m)] \\ &=\!=\!=: [\eta] \cdot \mathrm{vol}(F) \end{aligned} \tag{III.13}$$

so that indeed $P \circ f^* = \mathrm{vol}(F) \cdot \mathrm{id}$. Since both $H^q(T, \Omega_T^p), H^q(X, f^*\Omega_T^p)$ are finite dimensional vector spaces to complete the proof of our result it thus suffices to prove that f^* is an isomorphism. Then, (modulo a scalar factor) P will automatically be its inverse and, hence, an isomorphism itself.

But indeed, since every fibre F_t is Fano the Kodaira vanishing theorem yields

$$H^j(F_t, \mathcal{O}_{F_t}) = H^j(F_t, \mathcal{O}_{F_t}(-K_{F_t} + K_{F_t})) = 0, \quad \forall\, j > 0.$$

Thus, $R^j f_* \mathcal{O}_X = 0$ according to Grauert's Theorem IV.1.2 and, hence,

$$R^j f_* f^* \Omega_T^p = \Omega_T^p \otimes R^j f_* \mathcal{O}_X = 0, \quad \forall\, j > 0.$$

It follows immediately from the Leray spectral sequence that

$$f^* \colon H^q(T, \Omega_T^p) \to H^q(X, f^*\Omega_T^p).$$

is an isomorphism. Combining this with Eq. (III.13) we are done. □

Proposition III.2.14. *Let $f \colon X \to T$ be a holomorphic submersion of relative dimension m between compact Kähler manifolds. Assume that the natural short exact sequence*

$$0 \to f^*\Omega_T^1 \to \Omega_X^1 \to \Omega_{X/T}^1 \to 0$$

admits a global holomorphic splitting $s \colon \Omega_X^1 \to f^*\Omega_T^1$ (recall that this is always true provided that f is a flat fibre bundle).

Fix a Kähler form ω_X on X, consider the decomposition

$$[\omega_X] = [\omega_{X/T}] + a_T \in H^1(X, \Omega_X^1) = H^1(X, \Omega_{X/T}^1) \oplus H^1(X, f^*\Omega_T^1)$$

according to the splitting s (i.e. $a_T = H^1(s)([\omega_X])$) and let $\omega_T := f_*(\omega_X^{m+1})$ denote the Kähler form on T obtained from ω_X by integration along the fibres. Then,

$$a_T = \frac{1}{(m+1)! \cdot \mathrm{vol}(F)} \cdot [f^*\omega_T] \in H^1(X, \Omega_X^1). \tag{III.14}$$

Corollary III.2.15. *Let (X, ω_X) be a compact Kähler manifold with nef tangent bundle. Assume that X is of maximal irregularity so that the Albanese morphism $\alpha\colon X \to \mathrm{Alb}(X) =: T$ is a flat holomorphic fibre bundle whose typical fibre F is a Fano manifold.*

Then, there exists a natural isomorphism of affine bundles

$$Z_{\mathcal{T}_X,[\omega_X]} \cong Z_{\mathcal{T}_{X/T},[\omega_{X/T}]} \times_X Z_{\alpha^*\mathcal{T}_T,[\alpha^*\omega_T]} \cong Z_{\mathcal{T}_{X/T},[\omega_{X/T}]} \times_T Z_{\mathcal{T}_T,[\omega_T]}.$$

Here, by $[\omega_{X/T}]$ we denote the image of $[\omega_X]$ under the natural homomorphism

$$H^1(X, \Omega^1_X) \to H^1(X, \Omega^1_{X/T})$$

and we denote $\omega_T := \alpha_*(\omega_X^{m+1})$, where $m := \dim F$.

Proof. Since α is flat the short exact sequence

$$0 \to \alpha^*\Omega^1_T \to \Omega^1_X \to \Omega^1_{X/T} \to 0$$

splits. According to Proposition III.2.14 above, the decomposition of the cohomology class $[\omega_X]$ according to this splitting is given by

$$[\omega_X] = [\omega_{X/T}] + \lambda \cdot [\alpha^*\omega_T] \in \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}_X, \Omega^1_X) = \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}_X, \Omega^1_{X/T}) \oplus \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}_X, \alpha^*\Omega^1_T),$$

where $\lambda := \frac{1}{(m+1)!\cdot \mathrm{vol}(F)} > 0$ is some positive real number. In effect, an application of Proposition III.2.9 yields

$$Z_{\mathcal{T}_X,[\omega_X]} \cong Z_{\mathcal{T}_{X/T},[\omega_{X/T}]} \times_X Z_{\alpha^*\mathcal{T}_T,\lambda\cdot[\alpha^*\omega_T]}.$$

Since extensions only depend on their defining cohomology class up to scaling by Corollary III.1.9 it follows that

$$Z_{\mathcal{T}_X,[\omega_X]} \cong Z_{\mathcal{T}_{X/T},[\omega_{X/T}]} \times_X Z_{\alpha^*\mathcal{T}_T,[\alpha^*\omega_T]} \cong Z_{\mathcal{T}_{X/T},[\omega_{X/T}]} \times_T Z_{\mathcal{T}_T,[\omega_T]}.$$

Here, in the last step we used that we know from Proposition III.1.7 that there exists a natural identification $Z_{\alpha^*\mathcal{T}_T,[\alpha^*\omega_T]} \cong Z_{\mathcal{T}_T,[\omega_T]} \times_T X$. This concludes the proof. □

