Quillen's + and $S^{-1}S$ - constructions

§1 Quillen's +-construction

Let R be a ring with unit (possibly non-commutative)

Goal: Construct a topological space $K(R)$ whose homotopy groups recover the K-theory of R.

Recall: For all $n \geq 1$ the maps $GL_n(R) \to GL_{n+1}(R)$, $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$ fit into an inductive system. We define $GL(R) = \varinjlim_n GL_n(R)$ and $E(R) = [GL(R), GL(R)]$ $(= D(GL(R)))$.

Moreover, $E(R)$ is generated by the elementary matrices $e_{ij}(r)$, $1 \leq i \neq j \leq n$, $r \in R$.

We have defined $K_1(R) = GL(R)/E(R)$.

Recall: Given a group G, the classifying space BG is a topological space such that:

1) $\pi_1(BG) = G$;

2) $\pi_n(BG) = 0$ for all $n \geq 2$.

These properties characterize BG up to homotopy equivalence.

The goal is to modify the construction of BG for $G = GL(R)$ into some topological space $BGL(R)^+$ whose homotopy groups recover the K-groups of R.

Def: A "model for $BGL(R)^+$" is a CW-complex X together with a map $BGL(R) \to X$ s.t.

1) $\pi_1(X) \cong K_1(R)$;

2) the natural map $GL(R) = \pi_1(BGL(R)) \longrightarrow \pi_1(X)$ is surjective with kernel $E(R)$;

3) for all $K_1(R)$-modules M we have $H_*(BGL(R), M) \cong H_*(X, M)$.

We will see that any two models of $BGL(R)^+$ are homotopy equivalent, so we will write $BGL(R)^+$ to denote any such model.

Def: For $n \geq 1$ we set $K_n(R) := \pi_n(BGL(R)^+)$.

By definition, $\pi_1(BGL(R)^+)$ coincides with $K_1(R)$ defined previously. We will also see that $\pi_2(BGL(R)^+)$ coincides with the definition of $K_2(R)$ given two talks ago.

Def: $K(R) := K_0(R) \times BGL(R)^+$, where $K_0(R)$ is considered with the discrete topology.

$K(R)$ is a disjoint union of $K_0(R)$ copies of $BGL(R)^+$, so $\pi_0(K(R)) = K_0(R)$, while for $n \geq 1$ we have $\pi_n(K(R)) = \pi_n(BGL(R)^+) = K_n(R)$.

Remark: (functoriality)

$K_n : \mathrm{Ring} \to \mathcal{Ab}$, $BGL(-)^+ : \mathrm{Ring} \to h\mathcal{Top}$, $K(-) : \mathrm{Ring} \to h\mathcal{Top}$ are functors.
Notice that for the latter two we cannot land in $\mathcal{Top}$, because $BGL(R)^+$ is defined only up to homotopy. To recover a topological space, one should specify a construction of $BGL(R)^+$.

We will give a construction of $BGL(R)^+$, the so-called "+ construction". Before proceeding, we introduce some topological background.

Def: Let $f : E \to B$ be a continuous map of topological spaces.

- The "homotopy fiber" of f relative to a basepoint $b \in B$ is the space $F(f)$ of pairs (e, γ) with $e \in E$ and $\gamma : [0,1] \to B$ is a path from b to e.
  There is a natural projection $F(f) \to E$.
- A homotopy fibration sequence is a sequence $F \xrightarrow{g} E \xrightarrow{f} B$ of pointed topological spaces such that the natural map $F \to F(f)$, $x \mapsto (g(x), \text{constant path})$ is a homotopy equivalence.

Proposition: Let $F \to E \to B$ be a homotopy fibration sequence. Then there is a long exact sequence

$$\dots \to \pi_{n+1}(B) \xrightarrow{\partial} \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \dots$$
$$\dots \xrightarrow{\partial} \pi_1(F) \to \pi_1(E) \to \pi_1(B) \xrightarrow{\partial} \pi_0(F) \to \pi_0(E) \to \pi_0(B)$$

Def: We say that a topological space E is acyclic if $\tilde{H}_*(E, \mathbb{Z}) = 0$

Lemma: Let E be an acyclic space. Then E is connected, $\pi_1(E)$ is perfect and $H_2(\pi_1(E), \mathbb{Z}) = 0$.

Proof: $\tilde{H}_0(E, \mathbb{Z}) = 0$ means $H_0(E, \mathbb{Z}) = \mathbb{Z}$, so E must be connected. Also, recall that $H_1(E, \mathbb{Z}) = \pi_1(E) / [\pi_1(E), \pi_1(E)]$, so if $H_1(E, \mathbb{Z}) = 0$, then $\pi_1(E)$ is perfect.
Now let $\tilde{E}$ be the universal covering space of E.

Claim: the homotopy fiber of $E \to B\pi_1(E)$ is isomorphic to the canonical map $\tilde{E} \to E$.
Indeed, the long exact sequence of $F \to E \to B\pi_1(E)$ gives

$$\dots \to \underbrace{\pi_3(B\pi_1(E))}_{=0} \to \pi_2(F) \to \pi_2(E) \to \underbrace{\pi_2(B\pi_1(E))}_{=0} \to \pi_1(F) \to \pi_1(E) \xrightarrow{\sim} \underbrace{\pi_1(B\pi_1(E))}_{=\pi_1(E)} \to \pi_0(F) \to 1$$

This means that $\pi_1(F) = 1$ and $\pi_n(F) = \pi_n(E)$ for $n \neq 1$.

Since $\tilde{E}$ has this homotopy groups, it follows that $\tilde{E}$ is homotopic to F.

Back to our proof. Every fibration $f: E \to B$ satisfies the following spectral sequence, due to Serre: $E^2_{p,q} = H_p(B, H_q(F(f), \mathbb{Z})) \Rightarrow H_{p+q}(E, \mathbb{Z})$

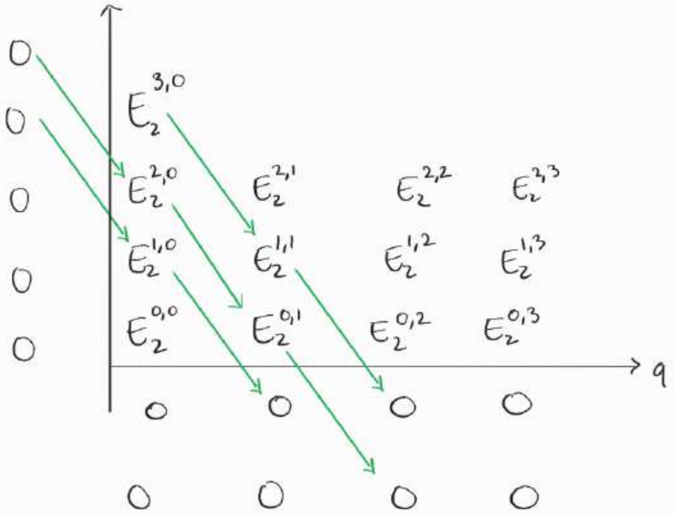
In our case: $E^2_{p,q} = H_p(\varpi_1(E), H_q(\tilde{E}, \mathbb{Z})) \Rightarrow H_{p+q}(E, \mathbb{Z})$.

