"The Grothendieck gp of rings and Schemes"

2 approaches to define the Grothendieck gp of some objects $\mathcal{O}$

It's the free abelian gp gen. by elements of $\mathcal{O}$
modulo relations gen. by

a monoidal operation on $\mathcal{O}$
(Need gp. completion of a monoid)

a certain notion of short exact sequences of $\mathcal{O}$
(exact + symmetric monoidal categories)

Def: The gp completion of an abelian monoid M is an abelian gp $M^{-1}M$, together with a universal map of monoids $[\ ]: M \to M^{-1}M$,
i.e. $\forall A$ ab. gp, $\forall$ monoid map $M \xrightarrow{\alpha} A$
$\exists!$ hom of ab. gps $M^{-1}M \xrightarrow{\tilde{\alpha}} A$
s.t

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "{[\ ]}"'] \arrow[dr, phantom, "\circlearrowleft"] & A \\ M^{-1}M \arrow[ur, dotted, "\exists! \tilde{\alpha}"'] & \end{tikzcd}$$

e.g: gp completion of $\mathbb{N}$ is $\mathbb{Z}$.

$$\begin{tikzcd} \mathbb{N} \arrow[r, "\alpha"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & A \\ \mathbb{Z} \arrow[ur, dotted, "\exists! \tilde{\alpha}"'] & \end{tikzcd}$$

where $\tilde{\alpha}(m-n) := \alpha(m) - \alpha(n)$ well defined

Q: Does gp completion always exist?

Yes! Every abelian monoid has a gp completion.

Recipe: M abelian monoid.

$F(M)$: free abelian gp on symbols $[m]$, $m \in M$

$R(M)$: subgp gen. by relations $[m+n] - [m] - [n]$.

Claim. $F(M) / R(M) = M^{-1}M$.

If $M \to N$ is a monoid map

Then

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N \arrow[r] & N^{-1}N \\ M^{-1}M \arrow[urr, dotted, "\exists!"'] & & \end{tikzcd}$$

extends uniquely by universality

$\rightsquigarrow$ gp completion = functor from abelian monoids to abelian gps.

It is the left adjoint to the forgetful functor as: $\mathrm{Hom}_{ab.mon.}(M, A) \simeq \mathrm{Hom}_{ab.gps}(M^{-1}M, A)$

Prop: M ab. monoid. Then:

(1) Every element of $M^{-1}M$ is of the form $[m]-[n]$ for some $m, n \in M$.

(2) $[m] = [n] \iff m+p = n+p$ for some $p \in M$

(3) $M^{-1}M$ is the set theoretic qt of $M \times M$ by the eq. relation $(m,n) \sim (m+p, n+p)$

Pf: (1) Every elt of any free ab. gp can be written as a difference of sums of generators. and mod $R(M)$ we have $[m_1]+[m_2]\dots$

$$\equiv [m_1+m_2+\dots]$$ ✓

(2) Suppose $[m]-[n] = 0$ in $M^{-1}M$

in $F(M)$: $[m]-[n] = \left(\sum_i [a_i+b_i]-[a_i]-[b_i]\right)$

$$- \left(\sum_j [c_j+d_j]-[c_j]-[d_j]\right)$$

$$\Longleftrightarrow [m]+\sum([a_i]+[b_i])+\sum[c_j+d_j] = [n]+\sum[a_i+b_i]+\sum([c_j]+[d_j])$$

In a free ab. gp two sums of gen. $\sum_{i\in I} x_i$, $\sum_{j\in J} y_j$ are equal iff $I=J$ and $y_i = x_{\sigma(i)}$ for some permutation σ.

$$\Longrightarrow m+\sum(a_i+b_i)+\sum(c_j+d_j) = n+\sum(a_i+b_i)+\sum(c_j+d_j)$$

∎

Example: G finite gp.

$\mathrm{Rep}_{\mathbb{C}}(G) = \{$ finite dim. rep. $\rho: G \to GL_n(\mathbb{C})$ up to iso $\}$

is an abelian monoid under $\oplus$

By Maschke's thm, $\underbrace{\mathbb{C}G}_{\text{gp algebra}}$ is semisimple and

$\mathrm{Rep}_{\mathbb{C}}(G) \simeq \mathbb{N}^r$, where r is the number of conjugacy classes of elements of G

$\rightsquigarrow$ (gp completion of $\mathrm{Rep}_{\mathbb{C}}(G)$) $\simeq \mathbb{Z}^r$ as ab. gps.

Rem: Let M be semiring (missing additive inverses)
Then $M^{-1}M$ (w/ respect to $+$) is a ring.

§ K_0 of a ring:

R a ring, $P(R)$ = iso classes of fin. gen. projective R-modules together w/ $\oplus$ and id 0
$P(R)$ is an abelian monoid.

The Grothendieck gp of R denoted $K_0(R)$ is the gp completion of $P(R)$.

If R is comm. $\Rightarrow$ $K_0(R)$ is a comm. ring with $1 = [R]$.

Because $P(R)$ is a com. semiring w/ product $\otimes_R$
(the axioms are clear)

e.g.: Let R be a field or a PID
$\Rightarrow$ Over R, every fin. gen. projective R-module is free $\Rightarrow$ $P(R) \cong \mathbb{N}$
$\Rightarrow$ $K_0(R) \cong \mathbb{Z}$

Q: Why do we restrict to fin. gen. proj. modules?