Proof. (of Proposition III.2.14)
We will verify Eq. (III.14) by an explicit calculation using Dolbeaut representatives. Indeed, $s\colon \Omega^1_X \to f^*\Omega^1_T$ induces maps of sections $s^{(0,1)}\colon \mathcal{A}^{0,1}(\Omega^1_X) \to \mathcal{A}^{0,1}(f^*\Omega^1_T)$ and the class

$$i_*(a_T) = i_*\left(H^1(s)\Big([\omega_X]\Big)\right) \in H^1(X, f^*\Omega^1_T) \overset{i_*}{\hookrightarrow} H^1(X, \Omega^1_X) \qquad \text{(III.15)}$$

is represented by the form $\zeta := i_*(s^{(0,1)}(\omega_X))$. Below, we will show that

$$f_*(\zeta \wedge \omega_X^m) = \frac{f_*(\omega_X^{m+1})}{m+1} \tag{III.16}$$

This will immediately yield the result because assuming Eq. (III.16) we compute

$$\begin{aligned} i_*(a_T) =: [\zeta] &\overset{\text{Corollary III.2.13}}{=\!=\!=} \frac{1}{\mathrm{vol}(F)} \cdot i_* \left[f^* f_* \left(\zeta \wedge \frac{\omega_X^m}{m!} \right) \right] \\ &\overset{\text{Eq. (III.16)}}{=\!=\!=} \frac{1}{\mathrm{vol}(F)} \cdot \frac{1}{(m+1)!} \cdot i_* \left[f^* f_* \left(\omega_X^{m+1} \right) \right] \\ &=\!=\!=: \frac{1}{\mathrm{vol}(F) \cdot (m+1)!} \cdot i_* \left[f^* \omega_T \right]. \end{aligned} \tag{III.17}$$

which, using that by Corollary III.2.13 i_* is injective, is the equation to prove. In conclusion, it remains to verify Eq. (III.16). To this end, fix a point $t \in T$ and vectors $v \in T_t^{(1,0)}T$, $w \in T_t^{(0,1)}T$. Let $\widetilde{V} := s^*(v), \widetilde{W} := s^*(w)$ be the differentiable vector fields along F_t induced by the dual splitting $s^*\colon f^*\mathcal{T}_T \hookrightarrow \mathcal{T}_X$. Then, $\widetilde{V}, \widetilde{W}$ are of type $(1,0)$ (respectively $(0,1)$) and lift v, w, i.e.

$$df(\widetilde{V}|_x) = v, \quad df(\widetilde{W}|_x) = w, \quad \forall\, x \in F_t.$$

By definition it holds that

$$\left(f_*\left(\zeta \wedge \omega_X^m\right)\right)(v,w) = \int_{F_t} \iota_{\widetilde{V},\widetilde{W}} \left(\zeta \wedge \omega_X^m\right), \tag{III.18}$$

$$\left(f_* \omega_X^{m+1}\right)(v,w) = \int_{F_t} \iota_{\widetilde{V},\widetilde{W}} \left(\omega_X^{m+1}\right) \tag{III.19}$$

and we need to prove the equality of both expressions (modulo a scalar factor). Clearly it suffices to prove equality of the integrands (as differential forms) and this is what we will do: Fix a point $x \in F_t$ and denote $\tilde{v} := \widetilde{V}|_x, \tilde{w} := \widetilde{W}|_x$.

Step 1: For all tangent vectors $v' \in T_x^{1,0}X, w' \in T_x^{0,1}X$ it holds that

$$\zeta(v',w') :\overset{\text{Eq. (III.15)}}{=\!=\!=} i_*\left(s^{(0,1)}\left(\omega_X\right)\right)(v',w') = \omega_X\left(s^*\left(df(v')\right), w'\right)$$

Indeed, if more generally $\phi\colon \mathcal{E} \to \mathcal{F}$ is any morphism between holomorphic vector bundles, then the induced map $\phi^{(0,1)}\colon \mathcal{A}^{0,1}(\mathcal{E}) \to \mathcal{A}^{0,1}(\mathcal{F})$ is determined by the rule $\phi^{(0,1)}(\sigma \otimes d\bar{z}) = \phi(\sigma) \otimes d\bar{z}$. Accordingly, if (z^j) are some local coordinates centred at $x \in F_t$ and if with respect to these coordinates $\omega_X = \sum h_{k,\ell}\, dz^k \wedge d\bar{z}^\ell$, then $s^{(0,1)}(\omega_X)$ is locally given by the expression

$$s^{(0,1)}(\omega_X) = s^{(0,1)}\left(\sum h_{k,\ell}\, dz^k \wedge d\bar{z}^\ell\right) = \sum h_{k,\ell}\, s\left(dz^k\right) \otimes d\bar{z}^\ell.$$

Similarly, $i_*\colon \mathcal{A}^{0,1}(f^*\Omega^1_T) \hookrightarrow \mathcal{A}^{0,1}(\Omega^1_X)$ is by construction the map induced by the bundle morphism $(df)^*\colon f^*\Omega^1_T \hookrightarrow \Omega^1_X$. In other words,

$$\begin{aligned} i_*\Big(s^{(0,1)}(\omega_X)\Big)(v',w') :&= \Big(\sum h_{k,\ell}\, df^*\Big(s\left(dz^k\right)\Big) \otimes d\bar{z}^\ell\Big)(v',w') \\ &= \sum h_{k,\ell}\, \Big((df^*\circ s)(dz^k)\Big)(v') \otimes d\bar{z}^\ell(w') \\ &= \sum h_{k,\ell}\, dz^k\Big(s^*(df(v'))\Big) \otimes d\bar{z}^\ell(w') \\ &= \Big(\sum h_{k,\ell}\, dz^k \otimes d\bar{z}^\ell\Big)\left(s^*(df(v')), w'\right) = \omega_X\left(s^*(df(v')), w'\right). \end{aligned}$$

Step 2: The following identity holds true:

$$\iota_{\widetilde{v},\widetilde{w}}(\zeta \wedge \omega_X^m)\Big|_{F_t} = \Big(\omega_X(\widetilde{v},\widetilde{w})\cdot\omega_X^m - \iota_{\widetilde{v}}(\omega_X)\wedge\iota_{\widetilde{w}}(\omega_X^m)\Big)\Big|_{F_t}.$$

Using the formula in Proposition IV.3.3 regarding contractions by vectors of wedge products we compute

$$\begin{aligned} \iota_{\widetilde{w}}\iota_{\widetilde{v}}(\zeta\wedge\omega_X^m) &= \iota_{\widetilde{w}}\Big(\iota_{\widetilde{v}}(\zeta)\wedge\omega_X^m + (-1)^2\,\zeta\wedge\iota_{\widetilde{v}}(\omega_X^m)\Big) \\ &= \zeta(\widetilde{v},\widetilde{w})\cdot\omega_X^m + (-1)\,\iota_{\widetilde{v}}(\zeta)\wedge\iota_{\widetilde{w}}(\omega_X^m) \\ &\quad + (-1)^2\,\iota_{\widetilde{w}}(\zeta)\wedge\iota_{\widetilde{v}}(\omega_X^m) + (-1)^4\,\zeta\wedge\iota_{\widetilde{v},\widetilde{w}}(\omega_X^m). \end{aligned} \tag{III.20}$$