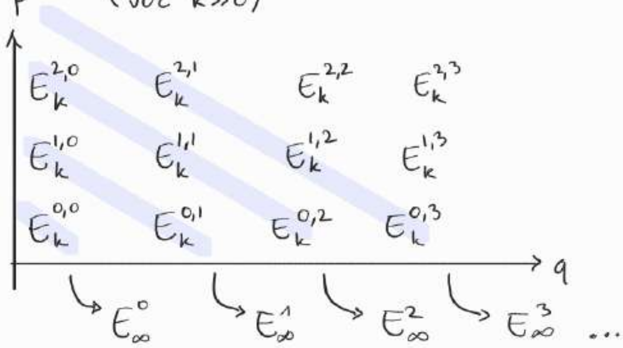
Looking at low degree terms, we can get a five-terms exact sequence out of this.

Just to show that I did my homework, here is how to obtain it.

Suppose you have a spectral sequence $E_2^{p,q} \Rightarrow E_\infty^{p+q}$.

The second page looks as follows:





Notice that:

- The differentials in and out of $E_2^{1,0}$ are zero. Thus, $E_2^{1,0}$ has converged. Since $E^1_\infty \cong E_k^{1,0} \oplus E_k^{0,1}$ for $k \gg 0$, we have an exact sequence $E^1_\infty \to E_2^{1,0} \to 0$. The kernel of the first map is $E_k^{0,1}$ for some k big enough
- $E_2^{0,1}$ has an arrowing going out to zero. Thus, on the next page we have $E_3^{0,1} = \operatorname{coker}(E_2^{2,0} \to E_2^{0,1})$. But on the third page $E_3^{0,1}$ is surrounded by zeroes, so it has converged. Completing the sequence of the previous point, we get a S.E.S. $0 \to E_3^{0,1} \to E^1_\infty \to E_2^{1,0} \to 0 \rightsquigarrow E_2^{2,0} \to E_2^{0,1} \to E^1_\infty \to E_2^{1,0}$ exact
- Arguing similarly, $E_3^{2,0} = \ker(E_2^{2,0} \to E_2^{0,1})$ has converged, so we have an exact sequence $0 \to E_3^{2,0} \to E_2^{2,0} \to E_2^{0,1}$. Combining this with the previous sequence and observing that E^2_∞ surjects onto $E_3^{2,0}$, we get

  $E^2_\infty \to E_2^{2,0} \to E_2^{0,1} \to E^1_\infty \to E_2^{1,0}$

In our case

$$H_2(E,\mathbb{Z}) \to H_2(\varpi_1(E), H_0(\tilde{E},\mathbb{Z})) \to H_0(\varpi_1(E), H_1(\tilde{E},\mathbb{Z})) \to H_1(E,\mathbb{Z}) \to H_1(\varpi_1(E), H_0(E,\mathbb{Z}))$$

Since $H_0(\tilde{E},\mathbb{Z})=\mathbb{Z}$, $H_1(\tilde{E},\mathbb{Z})=0$ ($\tilde{E}$ is simply connected), $H_2(E,\mathbb{Z})=0$ (E acyclic), $H_0(\varpi_1(E),H_1(\tilde{E},\mathbb{Z}))=H_1(\tilde{E},\mathbb{Z})^{\varpi_1(E)}=0$, we conclude that $H_2(\varpi_1(E),\mathbb{Z})=0$ $\square$

Def: Let X,Y be pointed connected CW-complexes. A map $f:X\to Y$ is called "acyclic" if the homotopy fiber of f is acyclic.

By the previous lemma, if $f:X\to Y$ is acyclic, then $F(f)$ is connected and $\varpi_1(F(f))$ is perfect. Moreover, recall that there is an exact sequence

$$\varpi_1(F(f))\to\varpi_1(X)\to\varpi_1(Y)\to\varpi_1(F(f))$$

It follows that the map $\varpi_1(X)\to\varpi_1(Y)$ is surjective with kernel a perfect group.

Def: Let X be a pointed connected CW-complex. Let P be a perfect normal subgroup of $\varpi_1(X)$. An acyclic map $f:X\to Y$ is called a "+-construction" on X (relative to P) if P is the kernel of $\varpi_1(X)\to\varpi_1(Y)$.

Theorem (Quillen) Let P be a perfect normal subgroup of $\varpi_1(X)$. Then:

1) There is a +-construction $f:X\to Y$ relative to P
2) Let $f:X\to Y$ be a +-construction relative to P. Let $g:X\to Z$ be a map such that $P\subseteq\ker(\varpi_1(X)\to\varpi_1(Z))$. Then there is a unique map, up to homotopy, $h:Y\to Z$ such that $h\circ f=g$
3) In particular, if g is another +-construction relative to P, then the map h in (2) is an equivalence.

Thus, every perfect normal subgroup P of $\varpi_1(X)$ determines a unique +-construction (up to homotopy)

Idea of the construction:

Let X be a pointed connected CW-complex. Let Y be a CW-complex obtained by glueing a 2-cell to every element of P (which we view as a loop in the 1-skeleton). This operation has the effect of contracting all loops in P, hence $\varpi_1(Y)=\varpi_1(X)/P$.

Notice that $\{\text{2-cells of } Y\}=\{\text{2-cells of } X\}\cup P$. Let e_p be the 2-cell of Y attached to $p\in P$.

Consider the cellular complex $\dots\to C_2(Y)\to C_1(Y)\to\dots$

The cells e_p do not lie in any boundary of any 3-cell by construction.

Moreover, we have a map $\partial_2 e_*: P\to C_1(Y)$, $p\mapsto\partial_2 e_p$. Notice that $\partial_2 e_p=p$ (which will then be written in terms of 1-cycles). Since $\partial_2 e_{pq}=e_p+e_q$, we see that this map is a group homomorphism. But $C_1(Y)$ is abelian, hence $[P,P]\subseteq\ker\partial_2 e_*$. Recall that P is perfect, so $P=[P,P] \in \ker \partial_2 e_*$. Extending this to $C_2(Y)$ by $\mathbb{Z}$-linearity gives that $\bigoplus_{p\in P} \mathbb{Z} e_p \subseteq \ker \partial_2$. This 2-cells are not in the image of ∂_3, hence $H_2(Y,\mathbb{Z}) = H_2(X,\mathbb{Z}) \oplus \bigoplus_{p\in P} \mathbb{Z} e_p$.

Since $\partial e_p = p$ is contractible, we may represent every class $[e_p] \in H_2(Y,\mathbb{Z})$ by a map $h_p : \mathbb{S}^2 \to Y$. Consider now the space Z obtained by glueing a 3-cell α_p for all $p \in P$ along the map h_p. We have

1) $\varpi_1(Z) = \varpi_1(Y) = \varpi_1(X)/P$, because the 1- and 2-skeleta of Y and Z are the same and the fundamental group only depends on these

2) $H_2(X,\mathbb{Z}) \cong H_2(Z,X)$

* surjectivity: take $h: \mathbb{S}^2 \to Z$, which we see as a cellular map, $\mathbb{S}^2$ consisting of a 1-cell and a 2-cell only. If the 2-cell of $\mathbb{S}^2$ is mapped to a 2-cell of X, then h actually lands in X, so it defines an element of $\varpi_2(X)$. Otherwise, the 2-cell of $\mathbb{S}^1$ lands in a 2-cell of the form e_p for some p. The presence of α_p then makes this 2-cell contractible, so $[h] \in \varpi_2(Z)$ vanishes.

