The Geography of Surfaces of General Type

ALGANT Master Thesis

Aryaman Patel

Supervised by Prof. Daniel Greb

Universiteit Leiden
Faculteit der Wiskunde en Natuurwetenschappen

Universität Duisburg-Essen
Fakultät für Mathematik

Declaration of Authorship

I hereby declare that this thesis is based on my own unaided work, unless stated otherwise. All references and verbatim extracts have been marked as such and I assure that no other sources have been used. Moreover, this thesis has not been part of any other exam in this or any other form.

Essen, Germany

Aryaman Patel

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Preface

The *Enriques-Kodaira classification* is a classification of compact complex surfaces into ten classes. For each class, the surfaces in the class can be parametrized by a moduli space. While the moduli spaces for most of the classes are well understood, the moduli space for the class of surfaces of general type is difficult to describe explicitly. This is one of the reasons why this class of surfaces is worth exploring. Some well-known examples of surfaces of general type include the Castelnuovo surfaces, fake projective planes (see Chapter 5), and products of two curves, each having genus at least 2. The Chern numbers of a compact complex surface of general type satisfy the inequality $c_1^2 \leqslant 3c_2$. This is now a well known classical result known as the *Bogomolov-Miyaoka-Yau inequality*, or BMY inequality for short. Surfaces of general type satisfying $c_1^2 = 3c_2$, i.e., the extreme case of the BMY inequality, are especially interesting. It was shown by Hirzebruch that every such surface occurs as the quotient of the unit ball in $\mathbb{C}^2$ by the free action of an infinite discrete group. These ball quotient surfaces have proven tricky to construct and have been a subject of great interest and research in recent years.

This thesis consists broadly of three parts. In the first part, we study the proof of the Bogomolov-Miyaoka-Yau inequality following the 1978 article of Miyaoka "On Chern numbers of surfaces of general type". We first discuss some preliminary results that will be used later in the thesis, and then make explicit and more accessible the proof of every result in Miyaoka's article. For this part, the main reference, apart from the article itself, is the book "Compact complex surfaces" by Barth, Hulek, Peters, and van de Ven (see [1]). This book is a detailed resource on the (classical) theory of surfaces, and we encourage the interested reader to follow it.

In the second part of the thesis, we discuss examples of surfaces of general type, focusing mainly on those that satisfy the equality $c_1^2 = 3c_2$. We study the construction of such surfaces starting from line arrangements in the projective plane $\mathbb{P}^2$. In this method, the desired surface is constructed as a 'good cover' of a blow up of the projective plane $\mathbb{P}^2$, branched along an arrangement of divisors. We describe the construction of a *Kummer covering* of a blow up of $\mathbb{P}^2$ branched along an arrangement of divisors arising from a line arrangement in $\mathbb{P}^2$, and derive conditions for such a surface to satisfy $c_1^2 = 3c_2$. This construction was introduced by Hirzebruch in his article "Arrangements of lines and algebraic surfaces", although it is not described in as much detail here. For this part of the thesis, we use the book "Complex ball quotients and line arrangements in the projective plane" by P. Tretkoff as the main reference.

The third part of the thesis is dedicated to an interesting class of surfaces of general type satisfying $c_1^2 = 3c_2$, the *fake projective planes*. A fake projective plane is so called because it has the same Betti numbers as the projective plane $\mathbb{P}^2$ but is not isomorphic to it. Fake projective planes have proven difficult to construct and have been studied extensively in recent years. Prasad and Yeung [11] have shown that many fake projective planes admit finite automorphism groups. Following this, Keum classified quotients of fake projective planes by the action of their finite automorphism groups, and their minimal resolutions of singularities, in his article "Quotients of fake projective planes". In the thesis we study this article and make explicit the proof of all results in it.

A natural question that arises is: Can one construct a fake projective plane starting from a line arrangement in $\mathbb{P}^2$, using the method mentioned above? For the complete quadrilateral arrangement discussed in the thesis, the answer turns out, unfortunately, to be no. However, this question is still worth exploring, possibly in a more general setting. Another interesting direction to pursue would be studying automorphism groups of surfaces of general type satisfying $c_1^2 = 3c_2$ constructed using line arrangements on $\mathbb{P}^2$. Are there any such surfaces admitting finite automorphism groups? If yes, what do their quotients and minimal resolutions look like?

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0 Notation

The following notation is used throughout the thesis.

$\mathcal{O}_X$: structure sheaf of the algebraic variety or complex space X.

$\det(\mathcal{F})$: determinant of the locally free sheaf (or vector bundle) $\mathcal{F}$.

$\widehat{\mathcal{F}}$: dual sheaf of the locally free sheaf $\mathcal{F}$.

$\mathcal{S}^n\mathcal{F}$: the n-th symmetric power of $\mathcal{F}$.

$\mathbb{P}(\mathcal{F})$: projective bundle associated to $\mathcal{F}$.

H: divisor associated to the tautological invertible sheaf on $\mathbb{P}(\mathcal{F})$.

$e(X)$: topological Euler characteristic (also called Euler number) of X.

If X is a complete variety, the following notation is used:

$c_i(\mathcal{F})$: the i-th Chern class of the locally free sheaf $\mathcal{F}$ on X.

$[\mathcal{L}]$: the Cartier divisor associated to an inverticle sheaf $\mathcal{L}$ on X.

$h^i(X, \mathcal{F})$: dimension of $H^i(X, \mathcal{F})$.

$\chi(X, \mathcal{F})$: the Euler characteristic of $\mathcal{F}$; by definition, $\chi(X, \mathcal{F}) = \sum_i (-1)^i h^i(X, \mathcal{F})$.

$|D|$: complete linear system associated to the divisor D on X.

If X is a complete smooth variety, we use the following notation:

Ω^1_X: locally free sheaf of holomorphic 1-forms on X.

K_X: canonical divisor on X; by definition, $\mathcal{O}_X(K_X) = \det\Omega^1_X$.

$c_i(X)$: the i-th Chern class of X; by definition, $c_i(X) = c_i(\widehat{\Omega^1_X}) = (-1)^i c_i(\Omega^1_X)$. Note that $c_2(X) = e(X)$ if X is a smooth surface.

$\mathrm{Pic}(X)$: Picard group of X i.e., the group of isomorphism classes of invertible sheaves on X.

$P_m(X)$: the m-th plurigenus $h^0(X, \mathcal{O}_X(mK_X))$ of X.

If X is a complete surface, we use the following notation:

$p_g(X)$: the geometric genus $h^2(X, \mathcal{O}_X)$ of X.

$q(X)$: the irregularity $h^1(X, \mathcal{O}_X)$ of X.

$p_a(X)$: the arithmetic genus $p_g(X) - q(X)$ of X.

1 Basic invariants of complex manifolds

In this section we discuss some important invariants associated to a complex manifold and how these change under birational transformations. By [1], Corollary III.4.4, every birational map is a composition of finitely many blow-ups, so it suffices to study how the invariants change under a blow-up at a single point. We begin with the following fundamental result from [1].

Theorem 1.1 ([1], I.9.1)**.** *Let X be a complex manifold of dimension $n \geqslant 2$, and $p : \bar{X} \to X$ the blow-up of X at a point x_0. Let $E = p^{-1}(x_0)$ be the exceptional divisor on $\bar{X}$, which is isomorphic to $\mathbb{P}^{n-1}$. Then*

1. *p induces an isomorphism between the fields of meromorphic functions on X and $\bar{X}$. In particular, if X (and hence $\bar{X}$) is compact, then X and $\bar{X}$ have the same algebraic dimension.*
2. *$p_*\mathcal{O}_{\overline{X}} = \mathcal{O}_X$ and $R^i p_*(\mathcal{O}_X) = 0$ for $i \geqslant 1$.*
3. *$p^* : H^i(X, \mathcal{O}_X) \to H^i(\bar{X}, \mathcal{O}_{\overline{X}})$ is an isomorphism for all $i \geqslant 0$.*
4. *$p^* : H^i(X, \mathbb{Z}) \to H^i(\widehat{X}, \mathbb{Z})$ is bijective for $i = 1$ and injective for $i = 2$. Furthermore,*
$$H^2(\bar{X}, \mathbb{Z}) \cong p^*(H^2(X, \mathbb{Z})) \oplus \mathbb{Z}\{e\},$$
where $e = c_1(\mathcal{O}_{\overline{X}}(E))$.
5. *For every $a \in H^2(X, \mathbb{Z})$, we have $p_! p^*(a) = a$.*
6. *$p^* : H^1(X, \mathcal{O}_X) \to H^1(\bar{X}, \mathcal{O}_{\overline{X}})$ is injective and thus $Pic(\bar{X})$ is isomorphic to the product of $Pic(X)$ and the infinite cyclic group generated by $\mathcal{O}_{\overline{X}}(E)$.*
7. *$\mathcal{O}_{\overline{X}}(K_{\widehat{X}}) = p^*(\mathcal{O}_X(K_X)) \otimes \mathcal{O}_{\overline{X}}((dim(X) - 1)E)$.*
8. *p induces an isomorphism $p^* : \Gamma(X, \mathcal{O}_X(mK_X)) \to \Gamma(\bar{X}, \mathcal{O}_{\overline{X}}(mK_{\overline{X}}))$ for all $m \geqslant 1$, so if X is compact, $P_m(\bar{X}) = P_m(X)$ for $m \geqslant 1$ and $\kappa(\bar{X}) = \kappa(X)$.*

For a complex surface X, the numbers $c_1^2(X)$ and $c_2(X)$ play a central role in the discussion that follows and so it is important to know how these numbers change if X is blown up at a point.

Lemma 1.2. *Let X be a smooth, connected complex surface, and let $p : \bar{X} \to X$ be the blow-up of X at a point x_0. Then we have*

$$c_1^2(\bar{X}) = c_1^2(X) - 1$$
$$c_2(\bar{X}) = c_2(X) + 1.$$

Proof. This is an easy consequence of Theorem 1.1 and Noether's formula. For a smooth complex surface Y, we have, by definition, $c_2(Y) = e(Y) = \sum_{i=0}^4 (-1)^i b_i(Y)$, where $b_i(Y)$ denotes the i-th Betti number of Y. Poincare duality implies that $b_i(Y) = b_{4-i}(Y)$, moreover if Y is connected, we have $b_0(Y) = b_4(Y) = 1$. Let $E = p^{-1}(x_0)$ be the exceptional divisor on $\bar{X}$. Then $E \cong \mathbb{P}^1$ and $\pi : \bar{X} \backslash E \to X \backslash \{x_0\}$ is a biregular map. Hence $\bar{X}$ is also a smooth, connected surface, and statement 4 of Theorem 1.1 implies that $b_1(\bar{X}) = b_1(X)$ and $b_2(\bar{X}) = b_2(X) + 1$. This implies that $c_2(\bar{X}) = e(\bar{X}) = \sum_{i=0}^4 b_i(\bar{X}) = 1 - b_1(X) + b_2(X) + 1 - b_3(X) + 1 = c_2(X) + 1$.
The holomorphic Euler characteristic $\chi(X, \mathcal{O}_X)$ of X is defined as $\chi(X, \mathcal{O}_X) = \sum_i (-1)^i h^1(X, \mathcal{O}_X)$. From statement 3 of Theorem 1.1 we get $\chi(X, \mathcal{O}_X) = \chi(\bar{X}, \mathcal{O}_{\overline{X}})$. Noether's formula says

$$\chi(X, \mathcal{O}_X) = \frac{1}{12}(c_1^2(X) + c_2(X)),$$

from which it follows that $c_1^2(X) + c_2(X) = c_1^2(\bar{X}) + c_2(\bar{X})$. Since we know that $c_2(\bar{X}) = c_2(X) + 1$, we get $c_1^2(\bar{X}) = c_1^2(X) - 1$. This proves the assertion. $\square$

Lemma 1.3. *Let X be a smooth projective surface. Then*

$$\chi(X, \mathcal{O}_X) = 1 + p_g(X) - q(X) = 1 + p_a(X).$$

If X is Kähler then

$$c_2(X) = 2 - 4q(X) + b_2(X).$$

Proof. Recall that $\chi(X, \mathcal{O}_X) = \sum_i (-1)^i h^i(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X)$. We also have $h^i(X, \mathcal{O}_X) = \dim H^i(X, \Omega_X^0) = h^{0,i}(X)$. Note that $h^{0,0} = b_0 = 1$, and $h^{0,1} = q(X)$ and $h^{0,2}(X) = p_g(X)$ by definition. Hence it follows that $\chi(X, \mathcal{O}_X) = 1 - q(X) + p_g(X)$.
If X is Kähler, then the first Betti number $b_1(X)$ is even i.e., $h^{0,1}(X) = h^{1,0}(X)$ (see [1], Theorem IV.3.1) and so $b_1(X) = b_3(X) = 2q(X)$. Thus it follows that $c_2(X) = \sum_{i=0}^4 b_i(X) = 2 - 4q(X) + b_2(X)$. This proves the assertion. $\square$

Remark 1.4. *Let X be a smooth complex surface. Any birational transformation of X is a composition of finitely many blow-ups, and so statement 3 of Theorem 1.1 implies that the holomorphic Euler characteristic $\chi(X, \mathcal{O}_X)$, geometric genus $p_g(X)$, and irregularity $q(X)$ are invariant under birational transformations of X. If in addition X is compact and connected, then statement 8 of Theorem 1.1 implies that the plurigenera $P_m(X)$ are also invariant under birational transformations of X for $m \geqslant 1$.*

For any compact, connected, oriented (not necessarily differentiable) manifold X, the index $\tau(X)$ is defined as follows. If $\dim(X) \not\equiv 0 \mod 4$, then set $\tau(X) = 0$. If $\dim(X) = 4m$, the cup product form defines on $H^{2m}(X, \mathbb{Q})$ a non-degenerate quadratic form $Q(X)$, and we set $\tau(X) = \tau(Q(X))$, i.e., $\tau(X) = b^+(X) - b^-(X)$, where $b^+(X)$ and $b^-(X)$ denote the number of positive and negative eigenvalues of Q respectively. Note that $b_{2m}(X) = b^+(X) + b^-(X)$. Writing $H^*(X, \mathbb{Q}) = \sum_i H^i(X, \mathbb{Q})$, we can make $H^*(X, \mathbb{Q})$ into a graded ring by means of the cup product. For any element $e \in H^*(X, \mathbb{Q})$, let $t_i(e)$ denote the component of e which is in dimension i. Thus, given an isomorphism $H^n(X, \mathbb{Q}) \cong \mathbb{Q}$, $t_n(e)$ is a rational number. Let $L(X) \in H^*(X, \mathbb{Q})$ denote the L-class of the tangent bundle of X (see [1], Chapter I.3 and references therein for a more detailed discussion). We now state an important result due to Hirzebruch.

Theorem 1.5 ([1], Theorem I.3.1 (Thom-Hirzebruch index theorem))**.** *Let X be a compact, connected, oriented differentiable manifold of dimension $4m$. Then,*

$$\tau(X) = t_{4m}(L(X)).$$

In particular, if $m = 1$, and X carries an almost-complex structure, then $\tau(X) = b^+(X) - b^-(X) = \frac{1}{3}(c_1^2(X) - 2c_2(X))$.

Lemma 1.6 ([1], Lemma IV.2.6)**.** *For every compact complex surface X the following inequalities hold:*

1. $2h^{1,0}(X) \leqslant h^{0,1}(X) + h^{1,0}(X) \leqslant 2h^{0,1}(X)$
2. $2p_g(X) \leqslant b^+(X)$.

Theorem 1.7 ([1], Theorem IV.2.7)**.** *Let X be a compact complex surface. Then*

1. *if $b_1(X)$ is even, then $h^{1,0}(X) = h^{0,1}(X)$ and $b^+(X) = 2p_g(X) + 1$;*

2. *if $b_1(X)$ is odd, then $h^{1,0}(X) = h^{0,1}(X) - 1$ and $b^+(X) = 2p_g(X)$;*

3. *$q(X)$ and $p_g(X)$ are topological invariants, $q(X)$ of the non-oriented, and $p_g(X)$ of the oriented underlying manifold.*

Proof. From the Thom-Hirzebruch theorem (Theorem 1.7) it follows that

$$b^+(X) - b^-(X) = \frac{1}{3}(c_1^2(X) - 2c_2(X)). \tag{1}$$

Using Noether's formula and Lemma 1.3, we have

$$1 - q(X) + p_g(X) = \frac{1}{12}(c_1^2(X) + c_2(X)). \tag{2}$$

Multiplying equation 2 by 4 and subtracting equation 1, we get

$$4 - 4q(X) + 4p_g(X) - b^+(X) + b^-(X) = c_2(X) = e(X) = 2 - 2b_1(X) + b^+(X) - b^-(X),$$

where we have used $b_2(X) = b^+(X) + b^-(X)$. Rearranging the terms in this equation, we get

$$(b^+(X) - 2p_g(X)) + (2q(X) - b_1(X)) = 1.$$

From Lemma 1.6 we know that each term in brackets in the left hand side of the above equation is a non-negative integer. Thus there are exactly two possibilities, which are statements 1 and 2 of the theorem. $\square$

Remark 1.8. *Noether's formula, together with statement 3 of Theorem 1.7 implies that $c_1^2(X)$ is a topological invariant of the underlying oriented manifold. This is also clear from the Thom-Hirzebruch index theorem.*

2 Some tools from algebraic geometry

2.1 Cyclic coverings

We begin with studying covering maps between complex spaces, because these will appear often in the discussion to follow. Although the meaning of "covering space of a topological space" depends largely on the context, we have the following broad definition.

Definition 2.1. *A covering space or cover of a topological space X is a topological space Y together with a continuous map $\pi : Y \to X$ such that every point $x \in X$ has an open neighbourhood $U_x \subset X$ such that $\pi^{-1}(U_x)$ is a disjoint union of open sets in Y, each of which is mapped homeomorphically onto U_x via π.*

We can modify Definition 2.1 to suit the situation we are in. For example, suppose X and Y are connected complex spaces and $\pi : X \to Y$ is a surjective holomorphic map such that all points $y \in Y$ have a connected neighbourhood V_y, with the property that $\pi^{-1}(V_y)$ is a disjoint union of open subsets of X, each of which is mapped isomorphically onto V_y via π. In this case X is called an analytic covering space of Y and $\pi : X \to Y$ is the covering map.
A covering map $\pi : X \to Y$ of topological spaces is called finite if for every point $y \in Y$, the fibre $\pi^{-1}(y)$ is a discrete, finite subset of X. The fibres are homeomorphic over each connected component of Y. If Y is connected then the degree of the covering map is defined as the cardinality of a fibre. Let X and Y be schemes. Then a covering map $\pi : X \to Y$ is called flat if it is flat as a morphism of schemes. Now suppose that X and Y are complex manifolds of the same dimension. A continuous map $\pi : X \to Y$ is a branched covering if, away from a closed subspace S of Y, the map $\pi : X \backslash \pi^{-1}(S) \to Y \backslash S$ is a covering map

as in Definition 2.1, and S has codimension at least 1 in Y. The subspace $S \subset Y$ is called the branch or ramification locus of π. Note that the cardinality of a fibre over any point in the branch locus is strictly less than the cardinality of a fibre over any point not in the branch locus.

A cyclic cover is a branched covering space for which the set of covering transformations forms a cyclic group. Cyclic coverings are a useful tool to construct new examples of surfaces. We study the construction of a cyclic cover of a complex manifold branched along a divisor. We first consider a local description of the m-fold cyclic covering of a variety branched along a divisor as given in [8]. Let X be an affine variety and let $s \in \mathbb{C}(X)$ be a non-zero regular function. The aim is to construct a variety Y on which the m-th root $\sqrt[m]{s}$ of s makes sense. To do this, we begin by taking the product $X \times \mathbb{A}^1$ of X and the affine line. Let t be the coordinate on $\mathbb{A}^1$ and let $Y \subset X \times \mathbb{A}^1$ be the subvariety defined by the equation $t^m - s = 0$.

$$\begin{tikzcd}
\{t^m - s = 0\} = Y \arrow[dr, "\pi"'] & \subset & X \times \mathbb{A}^1 \arrow[dl, "\mathrm{pr}_1"] \\
& X &
\end{tikzcd}$$

The natural mapping $\pi : Y \to X$ is a cyclic covering of X brached along the zero divisor D of s. Setting $s' = t|_Y \in \mathbb{C}(Y)$, we have the equality

$$(s')^m = \pi^* s$$

of functions on Y. Thus we have constructed the desired m-th root of s. Note that the function s' defines a divisor D' on Y which satisfies $\pi^* D = mD'$.

This local construction can be globalized by means of the following result, which is proposition 4.1.6 in [8].

Proposition 2.2 (Cyclic coverings)**.** *Let X be a variety and $\mathcal{L}$ a line bundle on X. Let m be a positive integer and let $s \in \Gamma(X, \mathcal{L}^{\otimes m})$ be a non-zero section defining a divisor D on X. Then there exists a finite flat covering $\pi : Y \to X$, where Y is a scheme with the property that the line bundle $\pi^*\mathcal{L}$ has a section*

$$s' \in \Gamma(Y, \pi^*\mathcal{L}) \quad \textit{with} \quad (s')^m = \pi^* s.$$

The zero divisor D' of s' maps isomorphically to D. Moreover, if X and D are non-singular, so too are Y and D'.

Proof. This can be proved by taking an affine open covering $\{U_i\}$ of X over which $\mathcal{L}$ is locally trivial, and carrying out the above local construction over each U_i. Since s is a section of the m-th tensor power of the line bundle $\mathcal{L}$, the resulting local coverings can be glued together. However, this local construction can be globalized in a more direct manner.

More formally, let L be the total space of the line bundle $\mathcal{L}$ and let $p : L \to X$ be the bundle projection. In other words, we have $L = \mathrm{Spec}_{\mathcal{O}_X} \mathrm{Sym}(\widehat{\mathcal{L}})$. Then, there is a tautological section $t \in \Gamma(L, p^*\mathcal{L})$. In fact, a section of $p^*\mathcal{L}$ is specified geometrically by giving for each point $a \in L$ a vector in the fibre of p over $x = p(a)$. But a itself is such a vector, and we set $t(a) = a$. More formally, t is determined by a homomorphism

$$\mathcal{O}_L \longrightarrow p^*\mathcal{L}$$

of $\mathcal{O}_L$-modules, or equivalently, by a mapping

$$\mathrm{Sym}_{\mathcal{O}_X}(\widehat{\mathcal{L}}) \longrightarrow \mathcal{L} \otimes \mathrm{Sym}_{\mathcal{O}_X}(\widehat{\mathcal{L}}) \tag{3}$$

of quasi-coherent sheaves on X. The term on the left in (3) is naturally a summand of the term on the right, and the map is the canonical inclusion.

The proposition now follows, by taking $Y \subset L$ to be the zero divisor of the section

$$t^m - p^*s \in \Gamma(L, p^*\mathcal{L}^{\otimes m}),$$

and $s' = t|_Y$. We then have a finite flat map $\pi : Y \to X$ such that $(s')^m = \pi^*s \in \Gamma(Y, \pi^*\mathcal{L}^{\otimes m})$, with $s' \in \Gamma(Y, \pi^*\mathcal{L})$. The other assertions of the proposition follow from the local construction. $\square$

As an application, we have the following result, which will be used in the proof of Bogomolov's lemma (see Theorem 3.13).

Lemma 2.3. *Let X be a non-singular projective variety and $\mathcal{L}$ a line bundle on X. Let n be a positive integer and let $f_1, f_2, f_3 \in \Gamma(X, \mathcal{L}^{\otimes n})$ be non-zero sections. Then there exists a scheme Y and a finite flat covering $\pi : Y \to X$ such that $\pi^* f_i = g_i^n$, where $g_i \in \Gamma(Y, \pi^*\mathcal{L})$ for $i = 1, 2, 3$.*

Proof. The sections f_1, f_2, and f_3 define divisors D_1, D_2, and D_3 respectively, on X. Let $\pi_1 : X_1 \to X$ be an n-sheeted cyclic covering of X branched along D_1 (see Proposition 2.2). Then, there exists a divisor D_1' on X_1 given by a section $f_1' \in \Gamma(X_1, \pi_1^*\mathcal{L})$ such that $\pi_1^* D_1 = nD_1'$, i.e., $\pi_1^* f_1 = (f_1')^n$. Let $D_2' = \pi_1^* D_2$ and $D_3' = \pi_1^* D_3$. Then D_2' and D_3' are divisors on X_1 defined by the sections $f_2' = \pi_1^* f_2$ and $f_3' = \pi_1^* f_3$ respectively, where $f_2', f_3' \in \Gamma(X_1, \pi_1^*\mathcal{L})$.
Now let $\pi_2 : X_2 \to X_1$ be an n-sheeted cyclic covering of X_1 branched along $D_2' = \pi_1^* D_2$. Then, there is a divisor D_2'' on X_2 given by a section $f_2'' \in \Gamma(X_2, \pi_2^*(\pi_1^*\mathcal{L}))$ such that $\pi_2^* D_2' = nD_2''$, i.e., $\pi_2^* f_2' = (f_2'')^n$. Let $D_1'' = \pi_2^* D_1'$ and $D_3'' = \pi_2^* D_3'$. Then D_1'' and D_3'' are divisors on X_2 defined by the sections $f_1'' = \pi_2^* f_1'$ and $f_3'' = \pi_2^* f_3'$ respectively, where $f_2'', f_3'' \in \Gamma(X_2, \pi_2^*(\pi_1^*\mathcal{L}))$.
Finally, let $\pi_3 : X_3 \to X_2$ be an n-sheeted cyclic covering of X_2 branched along $D_3'' = \pi_2^* D_3'$. As before, there is a divisor D_3''' on X_3 given by a section $f_3''' \in \Gamma(X_3, \pi_3^*(\pi_2^*(\pi_1^*\mathcal{L})))$ such that $\pi_3^* D_3'' = nD_3'''$, i.e., $\pi_3^* f_3'' = (f_3''')^n$. Let $D_1''' = \pi_3^* D_1''$ and $D_2''' = \pi_3^* D_2''$. Then D_1''' and D_2''' are divisors on X_3 defined by the sections $f_1''' = \pi_3^* f_1''$ and $f_2''' = \pi_3^* f_2''$ respectively, where $f_1''', f_2''' \in \Gamma(X_3, \pi_3^*(\pi_2^*(\pi_1^*\mathcal{L})))$.
Now take $Y = X_3$. Then $\pi = \pi_1 \circ \pi_2 \circ \pi_3 : Y \to X$ is a finite flat covering of X. Moreover, taking $g_i = f_i''' \in \Gamma(Y, \pi^*\mathcal{L})$ we get $\pi^* f_i = g_i^n \in \Gamma(Y, \pi^*\mathcal{L}^{\otimes n})$ for $i = 1, 2, 3$. Thus the assertion is proved. $\square$

2.2 Invariants of good covers

We now discuss coverings branched over subvarieties with transverse intersections. Let X be a complex surface i.e., a complex manifold of dimension 2. Let $\{D_i\}$, $i \in I$ for some finite index set I, be a set of smooth, one-dimensional irreducible subvarieties. In a neighbourhood of any point $p \in D_i$, we can choose local coordinates (u, v) on X such that D_i is given locally by the equation $u = 0$, and u is called a normal coordinate to D_i at p. We assume that for $i \neq j$, D_i and D_j intersect transversally i.e., for any $p \in D_i \cap D_j$, we have normal crossing at p. This means that we can assign local coordinates (u, v) at p such that D_i is given locally by $u = 0$ and D_j is given locally by $v = 0$. Moreover, we assume that no more than two of the D_i intersect at one point i.e., $\bigcup_i D_i$ consists of only ordinary double points.
Under these conditions, a good covering of X, as in [13], Def. 3.1, is given by the following definition.

Definition 2.4. *Let Y be a complex surface that is a finite covering $\pi : Y \to X$ of X. Suppose that π is branched along a system $\{D_i\}$ of one-dimensional subvarieties of X intersecting transversally. The covering is a good covering if, in addition, there are integers $N \geqslant 1$ and $b_i \geqslant 2$ for all $i \in I$, such that*
(i) for all $i \in I$, we have $b_i | N$ and there are N/b_i points of Y over each point of $D_i \backslash \bigcup_{i \neq j} D_i \cap D_j$.
(ii) for all $i, j \in I$, $i \neq j$ and $D_i \cap D_j \neq \varnothing$, we have $b_i b_j | N$ and there are $N/b_i b_j$ over each point of $D_i \cap D_j$.
(iii) over the points not appearing in (i) and (ii), there are N points of Y, and N is called the degree of this covering.