Now, according to *Step 1* it holds that

$$\zeta(v',-) = \omega_X(s^*(df(v')),-), \quad \forall\, v' \in T_x^{0,1}X. \tag{III.21}$$

In particular, if v' is tangent along the fibres, then $df(v') = 0$ and so $\iota_{v'}\zeta = 0$. This immediately implies that

$$\iota_{\widetilde{w}}(\zeta)|_{F_t} = \zeta|_{F_t} = 0. \tag{III.22}$$

On the other hand, consider the case $v' = \widetilde{v}$ in Eq. (III.21) above. Then,

$$s^*(df(\widetilde{v})) \xlongequal{df(\widetilde{v})=v} s^*(v) =: \widetilde{v}$$

by definition of $\widetilde{v}$. In view of Eq. (III.21) this implies that

$$\zeta(\widetilde{v},\widetilde{w}) = \omega_X(\widetilde{v},\widetilde{w}), \quad \iota_{\widetilde{v}}(\zeta) = \iota_{\widetilde{v}}(\omega_X). \tag{III.23}$$

Substituting the terms in Eq. (III.20) above using Eq. (III.22) and Eq. (III.23) we find

$$\iota_{\widetilde{v},\widetilde{w}}(\zeta\wedge\omega_X^m)\Big|_{F_t} = \Big(\omega_X(\widetilde{v},\widetilde{w})\cdot\omega_X^m - \iota_{\widetilde{v}}(\omega_X)\wedge\iota_{\widetilde{w}}(\omega_X^m) + 0\Big)\Big|_{F_t}.$$

which is the identity in question.

Step 3: It holds that $\iota_{\widetilde{v},\widetilde{w}}(\omega_X^{m+1}) = (m+1)\left(\omega_X(\widetilde{v},\widetilde{w})\cdot\omega_X^m - \iota_{\widetilde{v}}(\omega_X)\wedge\iota_{\widetilde{w}}(\omega_X^m)\right)$.

Using again Proposition IV.3.3 we compute

$$\begin{aligned}\iota_{\widetilde{v},\widetilde{w}}(\omega_X^{m+1}) &\overset{\text{Proposition IV.3.3(iii)}}{=\!=\!=} (m+1)\cdot\omega_X(\widetilde{v},\widetilde{w})\cdot\omega_X^m \\ &\qquad - m(m+1)\cdot\iota_{\widetilde{v}}(\omega_X)\wedge\iota_{\widetilde{w}}(\omega_X)\wedge\omega_X^{m-1} \\ &\overset{\text{Proposition IV.3.3(ii)}}{=\!=\!=} (m+1)\cdot\left(\omega_X(\widetilde{v},\widetilde{w})\cdot\omega_X^m - \iota_{\widetilde{v}}(\omega_X)\wedge\iota_{\widetilde{w}}(\omega_X^m)\right).\end{aligned}$$

This finishes the proof of *Step 3*.

Step 4: Conclusion.

Combining the results of *Step 2* and *Step 3* we find that

$$\iota_{\widetilde{v},\widetilde{w}}(s(\omega_X)\wedge\omega_X^m)\Big|_{F_t} = \frac{1}{m+1}\cdot\iota_{\widetilde{v},\widetilde{w}}(\omega_X^{m+1})\Big|_{F_t}.$$

Thus, the integrands in Eq. (III.18) and Eq. (III.19) above agree (up to scaling) and, hence,

$$f_*\left(s(\omega_X)\wedge\omega_X^m\right)(v,w) = \frac{(f_*\omega_X^{m+1})(v,w)}{m+1}, \quad \forall\, v\in T^{(1,0)}T,\ \forall\, w\in T^{(0,1)}T.$$

This proves Eq. (III.16) and, as discussed above in Eq. (III.17), the result immediately follows. □

Corollary III.2.15 yields a splitting $Z_X \cong Z_{X/T}\times_T Z_T$. Our next goal is to prove that the induced map $Z_X\to Z_T$ makes Z_X into a holomorphic fibre bundle with typical fibre Z_F. To this end, we first need to take a closer look at Z_F:

Proposition III.2.16. *Let* (F,ω_F) *be a compact Kähler manifold and denote by* $G := \mathrm{Aut}^0(F)$. *Then,* G *is a complex Lie group according to Theorem II.3.10 and*

(1) the natural action of G *on* $H^*(F,\mathbb{R})$ *is trivial.*
(2) If $H^1(F,\mathbb{R})=0$, *then the action of* G *on* F *extends naturally to an action by automorphisms of affine bundles on* $Z_{[\omega_F]}$.

Proof. Regarding the first statement, since G is a Lie group, $G=\mathrm{Aut}^0(F)$ is not only the connected component of the identity in $\mathrm{Aut}(F)$ but also the path-connected component. Thus, for any $g\in G$ there exists a (smooth) path from id_F to g in G. But such a path is nothing but a (smooth) homotopy between id_F and g, i.e. all maps in G are null homotopic. In particular, they induce the identity maps on de Rahm cohomology.

For the second statement, note that any element $g \in G$ naturally induces an isomorphism of affine bundles

$$g\colon Z_{[\omega_F]} \to g^* Z_{[\omega_F]} = Z_{[g^*\omega_F]}.$$

Since the action of G on $H^*(F, \mathbb{R})$ is trivial by item (1), in particular $[g^*\omega_F] = [\omega_F]$ for all $g \in G$. Hence, there exists *an* isomorphism of affine bundles $Z_{[g^*\omega_F]} \cong Z_{[\omega_F]}$. We claim, that in fact there exists only one such isomorphism. In particular, we may identify $Z_{[g^*\omega_F]}$ and $Z_{[\omega_F]}$ in a natural way and so the action of G on F lifts to Z_F as required.