* injectivity: take $h: \mathbb{S}^2 \to X$ such that $[h] \in \varpi_2(Z)$ vanishes. Then there is a homotopy $H: \mathbb{S}^2 \times I \to Z$ between h and the constant map to a point. We can view H as a cellular map. I is sent inside X because the 1-skeleta of X and Z coincide. Thus, $\partial(\mathbb{S}^2 \times I)$ is mapped entirely inside X (since h had image in X). The 3-cell of $\mathbb{S}^2 \times I$ is mapped to a 3-cell of Z, and the only ones that have boundary in X are the ones of X (not the α_p's, in which e_p appears).
This means that $H: \mathbb{S}^2 \times I \to Z$ actually takes values in X up to homotopy, hence $[h] \in \varpi_1(X)$ vanishes.

3) $H_3(X,\mathbb{Z}) \cong H_3(Z,X)$

* injectivity is clear: we do as above and we observe that the 4-cells of X are the same as the ones of Z.

* for surjectivity, we need to lift $h: \mathbb{S}^3 \to Z$ with the 3-cell of $\mathbb{S}^3$ being mapped to α_p. But the α_p's are contractible, because their boundary is e_p.

4) $H_n(X,\mathbb{Z}) \cong H_n(Z,\mathbb{Z})$ for all $n \geq 4$. This follows from the higher cells of X and Z being the same.
For general coefficients, you can imagine.

From the homotopy sequence of the fiber, it follows that $\pi_n(F(X \hookrightarrow Z)) = 0$ for $n \geq 2$ and $1 \to \pi_1(F(X \hookrightarrow Z)) \to \pi_1(X) \to \pi_1(Z) \to 1$ is exact, hence $\pi_1(F(X \hookrightarrow Z)) \cong P$.
It follows that $X \hookrightarrow Z$ is acyclic, so Z is a $+$-construction for X relative to P. $\square$

<u>Remark:</u> The sum of two perfect subgroups A, B of a group G is perfect.
(Because $\langle A, B \rangle = \langle [A,A], [B,B] \rangle \subseteq \langle [\langle A,B\rangle, \langle A,B\rangle], [\langle A,B\rangle,\langle A,B\rangle] \rangle = [\langle A,B\rangle, \langle A,B\rangle]$)
Thus, every group contains a maximal perfect subgroup, called the "perfect radical" of G.
This is also normal: every conjugate of the perfect radical is perfect, hence contained in the perfect radical, but conjugations is an isomorphism.
<u>Notation:</u> For any pointed connected CW-complex X, we denote by X^+ its plus construction with respect to the perfect radical of $\pi_1(X)$.

Recall our original goal: we wish to construct some space $BGL(R)^+$ such that
- $\pi_1(BGL(R)^+) \cong K_1(R)$ naturally;
- for all $K_1(R)$-modules M we have $H_*(BGL(R), M) \cong H_*(BGL(R)^+, M)$.

Since $\pi_1(BGL(R)) = GL(R)$ and $K_1(R)$ is the quotient of $GL(R)$ modulo a perfect normal subgroup $(E(R))$. Thus, the $+$-construction of $BGL(R)$ relative to $E(R)$ has fundamental group $K_1(R)$. We are left to see that the cohomological statement holds true. For this, we need to use acilycity of the canonical map of the $+$-construction.
<u>Lemma:</u> Let X and Y be CW-complexes. A map $f: X \to Y$ is acyclic if and only if for every $\pi_1(Y)$-module M we have $H_*(X, M) \cong H_*(Y, M)$.
<u>Proof:</u> Suppose f acyclic, so $F(f)$ has the homotopy type of a point. The composition $\pi_1(F(f)) \to \pi_1(X) \to \pi_1(Y)$ is zero (recall the long exact sequence for the homotopy fiber).
Given a $\pi_1(Y)$-module M, this is a $\pi_1(X)$-module by precomposing with $\pi_1(X) \to \pi_1(Y)$,

and $\varpi_1(F)$ acts trivially on M. By the universal coefficient theorem, we have

$$H_i(F(f),M) \cong H_i(F(f),\mathbb{Z}) \otimes_{\mathbb{Z}} M \oplus \operatorname{Tor}_1(H_{i-1}(F(f),\mathbb{Z}),M).$$
$$= \quad 0 \quad \oplus \operatorname{Tor}_1(H_{i-1}(F(f),\mathbb{Z}),M)$$

For $i=1$ we get $H_1(F(f),\mathbb{Z}) = \operatorname{Tor}_1(H_0(F(f),\mathbb{Z}),M) = \operatorname{Tor}_1(\mathbb{Z},M)=0$, and by induction this implies that $H_q(F(f),M)=0$ for $q>0$ and $H_0(F(f),M)=M$. In Serre's spectral sequence we then have

$$H_p(Y, H_q(F(f),M)) \Rightarrow H_{p+q}(X,M)$$

The E_2-page already converges and $H_p(Y,M) \cong H_p(X,M)$.

Conversely, suppose that for every $\varpi_1(Y)$-module M we have $H_*(X,M) \cong H_*(Y,M)$.

We deal first with the case $\varpi_1(Y)=0$. Consider the two Serre spectral sequences for $F(f)\to X \to Y$ and $* \to Y \xrightarrow{id} Y$, which converge to $H_*(X,M)$ and $H_*(Y,M)$ respectively. Let $E^2_{p,q} \Rightarrow E^\infty_{p+q}$ and $\tilde{E}^2_{p,q} \Rightarrow \tilde{E}^\infty_{p+q}$ be the corresponding spectral sequences.

The map $f: X\to Y$ gives a diagram
$$\begin{tikzcd} F(f) \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d, "f"] & Y \arrow[d, "id"] \\ * \arrow[r] & Y \arrow[r, "id"] & Y \end{tikzcd},$$

which induces a morphism at the level of spectral sequences. By the universal coefficient theorem, this morphism fits into short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & E^2_{p,0}\otimes E^2_{0,q} \arrow[r] \arrow[d, "f\otimes f"] & E^2_{p,q} \arrow[r] \arrow[d, "f"] & \operatorname{Tor}(E^2_{p-1,0}, E^2_{0,q}) \arrow[r] \arrow[d, "\operatorname{Tor}(f,f)"] & 0 \\ 0 \arrow[r] & \tilde{E}^2_{p,0}\otimes \tilde{E}^2_{0,q} \arrow[r] & \tilde{E}^2_{p,q} \arrow[r] & \operatorname{Tor}(\tilde{E}^2_{p-1,0}, \tilde{E}^2_{0,q}) \arrow[r] & 0 \end{tikzcd}$$

Then any two of the following implies the third one:

(1) $f: E^2_{p,0} \to E^2_{p,0}$ is an iso for all $p\geq 0$

(2) $f: E^2_{0,q} \to E^2_{0,q}$ is an iso for all $q\geq 0$

(3) $f: E^\infty_{p,q} \to E^\infty_{p,q}$ is an iso for all p,q.

In our case, (3) is ensured by our assumption $H_*(X,M)\cong H_*(Y,M)$. Moreover, we have $H_p(Y,H_0(F(f),M)) \cong H_p(Y,\mathbb{Z}) \cong H_p(Y,H_0(*,M))$, so also (2) holds. By (1) we conclude that $H_q(F(f),M) \cong H_q(*,M)=0$ for $q\geq 1$, hence $F(f)$ is acyclic.

