The Emerton-Gee stack
the rank one case

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1 Introduction and main statement

This talk is devoted to the study of the moduli stack of rank one étale (φ,Γ)-modules, following the work of M. Emerton and T. Gee in [2] (chapter 7 in particular) and the preprint [3].

We will start recalling and introducing some notation.

- p denotes an odd positive prime integer, we fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, with completion $\mathbb{C}_p$.
- E denotes a finite extension of $\mathbb{Q}_p$ (lying inside $\overline{\mathbb{Q}}_p$), $\mathcal{O}$ denotes the ring of integers of E with uniformizer π and residue field $\mathbb{F} = \mathcal{O}/(\pi)$.
- K denotes another finite extension of $\mathbb{Q}_p$ (lying inside $\overline{\mathbb{Q}}_p$), $\mathcal{O}_K$ denotes the ring of integers of K and k the residue field. We will assume that $\mathbb{F}$ contains k in the sequel.
- $G_K := \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ denotes the absolute Galois group of K and I_K the inertia subgroup, yielding a short exact sequence

$$1 \to I_K \to G_K \to (\mathrm{Frob}_k)^{\hat{\mathbb{Z}}} \to 1, \tag{1}$$

  where Frob_k denotes the *geometric* Frobenius of k, i.e. the *inverse* of the Frobenius automorphism $\varphi_q : k \to k$ given by raising to the p^f-th power where $q = p^f = \#(k)$.
- $G_K^{\mathrm{ab}} = \mathrm{Gal}(K^{\mathrm{ab}}/K)$ denotes the Galois group of the maximal abelian extension of K and I_K^{ab} denotes the image of I_K inside G_K^{ab}. Local class field theory produces an isomorphism $I_K^{\mathrm{ab}} \cong \mathcal{O}^\times$ and the short exact sequence

$$1 \to I_K^{\mathrm{ab}} \to G_K^{\mathrm{ab}} \to (\mathrm{Frob}_k)^{\hat{\mathbb{Z}}} \to 1. \tag{2}$$

- K_{cyc} denotes the cyclotomic $\mathbb{Z}_p$-extension of K, $\Gamma := \Gamma_K := \mathrm{Gal}(K_{\mathrm{cyc}}/K) \cong \mathbb{Z}_p$ its Galois group and k_∞ its residue field (note that this is a finite extension of k).
- $\mathbf{A}_K$ denotes the discrete valuation ring attached to K in the previous talks. More precisely, we saw the explicit construction of $\mathbf{A}_{\mathbb{Q}_p}$ as $(\mathbf{A}'_{\mathbb{Q}_p})^\Delta$ with

$$\Delta = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_{p,\mathrm{cyc}}) \cong \mathbb{Z}/(p-1)\mathbb{Z}$$

  and $\mathbf{A}'_{\mathbb{Q}_p}$ defined as the image of the embedding

$$\widehat{\mathbb{Z}_p((T))} \hookrightarrow W((\widehat{\mathbb{Q}_p(\mu_{p^\infty})})^\flat)$$

uniquely determined by sending T to $[\varepsilon]-1$ where $\varepsilon \in (\widehat{\mathbb{Q}_p(\mu_{p^\infty})})^\flat$ is given by a compatible system of p-th power roots of 1 and $[\cdot]$ denotes the Teichmüller lift. The construction of $\mathbf{A}_K$ is for a general K is slightly more technical, but essentially analogous (cf. [2], section 2.1). Recall that $\mathbf{A}_K$ is endowed with an action of Γ and with a Frobenius endomorphism φ. The two actions commute.

- If A is a π-adically complete $\mathcal{O}$-algebra we set

$$\mathbf{A}_{K,A} \coloneqq \mathbf{A}_K \hat{\otimes}_{\mathcal{O}} A \coloneqq \varprojlim_m (\varprojlim_n (\mathbf{A}_K^+/(\pi^m, T_K^n) \otimes_{\mathbb{Z}_p} A)[1/T_K])$$

  where T_K is a lift of a uniformizer for the residue field of $\mathbf{A}_K$ and $\mathbf{A}_K^+ = W(k_\infty)[[T_K]]$.

