Babyseminar W.S. 2025/26 - Talk 0

INTRODUCTION

1. What is K-theory

A family of abelian groups, indexed over $\mathbb{Z}$, associated to some mathematical object

$$\mathcal{E} \rightsquigarrow \{K_n(\mathcal{E})\}_{n\in\mathbb{Z}} \quad + \text{ some functoriality}$$

in fact, we will just see $n \in \mathbb{N}$. See later discussion *

main goal of the seminar: study their def and some properties

- ring with unit R
- scheme X
- category (symmetric monoidal, exact, Waldhausen, ∞)
- topological space

we just talk about it today, for: historical reasons, motivation for development of the subject, introduce some homotopical tools

- It is difficult to compute, but for finite fields is known (Talk 10)
- Has connections with Geometric Topology, Algebraic Geometry, Number Theory (Talk 11)

2. History

I) Grothendieck, late '50s: (Talk 1)

$$K(X) := \frac{\langle [E] \text{ isom. class of } \begin{smallmatrix} E \\ \downarrow \\ X \end{smallmatrix} \text{ algebraic vector bundle of finite rk} \rangle_{\mathbb{Z}}}{\langle [E]-[E']-[E''] \text{ for } 0\to E'\to E\to E''\to 0 \text{ SES} \rangle}$$

algebraic vector bundle of finite rk $=$ finite locally free $\mathcal{O}_X$-modules

$\langle \ \rangle_{\mathbb{Z}}$: free abelian group generated

$\langle \ \rangle$: subgroup generated

on also rings:

$[E][E'] = [E\otimes E']$

$[P][P'] = [P\otimes_R P']$

$$K_0(R) := \frac{\langle [P] \text{ isom. class of } P \text{ f.g. proj } R\text{-mod} \rangle_{\mathbb{Z}}}{\langle [P_1\oplus P_2]-[P_1]-[P_2] \rangle}$$

R commutative: $K_0(R) \cong K_0(\operatorname{Spec} R)$

because:

- alg. v.b. / Spec R $\rightsquigarrow$ f.g. proj R-mod
- any SES of f.g. proj R-mod splits, i.e.

$0\to P'\to P\to P''\to 0$ SES $\Rightarrow$ $P \cong P'\oplus P''$

(this is equivalent to the condition P'' proj.)

Relation to Intersection Theory: X smooth alg. variety $/k$ field

there exists a ring hom

$$ch: K_0(X) \longrightarrow CH^*(X)_{\mathbb{Q}} = \bigoplus_{i=0}^{\dim X} CH^i(X)\otimes_{\mathbb{Z}}\mathbb{Q}$$

Chern character Chow group $CH^i(X) = Z^i(X)/\sim_{rat}$

↑ algebraic cycles $Z^i(X) = \bigoplus_{Z \subset X \text{ integral closed}} \mathbb{Z}$

s.t. $ch_{\mathbb{Q}}: K_0(X)_{\mathbb{Q}} \xrightarrow{\cong} CH^*(X)_{\mathbb{Q}}$ is an iso. (Tsek 2-3)

II) Atiyah, Bott, Hirzebruch ... '60s: do the analogous replacing

X scheme $\rightsquigarrow$ X paracompact Hausdorff top sp.

$\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ algebraic v.b $\longrightarrow$ $\begin{matrix} E \\ \downarrow \\ X \end{matrix}$ real or complex topological v.b.

it's a sufficient condition for: any SES of real or complex topological v.b. splits.

$\rightsquigarrow$ topological K-theory:

$$KO(X) := \frac{\left\langle [E] \text{ isom. class of } \begin{matrix} E \\ \downarrow \\ X \end{matrix} \text{ real (complex) topological v.b. of finite rk} \right\rangle_{\mathbb{Z}}}{\langle [E_1 \oplus E_2] - [E_1] - [E_2] \rangle}$$

(KU)

In this topological context we have the following nice facts:

(consider the real case, but analogous result hold in the complex case)

↳ just denote it $K(X)$

- There exists a bijection:

$$VB^n_{\mathbb{R}}(X)^{iso} \longleftrightarrow [X, BO(n)]$$

$VB^n_{\mathbb{R}}(X)^{iso}$ = isomorphism classes of real vector bundles $/X$ of rank n

$[X, BO(n)]$ = homotopy classes of cont. maps $X \longrightarrow BO(n) := Grass^{\infty}_n = \varinjlim_{q\geq 0} Grass^q_n$

$Grass^q_n$ = {subspaces of $\mathbb{R}^q$ of dim n}

homotopy theory comes into play!

$$\begin{array}{ccc} f^*E_n & \longrightarrow & E_n \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{f} & BO(n) \end{array}$$

E_n universal v.b. of rank n

f^*E_n doesn't depend on the homotopy type of f!