In general, let $\tilde{Y}$ be the universal covering space of Y and consider $\tilde{X} = X\times_Y \tilde{Y}$. Then there are natural isomorphisms $H_*(\tilde{Y},\mathbb{Z}) \cong H_*(Y,\mathbb{Z}[\varpi_1(Y)])$ and $H_*(\tilde{X},\mathbb{Z}) \cong H_*(X,\mathbb{Z}[\varpi_1(Y)])$. Choosing $M=\mathbb{Z}[\varpi_1(Y)]$, by assumption $H_*(X,M)\cong H_*(Y,M)$

via f, hence the induced map $\tilde{f}: \tilde{X} \to \tilde{Y}$ gives isomorphisms on integral cohomology. But $\pi_1(\tilde{Y}) = 0$, hence the previous argument shows that $F(\tilde{f})$ is acyclic. Using the explicit description of $F(f)$ and the path-lifting property of universal coverings, one can see that $F(\tilde{f}) = F(f)$, which concludes the proof. $\square$

Choosing $X = BGL(R)$ and $P = E(R)$, we have shown that the $+$-construction X^+ of X relative to $BGL(R)$ is a $BGL(R)^+$-model. For coherence, we need to recover $K_2(R)$ as $\pi_2(X^+)$. This is our next goal.

<u>Recall</u>: (universal central extension)

Let G be a group. A central extension of G is an extension $1 \to A \to E \to G \to 1$ such that $A \subseteq Z(E)$. A universal central extension is initial among all central extensions of G. We have:

- Central extension of G by a fixed abelian group A are in bijection with $H^2(G, A)$.
- If G has a universal central extension X, then both X and G are perfect.
- If G is perfect, then it has a universal central extension
$$1 \to H_2(G, \mathbb{Z}) \to X \to G \to 1.$$
Moreover, a central extension X of G is universal if and only if $H_1(X, \mathbb{Z}) = H_2(X, \mathbb{Z}) = 0$.

<u>Proposition</u>: Let P be a perfect normal subgroup of a group G. Consider the corresponding $+$-construction $f: BG \to BG^+$. Then $\pi_1(F(f))$ is the universal central extension of P, and $\pi_2(BG^+) \cong H_2(P, \mathbb{Z})$

<u>Proof</u>: By the long exact sequence of $F(f)$, we have an exact sequence
$$\underset{\overset{\parallel}{0}}{\pi_2(BG)} \to \pi_2(BG^+) \to \pi_1(F(f)) \to \underset{\overset{\cong}{G}}{\pi_1(BG)} \to \underset{\overset{\cong}{G/P}}{\pi_1(BG^+)} \to 1$$

Blackbox: $\pi_2(BG^+)$ is mapped inside the center of $\pi_1(F(f))$ ($\pi_n(X) \to \pi_n(X, A)$ has image in the center)

We may then extract a central extension of P: $1 \to \pi_2(BG^+) \to \pi_1(F(f)) \to P \to 1$.

To check that this is universal, we need to see that $H_1(\pi_1(F(f)), \mathbb{Z}) = H_2(\pi_1(F(f)), \mathbb{Z}) = 0$

Since $F(f)$ is acyclic, $\pi_1(F(f))$ is perfect, so $H_1(\pi_1(F(f)), \mathbb{Z}) = 0$. We have also seen that

$H_2(\pi_1(F(f)), \mathbb{Z}) = 0$, which concludes the proof $\square$

Corollary. $K_2(R) = \pi_2(BGL(R)^+)$ is isomorphic to $H_2(E(R), \mathbb{Z})$.
Since the universal central extension of $E(R)$ is $St(R)$, we recover the description of $K_2(R)$ given in the previous talks.

Corollary: $\pi_3(BGL(R)^+) \cong H_3(St(R), \mathbb{Z})$.
Proof: See addendum at the end of the notes $\square$

§ The $S^{-1}S$-construction

Recall: A symmetric monoidal category is a category S endowed with a functor $\otimes : S \times S \to S$ with a unit object $1 \in S$ such that $\otimes$ is associative and commutative (i.e. there are natural isomorphisms of functors between $S \times S \to S, (a,b) \mapsto a \otimes b$ and $S \times S \to S$, $(a,b) \mapsto b \otimes a$, etc.)

Consider the space BS associated with a symmetric monoidal category $(S, \otimes)$. We then obtain a map $BS \times BS \cong B(S \times S) \xrightarrow{B\otimes} BS$, which turns BS into a commutative monoid in the homotopy category of topological spaces, in particular an H-space

Problem: BS might not be interesting: if 1 is initial, then BS is contractible.

Example: $S = \mathcal{F}in\,Set$, $\otimes = \sqcup$, then $\emptyset$ is initial

Def: We define $iso(S)$ to be the category with objects the ones of S and morphisms the isomorphisms of S. If S is symmetric monoidal, $iso(S)$ is also symmetric monoidal.
Thus, $iso(S)$ is also an H-space.

Example: *) For $iso(\mathcal{F}in\,Set)$, consider that the automorphism group of a set with n-elements is the symmetric group Sym_n. There are no morphisms between sets with different cardinality, so $iso(\mathcal{F}in\,Set) \simeq \bigsqcup_n Sym_n$. Thus: $B(iso(\mathcal{F}in\,Set)) \simeq \bigsqcup_n B(Sym_n)$

*) $S = \mathcal{P}(R)$, fin. gen.'ed proj. R-modules with $\oplus$ as product. 0 is an initial object, so BS is contractible. But $B(\mathrm{iso}(S)) = \coprod_{P \in \mathcal{P}(R)/\mathrm{iso}} B\,\mathrm{Aut}\,P$

Given S symmetric monoidal, BS is an H-space. In particular, $\pi_0(BS)$ is a monoid.

<u>Goal</u>: Construct a category $S^{-1}S$ such that $B(S^{-1}S)$ is a "group completion" of BS.

<u>Def</u>: Let S be symmetric monoidal. We define the category $S^{-1}S$ to have

- objects the ones of $S \times S$
- morphisms: the set of composites $(a,b) \xrightarrow{s\otimes} (s\otimes a, s\otimes b) \xrightarrow{(f,g)} (c,d)$ modulo the equivalence relation:
  $(a,b) \xrightarrow{s\otimes} (s\otimes a, s\otimes b) \xrightarrow{(f,g)} (c,d) \sim (a,b) \xrightarrow{s'\otimes} (s'\otimes a, s'\otimes b) \xrightarrow{(f',g')} (c,d)$ if and only if there is an iso $\alpha : s \xrightarrow{\sim} s'$ such that composition with $\alpha\otimes a, \alpha \otimes b$ sends f to f' and g to g'. Graphically:

$$\begin{tikzcd}
(a,b) \arrow[r, "s\otimes"] & (s\otimes a, s\otimes b) \arrow[r, "(f,g)"] \arrow[d, "\alpha\otimes a \times \alpha\otimes b"'] \arrow[dr, phantom, "\circlearrowleft"] & (c,d) \arrow[d, equal, "id"] \\
(a,b) \arrow[r, "s'\otimes"] & (s'\otimes a, s'\otimes b) \arrow[r, "(f',g')"] & (c,d)
\end{tikzcd}$$

This construnction is functorial for monoidal functors between symmetric monoidal categories.

There are two kinds of morphisms in $S^{-1}S$:

1) morphisms coming from the inclusion $S\times S \hookrightarrow S^{-1}S$, thus arrows of the form $(f,g) : (a,b) \to (c,d)$ with f,g in S
2) morphisms of the form $s\otimes : (a,b) \to (s\otimes a, s\otimes b)$. Notice that these are purely formal arrows.

Formally: $\mathrm{Hom}_{S^{-1}S}((a,b),(c,d)) = \{ (s,f) \mid s \in S,\ f : (s\otimes a, s\otimes b) \to (c,d) \in S\times S \} /\sim$.