The following result gives a local description of a good cover.

Lemma 2.5. *Let the setting be as above and let $\pi : Y \to X$ be a good cover of X as in Definition 2.4. Then we have*
(i) Centered at each point q of Y lying over a point of $D_i \backslash \bigcup_{i\neq j} D_i \cap D_j$, there are local coordinates (s,t) such that $u = s^{b_i}$, $v = t$ are local coordinates centered at $\pi(q)$, with u a normal coordinate to D_i at $\pi(q)$. The map π is given locally by the quotient of an open neighbourhood of q by the action of $\mathbb{Z}/b_i\mathbb{Z}$ by $(s,t) \mapsto (exp(2\pi im/b_i)s, t)$ for $m \in \mathbb{Z}/b_i\mathbb{Z}$;
(ii) Centered at each point q of Y lying over a point of $D_i \cap D_j$, there are local coordinates (s,t) such that $u = s^{b_i}$, $v = t^{b_j}$ are local coordinates centered at $\pi(q)$, with u a normal coordinate to D_i at $\pi(q)$ and v a normal coordinate to D_j at $\pi(q)$. The map π is given locally by the quotient of an open neighbourhood of q by the action of $(\mathbb{Z}/b_i\mathbb{Z}) \times (\mathbb{Z}/b_j\mathbb{Z})$ by $(s,t) \mapsto (exp(2\pi im/b_i)s, exp(2\pi in/b_j)t)$ for $m \in \mathbb{Z}/b_i\mathbb{Z}$, $n \in \mathbb{Z}/b_j\mathbb{Z}$;
(iii) At any point q of Y not appearing in (i) or (ii), the map π is locally biholomorphic.

We now derive expressions for the Euler number and the self-intersection number of the canonical divisor of a good covering, as given in [13]. These formulae will be useful in constructing examples of surfaces which satisy the extreme case of the Bogomolov-Miyaoka-Yau inequality. Recall that for a smooth complex surface X, the second Chern class equals the Eucler characteristic (also called the Euler number), i.e., $c_2(X) = e(X)$. Let the setting be as in Definition 2.4.

Lemma 2.6. *The Euler number of a good covering Y of X of degree N is given by*

$$\frac{e(Y)}{N} = \frac{c_2(Y)}{N} = c_2(X) - \sum_i x_i e(D_i) + \frac{1}{2}\sum_{i\neq j} x_i x_j D_i D_j, \tag{4}$$

where we set $x_i = 1 - \frac{1}{b_i}$ for all i.

Proof. We compute the Euler number of Y in two parts: first we compute the contribution by the complement of the ramification locus and then the contribution by the ramification locus.
The Euler characteristic of each divisor D_i on X is $e(D_i)$ and so the total Euler characteristic of all divisors is $\sum_i e(D_i)$. Note that in this sum we have counted each intersection point in $D_i \cap D_j$ for $i \neq j$ twice- once on D_i and once on D_j. The total number of intersection points is $\frac{1}{2}\sum_{i\neq j} D_i D_j$. Thus the Euler characteristic of the branch locus on X is $\sum_i e(D_i) - \frac{1}{2}\sum_{i\neq j} D_i D_j$. Since there are N points of Y above each point of X in the complement of the branch locus, the Euler characteristic of the complement of the ramification locus on Y is given by

$$N\Big(e(X) - \sum_i e(D_i) + \frac{1}{2}\sum_{i\neq j} D_i D_j\Big). \tag{5}$$

Recall that over each point of $D_i \backslash D_i \cap D_j$ for $i \neq j$ there are N/b_i points of Y i.e., over each point of each divisor D_i except the intersection points, there are N/b_i points of Y. Thus the contribution to the Euler characteristic from divisors minus intersection points is $\sum_i \frac{N}{b_i} e(D_i) - \sum_{i\neq j} \frac{N}{b_i} D_i D_j$. Over each intersection point in $D_i \cap D_j$ for $i \neq j$ there are $N/b_i b_j$ points of Y. Hence the contribution to the Euler characteristic by intersection points is $\frac{1}{2}\sum_{i\neq j} \frac{N}{b_i b_j} D_i D_j$. Thus the contribution of the ramification locus is

$$N\sum_i \frac{1}{b_i}\Big(e(D_i) - \sum_{j\neq i} D_i D_j\Big) + \frac{N}{2}\sum_{i\neq j} \frac{1}{b_i b_j} D_i D_j. \tag{6}$$

Summing the expressions 5 and 6 we get the Euler characteristic of Y, which is given by

$$c_2(Y) = N\Big(c_2(X) - \sum_i e(D_i) + \frac{1}{2}\sum_{i\neq j} D_i D_j\Big) + N\sum_i \frac{1}{b_i}\Big(e(D_i) - \sum_{j\neq i} D_i D_j\Big) + \frac{N}{2}\sum_{i\neq j} \frac{1}{b_i b_j} D_i D_j. \tag{7}$$

Rearranging terms in the right hand side of the equation 7 and writing

$$N \sum_{i \neq j} \frac{1}{b_i} D_i D_j = \frac{N}{2} \sum_{i \neq j} \frac{1}{b_i} D_i D_j + \frac{N}{2} \sum_{i \neq j} \frac{1}{b_j} D_i D_j,$$

we get

$$\begin{aligned} c_2(Y) &= N c_2(X) - N \sum_i \Big(1 - \frac{1}{b_i}\Big) e(D_i) + \frac{N}{2} \Big(\sum_{i \neq j} D_i D_j - \sum_{i \neq j} \frac{1}{b_i} D_i D_j - \sum_{i \neq j} \frac{1}{b_j} D_i D_j + \sum_{i \neq j} \frac{1}{b_i b_j} D_i D_j \Big) \\ &= N c_2(X) - N \sum_i \Big(1 - \frac{1}{b_i}\Big) e(D_i) + \frac{N}{2} \sum_{i \neq j} \Big(1 - \frac{1}{b_i}\Big)\Big(1 - \frac{1}{b_j}\Big) D_i D_j. \end{aligned}$$

Setting $x_i = 1 - \frac{1}{b_i}$ for all i and dividing the above equation by N, we get

$$\frac{c_2(Y)}{N} = c_2(X) - \sum_i x_i e(D_i) + \frac{1}{2} \sum_{i \neq j} x_i x_j D_i D_j,$$

which is the equality 4. $\square$

Let R denote the ramification divisor of π on Y. Then the canonical divisor of Y is given by $K_Y = \pi^* K_X + R$ (see for example equation 20 on p.53 in [1]). Note that R is the vanishing locus of the determinant of the Jacobian of π on Y. We now derive an expression for K_Y in terms of the branch locus on X, by following Lemma I.16.1 of [1].

Lemma 2.7. *Let the setting be as in Definition 2.4. Then the canonical divisor on Y is given by*

$$K_Y = \pi^*(K_X + \sum_i x_i D_i), \tag{8}$$

where $x_i = 1 - \frac{1}{b_i}$ for all i, as before.

Proof. Let $R = \sum_i r_i R_i$, where r_i are integers for all i, and the R_i are irreducible components of R such that $\pi(R_i) = D_i$ for all i. We know that at any point $y \in R_i$ such that $\pi(y) = x \in D_i \backslash D_i \cap D_j$ for $i \neq j$, the branching order of π is b_i. Let (u, v) be local coordinates on X centered at x such that D_i is given locally by the equation $u = 0$ at x. If R_i is given locally by the equation $t = 0$ at y, then we have $\pi^*(u) = t^{b_i}$. Setting $\omega = du \wedge dv$, we get $\pi^*(\omega) = b_i t^{b_i - 1} dt \wedge ds$, where $s = \pi^*(v)$. This shows that (t, s) is a local coordinate system at y and that the zero divisor of $\pi^*(\omega)$ is $(b_i - 1) R_i$. Thus we have $r_i = b_i - 1$ for all i. This implies that

$$K_Y = \pi^* K_X + \sum_i (b_i - 1) R_i. \tag{9}$$

Since D_i is locally given by the equation $u = 0$ at x, $\pi^* D_i$ is locally given by $\pi^*(u) = t^{b_i} = 0$ at y. Recall that R_i is locally given by the equation $t = 0$ at y, and hence $\pi^* D_i = b_i R_i$ as divisors on Y. Plugging this into the equation 9, we get

$$K_Y = \pi^* K_X + \sum_i \frac{b_i - 1}{b_i} \pi^* D_i.$$

Setting $x_i = 1 - \frac{1}{b_i} = \frac{b_i - 1}{b_i}$ as before, it follows that

$$K_Y = \pi^* K_X + \sum_i x_i \pi^* D_i = \pi^*(K_X + \sum_i x_i D_i),$$

which is the expression 8. $\square$

Since the degree of the map $\pi : Y \to X$ is N, it follows from Lemma 2.7 that the self intersection number of the canonical class on Y is given by

$$K_Y^2 = N(K_X + \sum_i x_i D_i)^2. \tag{10}$$

Expressing the formula 10 in terms of the first Chern numbers of X and Y, we have

$$\frac{c_1^2(Y)}{N} = \Big(K_X + \sum_i x_i D_i\Big)^2 = c_1^2(X) + 2\sum_i x_i K_X D_i + \sum_i x_i^2 D_i^2 + \sum_{i \neq j} x_i x_j D_i D_j. \tag{11}$$

In order to simplify this expression, we use the following well-known result.

Lemma 2.8 (Adjunction formula)**.** *Let X be a complex manifold of dimension 2 and D a smooth submanifold of X of dimension 1. Then*

$$e(D) = -(K_X D + D^2). \tag{12}$$

Proof. A proof of the adjunction formula is given in [3], Proposition V.1.5. ☐

Lemma 2.9. *Let the setting be as in Definition 2.4. Then the self-intersection number of the canonical divisor of Y is given by*

$$\frac{c_1^2(Y)}{N} = c_1^2(X) - 2\sum_i x_i(e(D_i) + D_i^2) + \sum_i x_i^2 D_i^2 + \sum_{i \neq j} x_i x_j D_i D_j. \tag{13}$$

Proof. We can add and subtract $2\sum_i x_i D_i^2$ from the right hand side of equation 11, which gives

$$\frac{c_1^2(Y)}{N} = c_1^2(X) + 2\sum_i x_i(K_X D_i + D_i^2) - 2\sum_i x_i D_i^2 + \sum_i x_i^2 D_i^2 + \sum_{i \neq j} x_i x_j D_i D_j.$$

Using the adjunction formula 12, we replace $K_X D_i + D_i^2$ in the above equality by $-e(D_i)$ to get the equality 13. This proves the assertion. ☐

2.3 Construction of a Kummer covering

Let $\widetilde{\mathbb{P}}^2$ denote the blow up of $\mathbb{P}^2$ at the r-fold ($r \geqslant 3$) intersection points of a line arrangement in $\mathbb{P}^2$ and Y a good covering of $\widetilde{\mathbb{P}}^2$ of degree N as before. A special case occurs when we take the ramification indices of all divisors on $\widetilde{\mathbb{P}}^2$ to be the same positive integer $n \geqslant 2$. In this case Y is called a *Kummer covering* of $\widetilde{\mathbb{P}}^2$. We derive the conditions necessary for such a surface Y to satisfy the equality $3c_2(Y) = c_1^2(Y)$ in section 7.4. Now we discuss the construction of a Kummer covering starting from a line arrangement in $\mathbb{P}^2$.
Consider an arrangement of k lines in $\mathbb{P}^2$, defined by the equations $l_1 = 0, l_2 = 0, ..., l_k = 0$. We assume that the arrangement does not form a pencil, i.e., not all of the k lines pass through a single point. Let $\phi : \mathbb{P}^2 \to \mathbb{P}^{k-1}$ be the map defined by sending a point $x \in \mathbb{P}^2$ to the point $(l_1(x) : l_2(x) : ... : l_k(x)) \in \mathbb{P}^{k-1}$. Note that this map is well defined because we have assumed that the arrangement is not a pencil i.e., the l_i's do not all simultaneously vanish, and they are all homogeneous of the same degree. Let $\nu : \mathbb{P}^{k-1} \to \mathbb{P}^{k-1}$ denote the Fermat covering of $\mathbb{P}^{k-1}$ by itself, i.e., the map given by sending a point $(x_0 : x_1 : ... : x_{k-1}) \in \mathbb{P}^{k-1}$ to the point $(x_0^n : x_1^n : ... : x_{k-1}^n)$, where we take n to be the ramification index assigned to each line in the arrangement. It is straightforward to see that the degree of this map is n^{k-1}. We now define the variety X as follows

$$\mathbb{P}^2 \times_{\mathbb{P}^{k-1}} \mathbb{P}^{k-1} \supset X = \{(x, y) \in \mathbb{P}^2 \times_{\mathbb{P}^{k-1}} \mathbb{P}^{k-1} \mid \phi(x) = \nu(y)\}.$$

Let $\pi : X \to \mathbb{P}^2$ denote projection in the first factor, i.e., the map defined by sending a point $(x, y) \in X$ to the point $x \in \mathbb{P}^2$. Then the following statement is clear.

Lemma 2.10. *The map $\pi : X \to \mathbb{P}^2$ defined above is a finite surjective map of degree n^{k-1}. Moreover, this map is branched exactly along the line arrangement on $\mathbb{P}^2$ with ramification index n along each line and n^r at each r-fold intersection point of the arrangement. Any point q in X lying above a point p in $\mathbb{P}^2$ is a singular point if and only if p is an r-fold intersection point of the line arrangement.*

Thus if the line arrangement on $\mathbb{P}^2$ has r-fold intersection points then X is not smooth. Let $\rho : \widetilde{\mathbb{P}}^2 \to \mathbb{P}^2$ denote the blow-up of $\mathbb{P}^2$ at the r-fold intersection points of the arrangement. Then the surface $\widetilde{\mathbb{P}}^2$ is smooth, and the new arrangement on $\widetilde{\mathbb{P}}^2$ consists of proper transforms of the lines of the arrangement and exceptional divisors corresponding to the blown up points. Note that this new arrangement of divisors consists of only simple normal crossings.

Proposition 2.11. *There is a smooth surface Y together with a birational map $\tau : Y \to X$, and a surjective morphism $\sigma : Y \to \widetilde{\mathbb{P}}^2$ such that the latter is a good covering of $\widetilde{\mathbb{P}}^2$ of degree n^{k-1} in the sense of Definition 2.4. Moreover, σ is branched along the new arrangement, and for each divisor D_i in the arrangement, the ramification index at each point in the set $D_i \backslash \bigcup_{i \neq j} D_i \cap D_j$ is n, and the ramification index at each intersection point $D_i \cap D_j$ is n^2.*

The situation is represented by the following commutative diagram.

$$\begin{tikzcd} X \arrow[d, "\pi"'] & Y \arrow[l, "\tau"'] \arrow[d, "\sigma"] \\ \mathbb{P}^2 & \widetilde{\mathbb{P}}^2 \arrow[l, "\rho"] \end{tikzcd}$$

Proof. The existence of Y essentially follows from the proof of [1], Theorem III.6.1. We blow up $\mathbb{P}^2$ at all the r-fold points of the line arrangement ($r \geqslant 3$) to get $\widetilde{\mathbb{P}}^2$ and a new arrangement which consists of only simple normal crossings. We form the fibre product $X \times_{\mathbb{P}^2} \widetilde{\mathbb{P}}^2$ and observe that it is normal, since it is the fibre product of normal varieties. We set $Y = X \times_{\mathbb{P}^2} \widetilde{\mathbb{P}}^2$, and get a birational map $\tau : Y \to X$ and a map $\sigma : Y \to \widetilde{\mathbb{P}}^2$, which is a finite covering branched along the new arrangement on $\widetilde{\mathbb{P}}^2$. Note that Y is a resolution of singularities of X, i.e., Y is smooth. Since $\pi : X \to \mathbb{P}^2$ has degree n^{k-1}, it follows that $\sigma : Y \to \widetilde{\mathbb{P}}^2$ also has degree n^{k-1}.

To prove the assertion about ramification indices, we consider the affine local picture. Let p be an r-fold point of the line arrangement on $\mathbb{P}^2$ as before, let $l_1 = 0, l_2 = 0, ..., l_r = 0$ be the equations of the lines passing through p and let $l_{r+1} = 0$ be a line not passing through p. Then we can take $\frac{l_1}{l_{r+1}}$ and $\frac{l_2}{l_{r+1}}$ as local coordinates in an open affine neighbourhood U centered at p. The blow up of $p = (0,0) \in U$ is given by

$$\Big\{\Big([u:v], \Big(\frac{l_1}{l_{r+1}}, \frac{l_2}{l_{r+1}}\Big)\Big) \in \mathbb{P}^1 \times U \mid u\Big(\frac{l_2}{l_{r+1}}\Big) = v\Big(\frac{l_1}{l_{r+1}}\Big)\Big\} \longrightarrow U$$
$$\Big([u:v], \Big(\frac{l_1}{l_{r+1}}, \frac{l_2}{l_{r+1}}\Big)\Big) \longmapsto \Big(\frac{l_1}{l_{r+1}}, \frac{l_2}{l_{r+1}}\Big).$$

The exceptional divisor is $\mathbb{P}^1 \times (0,0)$, and we have two affine neighbourhoods on the blow up, namely $\{u \neq 0\}$ and $\{v \neq 0\}$. Putting $u = 1$, we have $\frac{l_2}{l_{r+1}} = v(\frac{l_1}{l_{r+1}})$. If $\frac{l_1}{l_{r+1}} = 0$, then $\frac{l_2}{l_{r+1}} = 0$ and so the exceptional divisor is given by the equation $\frac{l_1}{l_{r+1}} = 0$ in this chart. The proper transform of the line $\frac{l_2}{l_{r+1}} = 0$ is given by $v = \frac{l_2}{l_1} = 0$ in this chart. Similarly, putting $v = 1$ gives $\frac{l_1}{l_{r+1}} = u(\frac{l_2}{l_{r+1}})$, which implies that $\frac{l_2}{l_{r+1}} = 0$ defines the exceptional divisor, and $u = \frac{l_1}{l_2} = 0$ defines the proper transform of the line $\frac{l_1}{l_{r+1}} = 0$ in this chart. Let E denote the exceptional divisor corresponding to p and let D_1 and D_2 denote the proper transforms in $\widetilde{\mathbb{P}}^2$ of the lines given by $l_1 = 0$ and $l_2 = 0$ respectively. In an open affine neighbourhood of the intersection point

$E \cap D_1$, we can take $(\frac{l_1}{l_2}, \frac{l_2}{l_{r+1}})$ as local coordinates and similarly at $E \cap D_2$ we can take $(\frac{l_2}{l_1}, \frac{l_1}{l_{r+1}})$ as local coordinates. We treat all other intersection points $E_i \cap D_j$ in the new arrangement on $\widetilde{\mathbb{P}}^2$ in the same way. Let q be a singular point in X lying above p. Similarly as before, in an open affine neighbourhood of q we can take $(\frac{l_1}{l_{r+1}})^{\frac{1}{n}} = 0, (\frac{l_2}{l_{r+1}})^{\frac{1}{n}} = 0, ..., (\frac{l_r}{l_{r+1}})^{\frac{1}{n}} = 0$ as local equations of curves $L_1', L_2', ..., L_r'$ passing through q. Thus the strict transform of the curve L_1' under the birational map $\tau : Y \to X$ is given locally by $(\frac{l_1}{l_{r+1}})^{\frac{1}{n}} = 0$ in an open affine subset V of Y. Since $\tau : Y \backslash \tau^{-1}(\mathrm{Sing}(X)) \to X \backslash \mathrm{Sing}(X)$ and $\rho : \widetilde{\mathbb{P}}^2 \backslash \rho^{-1}(\mathrm{Sing}(\mathbb{P}^2)) \to \mathbb{P}^2 \backslash \mathrm{Sing}(\mathbb{P}^2)$ are isomorphisms, where $\mathrm{Sing}(X)$ denotes the set of singular points of X and $\mathrm{Sing}(\mathbb{P}^2)$ denotes the set of singular points of the line arrangement on $\mathbb{P}^2$, we have $\sigma|_{Y \backslash \tau^{-1}(\mathrm{Sing}(X))} = (\rho^{-1} \circ \pi \circ \tau)|_{Y \backslash \tau^{-1}(\mathrm{Sing}(X))}$. This implies that the ramification index of σ along the proper transform D_1' of L_1' is n. Thus D_1' is locally given by the equation $(\frac{l_1}{l_2})^{\frac{1}{n}} = 0$ in V, which implies that the exceptional divisor E' arising from blowing up q is given locally by $(\frac{l_2}{l_{r_1}})^{\frac{1}{n}} = 0$ in V. It follows that the ramification index along E' is also n. Since the ramification index of π at q is n^r, we see that E' maps onto E via σ with degree n^{r-1}. At the intersection point $D_1' \cap E'$, we can take $((\frac{l_1}{l_2})^{\frac{1}{n}}, (\frac{l_2}{l_{r+1}})^{\frac{1}{n}})$ as local coordinates in V, which implies that the ramification index at the intersection point $D_1' \cap E'$ is n^2. We treat all other intersection points $E_i' \cap D_j'$ in the new arrangement on Y in the same way.
Now suppose that $p \in \mathbb{P}^2$ is a regular intersection point of the lines defined by $l_1 = 0$ and $l_2 = 0$, and $l_3 = 0$ is any other line not passing through p. Then in an open affine neighbourhood of p, the lines meeting at p can be given locally by $\frac{l_1}{l_3} = 0$ and $\frac{l_2}{l_3} = 0$, and we can take $(\frac{l_1}{l_3}, \frac{l_2}{l_3})$ as local coordinates at p. A point q lying above p in X is not singular and is the intersection point of the lines $(l_1)^{\frac{1}{n}} = 0$ and $(l_2)^{\frac{1}{n}} = 0$. In an open affine neighbourhood of q, these lines can be locally given by $(\frac{l_1}{l_3})^{\frac{1}{n}} = 0$ and $(\frac{l_2}{l_3})^{\frac{1}{n}} = 0$, so we can take $((\frac{l_1}{l_3})^{\frac{1}{n}}, (\frac{l_1}{l_3})^{\frac{1}{n}})$ as local coordinates at q. Similarly, it follows that in an open affine neighbourhood of the intersection point $D_1 \cap D_2$ on $\widetilde{\mathbb{P}}^2$, the proper transforms D_1 and D_2 can be given locally by $\frac{l_1}{l_3} = 0$ and $\frac{l_2}{l_3} = 0$ and we can take $(\frac{l_1}{l_3}, \frac{l_2}{l_3})$ as local coordinates at this point. In an open affine neighbourhood of the intersection point $D_1' \cap D_2'$ on Y, D_1' and D_2' can be given locally by $(\frac{l_1}{l_3})^{\frac{1}{n}} = 0$ and $(\frac{l_2}{l_3})^{\frac{1}{n}} = 0$, and so we can take $((\frac{l_1}{l_3})^{\frac{1}{n}}, (\frac{l_2}{l_3})^{\frac{1}{n}})$ as local coordinates at this point. Thus the ramification index of σ at the intersection point $D_1' \cap D_2'$ is n^2. The same argument holds for all intersection points $D_i' \cap D_j'$ on Y.
At any point not on the line arrangement on $\mathbb{P}^2$, the cover $\pi : X \to \mathbb{P}^2$ is unramified of degree n^{k-1}, hence at any point on $\widetilde{\mathbb{P}}^2$ not on the new arrangement, the cover $\sigma : Y \to \mathbb{P}^2$ is also unramified of degree n^{k-1}.
Thus the three conditions of Definition 2.4 are satisfied and we conclude that $\sigma : Y \to \widetilde{\mathbb{P}}^2$ is a good covering of degree n^{k-1} branched along the new arrangement on $\widetilde{\mathbb{P}}^2$, as claimed. $\square$

The discussion of Kummer coverings is continued in Section 4.5.

We conclude this discussion by deriving formulae for the Euler characteristic and self intersection number of an exceptional curve C arising from blowing up a singular point q on X lying above an r-fold intersection point p of the line arrangement on $\mathbb{P}^2$. We also state a result of Hirzebruch, in which he gives a classification of surfaces constructed using the method described above.

Lemma 2.12. *The Euler characteristic $e(C)$ of an exceptional curve C described above is given by*

$$e(C) = n^{r-1}(2-r) + rn^{r-2}.$$

Proof. We first determine the contribution to the Euler characteristic from the complement C' of the set S of intersection points on C. We know that away from the intersection points of C with the divisors D_i', C maps to an exceptional divisor $E \cong \mathbb{P}^1$ on $\widetilde{\mathbb{P}}^2$ with degree n^{r-1}. The number of intersection points on E equals r, so we have

$$e(C') = n^{r-1}(e(\mathbb{P}^1) - r) = n^{r-1}(2-r). \tag{14}$$

We know that at each intersection point of C with a divisor D'_i, the ramification index is n^2, so above each intersection point $E \cap D_i$ on $\widetilde{\mathbb{P}}^2$, there are n^{r-2} points of C. Since the number of intersection points on E is r, we have

$$e(S) = rn^{r-2}. \tag{15}$$

Thus summing the equations 14 and 15, we obtain the Euler charcteristic of C

$$e(C) = n^{r-1}(2-r) + rn^{r-2}.$$

This completes the proof. □

Lemma 2.13. *The self intersection number of such an exceptional curve C is given by*

$$C^2 = -n^{r-2}.$$

Proof. Let E be the exceptional curve on $\widetilde{\mathbb{P}}^2$ to which C is mapped via the map $\sigma : Y \to \widetilde{\mathbb{P}}^2$. Note that we have $E^2 = -1$, and since σ has degree n^{k-1}, we get

$$(\sigma^* E)^2 = \deg(\sigma) E^2 = -n^{k-1}. \tag{16}$$

There are n^{k-1-r} singular points of X above an r-fold intersection point of the line arrangement on $\mathbb{P}^2$, each of which when blown up, gives a copy of the exceptional curve C on Y. Thus $\sigma^* E$ consists of n^{k-1-r} disjoint copies of C. Together with the equality 16, this implies

$$C^2 = \frac{-n^{k-1}}{n^{k-1-r}} = -n^{r-2},$$

as claimed. □

For a line arrangement in $\mathbb{P}^2$ which is not a pencil, recall that k denotes the number of lines, t_r denotes the number of r-fold points, and n denotes the ramification index assigned to each divisor in the corresponding new arrangement on $\widetilde{\mathbb{P}}^2$. Consider the following condition

$$k \geqslant 6, \quad n \geqslant 2, \quad t_k = t_{k-1} = t_{k-2} = 0. \tag{17}$$

Now consider a line arrangement satisfying 17 which has exactly two singular intersection points p_1 and p_2, lying on a single line L, and suppose L contains no other intersection points. Let u and v denote the number of lines passing through p_1 and p_2. Then, such an arrangement satisfies

$$\begin{aligned} &u + v - 1 = k, \quad u \geqslant 4, \quad v \geqslant 4, \\ &t_u = t_v = 1, \quad t_2 = (u-1)(v-1), \quad t_r = 0 \text{ otherwise.} \end{aligned} \tag{18}$$

Let Y, $\widetilde{\mathbb{P}}^2$, and $\sigma : Y \to \widetilde{\mathbb{P}}^2$ be as before. We can now state the following classification result due to Hirzebruch [4].

Theorem 2.14. *Assume the arrangement satisfies 17 and is not of type 18. Then the surface Y is minimal i.e., does not contain (-1)-curves. For an arrangement of type 18, the divisor $\sigma^* L'$ on Y consists of n^{k-3} disjoint (-1)-curves (each with multiplicity n), where L' is the proper transform of the line L containing the two singular points of the arrangement. Blowing down these (-1)-curves gives a minimal surface Y_0, which is a product of two curves C_1, C_2 with Euler numbers*

$$e(C_1) = n^{u-1}(2-u) + un^{u-2}, \quad n^{v-1}(2-v) + vn^{v-2}.$$

All the surfaces Y arising from arrangements satisfying 17 are of general type for $n \geqslant 3$. For $k = 6$ and $n = 2$ the surface Y is a K3 surface and for $k \geqslant 7$ and $n = 2$ it is elliptic (of Kodaira dimension $\geqslant 0$), or of general type.