Regarding the claim, by construction any isomorphism as above is induced by an isomorphism of extensions or, in other words, by a commutative diagram as below:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_F^1 & \longrightarrow & V & \longrightarrow & \mathcal{O}_F & \longrightarrow & 0 \\ \| & & \| & & \downarrow{\scriptstyle\phi} & & \| & & \| \\ 0 & \longrightarrow & \Omega_F^1 & \longrightarrow & V & \longrightarrow & \mathcal{O}_F & \longrightarrow & 0 \end{array}$$

It is now easily verified by a diagram chase that any morphism ϕ making the above diagram commute is of the form $\phi = \mathrm{id} + \eta$, where

$$\eta \in \mathrm{Hom}(\mathcal{O}_F, \Omega_F^1) = H^0(F, \Omega_F^1).$$

But $\dim_{\mathbb{C}} H^0(F, \Omega_F^1) = \dim_{\mathbb{R}} H^1(F, \mathbb{R}) = 0$ by the Hodge decomposition. Thus, there is only one isomorphism of affine bundles $Z_{[g^*\omega_F]} \cong Z_{[\omega_F]}$ and we are done. ☐

Lemma III.2.17. *Let $f\colon X \to T$ be a holomorphic fibre bundle with structure group G and with typical fibre F. Suppose that X and T are compact Kähler and fix a Kähler metric ω_X on X. Suppose moreover that $G \subseteq \mathrm{Aut}^0(F)$ and that $H^1(F, \mathbb{C}) = 0$. Then, also*

$$f \circ p\colon Z_{X/T} := Z_{\mathcal{T}_{X/T}, [\omega_{X/T}]} \to X \to T$$

is a holomorphic fibre bundle. Its typical fibre is $Z_{\mathcal{T}_F, [\omega_X|_F]}$ and the structure group may be chosen to be G.

Note that G indeed acts on $Z_{\mathcal{T}_F, [\omega_X|_F]}$ by Proposition III.2.16 so that the assertion about the structure group of the bundle makes sense.

Proof. Since both $f\colon X \to T$ and $p\colon Z_{X/T} \to X$ are holomorphic fibre bundles, $f \circ p$ is at least a surjective holomorphic submersion. Moreover, it follows from the functoriality of the construction of $Z_{\_}$ (see Proposition III.1.7) that the fibre of $f \circ p$ over $t \in T$ is given by

$$(f \circ p)^{-1}(t) = p^{-1}(F_t) = Z_{X/T} \times_X F_t \xlongequal{\text{Proposition III.1.7}} Z_{\mathcal{T}_{X/T}|_{F_t}, \left[\omega_X|_{F_t}\right]} = Z_{\mathcal{T}_{F_t}, [\omega_X|_{F_t}]}.$$

Now, fix $t \in T$, denote $F := f^{-1}(t)$ and choose a sufficiently small open polydisc $t \in U \subset T$ so that $f^{-1}(U) \cong U \times F$ is trivial. We want to show that there exists an isomorphism of fibre bundles

$$Z_{X/T}|_U \cong U \times Z_{\mathcal{T}_F,[\omega_X|F]} \tag{III.24}$$

respecting the affine bundle structure on both sides. Indeed, since U is a polydisc it holds that $H^j(U, \mathbb{C}) = 0$ for all $j > 0$. Thus, according to the classical Künneth formula the map

$$pr_F^*\colon H^*(F, \mathbb{C}) \to H^*(U \times F,\ \mathbb{C})$$

is an isomorphism. Note that an inverse is clearly provided by the restriction map

$$\cdot|_{\{t\}\times F}\colon H^*(U \times F,\ \mathbb{C}) \to H^*(F, \mathbb{C}).$$

In particular, we find that

$$[\omega_X|_{U\times F}] = pr_F^*[\omega_X|_F]. \tag{III.25}$$

Using again the functionality of extensions and the fact that $\mathcal{T}_{U\times F/U} = pr_F^*\mathcal{T}_F$ we compute

$$\begin{aligned} Z_{\mathcal{T}_{X/T},[\omega_{X/T}]}\Big|_U = Z_{\mathcal{T}_{U\times F/U},[\omega_{X/T}]} &\xlongequal{\text{Eq. (III.25)}} Z_{pr_F^*\mathcal{T}_F, pr_F^*[\omega_F]} \\ &\xlongequal{\text{Proposition III.1.7}} pr_F^* Z_{\mathcal{T}_F,[\omega_F]} := U \times Z_{\mathcal{T}_F,[\omega_F]}. \end{aligned}$$

This proves Eq. (III.24) and, hence, that $f \circ p$ is a holomorphic fibre bundle with fibre Z_F.

The assertion about the structure group being G is clear, because we already saw as part of the proof of Proposition III.2.16 that given any $g \in G$, there is one and only one identification of Z_F and g^*Z_F as affine bundles. Hence, both $f\colon X \to T$ and $f \circ p\colon Z_{X/T} \to T$ are constructed using the same transition functions. □

Remark III.2.18. Record for later reference that both the bundles $f\colon X \to T$ and $f \circ p\colon Z_{X/T} \to T$ are constructed using the same transition functions. In particular, the first is flat if and only if the latter is so.

Corollary III.2.19. *Let $f\colon X \to T$ be a holomorphic fibre bundle. Assume that X and T are compact Kähler, fix a Kähler form ω_X on X and suppose that the typical fibre F of f is a Fano manifold. Suppose moreover that the structure group G of f is contained in* $\mathrm{Aut}^0(F)$ *and that the short exact sequence*

$$0 \to \mathcal{T}_{X/T} \to \mathcal{T}_X \to f^*\mathcal{T}_T \to 0$$

admits a global holomorphic splitting (which is satisfied if for example f is flat).

Then, there exists an isomorphism of affine bundles

$$Z_{\mathcal{T}_X,[\omega_X]} \cong Z_{\mathcal{T}_{X/T},[\omega_{X/T}]} \times_T Z_{\mathcal{T}_T,[\omega_T]}. \tag{III.26}$$

Here, $\omega_T := f_*(\omega_X^{m+1})$ is the Kähler form on T obtained from ω_X by integration along the fibres. Moreover, the projection map

$$\bar{f} \colon Z_{\mathcal{T}_X,[\omega_X]} \to Z_{\mathcal{T}_T,[\omega_T]}$$

makes Z_X into a (flat if f is flat) holomorphic fibre bundle over Z_T with fibre $Z_{F,[\omega_X|_F]}$ and structure group G.