A: (Use Eilenberg-Swindle Trick) (fraudulent scheme :-))

Let R^∞ be an infinitely gen. free module.
if $P \oplus Q = R^n$.

$$\Rightarrow \quad P \oplus R^\infty \cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \dots$$
$$\cong (P \oplus Q) \oplus (P \oplus Q) \oplus \dots$$
$$\cong R^\infty$$

$\Rightarrow \quad [P] = 0$ for every fin. gen. proj. module.

$$\Rightarrow \quad K_0(R) = 0$$

#

Some useful reduction properties:

(1) If $R = R_1 \times R_2 \Rightarrow K_0(R) \simeq K_0(R_1) \times K_0(R_2)$

(2) If $R = \varinjlim_{i \in I} R_i \Rightarrow K_0(R) \simeq \varinjlim_{i \in I} K_0(R_i)$

(3) If $I \subset R$ is a nilpotent ideal
$$\Rightarrow K_0(R) \simeq K_0(R/I).$$

(4) If R, S are two Morita equivalent rings, i.e R-mod and S-mod are equivalent as abelian categories, then $K_0(R) \simeq K_0(S)$.

e.g.: $R = M_n(S)$ is Morita eq. to S.
One can check that

$$\begin{aligned} \mathrm{Mod}(S) &\to \mathrm{Mod}(R) \\ X &\mapsto S^{n \times 1} \otimes_S X \end{aligned}$$

$$S^{1 \times n} \otimes_R X \longleftarrow X$$

gives an eq. of categories.

so that $K_0(S) \simeq K_0(M_n(S))$

§ K_0 of a scheme

Revisiting $K_0(R)$:

One can define the K_0 of any exact category.
(It's an additive category that can be embedded in an abelian one, and carries a notion of exact sequences)

- Let $\mathcal{C}$ be exact

$K_0(\mathcal{C})$ = ab. gp. w/ generators $[C]$ and relations $[C] = [B] + [D]$ for every short exact sequence $0 \to B \to C \to D \to 0$ in $\mathcal{C}$.

In fact, $P(R)$ is exact (can be embedded in R-mod)

As every short exact sequence of proj. modules splits $\Rightarrow$ $K_0(P(R)) = K_0(R)$

Recall from your alg. Geometry course:

If $X = \operatorname{Spec} R$.

$\Rightarrow$ $VB(X)$ (finite loc. free $\mathcal{O}_X$-modules) $\underset{\text{equiv.}}{\simeq} P(R)$

Now let X be any scheme

$\Rightarrow$ $VB(X)$ is an exact category (as a sub cat. of the ab. category of $\mathcal{O}_X$-modules)

The tensor product of v.b. is a a biexact functor and $[\mathcal{E}][\mathcal{F}] = [\mathcal{E} \otimes \mathcal{F}]$

comm. + assoc.

$\Rightarrow$ $K_0(X)$ is a comm. ring

so that K_0 is a functor from schemes to comm. rings.

Note that exact sequences of v.b. do not always split.

$\Rightarrow$ $VB(X)$ is not always a split category.

• More abstract perspective:

S symmetric monoidal category (i.e. it comes w/ a functor $\square : S \times S \to S$ + a distinguished object e + 4 basic natural "coherent" isoms: $e \square s \simeq s$, $s \square e \simeq s$,

$$s \square (t \square u) \simeq (s \square t) \square u$$

$$s \square t \simeq t \square s.)$$

S^{iso} = iso classes of objects in S.

$\Rightarrow$ S^{iso} is an abelian monoid w/ prod $\square$ + identity e

S s.m.

$\Rightarrow$ gp comp. of S^{iso} is the Grothendieck gp of S and is denoted by $K_0^{\square} S$.

e.g.: $P(R)$ is sym. monoidal under $\oplus$

$\Rightarrow \quad K_0(R) = K_0^{\oplus} P(R)$

§ The G_0 group:

(Rings) $\quad R$ noetherian

$M(R) =$ f.g. R-modules

$\xRightarrow{R \text{ noet}}$ $M(R)$ is abelian

$$G_0(R) := K_0\, M(R).$$

$\exists$ a natural map called the Cartan hom.

$$K_0(R) \to G_0(R)$$

$$[P] \mapsto [P] \qquad \text{(forget projectivity)}$$

(Schemes) $\quad X$ noet. scheme.

$M(X)$ coherent $\mathcal{O}_X$-modules

$G_0(X) := K_0\, M(X)$

(If $X = \operatorname{Spec} R \Rightarrow$ this agrees w/ def above)

Thm: If X is a separated regular noet. scheme $\Rightarrow$ every coh. $\mathcal{O}_X$-module has a finite resolution by vector bundles (Serre)

$\Rightarrow \quad K_0(X) \simeq G_0(X)$

Pf: (Sketch) Use the so-called "Resolution thm"

Basically, any coherent $\mathcal{O}_X$-module $\mathcal{F}$ fits into an exact sequence

$$0 \to \mathcal{E}_n \to \dots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

s.t $\mathcal{E}_i$ is a v.b / X

$$G_0(X) \ni [\mathcal{F}] = \sum_{i=0}^{n} (-1)^i [\mathcal{E}_i] \in K_0(X)$$

so every element in $G_0(X)$ comes from $K_0(X)$

$\boxtimes$

Rmks: • Sometimes the G_0-gp has better properties than the K_0-group

• For example, G_0 has pushforward for proper maps

if $f: X \to Y$ is proper, $\exists$ a well-defined pushforward map

$$f_*([\mathcal{F}]) = \sum_{i=0}^{\infty} (-1)^i [\underbrace{R^i f_* \mathcal{F}}_{\text{higher direct images are coherent.}}]$$

However, the direct image of a vector bundle is not always a vector bundle

so we cannot define $f_*([\mathcal{E}]) = [f_* \mathcal{E}]$