For a proof of this result we refer the reader to [4], p. 127.

2.4 Hirzebruch-Jung singularities

We now discuss exceptional curves on nonsingular surfaces, because they are of fundamental importance when studying resolutions of surface singularities. We refer to [1], Section III.2 for this part.
A compact, reduced, connected curve C on a nonsingular surface X is called exceptional, if there is a bimeromorphic map $\pi : X \to Y$ such that C is exceptional for π, i.e., if there is an open neighbourhood U of C in X, a point $y \in Y$, and a neighbourhood V of y in Y, such that π maps $U \backslash C$ biholomorphically onto $V \backslash \{y\}$, and $\pi(C) = y$. Exceptional curves are characterized by the following result, known as Grauert's criterion

Theorem 2.15 ([1], Theorem III.2.1)**.** *A reduced, compact, connected curve C with irreducible components C_i on a smooth surface is exceptional if and only if the intersection matrix $(C_iC_j)_{ij}$ is negative definite.*

The following three kinds of exceptional curves are important.
1. Exceptional curves of the first kind. These are non-singular rational curves with self-intersection -1. They are also known as (-1)-curves. The following result is a useful characterization of (-1)-curves.

Proposition 2.16 ([1], Proposition III.2.2)**.** *An irreducible curve $C \subset X$ is a (-1)-curve if and only if*

$$C^2 < 0 \quad \textit{and} \quad K_X C < 0.$$

2. Hirzebruch-Jung strings. These are unions $C = \bigcup_{i=1}^r C_i$ of smooth rational curves C_i such that

$$C_i^2 \leqslant -2 \ \text{ for all } \ i,$$
$$C_iC_j = 1 \ \text{ if } \ |i-j| = 1,$$
$$C_iC_j = 0 \ \text{ if } \ |i-j| \geqslant 2.$$

If $e_i = C_i^2$ then this configuration is visualized by the dual graph

The intersection matrix

$$\begin{bmatrix} e_1 & 1 & 0 & \dots \\ 1 & e_2 & 1 & \ddots \\ 0 & 1 & e_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

is negative definite. Concrete examples of such curves are easy to construct. The simplest Hirzebruch-Jung string is a smooth rational curve with self intersection -2. Such a curve is also known as a (-2)-curve.
3. A-D-E curves. These are exceptional curves $C = \bigcup C_i$ of which all irreducible components (-2)-curves. The inequality

$$(C_i + C_j)^2 = 2(C_iC_j - 2) < 0 \ \text{ for all } \ i \neq j$$

implies that $C_iC_j \leqslant 1$ i.e., two such curves can intersect in at most one point and then transversally. Since the intersection form of C is negative definite, it must be one of the forms described by Dynkin diagram A_n with $n \geqslant 1$, D_n with $n \geqslant 4$, or E_6, E_7, or E_8 (see [1], Section I.2). Hence these Dynkin diagrams are the dual graphs of these curves. Note that the curves A_n are Hirzebruch-Jung strings. A-D-E curves are characterized by the following result

Proposition 2.17 ([1], Proposition III.2.5)**.** *Let $C \subset X$ be an exceptional curve with $K_X C_i = 0$ for each irreducible component C_i of C. Then C is an A-D-E curve.*

We now turn our attention to singularities that arise from contracting Hirzebruch-Jung strings. These are known as Hirzebruch-Jung singularities or $A_{n,q}$ singularities. We are interested in a particular case of these singularities, namely, cyclic quotient singularities, which appear in the section 7. We refer to [1], Section III.5 for this part.

Let $C = \sum_{i=1}^{r} C_i$ with $C_i^2 = e_i \leqslant -2$ for all i, be a Hirzebruch-Jung string. For a sufficiently small $X \supset C$ there is a (closed, but not necessarily compact) smooth curve C_0 which intersects C_1 transversally in one point, without meeting any of the other curves C_i. Similarly, there is a curve C_{r+1} intersecting C_r transversally in one point which does not intersect any other curve C_i. The following graph represents the situation.

Let $n_i \in \mathbb{Z}$, $n_i \geqslant 0$ for all $i = 0, ..., r+1$. It follows from [1], Section 3 that there is a holomorphic function φ on X with divisor $(\varphi) = \sum_{i=0}^{r+1} n_i C_i$ if and only if

$$n_{k_1} + e_k n_k + n_{k+1} = (\varphi)C_k = \sum_{i=0}^{r+1} C_i C_k = 0$$

for all $k = 1, ..., r$. Given n_0 and n_1, the coefficients n_k, $k = 2, ..., r+1$ are determined uniquely by the recursion formula

$$n_k = |e_{k-1}| n_{k-1} - n_{k-2}. \tag{19}$$

If $n_0 \leqslant n_1$ then it follows by induction that $n_k \leqslant n_{k+1}$ for $k = 1, ..., r$. Thus if we determine integers μ_k, ν_k using the recursion formula 19, starting with the initial data $\mu_0 = 0$, $\mu_1 = 1$, and $\nu_0 = 1$, $\nu_1 = 1$, then for $k \geqslant 1$ the integers μ_k, ν_k will be positive. Hence, we have holomorphic functions g, h on X with divisors

$$(g) = \sum_{i=0}^{r+1} \mu_i C_i, \quad (h) = \sum_{i=0}^{r+1} \nu_i C_i.$$

Notice that the integers μ_k satisfy

$$\mu_2 = |e_1|, \quad \frac{\mu_3}{\mu_2} = |e_2| - \frac{1}{|e_1|}, \quad \frac{\mu_{k+1}}{\mu_k} = |e_k| - \frac{1}{|e_{k-1}| - \frac{1}{\dots - \frac{1}{|e_1|}}}.$$

The recursion formula 19 implies that $\gcd(\mu_{k+1}, \mu_k) = \gcd(\mu_k, \mu_{k-1}) = ... = \gcd(\mu_2, \mu_1) = 1$. It follows that μ_k and μ_{k+1} are coprime, so they may also be defined by the above continued fraction expansion. Putting $n' = \mu_{r+1}$, $q' = \mu_r$, the expansion

$$\frac{n'}{q'} = |e_r| - \frac{1}{|e_{r-1}| - \frac{1}{\dots - \frac{1}{|e_1|}}}$$

shows that the self intersection numbers e_i are determined by the two integers n' and q'. Finally, we define a divisor

$$(f) = \sum_{i=0}^{r+1} \lambda_i C_i,$$

where the integers λ_i satisfy the recursion formula 19, and $\lambda_{r+1} = 0$, $\lambda_r = 1$. Here the integers λ_i are exactly the integers μ_i we would have obtained if the index i started at the other end of the Hirzebruch-Jung string C. Setting $\lambda_1 = q$ and $\lambda_0 = n$, we have

$$\frac{n}{q} = |e_1| - \frac{1}{|e_2| - \frac{1}{\cdots - \frac{1}{|e_r|}}}.$$

We now show that the equality $\lambda_k + (n-q)\mu_k = n\nu_k$ holds for all $k = 0, ..., r+1$ using induction on k. It is easy to verify that it holds for $k = 0$ and 1. The induction hypothesis says that $\lambda_j + (n-q)\mu_j = n\nu_j$ for all $j \leqslant k-1$. Using equality 19 we have $\lambda_k = |e_{k-1}|\lambda_{k-1} - \lambda_{k-2}$, and similarly for μ_k and ν_k. Together with the induction hypothesis, this implies

$$\lambda_k + (n-q)\mu_k = n\nu_k, \tag{20}$$

for all $k = 0, ..., r+1$. Similarly, using induction on k, we get

$$\lambda_k \mu_{k+1} - \lambda_{k+1}\mu_k = n, \tag{21}$$

for all $k = 0, ..., r+1$. Putting $k = r$ in equation 21 gives $n' = n$, and from equation 20 we get $1 + (n-q)q' = n\nu_r$, i.e., $qq' = 1 + n(q' - \nu_r)$. Thus q' is the unique integer determined by $0 < q' < n$, and $qq' \equiv 1 \mod n$. For the functions f, g, and h defined earlier, equation 19 implies that

$$(fg^{n-q}) = (f) + (n-q)(g) = n(h) = (h^n).$$

Hence, the functions fg^{n-q} and h^n have the same zeros, and so fg^{n-q}/h^n is a function in $\Gamma(X, \mathcal{O}_X^*)$. Then we have the relation $fg^{n-q} = h^n$. In other words, by $w = h$, $z_1 = f$, and $z_2 = g$, X is mapped into the surface

$$W = \{(w, z_1, z_2) \in \mathbb{C}^3 \mid w^n = z_1 z_2^{n-q}\} \subset \mathbb{C}^3.$$

Theorem 2.18 ([1], Theorem III.5.1)**.** *For $0 < q < n$, n and q coprime, let $C \subset X$ be a Hirzebruch-Jung string with self intersection numbers e_i satisfying equation 19, and let $y \in Y$ be the singularity resulting from contracting C. Then this singularity is isomorphic to the unique singularity lying over $0 \in \mathbb{C}^3$ in the normalization of the surface W above.*

Remark 2.19. *This theorem shows in particular that the singularity $y \in Y$ (hence the embedding $C \subset X$) depends on n and q only. It is thus called the $A_{n,q}$ singularity.*

We now discuss a particular situation in which Hirzebruch-Jung singularities occur, namely cyclic quotient singularities. A cyclic quotient singularity is the quotient $X = \mathbb{C}^2/(\mathbb{Z}/n\mathbb{Z})$ of $\mathbb{C}^2$ by the action of a finite cyclic group $\mathbb{Z}/n\mathbb{Z}$, $n \in \mathbb{Z}$. We denote the elements of $\mathbb{Z}/n\mathbb{Z}$ by integers k, $0 \leqslant k < n$. Every linear action of $\mathbb{Z}/n\mathbb{Z}$ on $\mathbb{C}^2$ can be expressed, with respect to suitable coordinates (u_1, u_2), as

$$k \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} e^{2\pi i q_1 k/n} & 0 \\ 0 & e^{2\pi i q_2 k/n} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} e^{2\pi i q_1 k/n} u_1 \\ e^{2\pi i q_2 k/n} u_2 \end{bmatrix}$$

with integers q_1, q_2 satisfying $0 \leqslant q_i < n$ for $i = 1, 2$. The integers q_1, q_2 are determined uniquely up to ordering by the action, and are called the weights of the action. If one of them vanishes, the action is essentially one-dimensional and the quotient is smooth, so we exclude this possibility henceforth. Moreover, if $c = \gcd(n, q_1, q_2) > 1$, then the action of $\mathbb{Z}/n\mathbb{Z}$ can be considered as an action of $\mathbb{Z}/(n/c)\mathbb{Z}$. So we assume without loss in generality that $\gcd(n, q_1, q_2) = 1$. We use the following notation for $i = 1, 2$, as in [1], p.104. $d_i = \gcd(n, q_i)$, $n = n_i d_i$, $q_i = p_i d_i$, $m = \gcd(n_1, n_2)$, p_i' the integer with $p_i p_i' \equiv 1 \mod m$, $0 < p_i' < m$, and q the integer with $q \equiv p_1 p_2' \mod m$, $0 < q < m$.

Proposition 2.20 ([1], Proposition III.5.3). *The image of* $(0,0) \in \mathbb{C}^2$ *in the quotient* $\mathbb{C}^2/(\mathbb{Z}/n\mathbb{Z})$ *is a singularity of type* $A_{m,q}$.

By a result of H. Cartan, every action of a finite group on a manifold can be locally linearized. Applying this result together with Proposition 2.20, it follows that

Theorem 2.21 ([1], Theorem III.5.4). *If the finite cyclic group* G *acts on a smooth surface* X*, then the quotient* X/G *has only singularities of Hirzebruch-Jung type.*

A singularity of type $\frac{1}{n}(1,a)$ is the quotient $\mathbb{C}^2/(\mathbb{Z}/n\mathbb{Z})$, where the action is given, with respect to coordinates u_1, u_2 on $\mathbb{C}^2$, by $k(u_1,u_2) = (e^{2\pi ik/n}u_1, e^{2\pi ika/n})$. So this is a special case of a cyclic quotient singularity in which, using the notation above, $m = n$ and q is the integer with $q(a/\gcd(n,a)) \equiv 1 \mod n$. Since this is a singularity of type $A_{n,q}$, we know from earlier computations that it results from contracting a Hirzebruch-Jung string $C = \bigcup_{i=1}^r C_i$, with self intersections $C_i^2 = e_i$, $e_i \leqslant -2$, given by

$$\frac{n}{q} = |e_1| - \cfrac{1}{|e_2| - \frac{1}{\cdots - \frac{1}{|e_r|}}}.$$

For example a singularity of type $\frac{1}{n}(1,1)$ results from contracting a single curve C with self intersection $C^2 = -n$. We encounter these singularities again when we discuss quotients of fake projective planes in section 7.

3 The Bogomolov-Miyaoka-Yau inequality

The discussion that follows is based on the article of Miyaoka[10] in which he proves the inequality $c_1^2 \leqslant 3c_2$ of Chern numbers of surfaces of general type, which is now known as the Bogomolov-Miyaoka-Yau inequality.

3.1 Some facts about projective bundles

The setting is as follows. X is a smooth, complete variety and $\mathcal{F}$ is a locally free sheaf of rank r over X. $\mathbb{P}(\mathcal{F})$ denotes the projective bundle $\mathrm{Proj}(\bigoplus_{j=0}^r \mathcal{S}^j\mathcal{F})$, and $\pi : \mathbb{P}(\mathcal{F}) \to X$ the canonical projection. H denotes the divisor associated to the tautological invertible sheaf on $\mathbb{P}(\mathcal{F})$. The results appearing in this section are used as facts throughout Miyaoka's paper. We refer to [2], Chapter 9 for a more detailed discussion on projective bundles.

Lemma 3.1. *There are natural isomorphisms*

$$\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{F})}(nH) \cong \mathcal{S}^n\mathcal{F} \quad (n \geqslant 0),$$
$$R^i\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{F})}(nH) = 0 \quad (n \geqslant 0, i > 0).$$

Proof. Let U be an affine open subset of X over which $\mathcal{F}$ is trivial, i.e., $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus r}$. Then the natural maps $H^0(\pi^*\mathcal{S}^n\mathcal{F}|_{\pi^{-1}U}) \to H^0(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(nH)|_{\pi^{-1}U})$ are isomorphisms. By definition of the direct image functor, we have $H^0(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(nH)|_{\pi^{-1}U}) = H^0(\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{F})}(nH)|_U)$. Thus it follows that $\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{F})}(nH) \cong \mathcal{S}^n\mathcal{F}$.
We know that $H^i(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(nH)|_{\pi^{-1}U}) = 0$ for $i > 0$. By definition of higher direct images, we have $R^i\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{F})}(nH)|_U = H^i(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(nH)|_{\pi^{-1}U})$. This implies that $R^i\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{F})}(nH) = 0$ for $i > 0$. This completes the proof. □

Lemma 3.2. *Any divisor* $\mathbb{P}(\mathcal{F})$ *is linearly equivalent to some divisor of the form* $mH + \pi^*D$*, where* m *is an integer and* D *is a divisor on* X.

This result follows from the proof of [2], Theorem 9.6.
There are the following natural exact sequences of sheaves

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{F})} \to \pi^*\widehat{\mathcal{F}} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{F})}(H) \to \widehat{\Omega^1_{\mathbb{P}(\mathcal{F})/X}} \to 0 \tag{22}$$

$$0 \to \pi^*\Omega^1_X \to \Omega^1_{\mathbb{P}(\mathcal{F})} \to \Omega^1_{\mathbb{P}(\mathcal{F})/X} \to 0 \tag{23}$$

$$0 \to \mathcal{O}_{\mathbb{P}(\mathcal{F})}(-H) \to \pi^*\widehat{\mathcal{F}} \to Q \to 0 \tag{24}$$

where Q denotes the universal quotient bundle of rank r on $\mathbb{P}(\mathcal{F})$. The exact sequences (22) and (24) are known as the relative Euler sequence and the tautological exact sequence of $\mathbb{P}(\mathcal{F})$ respectively. Hence we get

Lemma 3.3. $K_{\mathbb{P}(\mathcal{F})} = \pi^*([\mathit{det}(\mathcal{F})] + K_X]) - rH$

Proof. From the short exact sequence (24), it follows that $\pi^*(\det(\widehat{\mathcal{F}})) = \mathcal{O}_{\mathbb{P}(\mathcal{F})}(-H) \otimes \det(Q)$. From (23) we get

$$\mathcal{O}_{\mathbb{P}(\mathcal{F})}(K_{\mathbb{P}(\mathcal{F})}) = \det(\Omega^1_{\mathbb{P}(\mathcal{F})/X}) \otimes \pi^*\det(\Omega^1_X) = \det(\Omega^1_{\mathbb{P}(\mathcal{F})/X}) \otimes \pi^* K_X. \tag{25}$$

Now, we use that $\widehat{\Omega^1_{\mathbb{P}(\mathcal{F})/X}} = \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}(\mathcal{F})}}(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(-H), Q) = \mathcal{O}_{\mathbb{P}(\mathcal{F})}(H) \otimes Q$, from which it follows that $\det(\widehat{\Omega^1_{\mathbb{P}(\mathcal{F})/X}}) = \mathcal{O}_{\mathbb{P}(\mathcal{F})}((r-1)H) \otimes \det(Q)$. Dualizing, and observing that $\pi^*(\det(\mathcal{F})) = \mathcal{O}_{\mathbb{P}(\mathcal{F})}(H) \otimes \det(\widehat{Q})$, we get

$$\det(\Omega^1_{\mathbb{P}(\mathcal{F})/X}) = \mathcal{O}_{\mathbb{P}(\mathcal{F})}(-(r-1)H) \otimes \det(\widehat{Q}) = \mathcal{O}_{\mathbb{P}(\mathcal{F})}(-rH) \otimes \pi^*\det(\mathcal{F}). \tag{26}$$

The equalities (25) and (26) together imply that

$$\mathcal{O}_{\mathbb{P}(\mathcal{F})}(K_{\mathbb{P}(\mathcal{F})}) = \mathcal{O}_{\mathbb{P}(\mathcal{F})}(-rH) \otimes \pi^*\det(\mathcal{F}) \otimes \pi^*\mathcal{O}_X(K_X).$$

Thus observing that $\pi^*\det(\mathcal{F}) = \pi^*\mathcal{O}_X([\det(\mathcal{F})])$, we get $K_{\mathbb{P}(\mathcal{F})} = \pi^*(K_X + [\det(\mathcal{F})]) - rH$.
This concludes the proof. □

Lemma 3.4 (Grothendieck)**.** *We have the following identity in the cohomology group* $H^{2r}(\mathbb{P}(\mathcal{F}), \mathbb{Z})$

$$\sum_{j=0}^{r} c_1^j(H)\pi^* c_{r-j}(\widehat{\mathcal{F}}) = 0$$

As a consequence, we have

Lemma 3.5. *If* $\mathit{dim}(X) = \mathit{rank}(\mathcal{F}) = 2$*, we have the following intersection table*

$$\begin{aligned} H^3 &= c_1^2(\mathcal{F}) - c_2(\mathcal{F}) \\ H^2\pi^* D &= [\mathit{det}(\mathcal{F})]D \\ H\pi^* D\pi^* D' &= DD', \end{aligned}$$

where D *and* D' *are divisors on* X*.*

3.2 A fundamental lemma

In the discussion that follows, $\mathcal{F}$ will be a locally free sheaf of rank 2 over a complete smooth surface.

Theorem 3.6 (Algebraic Index Theorem)**.** *Let* X *be a complete smooth surface and* D_1*,* D_2 *divisors on* X*. If* $D_1^2 > 0$ *and* $D_1 D_2 = 0$ *then we have* $D_2^2 \leqslant 0$*.*

Proof. Since X is a smooth projective manifold, it admits a metric form ω which is a $(1,1)$-form and is pointwise positive definite. Now define

$$H^2_{\text{prim}}(X,\mathbb{Q}) = \{[\alpha];\ [\alpha\wedge\omega]=0\} = [\omega]^\perp$$

leading to the orthogonal direct sum decomposition

$$H^2(X,\mathbb{Q}) = \mathbb{Q}\cdot[\omega]\oplus H^2_{\text{prim}}(X,\mathbb{Q}).$$

Then, the intersection product is negative definite on $H^2_{\text{prim}}(X,\mathbb{R})\cap H^{1,1}$. This follows from the fact that for any real $(1,1)$-form α with $\alpha\wedge\omega=0$ one has $\alpha\wedge\omega\leqslant 0$ with equality if and only if $\alpha=0$; and from the compatibility of the intersection product and the wedge product

$$\int_X \alpha\wedge\beta = [\alpha]\cdot[\beta]$$

for all closed 2-forms α,β (details omitted).

Note that $[\omega]\cdot[\omega]>0$. The above statements imply that the intersection product is negative definite on $[\omega]^\perp$ and so the signature of the intersection product is $(1,h^{1,1}-1)$ on $H^{1,1}$. Thus it either restricts non-degenerately to the Neron-Severi group of X (mod torsion) with signature $(1,\rho-1)$, where ρ is the rank of the Neron-Severi group, or it is semi-negative (with rank one annihilator). Since the Neron-Severi group always contains the class of an ample divisor, the second possibility is excluded.

Now the assertion of the theorem follows from the fact that two divisors D_1 and D_2 are homologically equivalent up to torsion if and only if they are numerically equivalent, i.e. $c_1(D_1)=c_1(D_2)$ if and only if $D_1E=D_2E$ for all divisors E, where $c_1(D_i)$ denotes the image of D_i under the first Chern class map for $i=1,2$. □

We obtain the following lemma as a corollary.

Lemma 3.7. *Let $\rho: X'\to X$ be a surjective morphism of complete smooth surfaces. Assume that $\rho(C_i)$ is a point on X, where C_i is a curve on X'. Then we have*

$$(\sum_i a_iC_i)^2\leqslant 0 \qquad (a_i\in\mathbb{Q}\ \ \forall i).$$

Proof. Since X is a complete non-singular algebraic surface, it is a projective variety. This implies that there exists an embedding $\phi: X\to\mathbb{P}^N$ for some $N\geqslant 0$. The pullback $\phi^*\mathcal{O}_{\mathbb{P}^N}(1)$ is an invertible sheaf associated to a very ample divisor L on X. Since L is very ample, we have $L^2>0$. Hence $|nL|$ $(n\gg 0)$ is a base point free linear system, and so we may assume that nL does not meet the finite subset $\bigcup\rho(C_i)$. Indeed, for any section $\sigma\in H^0(X,\mathcal{O}_X(nL))$, the zero divisor of σ defines an element of $|nL|$ and every element of $|nL|$ arises in this way. Since L is very ample, so is nL. Thus nL defines an embedding $\phi: X\to\mathbb{P}^N$. Now take a hyperplane H in $\mathbb{P}^N$ intersecting $\phi(X)$ but not meeting the the finite subset $\phi(\bigcup\rho(C_i))$. Then $\phi^*H\in|nL|$ and does not meet $\bigcup\rho(C_i)$. Since H is the vanishing locus of a section of $\mathcal{O}_{\mathbb{P}^N}(1)$, and since $\mathcal{O}_X(nL)=\phi^*\mathcal{O}_{\mathbb{P}^N}(1)$, ϕ^*H is the vanishing locus of a section in $H^0(X,\mathcal{O}_X(nL))$ and hence an element of $|nL|$. Without losing generality we may assume it to be nL.

Hence it follows that $C_i\rho^*L=0$ and that $(\sum_i a_iC_i)\rho^*L=0$. Moreover, we have $(\rho^*L)^2=dL^2>0$, where d is the mapping degree of ρ. Now the assertion $(\sum_i a_iC_i)^2\leqslant 0$ follows from Theorem 3.6. □

Lemma 3.8. *Let $\rho: X'\to X$ be a birational morphism of a complete surface X' onto a non-singular surface X. Then the image $\rho(\Sigma)$ of the singular locus Σ of X' is a finite subset of X.*

Proof. Let $\rho' : X'' \to X'$ be a desingularization of X'. Then $\rho \circ \rho' : X'' \to X$ is a birational morphism of complete non-singular surfaces.

$$X'' \xrightarrow{\rho'} X' \xrightarrow{\rho} X$$

This implies that $\rho \circ \rho'$ is a composition of quadratic transformations. Therefore there is a finite subset Δ of X such that $\rho \circ \rho' : X''\backslash(\rho \circ \rho')^{-1}(\Delta) \to X\backslash\Delta$ is an isomorphism. Let $\psi : X\backslash\Delta \to X\backslash(\rho \circ \rho')^{-1}(\Delta)$ be the inverse isomorphism. Then $\rho' \circ \psi$ is an isomorphism of $X\backslash\Delta$ onto an open subset of X'.

$$\begin{tikzcd}
X''\backslash(\rho \circ \rho')^{-1}(\Delta) \arrow[rr, "\sim", "\rho\circ\rho'"'] \arrow[dr, "\rho'"'] & & X\backslash\Delta \arrow[ll, bend right, "\sim"', "\psi"] \arrow[dl, dashed, bend left, "\rho'\circ\psi"] \\
& X'\backslash\rho^{-1}(\Delta) \arrow[ur, "\rho"] &
\end{tikzcd}$$

Since the desingularization map $\rho' : X'' \to X'$ is surjective, the image of $X''\backslash(\rho \circ \rho')^{-1}(\Delta)$ under ρ' is $X'\backslash\rho^{-1}(\Delta)$. Thus the map $\rho \circ \rho' : X''\backslash(\rho \circ \rho')^{-1}(\Delta) \to X\backslash\Delta$ factorizes as $X''\backslash(\rho \circ \rho^{-1})(\Delta) \to X'\backslash\rho^{-1}(\Delta) \to X\backslash\Delta$. We know that this composition is an isomorphism and that the first map ρ' is surjective, so it follows that $\rho' : X'' - (\rho \circ \rho')^{-1}(\Delta) \to X' - \rho^{-1}(\Delta)$ must be an isomorphism. Thus $\rho' \circ \psi : X\backslash\Delta \to X'\backslash\rho^{-1}(\Delta)$ is an isomorphism. Since X is smooth, we have $\Sigma \subset \rho^{-1}(\Delta)$ i.e., $\rho(\Sigma) \subset \Delta$. But Δ is a finite subset of X, so it follows that $\rho(\Sigma)$ is a finite set. This proves the lemma. $\square$

Recall that $\mathcal{F}$ denotes a locally free sheaf of rank 2 over a complete smooth algebraic surface X, $\pi : V = \mathbb{P}(\mathcal{F}) \to X$ the associated projective bundle and H the divisor associated to the tautological invertible sheaf on V. Then we have the following

Lemma 3.9 (Fundamental lemma)**.** *Assume that an irreducible effective divisor W on V is linearly equivalent to $H - \pi^*D$, where D is a divisor on X. Then we have the following inequality*

$$D[det(\mathcal{F})] \leqslant c_2(\mathcal{F}) + D^2.$$

Proof. Let $i : W \to V$ be the canonical injection. Then $\pi \circ i : W \to X$ is a birational morphism. Now W is possibly a singular surface but by Lemma 3.8 we know that the singular locus lies over a finite subset of X. On the other hand, Hironaka's theorem implies that there is a sequence of blow-ups

$$V_s \xrightarrow{\mu_s} V_{s-1} \xrightarrow{\mu_{s-1}} \ldots \to V_1 \xrightarrow{\mu_1} V_0 = V$$

of which each center is non-singular and lies over the singular locus of W, such that the proper transform W' of W is a non-singular subvariety in V_s. Set $\mu = \mu_1 \circ \cdots \circ \mu_s$ and let $E_1, ..., E_s$ be the exceptional divisors on V_s. Then W' is linearly equivalent to $\mu^*(H - \pi^*D) - a_iE_i$, where $a_i \in \mathbb{Z}$ $\forall i$. Letting $i' : W' \to V_s$ be the canonical injection, we infer that $(i')^*E_i$ is an effective divisor whose each component is mapped to a point via $\rho = \pi \circ \mu \circ i' : W' \to X$.

$$\begin{tikzcd}
W' \arrow[r, hook, "i'"] \arrow[rrr, bend left, "\rho=\pi\circ\mu\circ i'"] & V_s \arrow[r, "\mu"] & V = \mathbb{P}(\mathcal{F}) \arrow[r, "\pi"] & X
\end{tikzcd}$$