Proof. First of all, Eq. (III.26) has already been verified in Corollary III.2.15. Regarding the second assertion, note that $H^1(F, \mathbb{C}) = 0$ as F is Fano (cf. e.g. Example II.2.8). Thus, Lemma III.2.17 above applies and yields that

$$Z_{\mathcal{T}_{X/T},[\omega_{X/T}]} \to T$$

is a (flat; see Remark III.2.18) holomorphic fibre bundle with structure group G and fibre Z_F. But Eq. (III.26) just says that

$$\bar{f} \colon Z_{X,[\omega_X]} \to Z_{T,[\omega_T]}$$

is the pull back along $Z_T \to T$ of the bundle $Z_{X/T} \to T$. Hence, along with $Z_{X/T} \to T$ also $\bar{f}$ is a (flat) holomorphic fibre bundle with structure group G and fibre Z_F. □

The following trick may be used to show that the condition $G \subseteq \operatorname{Aut}^0(F)$ in Corollary III.2.19 above is essentially superfluous.

Proposition III.2.20. *Let $f \colon X \to T$ be a holomorphic fibre bundle with typical fibre F, where both X and T are compact complex manifolds. Suppose that the the group $\operatorname{Aut}(F)/\operatorname{Aut}^0(F)$ is finite (by Lemma II.3.14 this is satisfied for example if F is Fano). Then, there exists a finite étale cover $\widetilde{T} \to T$ such that the structure group of the holomorphic fibre bundle $X \times_T \widetilde{T} \to \widetilde{T}$ may be chosen to be contained in $\operatorname{Aut}^0(F)$.*

Proof. Let us abbreviate $G := \operatorname{Aut}(F)$ and $G^0 := \operatorname{Aut}^0(F)$. Since $G = \operatorname{Aut}(F)$ acts effectively on F, there exists a unique holomorphic principal G-bundle $\mathcal{G} \xrightarrow{\pi} T$ such that $X \xrightarrow{f} T$ is the associated bundle with typical fibre F. Then,

$$\widetilde{T} := \mathcal{G}/G^0 \to T$$

is a finite étale cover of T (since G/G^0 is finite by assumption) and by construction the structure group of the principal G-bundle $\mathcal{G} \times_T \widetilde{T} \to \widetilde{T}$ may be reduced to G^0. In effect, the same is true of the associated bundle $X \times_T \widetilde{T} \to \widetilde{T}$ and so we are done. □

We are now finally ready to prove the main result of this chapter:

Theorem III.2.21. *Let (X, ω_X) be a compact Kähler manifold with nef tangent bundle. If the weak Campana-Peternell conjecture Conjecture II.3.7 holds true then the canonical extension*

$$Z_{X,[\omega_X]}$$

is a Stein manifold.

Proof. According to Example II.2.8 there exists a finite étale cover $\pi\colon \widetilde{X} \to X$ of maximal irregularity. Then, by our main result on manifolds with nef tangent bundles Theorem II.4.14 the Albanese $\alpha\colon \widetilde{X} \to \mathrm{Alb}(\widetilde{X}) =: T$ is a flat holomorphic fibre bundle. Its fibres are Fano manifolds with nef (and, hence, assuming Conjecture II.3.7 also big) tangent bundle. Possibly replacing $\widetilde{X}$ by another finite étale cover we may moreover assume by Proposition III.2.20 above that the structure group G of α is contained in $\mathrm{Aut}^0(F)$. But in this situation Corollary III.2.19 applies to the compact Kähler manifold $(\widetilde{X}, \pi^*\omega_X)$ and shows that that there exists a natural map

$$\bar{\alpha}\colon Z_{\widetilde{X},[\pi^*\omega_X]} \to Z_{T,[\omega_T]} \tag{III.27}$$

making $Z_{\widetilde{X}}$ into a flat holomorphic fibre bundle with structure group $G \subseteq \mathrm{Aut}^0(F)$ and fibre

$$Z_{F,[\pi^*\omega_X|_F]}.$$

Here, ω_T in Eq. (III.27) above is some (explicitly determined) Kähler from on T. Note that by Proposition III.2.16 $\mathrm{Aut}^0(F)$ acts on Z_F so that we may well assume the structure group of $\bar{\alpha}$ to be $\mathrm{Aut}^0(F)$. Note moreover, that we already proved in Corollary III.2.3 that Z_T must be Stein as a canonical extension of a complex torus and we showed in Corollary III.2.6 that Z_F must be Stein as a canonical extension of a Fano manifold with big and nef tangent bundle.

In summary, $Z_{\widetilde{X}}$ is naturally a holomorphic fibre bundle over the Stein manifold Z_T. The typical fibre of this bundle is Z_F, a Stein manifold, and the structure group of the bundle may be chosen to be the connected group $\mathrm{Aut}^0(F)$. But it is a classical theorem by [MM60, Théorème 6.] that in this situation also the total space

$$Z_{\widetilde{X},[\pi^*\omega_X]}$$

of the bundle is Stein. Finally, since $\pi\colon \widetilde{X} \to X$ is finite étale Proposition III.1.15 yields that also $Z_{X,[\omega_X]}$ is Stein and so we are done. □

3 Manifolds whose Canonical Extensions are Stein

Summarising the results obtained by [GW20], [HP21] and Theorem III.3.1 we see that the only implication in Conjecture III.1.11 that is still open is the one that compact Kähler manifolds whose canonical extensions are Stein posses a nef tangent bundle. This problem seems to be very difficult however: For example, due to a counter-example in [HP21, Proposition 1.4.] a potential argument can not be entirely abstract and would have to take into account the specific geometry of the situation. Nevertheless, Höring-Peternell made some progress in this direction. In this section, we want to summarise some of their main results. We end this chapter by discussing some future directions.

Let us start straight away with the principal result of [HP21] which is also the only known result valid in any dimension:

Theorem III.3.1. (Höring-Peternell, [HP21, Corollary 1.7.])
Let X be a compact Kähler manifold. Assume that X admits some Kähler metric whose canonical extension is Stein. Then, $\mathcal{O}_{\mathbb{P}(\mathcal{T}_X)}(1)$ is pseudo-effective in the sense explained before Theorem II.6.4. We say that $\mathcal{T}_X$ is weakly pseudo-effective.

Now, having a weakly pseudo-effective tangent bundle is far weaker then possessing a nef or even just a strongly psef tangent bundle. In particular, one can not apply the structure theory discussed in Theorem II.6.4. However, using the weak pseudo-effectivity and some recent advances in foliation theory one can at least prove:

Theorem III.3.2. (Höring-Peternell, [HP21, Theorem 1.12.])
Let X be a projective manifold of dimension at most three which is not uniruled. Assume that X admits some Kähler metric whose canonical extension is Stein. Then, there exists a finite étale cover $T \to X$ of X by a torus.