<u>Remark</u>: $S^{-1}S$ is a symmetric monoidal category by $(a,b)\otimes(c,d) = (a\otimes c, b\otimes d)$. The functor $S \to S^{-1}S$, $a \mapsto (a,1)$ is monoidal. It follows that the induced map $BS \to B(S^{-1}S)$ is a map of H-spaces, and $\pi_0(S) \to \pi_0(S^{-1}S)$ is a map of monoids.

N.B.: If 1 is not initial in S, there may not be an arrow $1 \to s$, hence no arrow $a \cong 1\otimes a \to s\otimes a$!

The point of the construction is the following:
Given $(a,b) \in S^{-1}S$, there is an arrow

$$\varphi: (1,1) \xrightarrow{(a\otimes b)\otimes} (a\otimes b, a\otimes b) \xrightarrow{\mathrm{id}\times \mathrm{swap}} (a\otimes b, b\otimes a) = (a,b)\otimes(b,a) \qquad \in \mathrm{Hom}_{S^{-1}S}((1,1), (a\otimes b, b\otimes a))$$

This means the the objects $(1,1)$ and $(a,b)\otimes(b,a)$ are connected by an arrow.
Now $\varpi_0(S^{-1}S) = \{\text{objects of } \mathcal{C}\} / (\text{being connected by an arrow})$.
The presence of the morphism $(1,1) \longrightarrow (a,b)\otimes(b,a)$ implies that the class of (a,b) in $\varpi_0(BS^{-1}S)$ has inverse (b,a). In particular, $\varpi_0(BS^{-1}S)$ is an abelian group.

<u>Def</u>: Let S be a symmetric monoidal category in which every morphism is an iso.
The K-groups of S are defined as: $K_n^{\otimes}(S) = \varpi_n(BS^{-1}S)$.
We also write $K^{\otimes}(S) = B(S^{-1}S)$ (K-theory space of S).
Functoriality: a monoidal functor $S \to T$ induces a map $S^{-1}S \to T^{-1}T$, and hence $B(S^{-1}S) \to B(T^{-1}T)$, which leads to group homomorphisms $K_n^{\otimes}(S) \to K_n^{\otimes}(T)$.

<u>Lemma</u>: The abelian group $K_0^{\otimes}(S) = \varpi_0(S^{-1}S)$ is the group completion of the abelian monoid $\varpi_0(S) = S^{iso}$

<u>Proof</u>: Let A be the group completion of $\varpi_0(S)$, which comes with a map $\varpi_0(S) \to A$.
We construct a map $\alpha: \varpi_0(S^{-1}S) \to A$, $(m,n) \mapsto [m]-[n]$. This is a group homomorphism

$$\alpha((m,n)\otimes(m',n')) = \alpha(m\otimes m', n\otimes n') = [m\otimes m'] - [n\otimes n'] = [m]+[m']-[n]-[n'] = ([m]-[n]) + ([m']-[n']) =$$
$$= \alpha((m,n)) + \alpha((m',n')).$$

Also notice that this map is well defined: for all $s \in S$ we have an arrow $(a,b) \xrightarrow{s\otimes} (s\otimes a, s\otimes b)$ in $S^{-1}S$, so in $\varpi_0(S^{-1}S)$ the class of (a,b) equals the one of $(s\otimes a, s\otimes b)$. We have
$\alpha((s\otimes a, s\otimes b)) = [s\otimes a] - [s\otimes b] = [s]+[a]-[s]-[b] = [a]-[b] = \alpha((a,b))$.
One then checks that α is an inverse to the universal homomorphism $A \xrightarrow{\varphi} \varpi_0(S^{-1}S)$.
For example, $\alpha\circ\varphi$ has the property that for all $a \in S$ $[a] \mapsto (a,1) \mapsto [a] - 0 = [a]$, so $\alpha\circ\varphi = \mathrm{id}_A$ by the uniqueness in the universal property of A. Conversely, for $\varphi\circ\alpha$
$(a,b) \mapsto [a]-[b] \mapsto (a,1)\otimes(b,1)^{-1} = (a,1)\otimes(1,b) = (a,b)$, so $\varphi\circ\alpha = \mathrm{id}_{\varpi_0(S^{-1}S)}$ $\square$

Remark: Under some mild assumptions, one could also prove that the topological space $B(S^{-1}S)$ is a group completion of the H-space BS.

Theorem: Let S be a symmetric monoidal category. Suppose that:

1) every morphism in S is an isomorphism;

2) every map $\mathrm{Aut}_S(a) \longrightarrow \mathrm{Aut}_S(s\otimes a)$ is injective.

Then $B(S^{-1}S)$ is the group completion of the H-space BS.

§ Comparison of $+$ and $S^{-1}S$ construction

Given a ring with unit R, we have defined a topological space $K(R)$ whose homotopy groups define the K-theory of R. We have seen that $K(R) = K_0(R) \times BGL(R)^+$, where $BGL(R)^+$ is a $+$-construction of $BGL(R)$.

On the other hand, we had also defined the first K-groups of a symmetric monoidal category. For a ring R, we have seen that $K_0(R) = K_0(\mathrm{iso}(\mathcal{P}(R)))$, that is, $K_0(R)$ is the K_0-group of the category of finitely generated projective R-modules.

The $S^{-1}S$-construction defines higher K-groups for a symmetric monoidal category. We may thus take $S = \mathrm{iso}(\mathcal{P}(R))$, whose K_0 agrees with $K_0(R)$.

Goal: Show that $K_n(R)$ (defined via the $+$-construction) is isomorphic to $K_n^{\oplus}(S)$ for $S = \mathrm{iso}(\mathcal{P}(R))$ (defined via the $S^{-1}S$-construction).

Notice that the $+$-construction exhibit some kind of dicotomy between the K_n for $n \geq 1$ and K_0. Indeed, $K_n(R) = \pi_n(BGL(R)^+)$ for $n \geq 1$, but for $n=0$ we artificially consider the space $K(R) = K_0(R) \times BGL(R)^+$ (with $K_0(R)$ as a discrete space). This product does not alter the higher homotopy groups.

This subtlety also influences the comparison with the $S^{-1}S$-construction. We first study a smaller category $\mathcal{F}(R) \subseteq \mathrm{iso}\,\mathcal{P}(R)$ and prove that $B(\mathcal{F}(R)^{-1}\mathcal{F}(R)) \cong \mathbb{Z} \times BGL(R)^+$.

Then we show that $B(\mathcal{F}(R)^{-1}\mathcal{F}(R))$ shares the same homotopy groups as $B(\mathrm{iso}\mathcal{P}(R)^{-1}\mathrm{iso}\mathcal{P}(R))$ for $n \geq 1$ by a cofinality argument. We then check K_0 separately.

Def: Let $\mathcal{F}(R)$ be the category of based free R-modules, that has:

- objects: $\{0, R, R^2, \dots, R^n, \dots\}$
- morphisms: $\mathrm{Hom}_R(R^n, R^m) = \emptyset$ if $n \neq m$
  $\mathrm{Hom}_R(R^n, R^n) = GL_n(R)$

Remark: Suppose that R has IBN (invariant basis number), that is, if $R^n \cong R^m$ then $n = m$. Then $\mathcal{F}(R)$ is equivalent to the category of free R-modules with a fixed basis and basis-preserving homomorphisms.