Now ρ is a birational morphism of non-singular surfaces, hence ρ is a composition of quadratic transformations. Thus we have $K_{W'} = \rho^*K_X + \sum C_i$, where each C_i is a curve on W' for which $\rho(C_i)$ is a point on X. The equality $K_{W'} - \rho^*K_X = \sum C_i$ implies that

$$(K_{W'} - \rho^*K_X + c_i(i')^*E_i)^2 = (c_i(i')^*E_i + \sum C_j)^2 \leqslant 0,$$

since each term in the parentheses is a curve whose image under ρ is a point in X. Hence from Lemma 3.7 we get the inequality

$$(K_{W'} - \rho^* K_X + c_i (i')^* E_i)^2 \leqslant 0, \tag{27}$$

for any $c_i \in \mathbb{Q}$. Observing Lemma 3.3, we have $K_{\mathbb{P}(\mathcal{F})} = \pi^*([\det(\mathcal{F})] + K_X) - 2H$ since $\mathcal{F}$ is locally free of rank 2. Thus we have $K_{V_s} = \mu^* K_{\mathbb{P}(\mathcal{F})} + \sum_i b_i E_i$, where $b_i \in \mathbb{Z}$ and E_i are the exceptional curves. Hence we have the following equality

$$K_{V_s} = \mu^*(-2H + \pi^* K_X + \pi^*[\det(\mathcal{F})]) + \sum_i b_i E_i, \qquad (b_i \in \mathbb{Z} \ \ \forall i).$$

Hence by the adjunction formula $K_{W'} = (i')^*(K_{V_s} + W')$, and using $W' \sim \mu^*(H - \pi^* D) - a_i E_i$, we have

$$K_{W'} = (i')^*(-\mu^* H + \mu^* \pi^*(K_X + [\det(\mathcal{F})] - D) + \sum_i (b_i - a_i) E_i).$$

Replacing c_i by $a_i - b_i$ in inequality (27) we obtain the inequality

$$(K_{W'} - \rho^* K_X + c_i (i')^* E_i)^2 = k = ((i')^* \mu^*(-H + \pi^*[\det(\mathcal{F})] - \pi^* D))^2 \leqslant 0. \tag{28}$$

On the other hand, we have

$$\begin{aligned} k &= (\mu^*(-H + \pi^*[\det(\mathcal{F})] - \pi * D))^2 (\mu^* H - \mu^* \pi^* D - \sum_i b_i E_i) \\ &= (-H + \pi^*[\det(\mathcal{F})] - \pi^* D)^2 (H - \pi^* D) \\ &= H^3 + H^2 \pi^*(D - 2[\det(\mathcal{F})]) + H((\pi^*[\det(\mathcal{F})])^2 - (\pi^* D)^2). \end{aligned}$$

Applying Lemma 3.5, we get

$$k = c_1^2(\mathcal{F}) - c_2(\mathcal{F}) - [\det(\mathcal{F})]^2 + D[\det(\mathcal{F})] - D^2.$$

By definition we have $[\det(\mathcal{F})]^2 = c_1^2(\mathcal{F})$. Hence, from the inequality (28) we obtain $0 \geqslant k = -c_2(\mathcal{F}) + D[\det(\mathcal{F})] - D^2$ i.e., $D[\det(\mathcal{F})] \leqslant c_2(\mathcal{F}) + D^2$. This proves the lemma. ☐

3.3 Bogomolov's Lemma

Lemma 3.10. *Let X be a Kähler manifold. Then, for any $f \in H^0(X, \Omega^1_X)$, we have $df = 0$.*

Remark 3.11. *For compact Kähler manifolds there is the more general fact that $d\omega = 0$ for any global holomorphic p-form ω.*

Proof of Lemma 4.1. Let ω be a Kähler form on X, i.e. ω is a real closed (1,1)-form, and let r be the dimension of X. Since f is holomorphic 1-form, df is a holomorphic 2-form and $\sqrt{-1} df \wedge d\bar{f} \wedge (\bigwedge^{r-2} \omega)$ is a positive-semidefinite $2r$-form which is positive on non-empty open subsets of X unless $df \equiv 0$. On the other hand, Stokes' theorem implies that

$$\int_X \sqrt{-1} df \wedge d\bar{f} \wedge \Big(\bigwedge^{r-2} \omega\Big) = \int_X d\Big(\sqrt{-1} f \wedge d\bar{f} \wedge \Big(\bigwedge^{r-2} \omega\Big)\Big) = \int_{\partial X} \sqrt{-1} f \wedge d\bar{f} \wedge \Big(\bigwedge^{r-2} \omega\Big) = 0.$$

Thus, it follows that $df \equiv 0$, which proves the assertion. ☐

Remark 3.12. *Note that if the dimension of X is 2 then the Kähler condition is not necessary.*

A variant of the proof of Lemma 3.10 is given in [12]

Theorem 3.13 (Bogomolov's lemma). *Let X be a non-singular projective variety and $\mathcal{L}$ an invertible subsheaf of the cotangent sheaf Ω^1_X. Then any three global sections f_1, f_2, f_3 of $\mathcal{L}^{\otimes n}$, for any $n > 0$, are not algebraically independent of each other.*

Proof. We may assume $f_i \neq 0$, for $i = 1, 2, 3$. From Lemma 2.3 we know that there exists a finite flat covering $\beta : X' \to X$ of X such that $\beta^* f_i = g_i^n \in \Gamma(X', \beta^*\mathcal{L}^{\otimes n})$, where $g_i \in \Gamma(X', \beta^*\mathcal{L})$, for $i = 1, 2, 3$. Recall that the pullback of an invertible sheaf is invertible, and so $\beta^*\mathcal{L}$ is an invertible sheaf on X', and we have the chain of inclusions $\beta^*\mathcal{L} \subset \beta^*\Omega^1_X \subset \Omega^1_{X'}$. On a sufficiently small open subset U of X', $(\beta^*\mathcal{L})|_U$ is generated by a single section, say, λ. Thus we can write $g_i = h_i\lambda$ where h_i is a holomorphic function on U, for $i = 1, 2, 3$. Since each g_i is a global section of $\beta^*\mathcal{L} \subset \Omega^1_{X'}$, i.e. $g_i \in H^0(X', \Omega^1_{X'})$, we have from Lemma 3.10 that $dg_i = 0$ i.e. $d(\lambda h_i) = 0$ which implies $dh_i \wedge \lambda + h_i d\lambda = 0$ i.e., $dh_i \wedge \lambda = -h_i d\lambda$. Hence we have the following equality of rational forms

$$d\Big(\frac{h_i}{h_j}\Big) \wedge \lambda = \Big(\frac{h_i dh_j - h_j dh_i}{h_j^2}\Big) \wedge \lambda = \frac{-h_i h_j d\lambda + h_i h_j d\lambda}{h_j^2} = 0. \tag{29}$$

Note that at any point $x \in U$, the stalk $(\beta^*\mathcal{L})_x$ is generated by λ. Thus it follows from the equality (29), i.e. from $d(\frac{h_i}{h_j}) \wedge \lambda = 0$ that the 1-forms $d(\frac{h_i}{h_j})$ at x are contained in $(\beta^*\mathcal{L})_x$, which is a rank one subsheaf of $\Omega^1_{X',x}$.

We claim that if h_1,h_2, and h_3 are algebraically independent of each other, then so are $\frac{h_2}{h_1}$ and $\frac{h_3}{h_1}$. Suppose they are not, then there is a polynomial $P \in \mathbb{C}[t_1, t_2]$ such that $P(\frac{h_2}{h_1}, \frac{h_3}{h_1}) = 0$. Multiplying this equality by a large enough power of h_1, we get an equality of the form $Q(h_1, h_2, h_3) = 0$, where $Q \in \mathbb{C}[t_1, t_2, t_3]$. This is a contradiction to the assumption, and hence the claim holds. Let z_1, z_2 be local coordinates on U, then h_1, h_2, and h_3 being algebraically independent of each other means that the set

$$\Big\{\Big(\frac{h_2}{h_1}(z_1, z_2), \frac{h_3}{h_1}(z_1, z_2)\Big) \in \mathbb{C}^2 \;\Big|\; (z_1, z_2) \in U\Big\} \tag{30}$$

is not the vanishing locus of any polynomial in $\mathbb{C}[t_1, t_2]$. Thus the set (30) is an open subset of $\mathbb{C}^2$ and moreover, $(\frac{h_2}{h_1}, \frac{h_3}{h_1})$ are local coordinates on this open subset. This implies that $d(\frac{h_2}{h_1})$ and $d(\frac{h_3}{h_1})$ are linearly independent and generate a rank two subsheaf of $\Omega^1_{X',x}$, which is a contradiction.

Hence it follows that the h_i's are algebraically dependent. Recall that $\beta^* f_i = g_i^n = (\lambda h_i)^n$ for $i = 1, 2, 3$, which implies that the $\beta^* f_i$'s are algebraically dependent. Since $(\beta^* f_i)(x) = f_i(\beta(x))$ for all $x \in X'$, it follows that the f_i's are algebraically dependent. □

Definition 3.14. *The D-dimension $\kappa(D, X)$ is defined as follows*

$$\kappa(D, X) = \begin{cases} \big(\textit{transcendence degree over } \mathbb{C} \textit{ of the graded ring } R_D = \bigoplus_{j=0}^{\infty} H^0(X, \mathcal{O}_X(jD))\big) - 1, & \textit{if } R_D \neq \mathbb{C} \\ -\infty, & \textit{if } R_D = \mathbb{C}. \end{cases}$$

Theorem 3.13 can be reformulated as follows

Theorem 3.15. *Let X be a non-singular projective variety and $\mathcal{O}_X(D)$ a subsheaf of Ω^1_X. Then the D-dimension of X does not exceed 1.*

For a line bundle $\mathcal{L}$ (as in Theorem 3.13), let D be the associated divisor, then $\mathcal{L} \cong \mathcal{O}_X(D)$ and $\mathcal{L}^{\otimes n} \cong \mathcal{O}_X(nD)$. Thus if no three global sections of $\mathcal{L}^{\otimes n}$ are algebraically independent of each other (for any n), then the transcendence degree of the ring $R_D = \bigoplus_{j=0}^{\infty} H^0(X, \mathcal{O}_X(jD))$ over $\mathbb{C}$ does not exceed 2, i.e., the D-dimension of X does not exceed 1, and conversely.

Iitaka's theory of D-dimension implies that Theorem 3.15 is equivalent to

Theorem 3.16. *If $\mathcal{O}_X(D)$ is contained in the cotangent sheaf Ω^1_X of a projective variety X, then there exists a constant c such that $h^0(X, \mathcal{O}_X(D)) \leqslant cn$ for $n \gg 0$.*

Proof. A proof of this result is given in [1], Proposition VII.4.2. □

As a corollary to Theorem 3.16, we have

Lemma 3.17. *Let X be a complete smooth surface, and $\mathcal{L}$ an invertible sheaf generated by its global sections. If $\mathcal{O}_X(D)$ is contained in Ω^1_X, then we have either*

$$D[\mathcal{L}] \leqslant 0 \tag{31}$$

or,

$$D^2 \leqslant 0. \tag{32}$$

Moreover if D is effective, the inequality (32) holds.

Proof. We assume that $D[\mathcal{L}]$ is positive and prove (32). Recall that a line bundle is globally generated if and only if it is base point free. Since $|[\mathcal{L}]|$ is free from base points, we have

$$E[\mathcal{L}] \geqslant 0 \tag{33}$$

for any effective divisor E. Hence $|K_X - nD|$ must be empty for $n \gg 0$. Indeed, consider an effective divisor E in $|K_X - nD|$, then we have $E[\mathcal{L}] = (K_X - nD)[\mathcal{L}] = K_X[\mathcal{L}] - nD[\mathcal{L}]$. Since we have assumed $D[\mathcal{L}] > 0$, we have that $E[\mathcal{L}] < 0$ for $n \gg 0$, which is a contradiction to the inequality (33). This implies that there is no effective divisor in $|K_X - nD|$, i.e. $|K_X - nD|$ is empty.
Thus

$$h^2(X, \mathcal{O}_X(nD)) = h^0(X, \mathcal{O}_X(K_X - nD)) = 0$$

for large n, where the first equality follows from Serre duality. Hence we get the inequality

$$cn \geqslant h^0(X, \mathcal{O}_X(nD)) \geqslant \chi(X, \mathcal{O}_X(nD)) = \frac{1}{2}n^2D^2 + \text{linear term in } n,$$

where the first " $\geqslant$ " follows from Theorem 3.16 and the second " $\geqslant$ " follows from the Riemann-Roch theorem. But the left hand side of the inequality has cn and the right hand side has $\frac{1}{2}n^2D^2$ as the leading term and so $cn \geqslant \frac{1}{2}n^2D^2 + ...$ implies that we must have $D^2 \leqslant 0$. This is (32).
Now suppose that D is an effective divisor. If D is trivial, then the inequality (32) is automatically satisfied, so we may assume D is non-trivial. If $[\mathcal{L}]$ is a very ample divisor, then we have $D[\mathcal{L}] > 0$. Following the arguments in the proof of the first part of the lemma, we get the inequality (32). This completes the proof. □

3.4 Chern numbers of surfaces of non-negative Kodaira dimension

The Kodaira dimension $\kappa(X)$ of a non-singular complete variety is defined as the K_X-dimension $\kappa(K_X, X)$. If X is a surface, $R_{K_X} = \bigoplus_{j=0}^{\infty} H^0(X, \mathcal{O}_X(jK_X))$ is a finitely generated $\mathbb{C}$-algebra and is independent of a choice of model of X. Hence $\kappa(X)$ is a birational invariant of X. For a surface X with $\kappa(X) \geqslant 0$, the following facts are well-known

1. X has a unique minimal model
2. If $\bar{X} \to X$ is a generically surjective rational map, then $\kappa(\bar{X}) \geqslant \kappa(X)$.

3. If X is minimal, then there exists an integer $n > 0$, such that $\mathcal{O}_X(nK_X)$ is generated by global sections.

We give a proof of statement 1, and cite references for proofs of staments 2 and 3.

Proof. 1. This follows from [1], Theorem IV.4.5 and Proposition IV.4.6. We first show that every compact nonsingular surface X has a minimal model. Suppose that X contains a (-1)-curve C and let X' be obtained by contracting C. Now if X' contains a (-1)-curve C', we obtain another surface by contracting C', and so on. Repeating this process must lead to a surface without (-1)-curves after finitely many steps because, by statement 4 of Theorem 1.1, the second Betti number decreases by 1 for each blow down, and the second Betti number is always non-negative.
In order to show that all minimal models of X are isomorphic, we prove the following more general claim: *Let X, Y be two compact connected nonsingular surfaces and $f : X \to Y$ a birational map. If K_Y is nef, then f is a morphism. If in addition K_X is nef, f is an isomorphism.*
Suppose that $\sigma : \bar{X} \to X$ is the blow up of a point $p \in X$. Let $C \subset X$ be a curve in X containing p with multiplicity m, and let $\bar{C} \subset \bar{X}$ be the proper transform of C. Then, $\bar{C} = \sigma^*C - mE$, where E is the exceptional divisor corresponding to p. We have

$$K_{\overline{X}}\bar{C} = (\sigma^*K_X + E)(\sigma^*C - mE) = K_XC + m \geqslant K_XC. \tag{34}$$

Thus the number K_XC does not increase under blowing down, and if K_X is nef, any curve $\bar{C}$ on $\bar{X}$ with $K_{\overline{X}}\bar{C} \leqslant -1$ must be mapped to a point in X. If $f : X \to Y$ is not a morphism, then we blow X up until we get a morphism $f' : X' \to Y$. The morphism f' is composed of blow-ups and any curve C' in X' which arises from blowing up a point in X is mapped by f' to a curve C in Y. Thus C' is mapped to a curve $\bar{C}$ in $\bar{Y}$, where $\bar{Y} \to Y$ is the first blow up map in the decomposition of f'. From the equality 34 we get $-1 = K_{X'}C' \geqslant K_{\bar{Y}}\bar{C}$. Again using 34 we have $K_YC \leqslant K_{\bar{C}}\bar{C}$ i.e. $K_YC \leqslant -1$, which is not possible because K_Y is nef by assumption. Hence, f is a morphism. If K_X is also nef, then the inverse birational map of f is also a morphism and so f is an isomorphism.

2. This result is Theorem 6.10 in [14]. It follows from Theorem 2.5 and Lemma 6.3 in the same book. For a detailed proof, see [14], Lemma 6.3 on p. 66.

3. For a minimal surface X, proving that $\mathcal{O}_X(nK_X)$ is generated by global sections for some $n > 0$ is equivalent to proving that K_X is semi-ample i.e., the linear system $|nK_X|$ is base point free for some $n > 0$. This is a non-trivial result known as the Abundance theorem, a proof is given in [9], Theorem 1-5-6. □

Remark 3.18. *Note that statement 1 is not true if the Kodaira dimension is $-\infty$. For example, the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ in one point can be blown down to get $\mathbb{P}^2$.*

Proposition 3.19. *Let $\mathcal{F} \subset \Omega^1_X$ be a locally free sheaf of rank 2 on X and assume that $(det(\mathcal{F}))^{\otimes n}$ is generated by global sections for some $n > 0$. If $\mathcal{F} \otimes \mathcal{O}_X(-D)$ admits a non-trivial global section then the divisor D satisfies the following numerical condition*

$$D[det(\mathcal{F})] \leqslant max(c_2(\mathcal{F}), 0).$$

Remark 3.20. *Such a $\mathcal{F}$ exists on X if and only if we have $\kappa(X) \geqslant 0$. Indeed, the inclusion $\mathcal{F} \subset \Omega^1_X$ implies that $det(\mathcal{F}) \subset det(\Omega^1_X) = \mathcal{O}_X(K_X)$, and so $(det(\mathcal{F}))^{\otimes n} \subset \mathcal{O}_X(nK_X)$. Since $(det(\mathcal{F}))^{\otimes n}$ is generated by global sections by assumption, the inclusion $(det(\mathcal{F}))^{\otimes n} \subset \mathcal{O}_X(nK_X)$ implies that the transcendence degree of R_{K_X} over $\mathbb{C}$ is positive i.e., $\kappa(X) \geqslant 0$. Conversely if $\kappa(X) \geqslant 0$, then $\mathcal{F} = \rho^*\Omega^1_{X'}$ satisfies the condition of the above* *proposition, where $\rho : X \to X'$ is the canonical projection of X onto the minimal model X' of X. Indeed, if X' is smooth, then $\Omega^1_{X'}$ is is a locally free sheaf of rank 2 on X' and so $\rho^*\Omega^1_{X'}$ is a locally free sheaf of rank 2 on X. Since taking determinant commutes with pullback, we have $det(\rho^*\Omega^1_{X'}) = \rho^* det(\Omega^1_{X'}) = \rho^*\mathcal{O}_X(K_{X'})$. Thus $(det(\mathcal{F}))^{\otimes n} = (\rho^*\mathcal{O}_X(K_{X'}))^{\otimes n} = \rho^*\mathcal{O}_X(nK_{X'})$. Since X' is minimal, we know that $\mathcal{O}_X(nK_{X'})$ is generated by global sections (for large enough n) and so the pullback $\rho^*\mathcal{O}_X(nK_{X'})$ is a sheaf generated by global sections on X.*

Proof of Proposition 3.19. Let $\pi : V = \mathbb{P}(\mathcal{F}) \to X$ be the projective bundle associated to $\mathcal{F}$. Since $\mathcal{O}_X(-D)$ is locally free, the projection formula can be applied i.e., we have

$$\pi_*\mathcal{O}_V(H) \otimes \mathcal{O}_X(-D) \cong \pi_*(\mathcal{O}_V(H) \otimes \pi^*\mathcal{O}_X(-D)).$$

Now using $\pi_*\mathcal{O}_V(H) \cong \mathcal{F}$ from Lemma 3.1 and that $\pi^*\mathcal{O}_X(-D) = \mathcal{O}_V(-\pi^*D)$, it follows that

$$\mathcal{F} \otimes \mathcal{O}_X(-D) \cong \pi_*(\mathcal{O}_V(H) \otimes \mathcal{O}_V(-\pi^*D)) = \pi_*\mathcal{O}_V(H - \pi^*D).$$

By definition of the direct image functor, we have $H^0(X, \pi_*\mathcal{O}_V(H - \pi^*D)) \cong H^0(V, \mathcal{O}_V(H - \pi^*D))$, which gives a canonical isomorphism

$$H^0(X, \mathcal{F} \otimes \mathcal{O}_X(-D)) \cong H^0(V, \mathcal{O}_V(H - \pi^*D)).$$

Thus if $\mathcal{F} \otimes \mathcal{O}_X(-D)$ has a non-trivial global section, we have $|H - \pi^*D| \neq \varnothing$. Let W be an element of the linear system $|H - \pi^*D|$. A non-trivial global section of $\mathcal{F} \otimes \mathcal{O}_X(-D)$ corresponds to an injective map $i : \mathcal{O}_X \to \mathcal{F} \otimes \mathcal{O}_X(-D)$, so we obtain a short exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{i} \mathcal{F} \otimes \mathcal{O}_X(-D) \xrightarrow{p} \mathcal{G} \to 0$$

where $\mathcal{G} = (\mathcal{F} \otimes \mathcal{O}_X(-D))/\mathcal{O}_X$. Let $\mathcal{N}$ be the torsion subsheaf of $\mathcal{G}$ and let $\mathcal{L} = p^{-1}\mathcal{N}$ be its preimage in $\mathcal{F} \otimes \mathcal{O}_X(-D)$. Note that $\mathcal{L}$ is a line bundle and contains $\mathcal{O}_X$, which implies that $\mathbb{L}$ has a non-trivial global section. Thus $\mathcal{L} \cong \mathcal{O}_X(D')$, where D' is an effective divisor on X. There is an inclusion $j : \mathcal{L} \to \mathcal{F} \otimes \mathcal{O}_X(-D)$ which fits into a short exact sequence

$$0 \to \mathcal{L} \xrightarrow{j} \mathcal{F} \otimes \mathcal{O}_X(-D) \xrightarrow{q} \mathcal{H} \to 0$$

where $\mathcal{H}$ is a torsion free sheaf. The map j corresponds to a global section of $\mathcal{F} \otimes \mathcal{O}_X(-D) \otimes \widehat{\mathcal{L}} \cong \mathcal{F} \otimes \mathcal{O}_X(-D - D')$. It can be checked that this section has at most isolated zeros. Thus W can be decomposed as follows

$$W = W_0 + \pi^*D'$$

where W_0 is an irreducible effective divisor which is linearly equivalent to $H - \pi^*(D + D')$. Since $(\det(\mathcal{F}))^{\otimes n}$ is generated by global sections, it defines a morphism $\phi : X \to \mathbb{P}^N$, for some N, such that $(\det(\mathcal{F}))^{\otimes n} \cong \phi^*\mathcal{O}_{\mathbb{P}^N}(1)$. Now $\mathcal{O}_{\mathbb{P}^N}(1)$ is ample, so in particular it is nef. Hence the pullback $\phi^*\mathcal{O}_{\mathbb{P}^N}(1)$ is nef, which implies that $(\det(\mathcal{F}))^{\otimes n}$ is nef. Since D' is effective, we have $nD'[\det(\mathcal{F})] \geqslant 0$, i.e. $D'[\det(\mathcal{F})] \geqslant 0$. Hence writing $D'' = D + D'$, we get $D[\det(\mathcal{F})] \leqslant D''[\det(\mathcal{F})]$. Thus it is sufficient to prove $D''[\det(\mathcal{F})] \leqslant \max(c_2(\mathcal{F}), 0)$. From Lemma 3.9 it follows that

$$D''[\det(\mathcal{F})] \leqslant c_2(\mathcal{F}) + (D'')^2. \tag{35}$$

Let f be a non-trivial global section of $\mathcal{F} \otimes \mathcal{O}_X(-D'')$. Then multiplication by f, i.e., the map $\mathcal{O}_X(D'') \to \mathcal{F}$ defined on every open set $U \subset X$ by $(\mathcal{O}_X(D''))(U) \to (\mathcal{F} \otimes \mathcal{O}_X(-D''))(U)$, $s \mapsto s \cdot f|_U$, is injective. Since

$\mathcal{F} \subset \Omega^1_X$, it follows that $\mathcal{O}_X(D'')$ is a subsheaf of Ω^1_X in a canonical way. Thus $\mathcal{O}_X(D'')$ satisfies the conditions of Lemma 3.17.
Now from Lemma 3.17 it follows that $D''[\det(\mathcal{F})] \leqslant 0$ or $(D'')^2 \leqslant 0$. If $D''[\det(\mathcal{F})] \leqslant 0$ then we are done. If $D''[\det(\mathcal{F})] > 0$ then $(D'')^2 \leqslant 0$ and so from the inequality (35) it follows that $D''[\det(\mathcal{F})] \leqslant c_2(\mathcal{F})$. This proves the assertion. □

Proposition 3.21. *If $\kappa(X) \geqslant 0$ then $c_2(X) \geqslant 0$ and $\chi(X, \mathcal{O}_X) \geqslant 0$.*

Proof. Let X' be the minimal model of X. Then Remark 3.20 implies that if $\kappa(X) \geqslant 0$ then $\mathcal{F} = \rho^*\Omega^1_{X'} \subset \Omega^1_X$ is a locally free sheaf of rank 2 and $(\det(\mathcal{F}))^{\otimes n} = \rho^*\mathcal{O}_{X'}(nK_{X'})$ is generated by global sections for some $n > 0$. But this means that $(\rho^*(nK_{X'}))^2 = \deg(\rho)n^2K^2_{X'} \geqslant 0$, which implies $K^2_{X'} = c_1^2(X') \geqslant 0$. At first, assume that the irregularity $q(X')$ vanishes. Then from Lemma 1.3 we have $c_2(X') = 2 - 4q(X') + b_2(X') > 0$, where $b_2(X')$ denotes the second Betti number of X'. Hence we get

$$12\chi(X, \mathcal{O}_X) = 12\chi(X', \mathcal{O}_{X'}) = c_1^2(X') + c_2(X') > 0.$$

The first equality follows from the fact that χ is a birational invariant in characteristic 0 (see Remark 1.4) and the second equality comes from the Riemann-Roch theorem. Lemma 1.2 says that blowing up a point increases c_2 by 1. Thus we have $c_2(X) \geqslant c_2(X')$, and the assertion is proved.
Now assume that $q(X') > 0$. Then from the definition $q(X') = h^{1,0} = \dim(H^0(X, \Omega^1_{X'}))$, it follows that $\mathcal{F} = \Omega^1_{X'}$ admits a non-trivial global section. Now $\mathcal{F} = \Omega^1_{X'}$ implies that $\det(\mathcal{F}) = \mathcal{O}_{X'}(K_{X'})$ and $(\det(\mathcal{F}))^{\otimes n} = \mathcal{O}_{X'}(nK_{X'})$. Since $\kappa(X') \geqslant 0$ by assumption, we know that $\mathcal{O}_{X'}(nK_{X'})$ is generated by global sections for some $n > 0$ and so $\mathcal{F}$ satisfies the conditions of Proposition 3.19. Note that $\mathcal{F} = \Omega^1_{X'}$ is locally free because X' is smooth. Since $\mathcal{F}$ admits a non-trivial global section, we know that $|H| \neq \varnothing$. Let W be an element of this linear system. Then as in the proof of Proposition 3.19, W can be decomposed as $W = W_0 + \pi^*D''$, where W_0 is an effective and irreducible divisor linearly equivalent to $H - \pi^*D''$ and D'' is an effective divisor on X'. So from Lemma 3.9 it follows that $D''[\det(\mathcal{F})] \leqslant c_2(\mathcal{F}) + (D'')^2$ and since $(\det(\mathcal{F}))^{\otimes n}$ is globally generated for some $n > 0$ and D'' is effective, we have $D''[\det(\mathcal{F})] \geqslant 0$. Since $\mathcal{O}_{X'}(D'') \subset \Omega^1_{X'}$ as above, and D'' is effective, we can apply Lemma 3.17 to conclude that $(D'')^2 \leqslant 0$. Thus

$$c_2(X) \geqslant c_2(X') = c_2(\Omega^1_{X'}) = c_2(\mathcal{F}) \geqslant -(D'')^2 \geqslant 0.$$

Moreover, we have

$$12\chi(X, \mathcal{O}_X) = 12\chi(X', \mathcal{O}_{X'}) = c_1^2(X') + c_2(X') \geqslant 0.$$

This completes the proof. □

Corollary 3.22. *If $\kappa(X) = 2$ then $\chi(X, \mathcal{O}_X) > 0$. In other words, if X is a surface of general type, then the arithmetic genus $p_a(X)$ is non-negative.*

Proof. We claim that if X is a minimal surface of general type, then $c_1^2(X) > 0$. Let C be a smooth hyperplane section of X. Consider the exact sequence

$$0 \to \mathcal{O}_X(nK_X - C) \to \mathcal{O}_X(nK_X) \to \mathcal{O}_C(nK_X) \to 0,$$

and the associated long exact cohomology sequence. By [1], Theorem I.7.2, there is a $c > 0$ such that $h^0(X, \mathcal{O}_X(nK_X)) > cn^2$ for large n, while the Riemann-Roch theorem for curves implies that $h^0(C, \mathcal{O}_C(nK_X))$ grows linearly with n. Thus there is an $m > 0$ such that there is an effective divisor E in $|mK_X - C|$. Since X is minimal, K_X is nef, hence $K_XE \geqslant 0$. We have

$$m^2K^2_X = (mK_X)(E + C) \geqslant mK_XC = EC + C^2 \geqslant C^2 > 0,$$

i.e., $c_1^2(X) > 0$ as claimed.
From Proposition 3.21 we know that $c_2(X) \geqslant 0$. But this means that $\chi(X, \mathcal{O}_X) = \frac{1}{12}(c_1^2(X) + c_2(X)) > 0$. Since $\chi(X', \mathcal{O}_{X'}) = \chi(X, \mathcal{O}_X)$ for any surface X' birationally equivalent to X, the assertion follows. □

Now we study symmetric powers of a locally free subsheaf of the cotangent sheaf of X. The following result is a consequence of the "branched covering trick", [1], Theorem I.18.2. We give a slightly different proof here.