Note that this is precisely what we expect: The tangent bundle of X should be nef so that by Theorem II.4.14 X should either be an (étale quotient of a) torus or a holomorphic fibre bundle with fibre a Fano manifold. In particular, in the latter case X would be uniruled as Fanos are uniruled.

In the uniruled case however, little is known. Naturally, one would try to study the low dimensional case first. Indeed, recall that by Example III.1.13 Conjecture III.1.11 holds true for curves, although the proof was not entirely trivial. The case of surfaces is already much harder. Using the surface classification and Theorem III.3.2, [HP21] manage to verify the conjecture in most cases. However, some of the cases left open by their discussion may be considered to be among the more interesting ones (compare the discussion after Question III.3.6 below):

Theorem III.3.3. (Höring-Peternell, [HP21, Theorem 1.13.])
Let X be a smooth projective surface. Assume that there exists some Kähler class ω_X on X whose canonical extension is Stein. Then, one of the following holds true:

(1) X is an étale quotient of a complex torus.
(2) X is a homogeneous Fano surface, i.e. either $X = \mathbb{P}^2$ or $X = \mathbb{P}^1 \times \mathbb{P}^1$.
(3) $X = \mathbb{P}(\mathcal{E}) \to C$ is a ruled surface over a curve of genus $g(C) \geq 1$. Moreover, if $g(C) \geq 2$ then $\mathcal{E}$ must be semi stable.

Note that item (3) is not quite optimal: From our characterisation of projective manifolds with nef tangent bundle we know that a ruled surface $X = \mathbb{P}(\mathcal{E}) \to C$ over a curve of genus $g(C) \geq 1$ has a nef tangent bundle if and only if $g(C) = 1$ and $\mathcal{E}$ is semi stable (cf. the discussion in Chapter II. Section 1). Thus, we expect that in the other cases no canonical extension of X should be Stein. Indeed, we are able to rule out one more case; to this end, we need the following auxiliary result:

Proposition III.3.4. *Let $X = \mathbb{P}(\mathcal{E}) \to C$ be a ruled surface. If $\mathcal{E}$ is semi stable, then π is a flat fibre bundle.*

Proof. This fact is rather well-known, see for example [JR13, Theorem 1.5, Proposition 1.7.]. □

Lemma III.3.5. *Let $X = \mathbb{P}(\mathcal{E}) \xrightarrow{f} C$ be a ruled surface over a curve of genus $g(C) \geq 2$ defined by a semi stable vector bundle $\mathcal{E}$. Then, no canonical extension of X is Stein.*

Proof. Assume to the contrary that there exists a Kähler metric ω_X on X whose canonical extension Z_X is Stein. Now, by Proposition III.3.4 $\pi\colon X \to C$ is a flat fibre bundle and its typical fibre is $\mathbb{P}^1$ - a Fano manifold with connected automorphism group. In particular, Corollary III.2.19 applies in this situation and shows that we may also consider Z_X as a flat fibre bundle over Z_C with typical fibre $Z_{\mathbb{P}^1}$ and with the same transition functions as $X \to C$. Here, for the latter assertion we use Remark III.2.18 and the fact, that by Proposition III.2.16 the action of $\mathrm{Aut}(\mathbb{P}^1)$ on $\mathbb{P}^1$ lifts uniquely to $Z_{\mathbb{P}^1}$. In other words, if we denote by $\widetilde{C} \xrightarrow{p} C$ the universal cover of C and if $\rho\colon \pi_1(C) \to \mathrm{Aut}(\mathbb{P}^1)$ is a representation defining the flat bundle $X \to C$, then we may identify

$$Z_{X,[\omega_X]} \cong (Z_{\widetilde{C},[p^*\omega_C]} \times Z_{\mathbb{P}^1,[\omega_X|_{\mathbb{P}^1}]})/\pi_1(C). \tag{III.28}$$

Here, $\omega_C := f_*(\omega_X \wedge \omega_X)$ is the induced Kähler form on C. Now, recall again that by Proposition III.2.16 the action of $G := \mathrm{Aut}(\mathbb{P}^1)$ on $\mathbb{P}^1$ lifts (uniquely) to an action on $Z_{\mathbb{P}^1}$ and in fact both actions are transitive as follows from Remark III.2.7. In particular, it follows from the description in Eq. (III.28) that the semi simple group G acts from the left on Z_X and (using that the action on $Z_{\mathbb{P}^1}$ is transitive) the quotient is clearly given by

$$G \backslash Z_X \cong G \backslash (Z_{\widetilde{C}} \times Z_{\mathbb{P}^1})/\pi_1(C) = Z_{\widetilde{C}}/\pi_1(C) = Z_C.$$

Since quotients of Stein spaces by reductive groups are again Stein by [Sno82] it follows that also Z_C is Stein. But this contradicts Example III.1.13 as $g(C) \geq 2$. Thus, Z_X can not be Stein after all and we are done. □

The case of unstable ruled surfaces over elliptic curves however is still completely open:

Question III.3.6. *Let $X = \mathbb{P}(\mathcal{E}) \to E$ be a ruled surface over an elliptic curve defined by an unstable bundle $\mathcal{E}$ (so that according to Corollary II.5.10 $\mathcal{T}_X$ is not nef). Is it true, that no canonical extension of X Stein?*

This question is interesting because such surfaces lie on the boundary of what is known: One can show, that they belong to the very restricted class of surfaces whose tangent bundle is (strongly) pseudo-effective but not nef (compare the discussion in [HIM21]). Thus, an affirmative answer to Question III.3.6 would provide a serious indication towards the correctness of Conjecture III.1.11. On the other hand, it seems very much possible that the answer to Question III.3.6 may turn out to be negative. In this case, it would of course be interesting to see how much positivity exactly one can infer from the Steiness of canonical extensions.

Chapter IV

Appendix

1 Summary of important Results

Below, we collect some famous results from algebraic and complex geometry which are used throughout this work. They are of course all very well-known. However, I felt that in case the reader may feel a bit uncertain about the precise, most general preconditions of a theorem or if the attribution is a bit ambitious having the list below at hand might be convenient.