- (X, x_0) pointed top. sp $\quad i : \{x_0\} \hookrightarrow X$

$\rightsquigarrow$ $K(X) \xrightarrow{i^*} K(x_0) \overset{\mathrm{rk}}{\cong} \mathbb{Z}$ $\qquad \tilde{K}(X) := \ker(i^*)$

rank of the real v.b. over the point x_0

$[E] \longmapsto [i^*E]$

$[\varepsilon^n] \longleftarrow n \geq 0$

↳ trivial v.b. of rank n

$\rightsquigarrow$ $K(X) \cong \tilde{K}(X) \oplus \mathbb{Z}$

<u>Note</u> : $X \rightsquigarrow (X_+ = X \amalg *, *)$ pointed top. sp.

$\rightsquigarrow$ $K(X) \cong \tilde{K}(X_+)$

- We have $K(X) \xrightarrow{\mathrm{rk}} [X, \mathbb{Z}]$ $\qquad \hat{K}(X) := \ker(\mathrm{rk})$

$\rightsquigarrow$ $K(X) \cong \hat{K}(X) \oplus [X, \mathbb{Z}]$

<u>Note</u> : If X is connected, for any choice of $x_0 \in X$, $\mathrm{rk} = i^*$

so $\hat{K}(X) = \tilde{K}(X)$.

We can express $\hat{K}(X)$ in terms of homotopy:

$VB^n_{\mathbb{R}}(X)^{iso} \hookrightarrow VB^{n+1}_{\mathbb{R}}(X)^{iso} \quad \rightsquigarrow \quad \varinjlim_{n \geq 0} VB^n_{\mathbb{R}}(X)^{iso}$

$[E] \longmapsto [E \oplus \varepsilon^1]$

<u>Fact</u> : $VB^n_{\mathbb{R}}(X)^{iso} \longrightarrow \hat{K}(X)$ induces $\varinjlim_{n \geq 0} VB^n_{\mathbb{R}}(X)^{iso} \longrightarrow \hat{K}(X)$

$[E] \longmapsto [E] - [\varepsilon^n]$

If X is compact, it is an iso.

$$\rightsquigarrow \hat{K}(X) \cong \varinjlim_{n \geq 0} VB^n_{\mathbb{R}}(X) \cong \varinjlim_{n \geq 0} [X, BO(n)] \cong [X, \underbrace{\varinjlim_{n \geq 0} BO(n)}_{=: BO}]$$

use X compact

$\leadsto K(X) \cong \tilde{K}(X) \oplus [X, \mathbb{Z}] \cong [X, BO] \oplus [X, \mathbb{Z}] \cong [X, BO \times \mathbb{Z}]$.

↪ + the machinery of homotopy theory!

▶ This justifies the following definitions:

For any (X, x_0) paracompact Hausdorff pointed top. sp, define

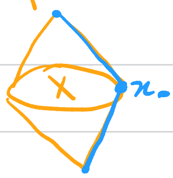
$$\tilde{K}^0(X) := [X, BO \times \mathbb{Z}]_* \cong [X, BO]$$

is $\tilde{K}(X)$ for X connected, compact

|| homotopy classes of pointed maps

$$\tilde{K}^{-n}(X) := \tilde{K}^0(\Sigma^n X) \qquad n \geq 0$$

Σ = suspension :



→ for ex. CW-complex

homotopy cat of good pointed top sp. — Obj: pointed top sp; Mor: homotopy classes of pointed maps

||

$\Sigma : HTop_* \longrightarrow HTop_*$

has a right adjoint

$HTop_* \longleftarrow HTop_* : \Omega$ loop space

Indeed $\underbrace{\Sigma = S^1 \wedge \_}_{\text{smash product}} \leadsto \underbrace{\Omega = Top_*(S^1, -)}_{\text{mapping space}}$

will be useful at some point in the seminar... ← Note: Homotopy groups of (X, x_0):

$$\pi_n(X) := [S^n, X]_* = [S^1 \wedge S^{n-1}, X]_* \cong [S^{n-1}, \Omega X]_* = \pi_{n-1}(\Omega X)$$

→ not pointed!