Observe that:

1) $\mathcal{F}(R)$ is symmetric monoidal with respect to the direct sum of R-modules.
2) If we interpret $GL_n(R)$ as a one-object category, then $\mathcal{F}(R)$ is equivalent to the category $\coprod_n GL_n(R)$.
3) The space $B\mathcal{F}(R)$ is homotopy equivalent to the disjoint union of classifying spaces $BGL_n(R)$ for all $n \geq 0$.
4) Suppose that R has IBN. Then $\mathcal{F}(R)$ is a full subcategory of $\mathrm{iso}\,\mathcal{P}(R)$.

Idea:

- Study the $S^{-1}S$ construction of $\mathcal{F}(R)$
  (intuitively, the appearance of $\coprod_n BGL_n(R)$ should draw a connection with $BGL(R)$)
- Observe that $\mathcal{F}(R)$ is cofinal in $\mathcal{P}(R)$, hence that the higher homotopy groups of their $S^{-1}S$-constructions are isomorphic.
- Deal with the K_0 separately.

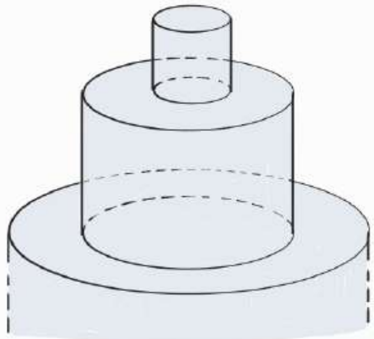
We start with the $S^{-1}S$ construction for $\mathcal{F}(R)$.

Theorem: Let $S = \mathcal{F}(R) \cong \coprod_n GL_n(R)$. Then $BS^{-1}S$ is the group completion of $BS = \coprod_n BGL_n(R)$ and $B(S^{-1}S) \cong \mathbb{Z} \times BGL(R)^+$.

Proof: The fact that $B(S^{-1}S)$ is the group completion of BS follows immediately from the previous theorem. For the second claim, we need to realize the identity component of $B(S^{-1}S)$ as a $+$-construction of $GL(R)$, so find an acyclic map $BGL(R) \longrightarrow B(S^{-1}S)^\circ$.
So, how to construct a map $BGL(R) \to B(S^{-1}S)$? In principle, we could give a monoidal functor $GL(R) \to S^{-1}S$. Ideally, $GL_n(R) \subseteq GL(R)$ should be mapped inside $\mathrm{Aut}_{S^{-1}S}((R^n, R^n))$. However, the category $GL(R)$ has just one object, which cannot be

sent simultaneously to each $(R^n, R^n) \in S^{-1}S$. For this reason, we need to give ourselves "a bit more space". The constructed map $BGL(R) \to B(S^{-1}S)$ will not come from a functor $GL(R) \to B(S^{-1}S)$.

We make $GL(R)$ a bit "thicker". The "mapping telescope" of $BGL(R) = \varinjlim_n BGL_n(R)$ is the topological space obtained from the disjoint union $\bigsqcup_n (BGL_n(R) \times [0,1])$ by glueing along the maps $BGL_n(R) \times \{1\} \to BGL_{n+1}(R) \times \{0\}$ induced by $GL_n(R) \to GL_{n+1}(R)$, $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. The picture is the following:



The mapping telescope of $BGL(R)$ is homotopy equivalent to $BGL(R)$: we may contract all the unit intervals and we are left with a union with glueing datum that coincides with the transition maps in the limit $\varinjlim_n BGL_n(R)$.

To define a map from the mapping telescope to $B(S^{-1}S)$ it is enough to give maps $BGL_n(R) \times I \to BS^{-1}S$ for all n which are compatible with the glueing condition.

This means that we need to give maps $\eta_n^{(0)}: BGL_n(R) \times \{0\} \to B(S^{-1}S)$, $\eta_n^{(1)}: BGL_n(R) \times \{1\} \to B(S^{-1}S)$ s.t.

1) there is a homotopy between them ; (so we get $BGL_n(R) \times I \to B(S^{-1}S)$)

2) $BGL_{n+1}(R) \times \{0\} \to B(S^{-1}S)$ restricts to $BGL_n(R) \times \{1\} \to B(S^{-1}S)$ via the map $GL_n(R) \to GL_{n+1}(R)$, $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. (so $\eta_{n+1}^{(0)}|_{BGL_n(R)} = \eta_n^{(1)}$).

Thus, we may as well define maps $\eta_n : BGL_n(R) \to B(S^{-1}S)$ such that $\eta_{n+1}|_{BGL_n(R)}$ is homotopic to η_n.

We may define these maps at the level of categories first. Consider the functor

$\eta_n: GL_n(R) \longrightarrow S^{-1}S$, objects: $* \longmapsto (R^n, R^n)$

$$\text{morphisms: } \begin{aligned} GL_n(R) &\longrightarrow Aut_{S^{-1}S}((R^n, R^n)) \\ g &\longmapsto (g, id) \end{aligned}$$

Each η_n induces a map $BGL_n(R) \to B(S^{-1}S)$. To give a homotopy between them it is enough to give a natural transformation $\varphi_n: \eta_n \to \eta_{n+1}|_{BGL_n(R)}$.

We consider: $\varphi_n: \eta_n \to \eta_{n+1}|_{BGL_n(R)}$ s.t. $\eta_n(*) = (R^n, R^n) \longrightarrow (R^{n+1}, R^{n+1}) = \eta_{n+1}(*)$ with $\varphi_n = \oplus (R,R)$.

This is indeed a natural transformation, because at the level of morphisms we have:

$$\begin{tikzcd}
GL_n(R) \arrow[r, "\eta_n"] \arrow[dr, bend right] & \mathrm{Aut}_{S^{-1}S}((R^n, R^n)) \arrow[d, "\oplus R"] \\
& \mathrm{Aut}_{S^{-1}S}(R^{n+1}, R^{n+1})
\end{tikzcd}$$

$$\begin{tikzcd}
g \arrow[r, mapsto] \arrow[dr, mapsto] & (g, id) \arrow[d, mapsto] \\
& \left(\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}, id\right) = (g \oplus id, id \oplus id)
\end{tikzcd}$$

and this diagram commutes.

Since $BGL(R) = \lim\limits_{\to n} BGL_n(R)$, the maps $\eta_n : GL_n(R) \to B(S^{-1}S)$ yield a map $BGL(R) \to B(S^{-1}S)$ which is well defined up to homotopy (because the compatibility between the η_n's is only up to homotopy). In order to work with an actual map, we replace $GL(R)$ by its mapping telescope. Since these spaces are homotopic, this will not harm our arguments.

Let T_R be the mapping telescope of $BGL(R)$. We have constructed a map $T_R \to B(S^{-1}S)$. The image of this map actually lies in the identity component $B(S^{-1}S)^\circ$ of $B(S^{-1}S)$. Indeed, T_R is connected and the constructed map preserves the group structure.

Now we want to show that the map $T_R \to B(S^{-1}S)$ is a $+$-construction. For this, we need the following fact: the $+$-construction $BGL(R) \to BGL(R)^+$ is universal for maps into H-spaces (Quillen). In particular, given $f : BGL(R) \to H$ with H an H-space, if $f_* : H_*(BGL(R), \mathbb{Z}) \to H_*(H, \mathbb{Z})$ is an iso, then f is acyclic and $H \cong BGL(R)^+$.

Thus, in our context we need to check that $H_*(BGL(R), \mathbb{Z}) \cong H_*(B(S^{-1}S), \mathbb{Z})$. Since we are working up to homotopy, we may as well replace T_R by $BGL(R)$. (The $+$-construction is well-defined up to homotopy anyway).