Lemma IV.1.1. (Zariski's main theorem, [Fis76, Lemma 1.23., Theorem 1.24.]) *Let $f\colon X \to T$ be a proper holomorphic map between complex manifolds. Then, the fibres of f are connected if and only if the natural map $\mathcal{O}_T \to f_*\mathcal{O}_X$ is an isomorphism.*

Theorem IV.1.2. (Grauerts thm. on direct image sheaves, [GPR94, Section III.4.2.]) *Let $f\colon X \to T$ be a flat holomorphic map between reduced complex analytic varieties (in all our applications, X and T will be smooth and f will be a submersion). Let $\mathcal{E}$ be a vector bundle on X. If for every point $t \in T$ it holds that*

$$H^j\left(X_t, \mathcal{E}|_{X_t}\right) = 0, \qquad \forall\, j > 0,$$

then all higher direct image sheaves $R^j f_*\mathcal{E} = 0$ vanish for $j > 0$, the direct image sheaf $f_*\mathcal{E}$ is a vector bundle and its fibres may be naturally identified to be

$$\mathcal{E}|_t = H^0\left(X_t, \mathcal{E}|_{X_t}\right), \qquad \forall\, t \in T.$$

Theorem IV.1.3. (Grauert's contraction theorem, [Gra62, Abschnitt 8.e.]) *Let X be a smooth projective surface and let C be an irreducible curve in X of negative self-intersection. Then, there exists a bimeromorphic map $\pi\colon X \to X'$ onto* *a normal complex analytic variety which is an isomorphism on $X \setminus C$ and contracts C to a point.*

Theorem IV.1.4. (Kawamata-Viehweg vanishing, [Laz04a, Example 4.3.7.])
Let X be a projective (!) manifold and let $\mathscr{L}$ be a nef line bundle on X. Then,

$$H^j(X, \mathscr{L} \otimes \mathcal{O}_X(K_X)) = 0, \qquad \forall\, j > \dim X - \nu(\mathscr{L}).$$

Here, $\nu(\mathscr{L})$ is the numerical dimension of $\mathscr{L}$ (see Definition II.4.10).

Theorem IV.1.5. (Griffiths vanishing theorem, [Laz04b, Example 7.3.3.])
Let X be a projective (!) manifold and let $\mathcal{E}$ be a big and nef vector bundle on X. Then,

$$H^j(F, \mathcal{O}_F(K_F) \otimes \operatorname{Sym}^k \mathcal{E} \otimes \det(\mathcal{E})) = 0, \qquad \forall\, k \geq 0, j > 0.$$

Remark IV.1.6. Theorem IV.1.5 includes the following famous special cases:

- $\mathcal{E}$ an ample vector bundle (original Griffiths vanishing)
- $\mathcal{E}$ an ample line bundle and $k = 0$ (Kodaira vanishing)
- $\mathcal{E}$ a big and nef line bundle and $k = 0$ (Kawamata-Viehweg vanishing)

Theorem IV.1.7. (Base-point free theorem, [KM98, Theorem 3.3.])
Let X be a smooth projective variety. Suppose that there exists a nef line bundle $\mathscr{L}$ on X and a natural number $k > 0$ such that

$$\mathscr{L}^{\otimes k} \otimes \mathcal{O}_X(-K_X)$$

is a big and nef. Then, $\mathscr{L}$ is semi ample.

Theorem IV.1.8. (Grothendieck-Riemann-Roch, [OTT85])
Let $f\colon X \to Y$ be a proper holomorphic submersion between complex manifolds. Then, for any coherent sheaf $\mathcal{F}$ on X it holds that

$$\sum_{j=0} (-1)^j \ \operatorname{ch}(R^j f_* \mathcal{F}) = f_*\Big(\operatorname{ch}(\mathcal{F}) \wedge \operatorname{td}(\mathcal{T}_{X/Y})\Big).$$

Here, one the left hand side f_* denotes the ordinary push forward of sheaves and on the right hand side f_* denotes integration along the fibres in the sense of Definition II.4.3.

2 Conventions regarding Projective Bundles

Let X be a complex analytic variety and let $\mathcal{E}$ be a holomorphic vector bundle on X. Since unfortunately there is no universally agreed upon convention regarding the definition of the projective bundle $\mathbb{P}(\mathcal{E})$, in the following we will explain in detail our conventions regarding it: Indeed, we will denote by $\mathbb{P}(\mathcal{E}) \xrightarrow{\pi} X$ the projective bundle of *hyperplanes* in $\mathcal{E}$. Equivalently, $\mathbb{P}(\mathcal{E}) \xrightarrow{\pi} X$ is the bundle of one-dimensional quotients of $\mathcal{E}$ or in other words $\mathbb{P}(\mathcal{E}^*) \to X$ is the bundle of *lines* in $\mathcal{E}$. With this convention, the bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is the tautological *quotient* line bundle of $\pi^*\mathcal{E}$.

We choose this convention because it makes the definition of the rational map associated to a linear series coordinate independent:

Definition IV.2.1. *Let X be a complex analytic variety and let $\mathscr{L}$ be a holomorphic line bundle on X such that $H^0(X, \mathscr{L}^{\otimes m}) \neq 0$ for some m. Then, the* rational holomorphic map associated to the linear series of $\mathscr{L}^{\otimes m}$ *is defined to be*

$$\phi_m \colon X \dashrightarrow \mathbb{P}(H^0(X, \mathscr{L}^{\otimes m})), \quad x \mapsto \{\sigma \in H^0(X, \mathscr{L}^{\otimes m}) | \sigma(x) = 0\},$$

wherever this makes sense.