Lemma 3.23. *Let $\pi : V = \mathbb{P}(\mathcal{F}) \to X$ be the projective bundle associated to a locally free sheaf $\mathcal{F}$ of rank 2 on X and let W be an element of the linear system $|mH - \pi^*D|$, where D is a divisor on X. Then there exists a surjective morphism β of a non-singular surface $\bar{X}$ onto X such that $\beta'^*W \subset \mathbb{P}(\beta^*\mathcal{F})$ can be decomposed as $W_1 + ... + W_m$, where W_i is an effective divisor on $\mathbb{P}(\beta^*\mathcal{F})$ linearly equivalent to $\bar{H} - \bar{\pi}^*D_i$, $\bar{H} = \beta'^*H$ and D_i is a divisor on $\bar{X}$. Here $\beta' : \bar{V} = \mathbb{P}(\beta^*\mathcal{F}) \to V$ and $\bar{\pi} : \bar{V} \to \bar{X}$ denote the morphism of projective bundles induced by β and the canonical projection respectively.*

Following is a diagram of the situation. The square is commutative.

$$\begin{tikzcd}
\mathbb{P}(\beta^*\mathcal{F}) \arrow[r, "\beta'"] \arrow[d, "\bar{\pi}"] & \mathbb{P}(\mathcal{F}) \arrow[d, "\pi"] \\
\bar{X} \arrow[r, "\beta"] & X
\end{tikzcd}$$

Proof. This result follows from the fact that the category of algebraic varieties over $\mathbb{C}$ and dominant rational maps between them is contravariantly equivalent to the category of finitely generated field extensions of $\mathbb{C}$. More specifically, every dominant map $\phi : Y \to Y'$ of varieties induces a morphism $\phi^* : \mathbb{C}(Y') \to \mathbb{C}(Y)$ of function fields and conversely, every morphism $\theta : K \to L$ of function fields induces a dominant rational map $\psi_\theta : Y \to Y'$ with $K \cong \mathbb{C}(Y')$ and $L \cong \mathbb{C}(Y)$.
Let x be the generic point of X. Then, then residue field $\mathbb{C}(x)$ is equal to the the function field $\mathbb{C}(X)$ of X. The fibre $\pi^{-1}(x)$ is a projective line over $\mathbb{C}(x)$. Since $W \in |mH - \pi^*D|$, W meets $\pi^{-1}(x)$ in m points and so $W_x = W \cap \pi^{-1}(x)$ is the vanishing locus of a homogeneous polynomial $f = \sum f_i T^i$ of degree m, where T is the coordinate on the projective line $\pi^{-1}(x)$ and $f_i \in \mathbb{C}(x)$. Now let K be the splitting field of f, then K is a finite field extension of $\mathbb{C}(X)$. Then by the contravariant equivalence of categories result, there exists a variety $\bar{X}$ and a rational map $\beta : \bar{X} \to X$ which is generically finite and dominant. This ensures that $\bar{X}$ is a surface and moreover that $\mathbb{C}(\bar{X}) = K$. A rational map of surfaces can be extended to a morphism after a sequence of blow-ups. Since blowing up preserves function fields up to isomorphism, we may assume that $\bar{X}$ is a non-singular surface and that $\beta : \bar{X} \to X$ is a surjective morphism. The morphism β induces a morphism $\beta' : \mathbb{P}(\beta^*\mathcal{F}) \to \mathbb{P}(\mathcal{F})$ of projective bundles (see figure above) and $W' = (\beta')^*W$ is an effective divisor on $\mathbb{P}(\beta^*\mathcal{F})$ in the linear system $|m\bar{H} - \bar{\pi}^*\beta^*D|$. Let $\bar{x}$ denote the generic point of $\bar{X}$. Then W' meets the fibre $\bar{\pi}^{-1}(\bar{x})$ in m points i.e., $W'_{\bar{x}} = W' \cap \bar{\pi}^{-1}(\bar{x})$ consists of m points and is the vanishing locus of β^*f. Since $\mathbb{C}(X) \to \mathbb{C}(\bar{X})$ is an inclusion, β^*f is just f i.e., $\beta^*f = \sum f_i(T')^i$, where T' is the coordinate on the projective line $\bar{\pi}^{-1}(\bar{x})$. Since f splits in $\mathbb{C}(\bar{X})$, β^*f can be expressed as a product of m linear polynomials and hence $W'_{\bar{x}}$ can be decomposed as $W'_{\bar{x}} = W_{\bar{x},1} + ... + W_{\bar{x},m}$, where each $W_{\bar{x},i}$ is the vanishing locus of a linear polynomial with coefficients in $\mathbb{C}(\bar{X})$. Since this decomposition holds over the generic point, it holds everywhere i.e., we can write $W' = (\beta')^*W = W_1 + ... + W_m$, where each W_i is an effective divisor in $|\bar{H} - \bar{\pi}^*D_i|$, where D_i is a divisor on $\bar{X}$ and the D_i are such that $\sum D_i \equiv \beta^*D$, where $\equiv$ denotes linear equivalence. This completes the proof. □

Theorem 3.24. *Let $\mathcal{F} \subset \Omega^1_X$ be a locally free sheaf of rank 2 on a complete non-singular surface X such that $(det(\mathcal{F}))^{\otimes n}$ is generated by global sections for some $n > 0$. If $\mathcal{S}^m\mathcal{F} \otimes \mathcal{O}_X(-D)$ has a non-trivial global section then the inequality*

$$D[det(\mathcal{F})] \leqslant max(mc_2(\mathcal{F}), 0)$$

holds.

Proof. Let f be a global section of $\mathcal{S}^m\mathcal{F} \otimes \mathcal{O}_X(-D)$. We again use that $\mathcal{O}_X(-D)$ is locally free and the projection formula to get an isomorphism

$$\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{F})}(mH) \otimes \mathcal{O}_X(-D) \cong \pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(mH) \otimes \pi^*\mathcal{O}_X(-D)).$$

Using $\pi_*\mathcal{O}_{\mathbb{P}(\mathcal{F})}(mH) \cong \mathcal{S}^m\mathcal{F}$ from Lemma 3.1 and that $\pi^*\mathcal{O}_X(-D) \cong \mathcal{O}_{\mathbb{P}(\mathcal{F})}(-\pi^*D)$, it follows that

$$\mathcal{S}^m\mathcal{F} \otimes \mathcal{O}_X(-D) \cong \pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(mH) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{F})}(-\pi^*D)) = \pi_*\mathcal{O}_{\mathbb{P}(\mathcal{F})}(mH - \pi^*D).$$

Since $H^0(X, \pi_*\mathcal{O}_{\mathbb{P}(\mathcal{F})}(mH - \pi^*D)) \cong H^0(\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(mH - \pi^*D))$, we get a canonical isomorphism

$$H^0(X, \mathcal{S}^m\mathcal{F} \otimes \mathcal{O}_X(-D)) \cong H^0(\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(mH - \pi^*D)).$$

Thus, f corresponds uniquely to a global section of $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(mH - \pi^*D)$ which defines an effective divisor linearly equivalent to $mH - \pi^*D$ i.e., a divisor W in the linear system $|mH - \pi^*D|$ on $\mathbb{P}(\mathcal{F})$. From the proof of Lemma 3.23 we know that there exists a surjective morphism $\beta : \bar{X} \to X$, where $\bar{X}$ is a non-singular surface whose function field $\mathbb{C}(\bar{X})$ is such that β^*f splits into linear factors in $\mathbb{C}(\bar{X})$. We can write

$$\beta^*f = f_1...f_m \in H^0(\bar{X}, \beta^*\mathcal{S}^m\mathcal{F} \otimes \mathcal{O}_{\overline{X}}(-\beta^*D)) = H^0(\bar{X}, \mathcal{S}^m\beta^*\mathcal{F} \otimes \mathcal{O}_{\overline{X}}(-\beta^*D)), \tag{36}$$

with $f_i \in H^0(\bar{X}, \beta^*\mathcal{F} \otimes \mathcal{O}_{\overline{X}}(-D_i))$ where D_i is a divisor on $\bar{X}$, for all i. The equality in (36) follows from the fact that pullback commutes with taking symmetric powers. Note that there are canonical injections $\beta^*\mathcal{F} \subset \beta^*\Omega^1_X \subset \Omega^1_{\overline{X}}$. Since pullback commutes with taking determinant, we have $(\det(\beta^*\mathcal{F}))^{\otimes n} \cong \beta^*(\det(\mathcal{F}))^{\otimes n}$. Moreover, $\beta^*(\det(\mathcal{F}))^{\otimes n}$ is generated by global sections because the pullback along a surjective morphism of a globally generated sheaf is globally generated. Since $\mathcal{F}$ is a locally free sheaf of rank 2, so is $\beta^*\mathcal{F}$, and $\beta^*\mathcal{F} \otimes \mathcal{O}_{\overline{X}}(-D_i)$ has a non-trivial global section, namely f_i, for all i. Thus, the conditions of Proposition 3.19 are satisfied. Applying Proposition 3.19, we get the inequalities

$$D_i[\det(\beta^*\mathcal{F})] \leqslant \max(c_2(\beta^*\mathcal{F}), 0), \tag{37}$$

for all i. Since $(\beta')^*W$ decomposes as $(\beta')^*W = W_1 + ... + W_m$, where each W_i is an effective divisor linearly equivalent to $\bar{H} - \bar{\pi}^*D_i$, $(\beta')^*W$ must be linearly equivalent to $m\bar{H} - \sum \bar{\pi}^*D_i = m\bar{H} - \bar{\pi}^* \sum D_i$. But W is linearly equivalent to $mH - \pi^*D$, which implies that $(\beta')^*W$ is linearly equivalent to $(\beta')^*(mH - \pi^*D) = m(\beta')^*H - (\beta')^*\pi^*D = m\bar{H} - \bar{\pi}^*\beta^*D$. So comparing, we see that $\bar{\pi}^*\beta^*D = \bar{\pi}^* \sum D_i$. Since $\bar{\pi} : \mathbb{P}(\beta^*\mathcal{F}) \to \bar{X}$ is surjective, $\bar{\pi}^* : \mathrm{Pic}(\bar{X}) \to \mathrm{Pic}(\mathbb{P}(\beta^*\mathcal{F}))$ is injective, and we get $\beta^*D = \sum D_i$. Thus, summing up the inequalities (37) for all i, we have

$$\beta^*D[\det(\beta^*\mathcal{F})] = \left(\sum D_i\right)[\det(\beta^*\mathcal{F})] \leqslant \max(mc_2(\beta^*\mathcal{F}), 0).$$

Letting d be the mapping degree of β, we have

$$\beta^*D[\det(\beta^*\mathcal{F})] = dD[\det(\mathcal{F})], \quad c_2(\beta^*\mathcal{F}) = dc_2(\mathcal{F}).$$

Thus $\beta^*D[\det(\beta^*\mathcal{F})] \leqslant \max(mc_2(\beta^*\mathcal{F}), 0) = \max(mdc_2(\mathcal{F}), 0)$. So $dD[\det(\mathcal{F})] \leqslant \max(mdc_2(\mathcal{F}), 0)$ implies $D[\det(\mathcal{F})] \leqslant \max(mc_2(\mathcal{F}), 0)$. This completes the proof. ∎

Theorem 3.25. *If X is a non-singular complete surface of general type then the inequality*

$$c_1^2(X) \leqslant 3c_2(X)$$

holds.

Proof. Suppose X is minimal and let X' be a surface birationally equivalent to X. We know from Lemma 1.2 that blowing up a point increases c_2 by 1, and so $c_2(X') \geqslant c_2(X)$. But we also know that $\chi(X, \mathcal{O}_X) = \chi(X', \mathcal{O}_{X'})$, which implies $c_1^2(X) + c_2(X) = c_1^2(X) + c_2(X')$. Now $c_2(X') \geqslant c_2(X)$ implies $c_1^2(X') \leqslant c_1^2(X)$. If X satisfies $c_1^2(X) \leqslant 3c_2(X)$, then $c_1^2(X') \leqslant c_1^2(X) \leqslant 3c_2(X) \leqslant 3c_2(X')$. Thus we may assume X to be minimal.
We consider the problem in two cases

1. $c_1^2(X) \leqslant 2c_2(X)$; in this case there is no question.

2. $c_1^2(X) > 2c_2(X)$.

In case 2, put

$$\alpha = \frac{c_2(X)}{c_1^2(X)} < \frac{1}{2},$$

and let $\delta > 0$ be a sufficiently small rational number. We claim that

$$h^0(X, \mathcal{S}^m\Omega_X^1 \otimes \mathcal{O}_X(-m(\alpha+\delta)K_X)) = 0,$$

where m is any positive integer such that $m(\alpha+\delta) \in \mathbb{Z}$. We know that if $\mathcal{F}$ is a locally free sheaf of rank 2 on X which satisfies the conditions of Theorem 3.24, and if $\mathcal{S}^m\mathcal{F} \otimes \mathcal{O}_X(-D)$ has a non-trivial global section, then the inequality $D[\det(\mathcal{F})] \leqslant \max(mc_2(\mathcal{F}), 0)$ holds. In this situation we have $D = m(\alpha+\delta)K_X$ and $\mathcal{F} = \Omega_X^1$, so that $\det(\mathcal{F}) = \mathcal{O}_X(K_X)$. Note that $m, \alpha, \delta > 0$, and since $\kappa(X) = 2$ and X is minimal, it follows from the proof of Corollary 3.22 that $K_X^2 > 0$. Hence we have $D[\det(\mathcal{F})] = m(\alpha+\delta)K_X^2 > 0$. Now

$$m(\alpha+\delta)K_X^2 = m\Big(\frac{c_2(X)}{c_1^2(X)} + \delta\Big)K_X^2 = mc_2(X) + m\delta K_X^2 = mc_2(\mathcal{F}) + m\delta K_X^2 > mc_2(\mathcal{F}).$$

Thus the inequality of Theorem 3.24 is not satisfied, which means $\mathcal{S}^m\Omega_X^1 \otimes \mathcal{O}_X(-m(\alpha+\delta)K_X)$ has no non-trivial global sections i.e., $h^0(X, \mathcal{S}^m\Omega_X^1 \otimes \mathcal{O}_X(-m(\alpha+\delta)K_X)) = 0$ as claimed.
Serre duality implies that

$$h^2(X, \mathcal{S}^m\Omega_X^1 \otimes \mathcal{O}_X(-m(\alpha+\delta)K_X)) = h^0(X, \mathcal{S}^m\Omega_X^1 \otimes \mathcal{O}_X(-m(1-\alpha-\delta)K_X - K_X)). \tag{38}$$

Since $\alpha < \frac{1}{2}$ and δ is small, we have $1-\alpha-\delta > \alpha$. We claim that

$$h^2(X, \mathcal{S}^m\Omega_X^1 \otimes \mathcal{O}_X(-m(\alpha+\delta)K_X)) = 0,$$

for any sufficiently large m. From the equality (38), this is equivalent to saying that $h^0(X, \mathcal{S}^m\Omega_X^1 \otimes \mathcal{O}_X(-m(1-\alpha-\delta)K_X - K_X)) = 0$. We take $\mathcal{F} = \Omega_X^1$ and $D = m(1-\alpha-\delta) + K_X$ and apply Theorem 3.24. In this case we have

$$D[\det(\mathcal{F})] = m(1-\alpha-\delta)K_X^2 + K_X^2 > m\alpha K_X^2 + K_X^2,$$

since $1-\alpha-\delta > \alpha$. Thus we have $m\alpha K_X^2 + K_X^2 = mc_2(\Omega_X^1) + K_X^2 > mc_2(\Omega_X^1)$, because $K_X^2 > 0$. The inequality of Theorem 3.24 is not satisfied, which implies that $h^0(X, \mathcal{S}^m\Omega_X^1 \otimes \mathcal{O}_X(-m(1-\alpha-\delta)K_X - K_X)) = 0$

for a sufficiently large m, as claimed. From the Hirzebruch-Riemann-Roch theorem it follows that the Euler characteristic $\chi(X, \mathcal{S}^m\Omega_X^1 \otimes \mathcal{O}_X(-m(\alpha+\delta)K_X))$ is non-positive. On the other hand, we know that

$$\chi(X, \mathcal{S}^m\Omega_X^1 \otimes \mathcal{O}_X(-m(\alpha+\delta)K_X)) = \chi(V, \mathcal{O}_V(m(H-(\alpha+\delta)\pi^*K_X)))$$

grows asymptotically as

$$\frac{1}{6}(H-(\alpha+\delta)\pi^*K_X)^3 m^3,$$

where V is the projective bundle $\mathbb{P}(\Omega_X^1)$ and H the tautological line bundle on V. Hence we obtain the inequality $(H-(\alpha+\delta)\pi^*K_X)^3 \leqslant 0$. Letting $\delta \to 0$, we have

$$\begin{aligned} 0 &\geqslant (H-\alpha\pi^*K_X)^3 && (39)\\ &= H^3 - 3H^2\alpha\pi^*K_X + 3H(\alpha\pi^*K_X)^2 - (\alpha\pi^*K_X)^3 && (40)\\ &= c_1^2(X) - c_2(X) - 3\alpha c_1^2(X) + 3\alpha^2 c_1^2(X). && (41)\end{aligned}$$

The equality (41) follows from applying Lemma 3.5 to the equality (40). Putting $c_2(X) = \alpha c_1^2(X)$ in equality (41), we get

$$\begin{aligned} 0 &\geqslant (1-\alpha-3\alpha+3\alpha^2)c_1^2(X)\\ &= (1-\alpha)(1-3\alpha)c_1^2(X).\end{aligned}$$

Since $\alpha < \frac{1}{2}$ and $c_1^2(X) > 0$, this implies that $3\alpha \geqslant 1$. Thus $c_1^2(X) \leqslant 3c_2(X)$. This proves the theorem. □

Following are some easy consequences.

Corollary 3.26. *If X is a surface of general type, then $c_2(X) > 0$.*

Proof. If X is a minimal surface of general type, then we know from the proof of Proposition 3.21 that $c_1^2(X) > 0$. From theorem 3.25 we have $c_1^2(X) \leqslant 3c_2(X)$, which implies that $c_2(X) > 0$. If X' is any surface of general type then $c_2(X') \geqslant c_2(X)$ and so the assertion follows. □

Corollary 3.27. *If X is a surface of general type, then the inequality $p_a(X) \geqslant \frac{1}{9}c_1^2(X) - 1$ holds.*

Proof. The arithmetic genus p_a of a complete smooth algebraic surface X is equal to

$$p_a(X) = \chi(X, \mathcal{O}_X) - 1.$$

Now $\chi(X, \mathcal{O}_X) = \frac{1}{12}(c_1^2(X) + c_2(X))$ implies that

$$p_a = \frac{1}{12}(c_1^2(X) + c_2(X)) - 1.$$

From Theorem 3.25 we have the inequality $c_1^2(X) \leqslant 3c_2(X)$ for X a surface of general type. Thus $c_2(X) \geqslant \frac{1}{3}c_1^2(X)$ and so $p_a(X) \geqslant \frac{1}{12}(c_1^2(X) + \frac{1}{3}c_1^2(X)) - 1$ i.e.,

$$p_a(X) \geqslant \frac{1}{9}c_1^2(X) - 1.$$

□

4 Examples and Applications

We now discuss some examples of surfaces of general type. The results discussed in subsections 4.1, 4.2, and 4.3 are those in subsections 6A, 6B, and 6C respectively, in Miyaoka's paper [10]. With the exception of subsection 4.3, we study surfaces of general type which satisfy the extreme case of the inequality of Theorem 3.25 i.e, surfaces whose Chern numbers satisfy $c_1^2 = 3c_2$. It is known that a surface X of general type satisfies this equality if and only if the universal cover of X is the complex unit ball $\mathbb{B}^2 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 < 1\}$.

4.1 Surfaces with $c_1^2 = 3c_2$ (construction by Borel and Hirzebruch)

The ball $\mathbb{B}^2$ is a bounded symmetric domain $U(2,1)/U(2) \times U(1)$. Suppose that a group G acts freely on B and that B/G is compact. Then $X = B/G$ is non singular algebraic surface of general type. Consider the associated compact symmetric space $B' = U(3)/U(2) \times U(1)$, which is also an algebraic variety, and is in fact isomorphic to the projective plane $\mathbb{P}^2$. Hirzebruch found that the Chern numbers of the two algebraic varieties X and B' are closely related.

Theorem 4.1 (Hirzebruch)**.** *There is a constant t associated to the group G such that*

$$c_1^2(X) = tc_1^2(\mathbb{P}^2), \quad c_2(X) = tc_2(\mathbb{P}^2).$$

Since K_X is ample, we have $c_1^2(X) > 0$. On the other hand, the equalities $c_1^2(\mathbb{P}^2) = 9$, $c_2(\mathbb{P}^2) = 3$ imply that $c_1^2(X) = 3c_2(X)$.

4.2 Surfaces of which the intersection matrices are positive definite

Let X be an algebraic surface and suppose that the intersection form on $H^2(X, \mathbb{Q})$ is positive definite. Then we have the following result.

Proposition 4.2. *We have the equality*

$$p_g(X) = q(X) = 0, \qquad b_2(X) = 1.$$

In particular, we have $c_1^2(X) = 9$ and $c_2(X) = 3$.

Proof. We set $h^{i,j}(X) = \dim H^j(X, \Omega_X^i)$, and let $b_2(X)$ denote the second Betti number of X. Note that $b_2(X) = h^{2,0}(X) + h^{1,1}(X) + h^{0,2}(X)$ and moreover that $h^{2,0}(X) = h^{0,2}(X)$ by Serre duality. We also have $p_g(X) = h^{2,0}(X)$, which gives $b_2(X) = 2p_g(X) + h^{1,1}(X)$. Recall that $b_2(X) = \dim H^2(X, \mathbb{Q})$ and let $\{\Gamma_1, ..., \Gamma_i, ..., \Gamma_{b_2}\}$ denote a basis of $H^2(X, \mathbb{Q})$. Define b^+ and b^- to be, respectively, the number of positive and negative eigenvalues of the symmetric matrix $(\Gamma_i\Gamma_j)_{ij}$, where $\Gamma_i\Gamma_j$ denotes the intersection product of Γ_i and Γ_j. Then from Theorem 1.5 and the paragraph preceding it, we have

$$b^+ - b^- = -\frac{2}{3}c_2 + \frac{1}{3}c_1^2 \tag{42}$$

$$b^+ + b^- = b_2. \tag{43}$$

Since we have assumed that the intersection form on $H^2(X, \mathbb{Q})$ is positive definite, we have that the matrix $(\Gamma_i\Gamma_j)_{ij}$ has no negative eigenvalues, i.e., $b^- = 0$. Then equation (43) implies $b^+ = b_2(X)$. Now, since X is a non-singular complex surface, we have $h^{1,0}(X) = h^{0,1}(X)$, which implies that $b_1(X) = h^{1,0}(X) + h^{0,1}(X)$ is even. Thus from statement 1 of Theorem 1.7, we have $b^+ = 2p_g(X) + 1$. It follows that $2p_g(X) + 1 =$

$2p_g(X)+h^{1,1}(X)$, i.e., $h^{1,1}(X)=1$. On the other hand, equation (42) implies that $b_2(X)=\frac{1}{3}(c_1^2(X)-2c_2(X))$. Hence, applying Theorem 3.25, i.e. the inequality $c_1^2 \leqslant 3c_2$, we have

$$2p_g(X)+1=b_2(X)\leqslant \frac{1}{3}c_2(X)=\frac{1}{3}(2-4q(X)+b_2(X)).$$

Using $b_2(X)=2p_g(X)+1$ again on the right hand side, we get

$$2p_g(X)+1\leqslant \frac{1}{3}(2-4q(X)+2p_g(X)+1)=1+\frac{1}{3}(2p_g(X)-4q(X)).$$

This gives $p_g(X)\leqslant -q(X)$. Since $p_g(X)=h^{2,0}(X)$ and $q(X)=h^{1,0}(X)$, $p_g(X)$ and $q(X)$ are both non-negative, which implies $p_g(X)=q(X)=0$. Thus $b_2(X)=h^{1,1}(X)=1$ and $c_2(X)=2-0+1=3$. From equation (42) we get $c_1^2(X)=9$. This proves the assertion. □

Corollary 4.3. *Let $k\mathbb{P}^2$ denote the connected sum of k copies of the complex projective plane $\mathbb{P}^2$. Then the topological manifold $k\mathbb{P}^2$ admits a complex structure if and only if $k=1$.*

Proof. We know that the Betti numbers b_0,b_1,b_2,b_3,b_4 of the complex projective plane $\mathbb{P}^2$ are $1,0,1,0,1$ respectively. Thus if a complex surface X is homeomorphic to $k\mathbb{P}^2$ then $b_1(X)=0$ and $b_2(X)=k$. Moreover, the intersection form on X is positive or negative definite according to the orientation of X. On the other hand, since $b_1(X)=0$, X is a Kähler manifold or a K3 surface. In any case, the intersection form is not negative definite. This proves the assertion. □

4.3 Surfaces with $c_1^2 \leqslant 2c_2$

Let (p,r) be a pair of integers. If $p+r\cong 0 \mod 12$ and $p\leqslant 2r$, then we can construct a surface X such that $c_1^2(X)=p$, and $c_2(X)=r$ as follows. Suppose q is an integer such that $12q=p+r$. Then from $p\leqslant 2r$ we get the inequalities $p\leqslant 8q$ and $r\geqslant 4q$. Let C be a smooth curve of genus $g(C)=|q|+1$, and set

$$C'=\begin{cases}\mathbb{P}^1 & \text{if } q<0,\\ \text{a smooth curve of genus 2} & \text{if } q\geqslant 0.\end{cases}$$

Consider the surface $X'=C\times C'$. Let $p_1: X'\to C$ and $p_2: X'\to C'$ denote the two projections. Then the canonical divisor of X' is given by $K_{X'}=p_1^*K_C+p_2^*K_{C'}$, where K_C and $K_{C'}$ are the canonical divisors of C and C' respectively (see Exercise II.1.5 in [3]). Thus we have

$$K_{X'}^2=(p_1^*K_C+p_2^*K_{C'})^2=(p_1^*K_C)^2+2(p_1^*K_C)(p_2^*K_{C'})+(p_2^*K_{C'})^2. \tag{44}$$

Note that $\deg(K_C)=2g(C)-2$, so the support of K_C consists of $2g(C)-2$ points on C. Similarly, the support of $K_{C'}$ consists of $2g(C')-2$ points on C'. Thus $p_1^*K_C\sim(2g(C)-2)(\{\mathrm{pt}\}\times C')\sim(2g(C)-2)C'$ and similarly $p_2^*K_{C'}\sim(2g(C')-2)(C\times\{\mathrm{pt}\})\sim(2g(C')-2)C$, where $\sim$ denotes numerical equivalence. Plugging this into equation 44, we get

$$K_{X'}^2=(2g(C)-2)C'^2+2(2g(C)-2)(2g(C')-2)CC'+(2g(C')-2)^2C^2. \tag{45}$$

Note that for any two points $P,Q\in C$, the fibres $p_1^*\{P\}$ and $p_1^*\{Q\}$ are algebraically equivalent, which implies they are numerically equivalent. Moreover, the fibres are disjoint. Since $C'\sim(\{P\}\times C')=p_1^*\{P\}\sim p_1^*\{Q\}=(\{Q\}\times C')$, we have $C'^2=(p_1^*\{P\})^2=(p_1^*\{P\})(p_1^*\{Q\})=(\{P\}\times C')(\{Q\}\times C')=0$. Similarly, we have $C^2=0$. The fibre of any point $P\in C$ meets the fibre of any point $P'\in C'$ at exactly one point in X, namely (P,P') and hence $CC'=1$. Thus it follows from equation 45 that

$$K_{X'}^2=2(2g(C)-2)(2g(C')-2)=8(g(C)-1)(g(C')-1). \tag{46}$$

The Euler characteristic of X is given by $e(X)e(C \times C') = e(C)e(C')$. We know that $e(C) = 2 - 2g(C)$ and similarly for C'. Thus we have $c_2(X') = e(X') = (2 - 2g(C))(2 - 2g(C')) = 4(g(C) - 1)(g(C') - 1)$. This implies that

$$c_2(X') = \frac{1}{2} K^2_{X'}. \tag{47}$$

From the equalities 46 and 47 it follows that $c_1^2(X') = 8q = 2c_2(X')$. Now let X be the surface which is the blow-up of X' at $8q - p$ points. Since blowing up a point decreases c_1^2 by 1 and increases c_2 by 1, we have $c_1^2(X) = 8q - (8q - p) = p$ and $c_2(X) = 4q + 8q - p = 12q - p = r$. Thus X is a surface with the desired Chern numbers. Moreover, since $\kappa(X) = \kappa(X') = \kappa(C) + \kappa(C')$ (see Theorem I.7.3 in [1]), we have

$$\kappa(X) = \begin{cases} -\infty & \text{if } q < 0, \\ 1 & \text{if } q = 0, \\ 2 & \text{if } q > 0. \end{cases}$$

4.4 The complete quadrilateral

It is well-known that the Chern numbers of the projective plane $\mathbb{P}^2$ are $c_1^2(\mathbb{P}^2) = 9$ and $c_2(\mathbb{P}^2) = 3$ i.e., they satisfy $c_1^2(\mathbb{P}^2) = 3c_2(\mathbb{P}^2)$. From Theorem 3.25 we have the inequality $c_1^2 \leqslant 3c_2$ for surfaces of general type. Our goal is to study surfaces of general type which satisfy the extreme case of this inequality i.e., surfaces for which the equality $c_1^2 = 3c_2$ holds. This motivates the following definition.