Here comes another result that we do not have time to see in full detail. For a symmetric monoidal category S, if (1) and (2) of the previous theorem are satisfied, then $H_*(B(S^{-1}S), \mathbb{Z})$ is the localization of $H_*(BS, \mathbb{Z})$ at $\varpi_0(BS)$ (here, we are identifying $H_0(BS, \mathbb{Z}) \cong \mathbb{Z}[\varpi_0(BS)]$ and we are observing that $\varpi_0(BS)$ is a multiplicatively closed subset, multiplication being given by its monoid structure).

In our case, $\varpi_0(BS) = \{0, R, R^2, \dots\}$, so $H_*(B(S^{-1}S), \mathbb{Z})$ is the localization of $H_*(BS, \mathbb{Z})$ at $\{0, R, R^2, \dots\}$, i.e. inverting R: $H_*(B(S^{-1}S), \mathbb{Z}) \cong H_*(BS, \mathbb{Z})\,[\,[R]^{-1}]$

Moreover, we have $\varpi_0(BS) = \mathbb{N}$, because $S = \bigsqcup_n GL_n(R)$. Since $\varpi_0(BS^{-1}S)$ is the group completion of the monoid $\varpi_0(BS)$, we have $\varpi_0(B(S^{-1}S)) = \mathbb{Z}$, hence

$H_*(B(S^{-1}S), \mathbb{Z}) \cong H_*(B(S^{-1}S)^\circ, \mathbb{Z}) \times \mathbb{Z}$.

We also notice that $H_*(BS,\mathbb{Z}) = H_*(\coprod_n BGL_n(R),\mathbb{Z}) = \prod_n H_*(BGL_n(R),\mathbb{Z})$.

Thus, at the level of homology:

$$\begin{tikzcd}
 & H_*(BGL(R),\mathbb{Z}) \arrow[d, phantom, sloped, "\cong"] & & & \Big(\overbrace{\prod_n H_*(BGL_n(R),\mathbb{Z})\Big)}^{H_*(BS,\mathbb{Z})}[[R]^{-1}] \\
H_*(BGL_n(R),\mathbb{Z}) \arrow[r] \arrow[d] & H_*(T_R,\mathbb{Z}) \arrow[rr] \arrow[dr] & & H_*(B(S^{-1}S),\mathbb{Z}) \arrow[ur, "\sim"] \arrow[dr, "\sim"] & \\
H_*(BGL_{n+1}(R),\mathbb{Z}) \arrow[ur] & & H_*(B(S^{-1}S)^\circ,\mathbb{Z}) \arrow[rr] & & H_*(B(S^{-1}S)^\circ,\mathbb{Z}) \times \mathbb{Z} \\
 & & x \arrow[rr, mapsto] & & (x,0)
\end{tikzcd}$$

We want to describe the image of $H_*(B(S^{-1}S)^\circ,\mathbb{Z})$ via
$H_*(B(S^{-1}S)^\circ,\mathbb{Z}) \longrightarrow H_*(BS^{-1}S,\mathbb{Z}) \cong H_*(BS,\mathbb{Z})[[R]^{-1}]$.

<u>Claim</u>: $H_*(B(S^{-1}S)^\circ,\mathbb{Z}) \cong \sum_n H_*(BGL_n(R),\mathbb{Z})\cdot [R]^{-n} \subseteq H_*(BS,\mathbb{Z})[[R]^{-1}]$

- We first observe that the map $H_*(BGL_n(R),\mathbb{Z}) \longrightarrow H_*(B(S^{-1}S)^\circ,\mathbb{Z}) \subseteq H_*(BS,\mathbb{Z})[[R]^{-1}]$ is given by $x \longmapsto x\cdot[R]^{-n}$.

  Indeed, the map $H_*(T_R,\mathbb{Z}) \to H_*(B(S^{-1}S),\mathbb{Z})$ is induced by the composition:
  $$\begin{aligned} GL_n(R) &\longrightarrow S \longrightarrow S^{-1}S \\ * &\longmapsto R^n \longmapsto (R^n,0) \end{aligned}$$
  On the other side, we have $\eta_n : GL_n(R) \longrightarrow S^{-1}S$, $* \longmapsto (R^n,R^n)$.
  There is a natural transformation between these two functors, given by:
  $(R^n,0) \xrightarrow{\oplus(0,R^n)} (R^n,R^n)$ (translation by $(0,R^n)$).
  On cohomology, this becomes multiplication by $[R]^n$. The presence of a natural transformation gives a commutative diagram
  $$\begin{tikzcd}
  H_*(BGL_n(R),\mathbb{Z}) \arrow[r] \arrow[d] & H_*(B(S^{-1}S),\mathbb{Z}) \cong H_*(BS,\mathbb{Z})[[R]^{-1}] \\
  H_*(B(S^{-1}S)^\circ,\mathbb{Z}) \arrow[r] & H_*(B(S^{-1}S),\mathbb{Z}) \arrow[u, "\cdot[R]^n"']
  \end{tikzcd}$$
  By reversing the arrows, we get $H_*(BGL_n(R),\mathbb{Z}) \longrightarrow H_*(B(S^{-1}S)^\circ,\mathbb{Z}) \subseteq H_*(BS,\mathbb{Z})[[R]^{-1}]$
  $$x \longmapsto x\cdot[R]^{-n}$$

- From the previous step it follows that $\sum_n H_*(BGL_n(R),\mathbb{Z})\cdot[R]^{-n} \subseteq H_*(B(S^{-1}S)^\circ,\mathbb{Z})$.
  Since $H_*(B(S^{-1}S)^\circ,\mathbb{Z}) \subseteq H_*(BS,\mathbb{Z})[[R]^{-1}]$, we may write every $y \in H_*(B(S^{-1}S)^\circ,\mathbb{Z})$ as $y = x\cdot[R]^{-n}$ for some $x \in H_*(BS,\mathbb{Z})$. But $B(S^{-1}S)^\circ$ is the connected component of the identity, and $H_*(BS,\mathbb{Z}) = \prod_n H_*(BGL_n(R),\mathbb{Z})$.
  Each factor corresponds to a different component, so for $x\cdot[R]^{-n}$ to lie in

the one of the identity we need to have $x \in H_*(BGL_n(R), \mathbb{Z})$.

•) $\sum_n H_*(BGL_n(R))[R]^{-n}$ is isomorphic to $\varinjlim_n H_*(BGL_n(R), \mathbb{Z})$ with transition maps given by multiplication by $[R]$. But these maps can be read already at a topological level by the transition maps of $BGL(R)$. Thus,

$$H_*(B(S^{-1}S)^0, \mathbb{Z}) \cong \sum_n H_*(BGL_n(R))[R]^{-n} \cong \varinjlim_n H_*(BGL_n(R), \mathbb{Z}) \cong H_*(BGL(R), \mathbb{Z})$$

□

Thus, we have seen that $BS^{-1}S \simeq \mathbb{Z} \times BGL(R)^+$ where $S = \mathcal{F}(R)$.

To recover the whole K-theory of R, we pass from $\mathcal{F}(R)$ to $\mathrm{iso}\,\mathcal{P}(R)$.

Def: Let S, T be symmetric monoidal categories, $f: S \to T$ a functor. We say that f is "cofinal" if for all $t \in T$ there is $t' \in T$ such that $t \otimes t'$ lies in the essential image of f.

Example: The natural inclusion $\mathcal{F}(R) \hookrightarrow \mathrm{iso}\,\mathcal{P}(R)$ is cofinal: for every f.g. proj. R-module P there is a f.g. proj. R-module P' such that $P \oplus P'$ is a free R-module.

