Seminar on The Local Langlands Conjecture for GL_2: Introduction

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1 The Local Langlands Conjecture

The Local Langlands Conjecture is a deep generalization of local class field theory. Let F be a non-archimedean local field with a finite residue field of characteristic p, i.e., a finite extension of $\mathbb{Q}_p$ or a formal Laurent series field $\mathbb{F}_{p^r}((t))$. Denote by $\mathcal{W}_F$ its Weil group, a topological group which is closely related to the absolute Galois group of F.

The classical local class field theory establishes a topological isomorphism between the multiplicative group $F^\times$ and the abelianization of the Weil group $\mathcal{W}_F^{\mathrm{ab}}$ via the reciprocity map:

$$\mathrm{rec}_F \colon F^\times \cong \mathcal{W}_F^{\mathrm{ab}}.$$

This leads to a one-to-one bijection between:

$$\{\text{continuous characters of } \mathcal{W}_F\} \leftrightarrow \{\text{continuous characters of } F^\times\}.$$

We rewrite the above as

$$\{\text{one-dimensional continuous representations of } \mathcal{W}_F\} \leftrightarrow \{\text{irreducible smooth representations of } GL_1(F)\}.$$

The aim of the Langlands Program is to extend this correspondence from GL_1 to the general linear group GL_n. Specifically, the Local Langlands Conjecture for GL_n asserts the existence of a unique bijection between two entirely different mathematical worlds:

- **The Galois Side:** Isomorphism classes of n-dimensional Frobenius-semisimple Weil-Deligne representations of the Weil group $\mathcal{W}_F$.
- **The Automorphic Side:** Isomorphism classes of irreducible smooth admissible representations of $GL_n(F)$

The correspondence is uniquely characterized by its compatibility with local L-functions, ε-factors, and other natural invariants.

2 Content of the Seminar

The local Langlands correspondence for GL_n is now a celebrated theorem, established through the work of several generations of mathematicians. For p-adic fields, the local Langlands correspondence for GL_n was established by Harris–Taylor [2] and Henniart [3]. A different proof was later given by Scholze [4].

In the seminar, we shall focus on the case $n = 2$ and follow the book of Colin J. Bushnell and Guy Henniart [1]. The GL_2 case is an excellent starting point for learning the Langlands correspondence: its inner workings are sufficiently complex to be deeply interesting, yet visible enough to illuminate what Bushnell and Henniart call "the miracle that is the Langlands correspondence".

Crucially, Bushnell and Henniart provide an approach by giving a **completely local and elementary** proof. By intentionally avoiding the heavy tools of harmonic analysis on adèle groups or base change, this text allows us to understand the correspondence through purely local methods.

The seminar is intended to provide students with a solid introduction to the local Langlands correspondence for GL_2. It also serves as a first step toward more advanced topics in representation theory, automorphic forms, and arithmetic geometry.

3 Structure and Prerequisites

The seminar is structured as a two-semester sequence. Both semesters will be self-contained.

- **Semester 1 (The Automorphic Side):** We build the smooth representation theory of $GL_2(F)$ from the ground up, exploring parabolic induction, Whittaker/Kirillov models, local L-functions, and the explicit construction of supercuspidal representations via the theory of types and strata.
- **Semester 2 (The Galois Side & The Bijection):** We develop the arithmetic side via the Weil group, examine Weil-Deligne representations, and explicitly assemble the Local Langlands Correspondence bijection.

We will organize the seminar to be highly accessible to first- and second-year master's students. Ph.D students who are interested in the Langlands program are also welcome. We only require the following as prerequisites:

1. Representation theory for finite groups (can be recalled quickly upon request).
2. Basic theory of local fields, e.g., discrete valuation rings, p-adic numbers.
3. Basic notions of point-set topology.

References

[1] C. J. Bushnell, G. Henniart, *The Local Langlands Conjecture for $GL(2)$*, Grundlehren der mathematischen Wissenschaften, Vol. 335, Springer-Verlag, Berlin, 2006.

[2] M. Harris, R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, Vol. 151, Princeton University Press, 2001.

[3] G. Henniart, *Une preuve simple des conjectures de Langlands pour $GL(n)$ sur un corps p-adique*, Inventiones Mathematicae, 139(2), 439–455, 2000.

[4] P. Scholze, *The local Langlands correspondence for GL_n over p-adic fields*, Inventiones Mathematicae, 192(3), 663–715, 2013.