Definition 4.4. *The proportionality deviation $Prop(Y)$ of a complex surface Y is given by*

$$Prop(Y) = 3c_2(Y) - c_1^2(Y).$$

For example, $\mathrm{Prop}(\mathbb{P}^2) = 0$. The following theorem gives a formula to compute the proportionality deviation for good coverings (see Definition 2.4) and is a useful computational tool.

Theorem 4.5. *Let the setting be as in Definition 2.4. Then for good coverings $\pi : Y \to X$ of degree N, the proportionality deviation is given by*

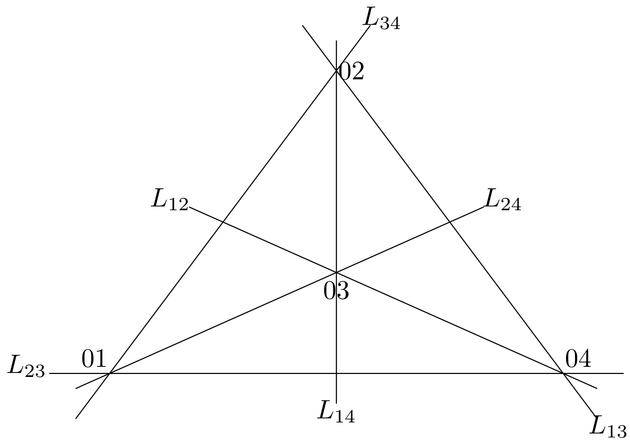
$$\frac{Prop(Y)}{N} = \frac{3c_2(Y) - c_1^2(Y)}{N} = 3c_2(X) - c_1^2(X) + \sum_i x_i(-e(D_i) + 2D_i^2) + \frac{1}{2}\sum_{i \neq j} x_i x_j D_i D_j - \sum_i x_i^2 D_i^2, \tag{48}$$

where the x_i's are real numbers given by $x_i = 1 - \frac{1}{b_i}$ for all i, and the b_i's are positive integers as in Definition 2.4.

Proof. The equality 48 follows from combining the equalities 4 and 13. $\square$

We now study a line arrangement in $\mathbb{P}^2$ which gives rise to a surface Y for which $\mathrm{Prop}(Y)$ vanishes. This arrangement is known as the complete quadrilateral and it is the arrangement of six lines having four triple intersection points, labelled $0i$ for $i = 1, 2, 3, 4$, no three of which are collinear (see figure below). Any four points with this property are equivalent up to a projective transformation. The six lines, labelled $L_{\alpha\beta}$ for

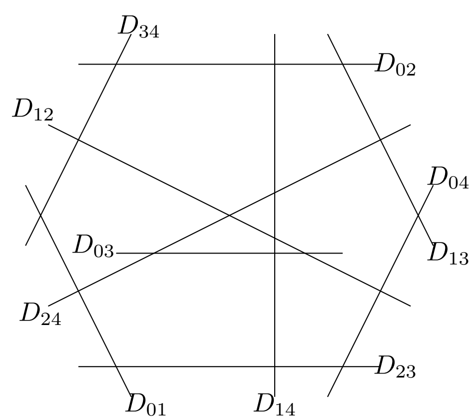
$\alpha, \beta \in \{1, 2, 3, 4\}$, are the six ways of connecting these four points by lines.



This arrangement has three double and four triple intersection points. Any three of its lines not having a common triple intersection point give an affine coordinate system on an open subset of $\mathbb{P}^2$. In suitable projective coordinates $(z_0 : z_1 : z_2)$, the arrangement is given by the equation

$$z_0 z_1 z_2 (z_2 - z_1)(z_2 - z_0)(z_0 - z_1) = 0.$$

We now construct a new surface X by blowing up the four triple intersection points of the complete quadrilateral on $\mathbb{P}^2$. Thus the surface X has ten divisors $D_{\alpha\beta}$, $\alpha, \beta \in \{0, 1, 2, 3, 4\}$, six of which are transforms of the original six lines of the arrangement and the other four are exceptional divisors corresponding to the blown-up points. For example, the divisor D_{12} is the proper transform of the line in the original arrangement passing through the points 03 and 04, while D_{0i} for $i = 1, 2, 3, 4$ is the divisor obtained by blowing up the point $0i$ (see figure below).



The ten divisors have only simple intersection points and there are fifteen such points.
Note that the intersection number of any two lines in the complete quadrilateral is 1. Since each line in the arrangement is isomorphic to $\mathbb{P}^1$, the self-intersection number of each line is 1. We now want to determine the intersection numbers of all divisors in the blown-up arrangement on X.

Lemma 4.6. *The intersection numbers of the ten divisors $D_{\alpha\beta}$ on X are given by*

$$D_{\alpha\beta} D_{\gamma\delta} = \begin{cases} 1 & \text{if } \{\alpha\beta\} \neq \{\gamma\delta\} \text{ and } \{\alpha\beta\} \cap \{\gamma\delta\} = \varnothing \\ 0 & \text{if } \{\alpha\beta\} \neq \{\gamma\delta\} \text{ and } \{\alpha\beta\} \cap \{\gamma\delta\} \neq \varnothing \\ -1 & \text{if } \{\alpha\beta\} = \{\gamma\delta\}. \end{cases} \tag{49}$$

Proof. We begin by computing the self-intersection numbers. We have $D_{0i}^2 = -1$ for $i = 1, 2, 3, 4$ since the D_{0i}'s are the exceptional divisors. For the other six $D_{\alpha\beta}$ which are proper transforms of the $L_{\alpha\beta}$, we have $D_{\alpha\beta}^2 = (\pi^* L_{\alpha\beta} - D_{0\gamma} - D_{0\delta})^2$, where $\{\alpha\beta\} \cap \{\gamma\delta\} = \varnothing$ and π denotes the projection of X onto $\mathbb{P}^2$. Since the exceptional divisors are disjoint, we have $D_{0i}D_{0j} = 0$ for $i \neq j$ and since $(\pi^* D)(\pi^* D') = DD'$ for any two divisors D, D' on $\mathbb{P}^2$, we have $(\pi^* L_{\alpha\beta})^2 = L_{\alpha\beta}^2 = 1$. By Serre's moving lemma we can move $L_{\alpha\beta}$ away from the points $0\gamma, 0\delta$ it passes through, and then pullback along π, from which it follows that $(\pi^* L_{\alpha\beta})D_{0\gamma} = (\pi^* L_{\alpha\beta})D_{0\delta} = 0$. Hence we get

$$D_{\alpha\beta}^2 = (\pi^* L_{\alpha\beta} - D_{0\gamma} - D_{0\delta})^2 = L_{\alpha\beta}^2 + D_{0\gamma}^2 + D_{0\delta}^2 = -1,$$

for $\alpha, \beta \in \{1, 2, 3, 4\}$. Thus we have $D_{\alpha\beta}^2 = -1$ for all $\alpha, \beta \in \{0, 1, ..., 4\}$.
Next, we compute intersection numbers $D_{\alpha\beta}D_{\gamma\delta}$, where $\{\alpha\beta\} \neq \{\gamma\delta\}$. We already know $D_{0i}D_{0j} = 0$ for $i \neq j$. Consider the intersection $D_{\alpha\beta}D_{0\gamma}$, where $D_{\alpha\beta}$ is not an exceptional divisor. We have $D_{\alpha\beta}D_{0\gamma} = (\pi^* L_{\alpha\beta} - D_{0\mu} - D_{0\nu})D_{0\gamma} = (\pi^* L_{\alpha\beta})D_{0\gamma} - D_{0\mu}D_{0\gamma} - D_{0\nu}D_{0\gamma}$. Thus if $\mu = \gamma$ or $\nu = \gamma$, the intersection number $D_{\alpha\beta}D_{0\gamma}$ equals 1, otherwise it equals 0. Finally, consider the intersection $D_{\alpha\beta}D_{\gamma\delta}$ where none of $D_{\alpha\beta}, D_{\gamma\delta}$ is an exceptional divisor. Hence we get

$$\begin{aligned} D_{\alpha\beta}D_{\gamma\delta} &= (\pi^* L_{\alpha\beta} - D_{0\mu} - D_{0\nu})(\pi^* L_{\gamma\delta} - D_{0\eta} - D_{0\rho}) \\ &= L_{\alpha\beta}L_{\gamma\delta} + D_{0\mu}D_{0\eta} + D_{0\mu}D_{0\rho} + D_{0\nu}D_{0\eta} + D_{0\nu}D_{0\rho}. \end{aligned}$$

If $\{\alpha\beta\} \cap \{\gamma\delta\} \neq \varnothing$ then the intersection number $D_{\alpha\beta}D_{\gamma\delta}$ equals 0, otherwise it equals 1. To summarize, we have $D_{\alpha\beta}D_{\gamma\delta} = 1$ if $\{\alpha\beta\} \cap \{\gamma\delta\} = \varnothing$ and $D_{\alpha\beta}D_{\gamma\delta} = 0$ if $\{\alpha\beta\} \cap \{\gamma\delta\} \neq \varnothing$, for all $\alpha, \beta, \gamma, \delta \in \{0, 1, ..., 4\}$. Hence, the intersection numbers are as given in 49 and the assertion is proved. □

Since each of the ten divisors $D_{\alpha\beta}$ on X is isomorphic to $\mathbb{P}^1$, we have $e(D_{\alpha\beta}) = 2$ and hence $-e(D_{\alpha\beta}) + 2D_{\alpha\beta}^2 = -4$ for all ten $D_{\alpha\beta}$. By Lemma 1.2, blowing up a point increases c_2 by 1 and decreases c_1^2 by 1. So we have

$$3c_2(X) - c_1^2(X) = 3(c_2(\mathbb{P}^2) + 4) - (c_1^2(\mathbb{P}^2) - 4) = 16.$$

Using Lemma 4.6 together with the formula 48 of Theorem 4.5, we conclude that a good covering Y of X of degree N satisfies

$$\frac{\mathrm{Prop}(Y)}{N} = 16 + \sum -4x_{\alpha\beta} + \sum x_{\alpha\beta}^2 + \frac{1}{2}\sum x_{\alpha\beta}\Big(\sum_{\{\alpha\beta\}\neq\{\gamma\delta\}} x_{\alpha\beta}D_{\alpha\beta}D_{\gamma\delta}\Big). \tag{50}$$

In order for Y to satisfy the equality $c_1^2(Y) = 3c_2(Y)$, we want the right hand side of the equality 50 to vanish. Thus we want real numbers $x_{\alpha\beta}$ which give $\mathrm{Prop}(Y) = 0$ when plugged into equation 50.
We now exhibit a case in which $\mathrm{Prop}(Y)$ vanishes. Let $\mu_0, \mu_1, \mu_2, \mu_3, \mu_4$ be real numbers such that $\mu_0 + ... + \mu_4 = 2$ and let $x_{\alpha\beta} = \mu_\alpha + \mu_\beta$. Then we have

$$\sum_{\alpha\beta} x_{\alpha\beta} = 8.$$

Putting this into the equality 50, we get

$$\frac{\mathrm{Prop}(Y)}{N} = -16 + \frac{1}{2}\sum x_{\alpha\beta}\Big\{\Big(\sum_{\{\alpha\beta\}\neq\{\gamma\delta\}} x_{\alpha\beta}D_{\alpha\beta}D_{\gamma\delta}\Big) + 2x_{\alpha\beta}\Big\}.$$

It is easily checked that for every $\alpha\beta$, the term in the parentheses $\{\cdot\}$ equals 4. Thus it follows that

$$\frac{\mathrm{Prop}(Y)}{N} = -16 + \frac{1}{2}\sum_{\alpha\beta} 4x_{\alpha\beta} = -16 + 16 = 0.$$

Note however, that we have the additional requirement $x_{\alpha\beta} = 1 - \frac{1}{b_{\alpha\beta}}$, where $b_{\alpha\beta}$ are positive integers as in Definition 2.4. We want solutions of $\mathrm{Prop}(Y) = 0$ with the additional constraint

$$b_{\alpha\beta} = \frac{1}{1 - x_{\alpha\beta}} = \frac{1}{1 - (\mu_\alpha + \mu_\beta)} \in \mathbb{Z}_{\geqslant 1}. \tag{51}$$

There are eight solutions of $\mathrm{Prop}(Y) = 0$ with the constraint 51, with $0 < \mu_\alpha < 1$ and $\sum \mu_\alpha = 2$ which are given in Table 3.1 in [13]. Each of these eight solutions gives a surface Y which, assuming it exists, is a ball quotient i.e., it is a surface of general type which satisfies the equality $c_1^2(Y) = 3c_2(Y)$. In his paper [5], Kato has proved that for an arbitrary line arrangement in $\mathbb{P}^2$, and any set of integers b_i assigned to the divisors D_i on X such that $b_i \geqslant 2$ for all i, there exists a good covering Y of X branched along the new arrangement of divisors on X, with ramification index b_i along the divisor D_i. Note that we have discussed the construction of a good cover in the special case $b_i = n$ for all i, $n \geqslant 2$, in section 2.3. In this case Y is called a Kummer covering of X.

In the complete quadrilateral case, the solution $\mu_\alpha = \frac{2}{5}$ for all α, and hence $b_{\alpha\beta} = 5$ for example, gives rise to a Kummer covering Y of X. We discuss this in more detail in the next section.

4.5 The case of a Kummer covering

Consider an arrangement of $k \geqslant 3$ lines $L_1, ..., L_k$ in $\mathbb{P}^2$ which are given by homogeneous linear equations $l_1 = 0, ..., l_k = 0$. Assume that not all of the lines pass through a single point, i.e., the arrangement is not a pencil. Let t_r denote the number of points in the arrangement through which r lines pass. Then the number of regular intersection points is t_2 and the number of singular intersection points is $\sum_{r\geqslant 3} t_r$. We blow up $\mathbb{P}^2$ at these $\sum_{r\geqslant 3} t_r$ singular points and get a smooth surface X. Let D_i denote the proper transform of L_i for $i = 1, ..., k$ and let E_j be the exceptional divisors corresponding to the blown-up points. Assign to each D_i the ramification $n_i = n$ and to each E_j the ramification $m_j = n$. We have described in section 2.3 the construction of a good covering Y of X (see Definition 2.4) of degree $N = n^{k-1}$ branched along each D_i with index $n_i = n$ and along each E_j with index $m_j = n$.

We know from statement 1 of Theorem 1.1 that $\mathbb{C}(X) = \mathbb{C}(\mathbb{P}^2)$, where $\mathbb{C}(X)$ and $\mathbb{C}(\mathbb{P}^2)$ denote the function fields of X and $\mathbb{P}^2$ respectively. Thus the quotient l_i/l_j of two linear polynomials in homogeneous coordinates is a meromorphic function on X. We consider all the n-th roots $\sqrt[n]{l_i/l_j}$, $i \neq j$. The covering Y is defined by the property that these n-th roots all become single valued on Y.

The function field of Y is given by

$$\mathbb{C}(Y) = \mathbb{C}(X)(\sqrt[n]{l_1/l_k}, ..., \sqrt[n]{l_{k-1}/l_k}).$$

Note that $\mathbb{C}(Y)$ is a Kummer extension of $\mathbb{C}(X)$ of degree n^{k-1}, and hence Y is a Kummer covering of X of degree n^{k-1}. More formally,

Definition 4.7. *A covering $\pi : Y \to X$ is called a Kummer covering if the function field of Y is a Kummer extension of the function field of X.*

Note that the existence of Y is guaranteed by the result of Kato [5], and also by the construction carried out in section 2.3.

Proposition 4.8. *The surface Y satisfies the following equation*

$$n^2\Big(\frac{3c_2(Y) - c_1^2(Y)}{N}\Big) = (n-1)^2(f_0 - k) - 2(n-1)(f_1 - 2f_0) + 4(f_0 - t_2). \tag{52}$$

where $f_0 = \sum_{r\geqslant 2} t_r$ and $f_1 = \sum_{r\geqslant 2} rt_r$.

Proof. Multiplying the formula 48 by n^2, we get

$$n^2\Big(\frac{3c_2(Y)-c_1^2(Y)}{N}\Big) = n^2(3c_2(X)-c_1^2(X)) + n^2\sum_i x_i(-e(D_i)+2D_i^2) + \frac{n^2}{2}\sum_{i\neq j} x_i x_j D_i D_j - n^2\sum_i x_i^2 D_i^2.$$

Note that the D_i in the above equation denote the proper transforms of the original lines in the arrangement and the exceptional divisors. Since the ramification indices b_i are equal to n for all i, we have $x_i = 1-\frac{1}{b_i} = 1-\frac{1}{n}$ for all i. Putting this into the above equation we get

$$\begin{aligned} n^2\Big(\frac{3c_2(Y)-c_1^2(Y)}{N}\Big) &= n^2(3c_2(X)-c_1^2(X)) + n(n-1)\sum_i(-e(D_i)+2D_i^2) + \frac{(n-1)^2}{2}\sum_{i\neq j} D_i D_j \\ &\quad - (n-1)^2\sum_i D_i^2. \end{aligned} \tag{53}$$

We compute each term in the right hand side of the equation 53. The surface X is a blow-up at all the singular intersection points of the arrangement i.e., points through which $r \geqslant 3$ lines pass. The number of such points is $\sum_{r\geqslant 3} t_r = f_0 - t_2$. By Lemma 1.2 we have

$$3c_2(X) - c_1^2(X) = 4(f_0 - t_2).$$

Each divisor D_i is isomorphic to $\mathbb{P}^1$ and so $e(D_i) = 2$ for all i. The total number of divisors is the number of lines in the original arrangement plus the number of singular points which is $k + f_0 - t_2$. Hence we get

$$\sum_i e(D_i) = 2(k + f_0 - t_2).$$

If D_i is an exceptional divisor, then $D_i^2 = -1$. If D_i is not an exceptional divisor, then $D_i^2 = (\pi^* L_i - E_{i_1} - ... - E_{i_r})^2 = (\pi^* L_i)^2 + E_{i_1}^2 + ... + E_{i_r}^2$, where $E_{i_1}, ..., E_{i_r}$ are the exceptional divisors which intersect D_i. The number of such divisors is equal to the number of singular points lying on L_i. Letting σ_i denote the number of singular points lying on L_i, we get $D_i^2 = 1 - \sigma_i$. Thus we have

$$\begin{aligned} \sum_i D_i^2 &= \sum_i (1-\sigma_i) - (f_0 - t_2) \\ &= k - \sum_i \sigma_i - (f_0 - t_2). \end{aligned}$$

Note that $\sum_i \sigma_i$ is the number of singular points of the arrangement counted with multiplicity. An r-fold intersection point is counted r times in the sum- once for each line it lies on. Hence $\sum_i \sigma_i = \sum_{r\geqslant 3} r t_r = f_1 - 2t_2$. Plugging this into the equation above, we get

$$\sum_i D_i^2 = k - (f_1 - 2t_2) - (f_0 - t_2) = k - f_0 - f_1 + 3t_2.$$

The intersection number $D_i D_j$ of two divisors is either 1 or 0. If D_i and D_j are proper transforms of lines in the original arrangement meeting at a regular point, or if one of D_i, D_j is an exceptional divisor corresponding to a singular point lying on a line of which the other divisor is the proper transform, then $D_i D_j = 1$. Otherwise $D_i D_j = 0$. It follows that

$$\sum_{i\neq j} D_i D_j = 2(f_1 - t_2).$$

Putting everything into the equation 53, we get

$$n^2\Big(\frac{3c_2(Y)-c_1^2(Y)}{N}\Big) = 4n^2(f_0 - t_2) + n(n-1)(-4f_0 - 2f_1 + 8t_2) + (n-1)^2(-k + f_0 + 2f_1 - 4t_2).$$

Finally, we write the above equation as a polynomial equation in $n-1$ and arrive at

$$n^2\Big(\frac{3c_2(Y)-c_1^2(Y)}{N}\Big) = (n-1)^2(f_0-k) - 2(n-1)(f_1-2f_0) + 4(f_0-t_2),$$

which is the equation 52. This proves the assertion. □

We want to apply this result to some known line arrangements in the projective plane. For the complete quadrilateral discussed in subsection 7.2, we have $k=6$ lines, $t_2=3$ regular intersection points and $t_3=4$ triple intersection points. This gives $f_0=7$ and $f_1=18$. Plugging this into the equation 52, we get

$$n^2\Big(\frac{\mathrm{Prop}(Y)}{N}\Big) = (n-5)^2.$$

Thus, we get $\mathrm{Prop}(Y)=0$ if $n=5$. In other words, the Kummer covering Y is a ball quotient (see subsection 4.1) for $n=5$. The degree of the covering is $N=n^{k-1}=5^5$.
For the Hesse arrangement (see [13], Chapter 5), we have $k=12$, $t_2=12$, $t_3=0$, and $t_4=9$, which gives $f_0=21$ and $f_1=60$. Putting this in equation 52, we have

$$n^2\Big(\frac{\mathrm{Prop}(Y)}{N}\Big) = 9(n-3)^2.$$

Thus a Kummer covering of the blown up Hesse arrangement is a ball quotient for $n=3$. The degree of the covering in this case is $N=n^{k-1}=3^{11}$.
For the Ceva(3) arrangement (see [13], Chapter 5) we have $k=9$, $t_2=0$, and $t_3=12$. This gives $f_0=12$ and $f_1=36$. Again, we put this into equation 53 and get

$$n^2\Big(\frac{\mathrm{Prop}(Y)}{N}\Big) = 3(n-5)^2.$$

The Kummer covering is a ball quotient for $n=5$ and the degree of the covering in this case is $N=n^{k-1}=5^8$. It easy to see that in each of the three examples, Y is a surface of general type, because we have $c_1^2(Y)>9$ for each example. For the complete quadrilateral we have $c_1^2(Y)=3^2\cdot 5^4$, for the Hesse arrangement we have $c_1^2(Y)=2^4\cdot 3^{11}$, and for the Ceva(3) arrangement we have $c_1^2(Y)=3^2\cdot 5^6\cdot 37$.

5 Fake projective planes

In this section we discuss the results in the article of Keum [6]. In this paper he classifies all possible structures of surfaces which are quotients of fake projective planes by their finite automorphism groups, and their minimal resolutions. He first considers the case when the automorphism group is of prime order and proves the following

Theorem 5.1. *Let G be a group of automorphisms of a fake projective plane X. Let $Z=X/G$ and $\nu: Y\to Z$ be a minimal resolution. Then the following claims hold*

1. *If the order of G is 3, then Z has three singular points of type $\frac{1}{3}(1,2)$, and Y is a minimal surface of general type with $K_Y^2=3$, $p_g(Y)=0$.*

2. *If the order of G is 7, then Z has three singular points of type $\frac{1}{7}(1,3)$, and Y is a minimal elliptic surface of Kodaira dimension 1 with two multiple fibres. The pair of multiplicities is one of the following three cases: (2,3), (2,4), (3,3).*

5.1 Preliminary results

A fake projective plane is a compact complex surface which has the same Betti numbers as the complex projective plane $\mathbb{P}^2$, but is not isomorphic to it. There are a number of equivalent characterizations of fake projective planes, some of which are as follows.

Theorem 5.2. *A non-singular compact complex surface X with $b_1(X) = 0$, $b_2(X) = 1$ is a fake projective plane if one the following holds true:*

1. *X is not isomorphic to $\mathbb{P}^2$.*
2. *X is not homeomorphic to $\mathbb{P}^2$.*
3. *X is not homotopy equivalent to $\mathbb{P}^2$.*
4. *The fundamental group $\pi_1(X)$ is an infinite group.*
5. *The universal cover of X is a two-dimensional complex ball $\mathbb{B}^2 \subset \mathbb{C}^2$, and $X \cong \mathbb{B}^2/\pi_1(X)$, where $\pi_1(X) \subset PU(2,1)$.*
6. *The canonical divisor K_X is ample.*
7. *K_X is ample, $p_g(X) = q(X) = 0$, and $K_X^2 = c_1^2(X) = 3c_2(X) = 9$.*

We begin with the following fundamental result.