With our conventions, the relative Euler sequence takes the following shape:

Proposition IV.2.2. (Relative Euler sequence)
Let X be a complex analytic variety and let $\mathcal{E}$ be a holomorphic vector bundle on X. Then, there exist natural short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) & \to & \pi^*\mathcal{E}^* & \to & \mathcal{T}_{\mathbb{P}(\mathcal{E})/X} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) & \to & 0 \\ 0 & \to & \mathcal{T}_{\mathbb{P}(\mathcal{E})/X} & \to & \mathcal{T}_{\mathbb{P}(\mathcal{E})} & \to & \pi^*\mathcal{T}_X & \to & 0 \end{array}$$

of vector bundles on $\mathbb{P}(\mathcal{E})$. In particular, taking determinants in both sequences we find the following formula for the canonical line bundle on $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$:

$$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(K_{\mathbb{P}(\mathcal{E})}) = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-\mathrm{rk}(\mathcal{E})) \otimes \pi^*\Big(\mathcal{O}_X(K_X) \otimes \det(\mathcal{E})\Big).$$

Proposition IV.2.3. *Any short exact sequence $0 \to \mathcal{K} \to \mathcal{E} \xrightarrow{p} \mathcal{Q} \to 0$ of holomorphic vector bundles gives rise to a natural closed embedding $\mathbb{P}(\mathcal{Q}) \overset{\iota}{\hookrightarrow} \mathbb{P}(\mathcal{E})$ of complex analytic varieties. The normal bundle is given by*

$$\mathcal{N}_{\mathbb{P}(\mathcal{Q})/\mathbb{P}(\mathcal{E})} = \pi^*(\mathcal{K}^*) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{Q})}(1).$$

3 Multilinear Algebra and Positivity of Forms

3.1 Conventions regarding the Wedge Product

As with projective bundles, one may find several definition of the wedge product in the literature which only agree up to a scalar factor. The following one is the definition usually encountered in differential geometry (see for example [Lee13]):

Definition IV.3.1. *Let V be a vector space over $\mathbb{C}$ and let T be a multilinear form on V of degree k. The* alternation $\mathrm{Alt}(T) \in \bigwedge^k V^*$ *of T is defined by the rule*

$$Alt(T)(v_1, \dots, v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} T(v_{\sigma(1)}, \dots, v_{\sigma(k)}), \quad \forall\, v_1, \dots, v_k \in V.$$

Definition IV.3.2. *Let V be a vector space over $\mathbb{C}$ and let φ, ψ be skew symmetric forms on V of degree k and ℓ respectively. Then, the form $\varphi \wedge \psi \in \bigwedge^{k+\ell} V^*$ is defined to be*

$$\varphi \wedge \psi := \frac{(k+\ell)!}{k!\ell!} \mathrm{Alt}(\varphi \otimes \psi).$$

While we are at it, let us also state the following formulae used in the main text:

Proposition IV.3.3. *Let $\varphi \in \bigwedge^k V^*$, $\psi \in \bigwedge^\ell V^*$ and $\omega \in \bigwedge^{2k} V^*$ be skew-symmetric forms on V of the indicated degree. Then, for all vectors $v, w \in V$ the following identities are satisfied:*

$$\begin{aligned} \iota_v(\varphi \wedge \psi) &= \iota_v(\varphi) \wedge \psi + (-1)^k \varphi \wedge \iota_v(\psi), \\ \iota_v(\omega^m) &= m \cdot \iota_v(\omega) \wedge \omega^{m-1}, \\ \iota_w \iota_v(\omega^m) &= m \cdot \iota_w \iota_v(\omega) \wedge \omega^{m-1} - m(m-1) \iota_v(\omega) \wedge \iota_w(\omega) \wedge \omega^{m-1}. \end{aligned}$$

Here, as per usual ι_v is the contraction by v: $\iota_v \varphi = \varphi(v, -)$.

Proof. The first identity is proved in [Lee13, Lemma 14.13.]. The second formula clearly follows from the first one by an induction argument (note that we assumed ω to be of even degree to avoid worries about the correct signs). Finally, the third one is obtained by applying the first identity to the second one. □

3.2 Positivity of Forms

On the exterior algebra of a complex vector space there is a natural notion of *positivity* of forms. Since this material is not typically covered in courses outside of complex geometry we want to recall these notions as well. This has the additional benefit of clarity since the notions are, again, not entirely universal.

Throughout this subsection we fix a complex vector space V of (complex) dimension n. As per usual, we denote by $I \colon V \to V$ the multiplication-by-i map.

Definition IV.3.4. *A real form $\eta \in \bigwedge^{k,k} V$ is called* (strongly) positive *if there exist complex linear forms $\beta_1, \dots, \beta_k$ such that*

$$\eta = (i\beta_1 \wedge \bar{\beta}_1) \wedge \cdots \wedge (i\beta_k \wedge \bar{\beta}_k).$$

In this case, we denote $\eta \geq 0$.

Remark IV.3.5.
(i) Note that in this language $0 \in \bigwedge^{k,k} V$ is a positive form. In this sense, it would perhaps be better to speak of semi positive forms but unfortunately this terminology has stuck.
(ii) Clearly, positive forms are necessarily real.

The following proposition is immediate

Proposition IV.3.6. *Wedge products of positive forms are positive.*

Most often we will consider $(1,1)$-forms. Here, the notion of positivity admits many reformulations:

Proposition IV.3.7. *Let $\eta = i \sum h_{k,\ell}\, dz^k \wedge d\bar{z}^\ell$ be a real $(1,1)$-form on $\mathbb{C}^n$. Then, the following are equivalent:*

(i) The $(1,1)$-form η is strongly positive,
(ii) the matrix $(h_{k,\ell})$ is positive semi definite,
(iii) it holds that $(-i) \cdot \eta(v, \bar{v}) \geq 0$ for all $v \in V^{\mathbb{C}}$,
(iv) it holds that $\eta(v, Iv) \geq 0$ for all $v \in V$.

In case the matrix $(h_{k,\ell})$ is even positive definite, we say that η is strictly positive *and write $\eta > 0$.*

Proof. The equivalence of (*i*) and (*ii*) is proved for example in [Dem12, Corollary III.1.7.]. Moreover, the equivalence of (*ii*) and (*iii*) is essentially tautologous and the equivalence of (*iii*) and (*iv*) is just an exercise in unravelling definitions. ☐

Example IV.3.8. Let $g\colon V \times V \to \mathbb{R}$ be a real inner product on V. Then, the fundamental form $\omega(-,-) = g(I-,-)$ associated to g is a strictly positive $(1,1)$-form. Consequently, $\omega_X^n \geq 0$ is positive. In fact, it is straightforward to verify that $\omega_X^n = n! \cdot \mathrm{vol}_g(X)$.

Finally, this theory clearly generalises to the setting of differential forms:

Definition IV.3.9. *Let X be a complex manifold. A differential form $\eta \in \mathcal{A}_X^{k,k}$ on X is called* positive *if it is point-wise a positive form.*

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