For $A \subset X$ closed $\leadsto X/A := \begin{cases} X_+ & A = \emptyset \\ C_i & A \neq \emptyset \end{cases}$ is a pointed top. sp.

$$\leadsto K^n(X, A) := \tilde{K}^n(X/A) \qquad n \geq 0$$

Notation : $K^n(X) := K^n(X, \emptyset) \cong \tilde{K}^n(X_+)$

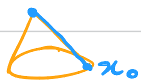
The machinery of homotopy theory allows to get the following:

$A_+ \xrightarrow{i_+} X_+ \longrightarrow C_i$ is an homotopy cofiber

Notice $C_i \simeq C_{i_+}$

⌈ in general, given $f : (X, x_0) \to (Y, y_0)$ pointed map, an homotopy fiber is

$X \xrightarrow{f} Y \longrightarrow C_f$ $\qquad C_f := CX \cup_f Y =$

$CX =$  cone

It is s.t. for any (Z, z_0)

$$\dots \to [\Sigma^2 X, Z_*] \to [\Sigma C_f, Z]_* \to [\Sigma Y, Z]_* \to [\Sigma X, Z]_* \to [C_f, Z]_* \to [Y, Z]_* \to [X, Z]_*$$

is long exact sequence of pointed sets (of abelian groups, if they are) ⌋

Taking $[-, BO\times\mathbb{Z}]_*$, we get a long exact sequence of abelian groups:

$$\begin{array}{ccccccccccc} \cdots\to \tilde{K}^{-1}(C_i) & \to & \tilde{K}^{-1}(X_+) & \to & \tilde{K}^{-1}(A_+) & \to & \tilde{K}^0(C_i) & \to & \tilde{K}^0(X_+) & \to & \tilde{K}^0(A_+) \\ \| & & \| & & \| & & \| & & \| & & \| \\ K^{-1}(X,A) & & K^{-1}(X) & & K^{-1}(A) & & K^0(X,A) & & K^0(X) & & K^0(A) \end{array}$$

Moreover, $K(X,A)$ defines a functor $K: \mathrm{Pair} \to \mathrm{Ab}$ satisfying all the properties of a generalized cohomology theory. → this has a precise def... just say that the main feature is to have the above long exact sequences

- (real) Bott periodicity theorem:
  | There is a pointed homotopy equivalence $\Omega^8 BO \simeq BO\times\mathbb{Z}$

$\rightsquigarrow$ $\tilde{K}^n(X) = [X, BO\times\mathbb{Z}]_* \cong [X, \Omega^8 BO]_* \cong [\Sigma^8 X, BO]_* \cong$

since $\Sigma^8 X$ is connected and $BO\times\mathbb{Z} = \coprod_{\mathbb{Z}} BO$, pointed maps $\Sigma^8 X \to \coprod_{\mathbb{Z}} BO$ have image in the component $BO^{(0)}$, so it is the same of a pointed map $\Sigma^8 X \to BO$.

$\cong [\Sigma^8 X, BO\times\mathbb{Z}]_* \cong \tilde{K}^{n+8}(X)$

This allow to extend the definition of $\tilde{K}^n(X)$, $K^n(X)$ for all $n\in\mathbb{Z}$, s.t. they are periodic of period 8.

For the complex case everything is analogous. The complex Bott periodicity states that there exists a pointed homotopy equivalence $\Omega^2 BU \simeq BU\times\mathbb{Z}$.

$\rightsquigarrow$ Q: Can be defined higher K-groups, for $n\in\mathbb{Z}$, also for schemes and rings, which fit in long exact sequences?

• Bass : $K_1(R) := GL(R)^{ab} \cong GL(R)/E(R)$ $\qquad GL(R) = \varinjlim_n GL_n(R)$

(Talk 4)

are definitions just for rings!

Milnor : $K_2(R) := \ker(St(R) \to E(R))$

For $I \subset R$ ideal, where also defined the relative groups $K_i(R,I)$ $i=0,1,2$, and proved that they fit in the exact sequence

$$K_2(R,I) \to K_2(R) \to K_2(R/I) \to K_1(R,I) \to K_1(R) \to K_1(R/I) \to K_0(R,I) \to K_0(R) \to K_0(R/I)$$

"localization sequence"

doesn't end with 0 in general!

Bass also defined negative K-theory of rings $K_n(R)$ and $K_n(R,I)$ for $n<0$ which allow to continue the exact sequence on the right.
For R regular noetherian, they all vanish and the exact sequence end with $\to 0$ on the right.

* I decided to not put this in the program because the general definition of negative K-groups (which we use to define for example negative K-groups of a scheme) involve more complicated machinery (topological spectra of topological spaces and their homotopy groups π_n, $n \in \mathbb{Z}$)
For X regular noetherian schemes they vanish.
Anyway, be aware that they exist!