Lemma 5.3. *Let X be a fake projective plane and C a smooth curve on X. Then $e(C) \leqslant -4$, or equivalently, $g(C) \geqslant 3$.*

Proof. Since X is a fake projective plane, we have by definition that $b_0(X) = 1$, $b_1(X) = 0$, $b_2(X) = 1$, $b_3(X) = 0$, $b_4(X) = 1$. Hence we have the equality

$$c_2(X) = \chi(X) = \sum_i (-1)^i b_i(X) = 3.$$

It is known that a complex surface which has even b_1 is Kähler. Since $b_1(X) = 0$, we conclude that X is Kähler. From Poincare duality and the Hodge decomposition theorem, we know that $b_k = \sum_{i+j=k} h^{i,j}$, where $h^{i,j}$ are the Hodge numbers of X. Note that all the $h^{i,j}$ are non-negative integers and we have $h^{i,j} = h^{j,i}$ by Hodge symmetry. This implies that $h^{0,0} = h^{1,1} = h^{2,2} = 1$ are the only non-zero Hodge numbers. The arithmetic genus of X is given by $p_a(X) = h^{2,0} - h^{1,0} = 0$. Since $\chi(\mathcal{O}_X) = 1 + p_a(X)$, it follows that $\chi(\mathcal{O}_X) = 1$. Applying Noether's formula, we get $12 = c_1^2(X) + c_2(X)$, which implies that $c_1^2(X) = 9$. Hence X satisfies $c_1^2(X) = 3c_2(X)$. Since $h^{1,1} = 1$, we have that the Picard number of X is 1. Let H denote the generator of the Neron-Severi group of X, then the canonical divisor K_X is a multiple of H. Any divisor L on X is ample if and only if nL is ample for some $n \geqslant 1$. Since L is a multiple of H, we can assume H to be ample. Hence $H^2 = 1$. Now $K_X^2 = 9$ implies that $K_X = \pm 3H$. If K_X were a negative multiple of H, then by a result of Hirzebruch and Kodaira, X would be biholomorphic to $\mathbb{P}^2$ (see Theorem 3 in [15]). Thus it follows that $K_X = 3H$.
Let C be a smooth curve on X, then $C \sim mH$ for some positive integer m, where $\sim$ denotes numerical equivalence. Applying the adjunction formula, we get

$$e(C) = 2 - 2g(C) = -C(C + K_X) = -(m^2 + 3m) \leqslant -4.$$

This implies that $2 - 2g(C) \leqslant -4$, i.e. $g(C) \geqslant 3$. Hence the assertion is proved. $\square$

A normal projective complex surface is called a $\mathbb{Q}$-homology $\mathbb{P}^2$ if it has the same Betti numbers as the complex projective plane $\mathbb{P}^2$. If a $\mathbb{Q}$-homology $\mathbb{P}^2$ is non-singular, then it is either $\mathbb{P}^2$ or a fake projective plane, by Theorem 5.2.

Proposition 5.4. *Let X be a $\mathbb{Q}$-homology $\mathbb{P}^2$ with quotient singularities only and suppose that X admits a finite group G of automorphisms. Then the quotient X/G is again a $\mathbb{Q}$-homology $\mathbb{P}^2$ with quotient singularities only.*
In particular, $p_g(X/G) = q(X/G) = 0$, $c_2(X/G) = 3$, and $\chi(\mathcal{O}_{X/G}) = 1$.

Proof. The canonical map $\pi : X \to X/G$ is finite and surjective. Hence it follows that the pullback map $\pi^* : H^i(X/G, \mathbb{C}) \to H^i(X, \mathbb{C})$ is injective, for all i. Since $H^i(X/G, \mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}(X/G)$ and $H^i(X, \mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}(X)$, the map π^* is injective on the level of each $H^{p,q}$ in the Hodge decomposition. Recall that $p_g(X) = \dim H^{2,0}(X) = h^{2,0}$ and $q(X) = \dim H^{1,0}(X) = h^{1,0}$. Since $p_g(X) = q(X) = 0$, the injectivity of π^* implies that $p_g(X/G) = q(X/G) = 0$. Thus we have

$$\chi(\mathcal{O}_{X/G}) = 1 - p_g(X/G) + q(X/G) = 1.$$

Since $b_2(X) = 1$, it follows that $b_2(X/G) = 1$. Recall that $c_2(X/G) = 2 - 4q(X/G) + b_2(X/G) = 3$. This concludes the proof. $\square$

Next, we consider fake projective planes with an automorphism of prime order.

Proposition 5.5. *Let X be a fake projective plane with an automorphism σ. Assume that the order of σ is a prime number p. Let $Z = X/\langle\sigma\rangle$ and let $\nu : Y \to Z$ be a minimal resolution. Then the following holds.*

1. *Z is a $\mathbb{Q}$-homology $\mathbb{P}^2$ with K_Z ample.*
2. *$p_g(Y) = q(Y) = 0$.*
3. *$K_Z^2 = 9/p$.*
4. *The fixed point set X^σ consists of three points.*

Proof. We know from Proposition 5.4 that the quotient surface Z has the same Betti numbers as X and hence is a $\mathbb{Q}$-homology $\mathbb{P}^2$. Moreover, we have $p_g(Z) = q(Z) = 0$. Note that the resolution of singularities $\nu : Y \to Z$ is a birational morphism and the irregularity q and geometric genus p_g are birational invariants. The latter statement follows from the fact that $R^1\nu_*\mathcal{O}_Y = 0$ since Z has only rational singularities, and that 2-forms can be extended along the resolution i.e., $\nu_*\Omega_Y^2 \cong \Omega_Z^2$. Thus it follows that Y also satisfies $p_g(Y) = q(Y) = 0$. This is (2).
Next, we prove (4). Suppose the fixed point locus X^σ consists of n curves $C_1, ..., C_n$ and m isolated points. This implies that the quotient surface Z has m singular points. Since each point has Euler characteristic 1, we have $e(X^\sigma) = m + \sum_{i=1}^n e(C_i)$. Note that since σ has order p, the degree of the quotient map $\pi : X \to Z$ is p. Using a Hurwitz type formula for surfaces, we get

$$e(X) = pe(Z) - (p-1)\Big(m + \sum_{i=1}^n e(C_i)\Big). \tag{54}$$

We know from Proposition 5.4 that $e(X) = e(Z) = 3$. This together with the equality 54 gives

$$m + \sum_{i=1}^n e(C_i) = 3. \tag{55}$$

Let $\widetilde{C_i}$ denote the normalization of the curve C_i on X. Then it is known that $\chi(C_i) = \chi(\widetilde{C_i}) - r_i$, where r_i is the number of nodal points on C_i. In particular, $\chi(C_i) \leqslant \chi(\widetilde{C_i})$. Thus observing equation 55 we may assume that all C_i are smooth curves. From Lemma 5.3 we know that $e(C_i) \leqslant -4$ for all $i = 1, ..., n$ and so equality 55 implies that

$$m \geqslant 3 + 4n. \tag{56}$$

It can be concluded from the orbifold Bogomolov-Miyaoka-Yau inequality that a $\mathbb{Q}$-homology $\mathbb{P}^2$ with quotient singularities only cannot have more than five singular points, i.e. $m \leqslant 5$ (see for example [7]). Then the inequality 56 implies that $n = 0$ and from 55 we get $m = 3$ i.e., the fixed point locus X^σ consists of three isolated points. This is (4).

Note that X is a good cover of Z in the sense of Definition 2.4 and hence we have the following relation between the canonical divisors K_X and K_Z (see equation 10).

$$K_X = \pi^*\Big(K_Z + \sum_i x_i D_i\Big),$$

where the numbers x_i are as defined earlier and D_i are ramification divisors on Z. Although the notion of a good cover was defined for smooth surfaces, Z has no codimension 1 singularities, and so the above formula is valid. Since the ramification locus consists only of three isolated points, we have $K_X \sim \pi^* K_Z$, where $\sim$ denotes numerical equivalence. Thus we have $K_X^2 = (\pi^*K_Z)^2 = \deg(\pi) K_Z^2 = pK_Z^2$ i.e., $K_Z^2 = K_X^2/p = 9/p$. This is (3).

Since K_X is ample, it follows from the Nakai-Moishezon criterion (or Corollary 1.2.24 in [8]) that π^*K_Z is ample. This further implies that K_Z is ample on Z (see Corollary 1.2.28 in [8]) and hence we have the second part of assertion (1). This concludes the proof. $\square$

Corollary 5.6. *Let the setting be as in Proposition 5.5. Then $p \neq 2$.*

Proof. Suppose $p = 2$. Then $Z = X/(\mathbb{Z}/2\mathbb{Z})$ has only singularities of type $\frac{1}{2}(1,1)$. We know that the exceptional divisor corresponding to such a singularity is a (-2)-curve. Thus $K_Y = \nu^*K_Z + \sum_i k_i E_i$, where $\nu : Y \to Z$ is the minimal resolution, and each E_i is a (-2)-curve. Applying the adjunction formula to E_i, we see that $2 = -(K_Y E_i + E_i^2) = 2k_i + 2$, which gives $k_i = 0$ for all i. This implies that $K_Y \sim \nu^* K_Z$. Since Y is smooth, we know that $c_2(Y) = e(Y) = \sum_i (-1)^i b_i(Y)$, and so $c_2(Y)$ is an integer. Moreover, $\chi(Y, \mathcal{O}_Y) = \sum_i (-1)^i h^i(Y, \mathcal{O}_Y)$ is also an integer. Hence using Noether's formula, it follows that $c_1^2(Y) = K_Y^2 = 12\chi(Y, \mathcal{O}_Y) - c_2(Y)$ is an integer. However, since $K_Y \sim \nu^*K_Z$, statement 3 of Proposition 5.5 implies that $K_Y^2 = K_Z^2 = 9/p = 9/2$, which is not an integer. This is a contradiction, hence $p \neq 2$. $\square$

Prasad and Yeung [11] have given precise possible values for the order p of the automorphism σ. According to their result, $p = 3$ or 7. The goal is to determine in each case the types of singularities of the quotient surface Z, using the holomorphic Lefschetz fixed point formula.

Lemma 5.7. *Let S be a compact complex manifold of dimension 2 with $p_g(S) = q(S) = 0$. Assume that S admits an automorphism σ of prime order p. Let r_i for $(1 \leqslant i \leqslant p-1)$ be the number of isolated fixed points of σ which give singularities of type $\frac{1}{p}(1,i)$ on the quotient surface. Let $C_1, ..., C_k$ be one-dimensional components of the fixed locus S^σ. Then*

$$1 = \sum_{j=1}^{k} \Big(\frac{1 - g(C_j)}{2} + \frac{(p+1)C_j^2}{12}\Big) + \sum_{i=1}^{p-1} a_i r_i$$

where

$$a_i = \frac{1}{p-1}\Big(\sum_{j=1}^{p-1} \frac{1}{(1-\zeta^j)(1-\zeta^{ij})}\Big)$$

with ζ *a primitive* p*-th root of 1. For example,* $a_1 = (5-p)/12$ *and* $a_2 = (11-p)/24$.

Proof. This follows from the original holomorphic Lefschetz fixed point formula. A proof is given in [16], Lemma 1.6 and it is valid for all two-dimensional complex manifolds with $p_g = q = 0$. ☐

5.2 The case G contains a normal subgroup of order 3

We now consider fake projective planes X which admit automorphisms of order 3.

Proposition 5.8. *Let* σ *be an automorphism of a fake projective plane* X *of order 3. Let* $Z = X/\langle\sigma\rangle$ *and let* $\nu : Y \to Z$ *be a minimal resolution. Then* Z *has three singularities of type* $\frac{1}{3}(1,2)$*, and* Y *is a minimal surface of general type with* $K_Y^2 = 3$*,* $p_g(Y) = 0$.

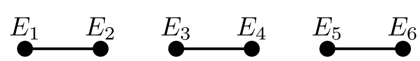
Proof. We know from statement (2) of Proposition 5.5 that $p_g(Y) = q(Y) = 0$ and since σ has order $p = 3$, it follows from statement (3) of Proposition 5.5 that $K_Z^2 = 9/3 = 3$. From statement (4) of the same proposition we know that the fixed point locus X^σ consists of three points. Suppose that Z has r_i singular points of type $\frac{1}{p}(1,i)$. Since X^σ consists of only three isolated points and no curves, the formula in Lemma 5.7 gives $1 = a_1r_1 + a_2r_2$ where

$$a_1 = \frac{1}{2}\Big(\frac{1}{(1-\omega)^2} + \frac{1}{(1-\omega^2)^2}\Big) = \frac{1}{6}$$
$$a_2 = \frac{1}{2}\Big(\frac{1}{(1-\omega)(1-\omega^2)} + \frac{1}{(1-\omega^2)(1-\omega)}\Big) = \frac{1}{3}.$$

Thus we have

$$1 = \frac{1}{6}r_1 + \frac{1}{3}r_2. \tag{57}$$

Since the total number of singular points is 3, we have $r_1 + r_2 = 3$. Together with the equality 57 this gives $r_1 = 0$ and $r_2 = 3$. Thus we conclude that Z has three singular points of type $\frac{1}{3}(1,2)$. Note that each of these three singularities is of type $A_{3,2}$ and since $\frac{3}{2} = 2 - \frac{1}{2}$, the exceptional divisor is a Hirzebruch-Jung string consisting of two curves intersecting in one point, each having self intersection -2. The Dynkin diagram of the exceptional divisors is



This implies that $K_Y = \nu^*K_Z + k_1E_1 + ... + k_6E_6$, where each E_i is a (-2)-curve. Applying the adjunction formula to E_1, we see that $2 = -(K_YE_1 + E_1^2) = 2k_1 - k_2 + 2$, which gives $2k_1 - k_2 = 0$. Similarly for E_2 we get $2 = -(K_YE_2 + E_2^2) = 2k_2 - k_1 + 2$ i.e., $2k_2 - k_1 = 0$. This gives $k_1 = k_2 = 0$, and similarly we get $k_3 = k_4 = k_5 = k_6 = 0$. It follows that $K_Y \sim \nu^*K_Z$, and hence $K_Y^2 = K_Z^2 = 3$. From statement (1) of Proposition 5.5 we know that K_Z is ample, which implies it is semi-ample. This means that $|mK_Z|$ is base point free for some integer $m > 0$. Let $y \in Y$ be any point. Then there exists a divisor $D \in |mK_Z|$ not containing $z = \nu(y)$. It follows that $\nu^*D \in |m\nu^*K_Z|$ does not contain y. Thus $|m\nu^*K_Z|$ is base point free and so ν^*K_Z is semi-ample. In particular, this means that ν^*K_Z is nef and since $K_Y \sim \nu^*K_Z$, it follows that K_Y is nef. If Y contains a (-1)-curve C, then by the adjunction formula 12 we have that $2 = -(K_YC + C^2)$, i.e. $K_YC = -1$, which is a contradiction. Hence Y is a minimal surface. Since $c_1^2(Y) = K_Y^2 = 3$, it follows from the Enriques-Kodaira classification theorem that Y must be of general type. ☐

Corollary 5.9. *Let X be a surface whose automorphism group G is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$. Let $Z = X/G$ and let $\nu : Y \to Z$ be a minimal resolution. Then Z has four singular points of type $\frac{1}{3}(1,2)$, and Y is a minimal surface of general type with $K_Y^2 = 1$, $p_g(Y) = 0$.*

Proof. We write the group $(\mathbb{Z}/3\mathbb{Z})^2$ as $(\mathbb{Z}/3\mathbb{Z})^2 = \{(a,b) \mid a,b \in \{0,1,2\}\}$. It has four subgroups isomorphic to $\mathbb{Z}/3\mathbb{Z}$, namely $G_1 = \{(0,0),(1,0),(2,0)\}$, $G_2 = \{(0,0),(0,1),(0,2)\}$, $G_3 = \{(0,0),(1,1),(2,2)\}$, and $G_4 = \{(0,0),(1,2),(2,1)\}$. Each subgroup G_i is generated by an automorphism of X of order 3. We know from Proposition 5.8 that the fixed point locus of an automorphism of order 3 consists of exactly three points, which correspond to three singularities of type $\frac{1}{3}(1,2)$ on the quotient surface Z. Hence, each subgroup G_i fixes three isolated points of X corresponding to singularities of type $\frac{1}{3}(1,2)$ on Z.
The stabilizer of any point $x \in X$ is either trivial- in which case the G-orbit of x corresponds to a smooth point in the quotient Z; or isomorphic to $\mathbb{Z}/3\mathbb{Z}$, or isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$. Let $x \in X$ be a point whose stabilizer is G_i, for some $i \in \{1,2,3,4\}$ i.e., it is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. Then the G-orbit of x consists of three points. Let y and z be the other two points of X fixed by G_i. Then we claim that y and z are also the other two points in the orbit of x. Indeed, let σ be an element in $G\backslash G_i$ and let $x' = \sigma x$. Then for a non-trivial element $\rho \in G_i$, we have $\rho x' = \rho\sigma x = \sigma\rho x = \sigma x = x'$, where the second equality follows from the fact that G is an abelian group. Thus ρ fixes x', and since G_i is generated by ρ, G_i fixes x'. This implies that x' must be either y or z. Note that if $\sigma x = y$, then $\sigma^2 x = z$ because $\sigma^2 x = x$ or $\sigma^2 x = y$ would mean $\sigma \in G_i$, a contradiction to the assumption that $\sigma \in G\backslash G_i$. This proves the claim. Now suppose x is a point whose stabilizer is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$. Then the orbit of x consists of only the point x. Moreover, x is fixed by each subgroup G_i, for $i = 1,2,3,4$. Each G_i fixes three points of X, which belong to the same G-orbit. However, since the orbit of x contains only one point, each G_i fixes a single point of X. This is a contradiction to statement (4) of Proposition 5.5 and so it follows that no point of X has stabilizer isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$. Thus there are 12 points, each of whose stabilizers is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. The fixed point locus of each G_i consists of three points in the same orbit, corresponding to a singular point of Z of type $\frac{1}{3}(1,2)$, and hence Z has 4 singular points of type $\frac{1}{3}(1,2)$.
The canonical divisor K_Z of Z is $\mathbb{Q}$-Cartier and since the ramification locus of the quotient map $\pi : X \to Z$ consists of only isolated points, we have

$$K_X \sim \pi^* K_Z.$$

Thus we have $K_Z^2 = K_X^2/\deg(\pi) = 9/9 = 1$. Statement (1) of Proposition 5.5 implies that K_Z is ample, and since Z has only singularities of type $\frac{1}{3}(1,2)$, it follows similarly as in the proof of Proposition 5.8 that $K_Y \sim \nu^* K_Z$. By the same argument as in the proof of Proposition 5.8, we have that K_Y is nef. Hence Y is a minimal surface satisfying $K_Y^2 = K_Z^2 = 1$ and $p_g(Y) = 0$. This completes the proof. $\square$

According to the results of [11], many fake projective planes admit an automorphism of order 3. Thus by taking quotients of such fake projective planes by the group generated by an order 3 automorphism, new examples of minimal surfaces of general type Y satisfying $K_Y^2 = 3$, $p_g(Y) = 0$ can be obtained.

5.3 The case G contains a normal subgroup of order 7

In this part we prove the following result and also the classification result of Keum (Proposition 4.6 in [6]).

Proposition 5.10. *Let σ be an automorphism of order 7 of a fake projective plane X. Let $Z = X/\langle\sigma\rangle$ and let $\nu : Y \to Z$ be a minimal resolution. Then Z has three singular points of type $\frac{1}{7}(1,3)$ and $K_Y^2 = 0$.*

To prove this we need the following three lemmas.

Lemma 5.11. *Let σ be an automorphism of order 7 of a fake projective plane X. Let $Z = X/\langle\sigma\rangle$ and let $\nu : Y \to Z$ be a minimal resolution. Then Z has either three singular points of type $\frac{1}{7}(1,3)$, or two singular points of type $\frac{1}{7}(1,4)$, and one singular point of type $\frac{1}{7}(1,6)$.*

Proof. We know from Proposition 5.5 that the fixed point locus X^σ consists of three isolated points and no curves. Suppose Z has r_i singular points of type $\frac{1}{7}(1,i)$. From the formula in Lemma 5.7 we have $1 = \sum_{i=1}^{6} a_i r_i$. After computing the coefficients a_i, we get

$$-r_1 + r_2 + 2r_3 + r_4 + 2r_5 + 4r_6 = 6. \tag{58}$$

Since the total number of singular points is three, we also have $\sum_i r_i = 3$. Adding this to the equation 58, we get

$$2(r_2 + r_4) + 3(r_3 + r_5) + 5r_6 = 9. \tag{59}$$

Now if $r_6 = 0$, then from the equality 59 it follows that $r_2 + r_4 = 0$ and $r_3 + r_5 = 3$, hence we get three singular points of type $\frac{1}{7}(1,3) = \frac{1}{7}(1,5)$. If $r_6 = 1$ then it follows that $r_2 + r_4 = 2$ and $r_3 + r_5 = 0$, hence we get one singular point of type $\frac{1}{7}(1,6)$ and two singular points of type $\frac{1}{7}(1,2) = \frac{1}{7}(1,4)$. This exhausts all possibilities and so the assertion is proved. □

Next, we exclude one of the two possible cases in Lemma 5.11.

Lemma 5.12. *Let σ be an automorphism of order 7 of a fake projective plane X. Then σ cannot have a fixed point corresponding to a singularity of type $\frac{1}{7}(1,4)$ on the quotient $Z = X/\langle\sigma\rangle$.*

Proof. Observe that any automorphism ρ of X in $\langle\sigma\rangle$ can be lifted to an automorphism $\bar{\rho}$ of the universal cover $\mathbb{B}^2$. However, $\bar{\rho}$ is not necessarily unique and there is no canonical way to choose such a $\bar{\rho}$. Every such $\bar{\rho}$ satisfies $\rho \circ \pi = \pi \circ \bar{\rho}$, where $\pi : \mathbb{B}^2 \to X$ denotes the projection map. It is easy to check that the set of all automorphisms $\bar{\rho}$ of $\mathbb{B}^2$ which are lifts of automorphisms of X in $\langle\sigma\rangle$ i.e., those which satisfy $\rho \circ \pi = \pi \circ \bar{\rho}$, $\rho \in \langle\sigma\rangle$, form a subgroup H of $\mathrm{Aut}(\mathbb{B}^2) = \mathrm{PU}(2,1)$. We now show that the orbit spaces $\mathbb{B}^2/H$ and $X/\langle\sigma\rangle$ are isomorphic. Consider the map $\phi : \mathbb{B}^2/H \to X/\langle\sigma\rangle$ defined by $H\bar{x} \mapsto \langle\sigma\rangle\pi(\bar{x})$. Note that ϕ is well-defined because for any two points $\bar{x}$ and $\bar{y}$ in $\mathbb{B}^2$ belonging to the same H-orbit, there is a $\bar{\rho} \in H$ such that $\bar{\rho}\bar{x} = \bar{y}$, and $\rho \circ \pi = \pi \circ \bar{\rho}$ for some $\rho \in \langle\sigma\rangle$. This means that $\rho\pi(\bar{x}) = \pi(\bar{\rho}\bar{x}) = \pi(\bar{y})$ i.e., $\pi(\bar{x})$ and $\pi(\bar{y})$ belong to the same $\langle\sigma\rangle$-orbit. In other words, $H\bar{x}$ and $H\bar{y}$ have the same image in $X/\langle\sigma\rangle$ via ϕ. Suppose $H\bar{x}$ and $H\bar{y}$ map to the same orbit $\langle\sigma\rangle x$ in $X/\langle\sigma\rangle$ via ϕ. This means that $\langle\sigma\rangle\pi(\bar{x}) = \langle\sigma\rangle\pi(\bar{y})$ i.e., $\pi(\bar{x})$ and $\pi(\bar{y})$ belong to the same orbit in $X/\langle\sigma\rangle$. This implies that there is a $\rho \in \langle\sigma\rangle$ such that $\rho\pi(\bar{x}) = \pi(\bar{y})$. Hence there is a $\bar{\rho} \in H$ such that $\rho \circ \pi = \pi \circ \bar{\rho}$ and $\bar{\rho}\bar{x} = \bar{y}$. It follows that $\bar{x}$ and $\bar{y}$ belong to the same H-orbit i.e. $H\bar{x} = H\bar{y}$ in $\mathbb{B}^2/H$, so ϕ is injective. Let $\langle\sigma\rangle x$ be any orbit in $X/\langle\sigma\rangle$. Let $\bar{x}$ be any point in the fibre $\pi^{-1}(x) \subset \mathbb{B}^2$. Then the orbit $H\bar{x}$ in $\mathbb{B}^2/H$ maps to $\langle\sigma\rangle\pi(\bar{x}) = \langle\sigma\rangle x$ via ϕ, so ϕ is surjective as well. Define the inverse map of ϕ by $\psi : X/\langle\sigma\rangle \to \mathbb{B}^2/H$, $\langle\sigma\rangle x \mapsto H\bar{x}$, where $\bar{x}$ is any point in the fibre $\pi^{-1}(x) \subset \mathbb{B}^2$. Note that ψ is well-defined because it is easy to check that all points in the same fibre belong to the same H-orbit, and for any two points x, y in X, the fibres $\pi^{-1}(x)$ and $\pi^{-1}(y)$ belong to the same H-orbit in $\mathbb{B}^2/H$. It is also easily verified that $\phi \circ \psi = \mathrm{id}_{X/\langle\sigma\rangle}$ and $\psi \circ \phi = \mathrm{id}_{\mathbb{B}^2/H}$. Thus ϕ and ψ are isomorphisms and we have $Z = X/\langle\sigma\rangle \cong \mathbb{B}^2/H$. Suppose that σ has a fixed point corresponding to a singularity of type $\frac{1}{7}(1,4)$. Then the group H contains a matrix M which diagonalizes as

$$M = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha\zeta & 0 \\ 0 & 0 & \alpha\zeta^4 \end{bmatrix}$$

where $\zeta = e^{2\pi i/7}$ is a primitive 7th root of unity and $\alpha \in \mathbb{C}$ is a complex number.
Following the notation of [11], we may choose the matrix M to be in $\bar{\Gamma}$, which is contained in a rank 3 division algebra over the field denoted by l. Hence the numbers

$$\operatorname{trace}(M) = \alpha(1 + \zeta + \zeta^4) \quad \text{and} \quad \det(M) = \alpha^3\zeta^5$$

must both belong to l. Thus l, being a field, contains $\operatorname{trace}(M)^3/\det(M)$, which is given by

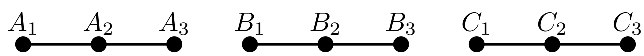
$$(1 + \zeta + \zeta^4)^3/\zeta^5 = 6(\zeta + \zeta^{-1})^3 + (\zeta + \zeta^{-1})^2 - 15(\zeta + \zeta^{-1}) + 5.$$

The field generated by this number over $\mathbb{Q}$ is $\mathbb{Q}[\zeta + \zeta^{-1}]$, and this must be contained in l. However, none of the cases on the final list of Prasad and Yeung[11] has such an l. Thus, Z does not have a fixed point singularity of the type $\frac{1}{7}(1, 4)$, and the assertion is proved. $\square$

A singularity of type $\frac{1}{7}(1, 3)$ is an $A_{7,5}$ singularity. The continued fraction expansion of $\frac{7}{5}$ is

$$\frac{7}{5} = 2 - \cfrac{1}{2 - \frac{1}{3}}$$

and so a singularity of type $\frac{1}{7}(1, 3)$ results from the contraction of a Hirzebruch-Jung string consisting of three rational curves, two of which have self intersection -2, and one has self intersection -3. When Z has three singularities of type $\frac{1}{7}(1, 3)$, we denote by A_1, A_2, A_3; B_1, B_2, B_3; C_1, C_2, C_3 the exceptional curves of the minimal resolution $\nu : Y \to Z$ whose Dynkin diagrams are



We take A_1, A_2, B_1, B_2, C_1, C_2 to be the (-2)-curves, and A_3, B_3, C_3 to be the (-3)-curves.

Lemma 5.13. *Assume that Z has three singularities of type $\frac{1}{7}(1, 3)$ and let $\nu : Y \to Z$ be the minimal resolution. Then*

$$K_Y \sim \nu^* K_Z - \frac{1}{7}(A_1 + 2A_2 + 3A_3) - \frac{1}{7}(B_1 + 2B_2 + 3B_3) - \frac{1}{7}(C_1 + 2C_2 + 3C_3).$$

In particular, $K_Y^2 = 0$.