Lemma: Let $f: S \to T$ be a functor of symmetric monoidal categories. Suppose that for all $s \in S$ $\mathrm{Aut}_S(s) \cong \mathrm{Aut}_T(f(s))$. Then the basepoint components of $K(S)$ and $K(T)$ are homotopy equivalent. In particular, $K_n(S) \cong K_n(T)$ for all $n \geq 1$.

Proof: Apparently:

$$H_*(B(S^{-1}S)^0, \mathbb{Z}) \cong \underbrace{\varinjlim_{s \in S}}_{\text{colimit over the translation category of } \pi_0(S)} H_*(B\,\mathrm{Aut}(s), \mathbb{Z}) = \lim_{s \in S} H_*(B\,\mathrm{Aut}_T(f(s)), \mathbb{Z}) \cong \lim_{t \in T} H_*(B\,\mathrm{Aut}_T(t), \mathbb{Z}) \cong H_*(BT^{-1}T, \mathbb{Z})$$

Then $B(S^{-1}S)^0$ and $B(T^{-1}T)^0$ are two connected H-spaces with the same homology, hence they are homotopy equivalent.

□

Corollary: Let $S = \mathrm{iso}\,\mathcal{P}(R)$. Then $B(S^{-1}S) \simeq K_0(R) \times BGL(R)^+$.

Proof: We know that $B(\mathcal{F}(R)^{-1}\mathcal{F}(R)) \simeq \mathbb{Z} \times BGL(R)^+$, and $\pi_0(S) = K_0(R)$. Since $\mathcal{F}(R)$ is cofinal in $\mathrm{iso}\,\mathcal{P}(R)$ and the functor $\mathcal{F}(R) \hookrightarrow \mathrm{iso}\,\mathcal{P}(R)$ is full, we conclude from the previous Lemma.

□

<u>Addendum</u>: How to see that $K_3(R) = H_3(St(R), \mathbb{Z})$.

Let G be a group, P its maximal perfect normal subgroup. Let $1 \to A \to S \to P \to 1$ be the universal central extension of P.

First, we observe that there is a homotopy fibration $BA \to BS^+ \to BP^+$. Indeed:

1) There is a homotopy fibration $BA \to BS \to BP$. This can be checked easily by looking at homotopy groups, using that for any group K $\pi_1(BK) = K$ and $\pi_n(BK) = 0$ for $n \geq 2$.

2) Functoriality of the + - construction gives maps

$$\begin{array}{ccccc} BA & \to & BS & \to & BP \\ & & \downarrow & & \downarrow \\ F & \longrightarrow & BS^+ & \longrightarrow & BP^+ \end{array}$$

where F is the fiber of $BS^+ \to BP^+$

3) $h\mathcal{T}op$ is triangulated and $BA \to BS \to BP$, $F \to BS^+ \to BP^+$ are exact triangles, hence we can complete the above diagram to

$$\begin{array}{ccccc} BA & \to & BS & \to & BP \\ \downarrow & & \downarrow & & \downarrow \\ F & \longrightarrow & BS^+ & \longrightarrow & BP^+ \end{array}$$

4) We prove that $F \cong BA^+$.

*) Since the two rows in the diagram are exact triangles, they yield long exact sequences in cohomology. The vertical maps $BS \to BS^+$, $BP \to BP^+$ yield isomorphisms in cohomology, hence also the map $BA \to F$ must induce isos in cohomology (by the five-lemma)

*) The + construction of A is relative to its maximal perfect subgroup, which is 0 because A is abelian. Hence, we need to show that $\pi_1(F) \cong A$. We observe that we have a long exact sequence

$$\pi_2(BS^+) \longrightarrow \pi_2(BP^+) \to \pi_1(F) \to \pi_1(BS^+) \to \pi_1(BP^+) \to 1$$

Since S is perfect, its + construction is relative to S itself, so $\pi_1(BS^+) = S/S = 1$.
By Hurewicz's theorem, we get $\pi_2(BS^+) = H_2(BS^+, \mathbb{Z})$, which coincides with $H_2(BS, \mathbb{Z})$ because $BS \to BS^+$ is acyclic. Moreover $H_2(BS, \mathbb{Z}) = H_2(S, \mathbb{Z}) = 0$ by the properties of universal central extensions. As a result, $\pi_2(BS^+) = 1$.
By the same argument we get $\pi_2(BP^+) = H_2(BP^+, \mathbb{Z}) = H_2(BP, \mathbb{Z}) = H_2(P, \mathbb{Z}) = A$.
Thus, $\pi_1(F) \cong A$, as desired.

But $\pi_1(BA) \cong A$, hence $BA^+ = BA$. This proves that $BA \to BS^+ \to BP^+$ is a homotopy fibration. Since $\pi_n(BA) = 0$ for $n \geq 2$, it follows that $\pi_n(BS^+) = \pi_n(BP^+)$ for all $n \geq 3$.

Now we observe that $\pi_n(BG^+) \cong \pi_n(BP^+)$ for $n \geq 3$. We actually show more:
BP^+ is homotopy equivalent to the universal covering space of BG^+.

Since P is perfect, we have $\pi_1(BP^+) = 0$, so BP^+ is simply connected. We only need to show that BP^+ is a covering space of BG^+.

We first show that $BP \to BG$ is a covering space of BG, up to homotopy.

<u>Claim:</u> The homotopy fiber of $BP \to BG$ is the discrete set G/P.

Since G/P is discrete, it is not difficult to see that this implies that $BP \to BG$ is a covering space (with each fiber isomorphic to G/P). To describe the homotopy fiber of $BP \to BG$ we use Quillen's Theorem B. Consider the inclusion functor $i : P \hookrightarrow G$, seen as one-object categories. Then the comma category $i/*$ has objects maps $* \to *$ in G, thus one for each element of G, and morphisms given by diagrams of the form

$$\begin{tikzcd} * \arrow[rr, "p"] \arrow[dr, "g"'] & \arrow[d, phantom, "\circlearrowleft"] & * \arrow[dl, "g'"] \\ & * & \end{tikzcd}$$

, that is, $\mathrm{Hom}_{i/*}(g, g') = \begin{cases} \emptyset & \text{if } g \not\equiv g' \pmod P \\ * & \text{if } g \equiv g' \pmod P \end{cases}$

(given $g, g' \in G$, i.e. objects of $i/*$, there is at most one $p \in P$ such that $g = pg'$).

Thus, $i/*$ is the category with objects G and morphisms only between objects in the same residue class modulo P, which compose in the unique possible way. It is clear that $B(i/*)$ is the discrete space G/P. Moreover, by Quillen's theorem B, it follows that this is precisely the homotopy fiber of $BP \to BG$.

Thus, $BP \to BG$ is a covering space. We do the same trick as before:

$$\begin{tikzcd} G/P \arrow[r] \arrow[d] & BP \arrow[r] \arrow[d] & BG \arrow[d] \\ F \arrow[r] & BP^+ \arrow[r] & BG^+ \end{tikzcd}$$

The map $G/P \to F$ preserves homology, hence F has trivial homology in degree ≥ 1. This means that F is discrete, so $BP^+ \to BG^+$ is also a covering space.

But $\pi_1(BP^+) = 0$, so $BP^+ \to BG^+$ is the universal covering space.

In particular, $\pi_n(BP^+) = \pi_n(BG^+)$ for all $n \geq 2$. $\square$