- Quillen : definition of all higher algebraic K-groups
  general idea : in order to obtain localization sequence, exploit topological machinery of homotopy fiber to produce long exact sequences :

dual of homotopy cofiber explained above!

homotopy fiber :

$f:(X,x_0)\to(Y,y_0)$ cont map $\rightsquigarrow$ $F_f=\{(x,\gamma)\in X\times Y^{[0,1]} \mid \gamma(1)=f(x)\}$

$\rightsquigarrow$ $F_f\to X\to Y$ is s.t. $\forall (Z,z_0)$

$$\dots\to[Z,\Omega^2 X]_*\to[Z,\Omega F_f]_*\to[Z,\Omega Y]_*\to[Z,\Omega X]_*\to[Z,F_f]_*\to[Z,X]_*\to[Z,Y]_*$$

is a long exact sequence of pointed sets.

Taking $(Z,z_0)=(S^0,\circ)$, we get (using the adjunction $\Sigma\dashv\Omega$)

$$\dots\to\pi_2(X)\to\pi_1(F_f)\to\pi_1(Y)\to\pi_1(X)\to\pi_0(F_f)\to\pi_0(Y)\to\pi_0(X)$$

The idea to define higher K-groups of R is to construct a topological space

$K(R)$ K-space of R

and define

$$K_n(R):=\pi_n(K(R))$$

* Topological spectra are a gadget for which negative homotopy groups are defined and allow to extend the long exact sequence on the right.

1st outcome : +-construction (Talk 6)

R ring $\rightsquigarrow$ $BGL(R)^+$ $\rightsquigarrow$ $K_n(R):=\pi_n(BGL(R)^+\times K_0(R))$

$K(R):=BGL(R)^+\times K_0(R)$ $n\geq 0$

It is an ad-hoc construction that allows to recover $K_0(R)$ $K_1(R)$ $K_2(R)$

For more details wait the presentation of the corresponding talk

But for schemes? And more general constructions?
Already in the def of K_0, we can see that $K_0(X)$ is an example of $K_0(\mathcal{A})$ for $\mathcal{A}$ an exact category ($\mathcal{A} = VB(X)$ cat. of algebraic vector bundles $/X$)

intuitively, a category with a notion of exact sequences.

2^{nd} outcome: Q-construction (Talk 7)

$$\mathcal{A} \text{ exact category} \rightsquigarrow \underset{\underset{BQ\mathcal{A}}{\shortparallel}}{K(\mathcal{A})} \rightsquigarrow K_n(\mathcal{A}) := \pi_n(K(\mathcal{A})) \quad n \geq 0$$

this "B" is the classifying space construction of small cat. (Talk 5)

For $\mathcal{A} = P(R)$ cat. of f.g. proj. R-mod, we get back the higher K-groups defined with the +-construction; this is the "+ = Q"-Theorem.
It is a corollary of a more general theorem that compares the Q-construction to the $S^{-1}S$-construction (Talk 6)

- Waldhausen:

Already for the definition of K_0, one can see that should exist a notion of "K-groups of a derived category".

3^{rd} outcome: S-construction

$$\mathcal{C} \text{ Waldhausen cat.} \rightsquigarrow \underset{\underset{BwS\mathcal{C}}{\shortparallel}}{K(\mathcal{C})} \rightsquigarrow K_n(\mathcal{C}) := \pi_n(K_n(\mathcal{C})) \quad n \geq 0$$

intuitively, a category with w.e. and cofibration.
Typical example: $Ch(\mathcal{A})$ category of chain complexes of an abelian category $\mathcal{A}$.

Also this can be proved to coincide with K-groups obtained with the Q-construction for exact categories (any exact category has a natural structure of Waldhausen category)

This construction was used by Thomason to prove the localization sequence for K-groups of a scheme:

X qcqs, $U \overset{\circ}{\subset} X$ qc, $Z := X \setminus U$

there is a long exact sequence

$$\dots \to K_n(X,Z) \to K_n(X) \to K_n(U) \to K_{n-1}(X,Z) \to \dots$$

$$\to K_0(X,Z) \to K_0(X) \to K_0(U)$$ → doesn't end with 0 in general!

* Thomason also defined a topological spectrum from Waldhausen construction (from Quillen construction they weren't able to define it!) and extended the above localization sequence also for negative degrees.

This gives a motivation for Waldhausen construction, even if we won't see that in the seminar.

However I think that Waldhausen construction will be useful to understand the modern point of view on K-theory with the language of ∞-categories. (Talk 12)