Proof. We have $K_Y = \nu^* K_Z + D$, where D is the exceptional divisor resulting from the three singular points of type $\frac{1}{7}(1, 3)$. Hence D is a $\mathbb{Q}$-linear combination of the curves A_i, B_i, C_i for $i = 1, 2, 3$. The coefficients can be uniquely determined by applying the adjunction formula to each curve. We carry out the computations for A_1, A_2, and A_3, the procedure being identical for the other six curves. Let a_1, a_2, and a_3 be the coefficients of A_1, A_2, and A_3 respectively, then applying the adjunction formula to A_1, we get $2 = -(K_Y A_1 + A_1^2) = 2a_1 - a_2 + 2$, which gives $2a_1 - a_2 = 0$. Applying it to A_2 we get $2 = -(K_Y A_2 + A_2^2) = 2a_2 - a_1 - a_3 + 2$, which gives $2a_2 - a_1 - a_3 = 0$. Similarly, for A_3 we have $2 = -(K_Y A_3 + A_3^2) = 3a_3 - a_2 + 3$, i.e., $3a_3 - a_2 = -1$. Solving the three equations for a_1, a_2, and a_3 simultaneously, we get $a_1 = -\frac{1}{7}$, $a_2 = -\frac{2}{7}$ and $a_3 = -\frac{3}{7}$. In the same way, letting b_1, b_2, b_3, c_1, c_2, c_3 denote the coefficients of B_1, B_2, B_3, C_1, C_2, C_3 respectively, we get $b_1 = c_1 = -\frac{1}{7}$, $b_2 = c_2 = -\frac{2}{7}$, and $b_3 = c_3 = -\frac{3}{7}$. Putting everything together, we have

$$K_Y \sim \nu^* K_Z - \frac{1}{7}(A_1 + 2A_2 + 3A_3) - \frac{1}{7}(B_1 + 2B_2 + 3B_3) - \frac{1}{7}(C_1 + 2C_2 + 3C_3).$$

It follows that

$$K_Y^2 = (\nu^* K_Z)^2 + \frac{1}{49}(A_1 + 2A_2 + 3A_3)^2 + \frac{1}{49}(B_1 + 2B_2 + 3B_3)^2 + \frac{1}{49}(C_1 + 2C_2 + 3C_3)^2. \tag{60}$$

From statement 3 of Proposition 5.5, we have $(\nu^*K_Z)^2 = K_Z^2 = 9/7$. Moreover, we have

$$\frac{1}{49}(A_1+2A_2+3A_3)^2 = \frac{1}{49}(B_1+2B_2+3B_3)^2 = \frac{1}{49}(C_1+2C_2+3C_3)^2 = -\frac{3}{7}.$$

Plugging everything into equation 60 we get $K_Y^2 = \frac{9}{7} - \frac{3}{7} - \frac{3}{7} - \frac{3}{7} = 0$, and the assertion is proved. $\square$

Thus the proof of Proposition 5.10 is complete. To complete the proof of Theorem 5.1 it suffices to prove the following result

Proposition 5.14. *Assume that Z has three singular points of type $\frac{1}{7}(1,3)$. Then there are the following three possibilities*

1. *Y is a minimal elliptic surface of Kodaira dimension 1 with two multiple fibres with multiplicities 2 and 3.*
2. *Y is a minimal elliptic surface of Kodaira dimension 1 with two multiple fibres with multiplicities 2 and 4.*
3. *Y is a minimal elliptic surface of Kodaira dimension 1 with two multiple fibres with multiplicities 3 and 3.*

The proof of Proposition 5.14 consists of several lemmas.

Lemma 5.15. *Assume that Z has three singularities of type $\frac{1}{7}(1,3)$. Then we have the following*

1. *$-mK_Y$ is not effective for any positive integer m.*
2. *The Kodaira dimension of Y is at least 1.*

Proof. We know that $K_Y = \nu^*K_Z + D$, where D is the exceptional divisor which was computed in Lemma 5.13. Then for $m \geqslant 1$, we have

$$(\nu^*K_Z)(-mK_Y) = (\nu^*K_Z)(-m(\nu^*K_Z + D)) = -m(\nu^*K_Z)^2 = -mK_Z^2 = -\frac{9}{7}m < 0.$$

Since K_Z is ample, ν^*K_Z is nef, and so $(\nu^*K_Z)(-mK_Y) < 0$ implies that $-mK_Y$ cannot be effective for $m \geqslant 1$. This proves statement 1.
We know from statement 2 of Proposition 5.5 that $p_g(Y) = q(Y) = 0$, and by Proposition 5.10 we have $K_Y^2 = 0$. Thus if Y has Kodaira dimension $\leqslant 0$, then by the Enriques-Kodaira classification theorem, Y is either a rational surface or an Enriques surface. We know from Proposition 5.4 that $e(Z) = 3$. Since Z has three singular points, each of whose exceptional divisor consists of three rational curves on Y, we have $e(Y) = c_2(Y) = e(Z) + 9 = 12$. Since $c_1^2(Y) = K_Y^2 = 0$, applying Noether's formula gives $\chi(\mathcal{O}_Y) = 1$. The Riemann-Roch theorem says that

$$h^0(D) - h^1(D) + h^0(K_Y - D) = \chi(\mathcal{O}_Y) + \frac{1}{2}(D^2 - K_Y D),$$

where D is a divisor on Y. Taking $D = kK_Y$ for $k \geqslant 2$, we have that $h^0(K_Y - D) = h^0((1-k)K_Y)$. We know from statement 1 that $-mK_Y$ is not effective for $m \geqslant 1$ and so $h^0((1-k)K_Y) = 0$ for $k \geqslant 2$. Thus the Riemann-Roch equation becomes

$$h^0(kK_Y) = 1 + h^1(kK_Y) \geqslant 1.$$

This implies that Y is not a rational surface. Since we have $(\nu^*K_Z)K_Y = (\nu^*K_Z)^2 = K_Z^2 = 9/7 > 0$, K_Y is not numerically trivial. Hence, Y is not an Enriques surface either. Thus the Kodaira dimension of Y must be $\geqslant 1$. This proves statement 2. $\square$

Note that $\mathrm{Pic}(Y) \cong H^2(Y, \mathbb{Z})$. Let $\mathrm{Pic}(Y)_f$ denote $\mathrm{Pic}(Y)$ modulo torsion. Then with the intersection pairing $\mathrm{Pic}(Y)_f$ becomes a lattice.

Lemma 5.16. *Assume that Z has three singularities of type $\frac{1}{7}(1,3)$. Then one can choose two $\mathbb{Q}$-divisors*

$$L = \frac{1}{7}(A_1 + 2A_2 + 3A_3) + \frac{2}{7}(B_1 + 2B_2 + 3B_3) + \frac{4}{7}(C_1 + 2C_2 + 3C_3),$$
$$M = \frac{1}{3}\nu^* K_Z - \frac{2}{7}(B_1 + 2B_2 + 3B_3) + \frac{1}{7}(C_1 + 2C_2 + 3C_3),$$

such that the lattice $Pic(Y)_f$ is generated over $\mathbb{Z}$ by the numerical equivalence classes of M, L, and the eight curves $A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3$.

Proof. We know from Lemma 5.13 that $K_Y^2 = 0$. Hence Noether's formula gives $c_2(Y) = 12$, which implies that the rank of $\mathrm{Pic}(Y)_f$ is 10. Since $\mathrm{Pic}(Y)_f$ contains an element having self intersection -3, e.g. the curve A_3, it is unimodular and of signature (1,9).
Let R be the sublattice of $\mathrm{Pic}(Y)_f$ generated by the numerical equivalence classes of the nine curves $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3$. Let $\bar{R}$ and $R^\perp$ denote its primitive closure and orthogonal complement respectively. Note that the rank of $R^\perp$ is 1.
For an integral lattice N, let $\mathrm{disc}(N)$ denote the discriminant group of N, defined as

$$\mathrm{disc}(N) = \mathrm{Hom}(N, \mathbb{Z})/N.$$

We have $\mathrm{disc}(R) \cong (\mathbb{Z}/7\mathbb{Z})^3$. More precisely,

$$\mathrm{disc}(R) = \langle \frac{1}{7}(A_1 + 2A_2 + 3A_3), \frac{1}{7}(B_1 + 2B_2 + 3B_3), \frac{1}{7}(C_1 + 2C_2 + 3C_3) \rangle.$$

Note that the length, i.e. the minimum number of generators of $\mathrm{disc}(R)$ is 3. Since the lattice $\mathrm{Pic}(Y)_f$ is unimodular, $\mathrm{disc}(\bar{R})$ is isomorphic to $\mathrm{disc}(R^\perp)$ which is of length 1. Hence R must be of index 7 in $\bar{R}$, and the generator of $\bar{R}/R$ is of the form

$$L = \frac{1}{7}(A_1 + 2A_2 + 3A_3) + \frac{a}{7}(B_1 + 2B_2 + 3B_3) + \frac{b}{7}(C_1 + 2C_2 + 3C_3).$$

Since both LK_Y and K_Y^2 must be integers, it follows that $(a, b) = (2, 4)$ or (4,2) modulo 7. Thus, up to interchanging the B_i's with the C_i's, the divisor L has been determined uniquely modulo R.
Now we have $\mathrm{disc}(\bar{R}) = \mathrm{disc}(R^\perp) \cong \mathbb{Z}/7\mathbb{Z}$. Note that the integral divisor $7\nu^* K_Z$ belongs to $R^\perp$ and $(7\nu^* K_Z)^2 = 7 \cdot 3^2$. Thus $R^\perp$ is generated by $\frac{7}{3}\nu^* K_Z$, hence

$$\mathrm{disc}(R^\perp) = \Big\langle \frac{1}{3}\nu^* K_Z \Big\rangle.$$

On the other hand,

$$\mathrm{disc}(\bar{R}) = \langle L \rangle^\perp / \langle L \rangle = \Big\langle \frac{3}{7}(B_1 + 2B_2 + 3B_3) + \frac{2}{7}(C_1 + 2C_2 + 3C_3) \Big\rangle,$$

where $\langle L \rangle = \bar{R}/R$ is the isotropic subgroup of $\mathrm{disc}(R)$ generated by L modulo R and $\langle L \rangle^\perp$ is its orthogonal complement in $\mathrm{disc}(R)$ with respect to the discriminant quadratic form on $\mathrm{disc}(R)$. Thus the extension of index 7 $\bar{R} \oplus R^\perp \subset \mathrm{Pic}(Y)_f$ is given by an element of the form

$$M = \frac{1}{3}\nu^* K_Z + a\Big\{\frac{3}{7}(B_1 + 2B_2 + 3B_3) + \frac{2}{7}(C_1 + 2C_2 + 3C_3)\Big\}.$$

Since MK_Y is an integer, we see that $a = 4$ modulo 7. This determines the divisor M uniquely modulo R. ☐

Lemma 5.17. *Assume that Z has three singularities of type $\frac{1}{7}(1,3)$. Then Y does not contain a (-1)-curve (a smooth rational curve with self intersection -1) E satisfying* $0 < E(\nu^* K_Z) < 9/7$.

Proof. Suppose that Y does contain such a (-1)-curve E. From Lemma 5.16 we know the generators over $\mathbb{Z}$ of $\mathrm{Pic}(Y)_f$, so we write

$$E \sim mM - dL + a_2A_2 + a_3A_3 + b_1B_1 + b_2B_2 + b_3B_3 + c_1C_1 + c_2C_2 + c_3C_3$$

where the coefficients are all integers. The above expression for E and the expressions for M and L from Lemma 5.16 imply that $E(\nu^*K_Z) = mM(\nu^*K_Z) = 3m/7$. Thus the condition $0 < E(\nu^*K_Z) < 9/7$ is equivalent to $0 < m < 3$ i.e., $1 \leqslant m \leqslant 2$ because m is an integer. Hence there are two cases: $m = 1$ or 2.
1. Assume that $m = 1$. Then the expression for E becomes

$$E \sim M - d + La_2A_2 + a_3A_3 + b_1B_1 + b_2B_2 + b_3B_3 + c_1C_1 + c_2C_2 + c_3C_3.$$

Computing the intersection number of E with each of the nine curves $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3$, and noting that this number must be non-negative, we get the following system of nine inequalities.

$$\begin{array}{lll} 0 \leqslant EA_1 = a_2, & 0 \leqslant EA_2 = -2a_2 + a_3, & 0 \leqslant EA_3 = d + a_2 - 3a_3, \\ 0 \leqslant EB_1 = -2b_1 + b_2, & 0 \leqslant EB_2 = b_1 - 2b_2 + b_3, & 0 \leqslant EB_3 = 2 + 2d + b_2 - 3b_3, \\ 0 \leqslant EC_1 = -2c_1 + c_2, & 0 \leqslant EC_2 = c_1 - 2c_2 + c_3, & 0 \leqslant EC_3 = -1 + 4d + c_2 - 3c_3. \end{array}$$

Using the expression for K_Y derived in Lemma 5.13 and applying the adjunction formula to E, we get the following equality

$$-1 = EK_Y = -3d + a_3 + b_3 + c_3. \tag{61}$$

From the system of nine inequalities, we obtain the following three inequalities

$$a_3 \leqslant \frac{2}{5}d, \quad b_3 \leqslant \frac{3}{7}(2 + 2d), \quad c_3 \leqslant \frac{3}{7}(-1 + 4d). \tag{62}$$

Indeed, eliminating a_1 and a_2 from the second and third inequalities of the system of nine inequalities, we arrive at the first inequality of 62. Eliminating b_1 and b_2 from the fourth, fifth, and sixth inequalities in the system, we get the second inequality of 62. Similarly, eliminating c_1 and c_2 from the seventh, eighth, and ninth inequalities of the system, we get the third inequality of 62.
From the first three inequalities of the system of nine inequalities, we have

$$d \geqslant -a_2 + 3a_3 = 3(-2a_2 + a_3) + 5a_2 \geqslant 5a_2 \geqslant 0.$$

Plugging the inequalities in 62 in the equality 61 gives

$$3d - 1 = a_3 + b_3 + c_3 \leqslant \frac{2}{5}d + \frac{3}{7}(2 + 2d) + \frac{3}{7}(-1 + 4d),$$

which simplifies to give $d \leqslant 50$. Thus we obtain the following bound on d

$$0 \leqslant d \leqslant 50. \tag{63}$$

Since E is a (-1)-curve, we have $E^2 = -1$. Together with the equality 61 this implies $E^2 = EK_Y$. Writing this equality in terms of the coefficients, we get

$$1 + 3d^2 + 2d = (4 + 2d)b_3 + (6d - 2)c_3 + (a_2A_2 + a_3A_3)^2 + \Big(\sum_{i=1}^{3} b_iB_i\Big)^2 + \Big(\sum_{i=1}^{3} c_iC_i\Big)^2. \tag{64}$$

We have the following inequalities for the last three terms in the right hand side of the equality 64

$$(a_2A_2 + a_3A_3)^2 = -2a_2^2 + 2a_2a_3 - 3a_3^2 = -2\Big(a_2 - \frac{1}{2}a_3\Big)^2 - \frac{5}{2}a_3^2 \leqslant -\frac{5}{2}a_3^2,$$

$$\Big(\sum_{i=1}^{3} b_iB_i\Big)^2 = -2\Big(b_1 - \frac{1}{2}b_2\Big)^2 - \frac{3}{2}\Big(b_2 - \frac{2}{3}b_3\Big)^2 - \frac{7}{3}b_3^2 \leqslant -\frac{7}{3}b_3^2,$$

$$\Big(\sum_{i=1}^{3} c_iC_i\Big)^2 = -2\Big(c_1 - \frac{1}{2}c_2\Big)^2 - \frac{3}{2}\Big(c_2 - \frac{2}{3}c_3\Big)^2 - \frac{7}{3}c_3^2 \leqslant -\frac{7}{3}c_3^2.$$

Plugging these inequalities into the equality 64, we arrive at the following inequality

$$1 + 3d^2 + 2d \leqslant -\frac{5}{2}a_3^2 - \frac{7}{3}b_3^2 - \frac{7}{3}c_3^2 + (4 + 2d)b_3 + (6d - 2)c_3. \tag{65}$$

We claim that there are no integers a_3, b_3, c_3, d satisfying the conditions 62, 63, and 65. Keum proves this claim in his paper in the following way. First, he obtains a list of quadruples (d, a_3, b_3, c_3) which solve the equality 61 under the constraints given by 62 and 63 i.e., for each value of d in 63, the equation $3d - 1 = a_3 + b_3 + c_3$ is solved in the range given by the inequalities in 62. A list of solutions is generated by a computer program, which is given in [6], p.14. It turns out that none of the solutions on this list satisfies the inequality 65.
Now assume that $m = 2$. In this case we have

$$E \sim 2M - dL + a_2A_2 + a_3A_3 + b_1B_1 + b_2B_2 + b_3B_3 + c_1C_1 + c_2C_2 + c_3C_3.$$

Similarly as in the previous case, we obtain a system of nine inequalities in the nine coefficients in the expression for E

$$\begin{aligned}
&0 \leqslant EA_1 = a_2, && 0 \leqslant EA_2 = -2a_2 + a_3, && 0 \leqslant EA_3 = d + a_2 - 3a_3,\\
&0 \leqslant EB_1 = -2b_1 + b_2, && 0 \leqslant EB_2 = b_1 - 2b_2 + b_3, && 0 \leqslant EB_3 = 4 + 2d + b_2 - 3b_3,\\
&0 \leqslant EC_1 = -2c_1 + c_2, && 0 \leqslant EC_2 = c_1 - 2c_2 + c_3, && 0 \leqslant EC_3 = -2 + 4d + c_2 - 3c_3.
\end{aligned}$$

Applying the adjunction formula to E, we again have

$$-1 = EK_Y = -3d + a_3 + b_3 + c_3.$$

Applying the same procedure as done in the case $m = 1$, we get the following three inequalities

$$a_3 \leqslant \frac{2}{5}d, \quad b_3 \leqslant \frac{3}{7}(4 + 2d), \quad c_3 \leqslant \frac{3}{7}(-2 + 4d). \tag{66}$$

In this case we obtain the following bound for d

$$0 \leqslant d \leqslant 65. \tag{67}$$

We also have the following analog of the inequality 65 in the previous case

$$7 + 3d^2 + 2d \leqslant -\frac{5}{2}a_3^2 - \frac{7}{3}b_3^2 - \frac{7}{3}c_3^2 + (8 + 2d)b_3 + (6d - 4)c_3. \tag{68}$$

The same argument as in the case $m = 1$ shows that there are no solutions satisfying the inequalities 66, 67, 68, and the equality 61. Thus we conclude that there is no (-1)-curve on E satisfying $0 < E(\nu^*K_Z) < 9/7$, which proves the assertion. $\square$

Lemma 5.18. *Assume that Z has three singularities of type $\frac{1}{7}(1,3)$. Then Y is a minimal surface of Kodaira dimension 1.*

Proof. From Proposition 5.10 we know that $K_Y^2 = 0$. Moreover, statement 2 of Lemma 5.15 says that the Kodaira dimension of Y is at least 1 i.e., it is either 1 or 2. Suppose Y is not minimal. If Y has Kodaira dimension 1 then the minimal model Y' of Y has $c_1^2(Y') = K_{Y'}^2 > 0$. However, this is not possible since the Enriques Kodaira classification theorem states that a minimal surface of Kodaira dimension 1 must have $c_1^2 = 0$. Hence Y is a surface of general type. Let $\mu : Y \to Y'$ be a birational morphism to the minimal model Y' of Y. Note that μ contracts all (-1)-curves of Y and hence $K_Y = \mu^* K_{Y'} + \sum_i E_i$, where the E_i's are effective (not necessarily irreducible) divisors satisfying $E_i^2 = -1$, $E_i E_j = 0$ for $i \neq j$. Since Y' is minimal, $K_{Y'}$ is nef and so some positive multiple of $\mu^* K_{Y'} = K_Y - \sum_i E_i$ is effective. Since $\nu^* K_Z$ is nef, we have

$$(\mu^* K_{Y'})(\nu^* K_Z) = (K_Y - \sum_i E_i)(\nu^* K_Z) \geqslant 0. \tag{69}$$

Furthermore, since each E_i is a (-1)-curve, applying the adjunction formula gives $2 = -(K_Y E_i - 1)$ i.e, $K_Y E_i = -1$ for all i. This implies that $\mu^* K_{Y'} = (K_Y - \sum_i E_i)^2 = K_Y^2 - 2\sum_i K_Y E_i + \sum_i E_i^2 > 0$. Thus by the Algebraic index theorem 3.6, we have

$$(K_Y - \sum_i E_i)(\nu^* K_Z) \neq 0. \tag{70}$$

The inequalities 69 and 70 together imply that

$$(K_Y - \sum_i E_i)(\nu^* K_Z) > 0. \tag{71}$$

Let E be a (-1)-curve on Y. Note that E is not contracted by ν because ν contracts only the Hirzebruch-Jung strings on Y corresponding to the three singularities of type $\frac{1}{7}(1,3)$ on Z. Since E is effective and $\nu^* K_Z$ is nef, we have

$$E(\nu^* K_Z) > 0. \tag{72}$$

On the other hand, from inequality 71 we have $K_Y(\nu^* K_Z) > (\sum_i E_i)(\nu^* K_Z) > E(\nu^* K_Z)$. Recall that $K_Y = \nu^* K_Z + D$, where D is the exceptional divisor computed in Lemma 5.15 and so $K_Y(\nu^* K_Z) = (\nu^* K_Z + D)(\nu^* K_Z) = (\nu^* K_Z)^2 = K_Z^2 = 9/7$. Together with inequality 72, we get

$$0 < E(\nu^* K_Z) < \frac{9}{7}.$$

However, we know from Lemma 5.17 that such a (-1)-curve does not exist on Y. Thus we conclude that Y does not contain (-1)-curves and hence Y is minimal. Since $c_1^2(Y) = K_Y^2 = 0$, it follows that Y has Kodaira dimension 1. This proves the assertion. $\square$

Proof of Proposition 5.14. We know from Lemma 5.15 and Lemma 5.18 that the Y is a minimal elliptic surface of Kodaira dimension 1. It remains to prove the assertion about the multiplicities of multiple fibres. Let $|F|$ denote the linear system associated with the general fibre F of the elliptic fibration. Then we have

$$F \sim nK_Y, \tag{73}$$

where n is a positive rational number. Note that Y contains a curve with self intersection -3, for example the curve A_3. Using the expression for K_Y in Lemma 5.13, we compute $A_3 K_Y = 1$. Note that since A_3 and F are both irreducible effective divisors on Y, the intersection number $A_3 F$ must be an integer. From 73, it follows that $A_3 F = nA_3 K_Y = n$, hence n must be an integer.

Let $m_1 F_1$, $m_2 F_2$,..., $m_r F_r$ be multiple fibres of the elliptic fibration having multiplicities m_1, m_2,..., m_r

respectively. Since Y is not a rational surface, we have $r \geqslant 2$. By the canonical bundle formula for elliptic fibrations ([1], Theorem V.12.1), we have

$$K_Y = -F + \sum_{i=1}^{r}(m_i - 1)F_i \equiv (r-1)F - \sum_{i=1}^{r} F_i, \tag{74}$$

where $\equiv$ denotes linear equivalence. Note that each m_iF_i is linearly equivalent to F. Taking the intersection product of the right hand side of 74 with A_3 and dividing by n, we get

$$\frac{1}{n} = r - 1 + \sum_{i=1}^{r} \frac{1}{m_i}. \tag{75}$$

Since $m_i \geqslant 2$ for all i, we have $\sum_{i=1}^{r} \frac{1}{m_i} \leqslant \frac{r}{2}$. Now equation 75 implies that $r \leqslant 3$ if $n = 2$, and because $r \geqslant 2$, we have $r = 2$ if $n \geqslant 3$. Since $A_3F = n = m_iA_3F_i$, and A_3F_i is an integer for all i, it follows that each m_i divides n. Then analysing 75 further shows that if $n = 2$, then $m_1 = m_2 = m_3 = 2$; if $n = 3$, then $m_1 = m_2 = 3$; if $n = 4$, then $m_1 = 2$, $m_2 = 4$; if $n = 6$, then $m_1 = 2$, $m_2 = 3$; if $n = 5$ or $n \geqslant 7$, then there is no solution for the m_i's. The case $n = 2$ and $m_1 = m_2 = m_3 = 2$ implies that there is a degree 2 map $A_3 \to \mathbb{P}^1$ branched at the three points over which the singular fibres lie, which is not possible. This completes the proof of Proposition 5.14. $\square$

Corollary 5.19. *Let X be a fake projective plane with $G = Aut(X) \cong 7:3$, the unique non-abelian group of order 21. Let $W = X/G$ and let $\nu : V \to W$ be a minimal resolution. Then W has three singular points of type $\frac{1}{3}(1,2)$ and one singular point of type $\frac{1}{7}(1,3)$. Furthermore, V is a minimal elliptic surface of Kodaira dimension 1 with two multiple fibres and four reducible fibres of type I_3. The pair of multiplicities is the same as that of the minimal resolution of the order 7 quotient of X.*

Proof. We can write G as

$$G = \langle \sigma, \tau \mid \sigma^7 = \tau^3 = 1,\ \tau\sigma\tau^{-1} = \sigma^2 \rangle.$$

Let $Z = X/\langle\sigma\rangle$ be the order 7 quotient of X and let Y be a minimal resolution of Z. We know from Proposition 5.14 that Y is a minimal elliptic surface of Kodaira dimension 1 with three singular points of type $\frac{1}{7}(1,3)$. It is straightforward to check that any two points $x_1, x_2 \in X$ belong to the same σ-orbit if an only if $\tau(x_1)$ and $\tau(x_2)$ belong to the same σ-orbit. Thus the automorphism τ induces a well-defined automorphism $\bar{\tau}$ of Z defined by $\bar{\tau}(z) = (\pi \circ \tau)(z)$ for all $z \in Z$, where $\pi : X \to Z$ denotes the canonical projection. Moreover, the three singular points of type $\frac{1}{7}(1,3)$ belong to the same $\bar{\tau}$-orbit in Z. We know from 5.8 that every order 3 subgroup of G has three fixed points corresponding to three singularities of type $\frac{1}{3}(1,2)$. A non singular point cannot have stabilizer isomorphic to $7:3$, thus $W = Z/\langle\bar{\tau}\rangle$ has three singular points of type $\frac{1}{3}(1,2)$ and one singular point of type $\frac{1}{7}(1,3)$.
Note that the canonical divisor K_W of W is an ample $\mathbb{Q}$-Cartier divisor, and we have

$$K_W^2 = \frac{K_X^2}{|G|} = \frac{3}{7}.$$

From the proofs of Proposition 5.8 and Lemma 5.13 it follows that the canonical divisor K_V of V is given by

$$K_V = \nu^*K_W - \frac{1}{7}(A_1 + 2A_2 + 3A_3),$$

where the divisors A_1, A_2, and A_3 are as in Lemma 5.13. This implies that $K_V^2 = 0$. We know from Proposition 5.14 that Y has Kodaira dimension 1, hence V has Kodaira dimension $\leqslant 1$. Note that the action of $\overline{\tau}$ on Z lifts to Y. Let $W' = Y/\langle\overline{\tau}\rangle$ and let $f : Y \to W'$ denote the canonical projection. We know from Proposition 5.14 that K_Y is nef, and since f is branched at three isolated points, we have $K_Y \sim f^*K_{W'}$. Thus $K_{W'}$ is nef. Since W' has three singular points of type $\frac{1}{3}(1,2)$, and since V is the minimal resolution of W' it follows that K_V is also nef. This implies that V is a minimal surface of Kodaira dimension $\geqslant 0$. Note that

$$(\nu^*K_W)K_V = (\nu^*K_W)(\nu^*K_W - \frac{1}{7}(A_1 + 2A_2 + 3A_3)) = K_W^2 = \frac{3}{7} > 0.$$

Thus K_V is not numerically trivial. This proves that V has Kodaira dimension 1. The elliptic fibration on V is given by a multiple of K_V.

Now V has nine smooth rational curves coming from the resolution $\nu : V \to W$. The eight (-2)-curves among them must be contained in fibres of the elliptic fibration. This is possible only if the fibres are the union of four reducible fibres of type I_3 since V has Picard number 10.

Since $W' = Y/\langle\overline{\tau}\rangle$, Y is a cover of W' of degree 3 branched along the three singularities of W' of type $\frac{1}{3}(1,2)$, corresponding to the three fixed points of the $\overline{\tau}$-action on Y. Note that W' has the structure of an elliptic fibration with a (-3)-curve that is a multi-section. The (-3)-curve on W' splits in Y giving three (-3)-curves, thus the elliptic fibres of W' do not split in Y. The fibre containing one of the singular points of W' gives a fibre of type I_1, the fibre of type I_3 gives a fibre of type I_9, and the multiple fibres give multiple fibres of the same multiplicities.

This completes the proof. □

From the proof of Corollary 5.19, we get

Corollary 5.20. *Let X be a fake projective plane with $Aut(X) \cong 7 : 3$. Let $G \cong \mathbb{Z}/7\mathbb{Z} \subset Aut(X)$, $Z = X/G$, and $\nu : Y \to Z$ a minimal resolution. Then the elliptic fibration of Y has three singular fibres of type I_1 and one reducible fibre of type I_9.*

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