- $\mathcal{X}_{K,d} = \mathcal{X}_d$ is the moduli stack of projective étale (φ,Γ)-modules of rank d defined over $\mathrm{Spf}(\mathcal{O})$.

  Recall that, for every π-adically complete $\mathcal{O}$-algebra A, $\mathcal{X}_d(\mathrm{Spf}(A))$ is defined as the groupoid of projective étale (φ,Γ)-modules with A-coefficients (recall that by definition such modules are projective $\mathbf{A}_{K,A}$-modules of constant rank d). This determines uniquely the stack $\mathcal{X}_d$, since for a stack over $\mathrm{Spf}(\mathcal{O})$ it is enough to specify the groupoid $\mathcal{X}_d(\mathrm{Spec}(A))$ for A any $\mathcal{O}$-algebra where π is nilpotent.

- X_1 is the sheaf on $Sch/\mathrm{Spec}(\mathcal{O})_{fppf}$, lying over $\mathrm{Spf}(\mathcal{O})$, which is uniquely determined by the fact that $X_1(\mathrm{Spf}(A))$ is given the continuous characters $W_K \to A^\times$, where A is any π-adically complete $\mathcal{O}$-algebra (where on A we consider the π-adic topology). It is actually an honest formal scheme, as we will check below.

- $\widehat{\mathbb{G}}_m$ will denote the π-adic completion of $\mathbb{G}_m = \mathrm{Spec}(\mathcal{O}[T,T^{-1}])$, i.e. more concretely $\widehat{\mathbb{G}}_m = \mathrm{Spf}(\mathcal{O}\langle T,T^{-1}\rangle)$ where $\mathcal{O}\langle T,T^{-1}\rangle$ it the ring of Laurent series with coefficients in $\mathcal{O}$ converging on $\{z \in \mathbb{C}_p \mid |z| = 1\}$ (and we take the formal completion with respect to the ideal (π)).

After having introduced the notation, we are ready to state the main result of today's talk.

Theorem 1 (Prop. 7.2.17 in [2], Corollary 3.2 in [3])**.** *There exists a morphism of stacks over* $\mathrm{Spf}(\mathcal{O})$

$$X_1 \longrightarrow \mathcal{X}_1 \tag{3}$$

inducing an isomorphism

$$[X_1 / \widehat{\mathbb{G}}_m] \xrightarrow{\sim} \mathcal{X}_1 \tag{4}$$

where the action of $\widehat{\mathbb{G}}_m$ on X_1 is taken to be trivial in the formation of the quotient stack.

2 Construction of the presentation morphism

In this section we describe the construction of the morphism in (3) and discuss its first properties.

It is convenient to recall here the key result relating Galois representations and (φ,Γ)-modules with coefficients in the *good* cases.

Theorem 2. *If A is a finite artinian $\mathcal{O}$-algebra (or $A = \overline{\mathbb{F}}_p$) and d is a positive integer, there is an equivalence of categories*

$$\left\{ \begin{array}{c} \textit{continuous } G_K\textit{-representations in} \\ \textit{finite free } A\textit{-modules of rank } d \end{array} \right\} \underset{T_A(-)}{\overset{D_A(-)}{\rightleftarrows}} \left\{ \begin{array}{c} \textit{projective étale } (\varphi,\Gamma)\textit{-modules} \\ \textit{of rank } d \textit{ with } A\textit{-coefficients} \end{array} \right\}$$

where

$$D_A(V) = (W(\mathbb{C}_p^\flat)_A \otimes_A V)^{G_{K_{\mathrm{cyc}}}} \qquad \textit{and} \qquad T_A(M) = (W(\mathbb{C}_p^\flat)_A \otimes_{\mathbf{A}_{K,A}} M)^{\varphi=1}.$$

As a first remark, note that fixing a choice of a lift $\sigma_K \in G_K$ of Frob_k (i.e. choosing compatible splittings of the exact sequences (1) and (2)) is equivalent to fixing an isomorphism of noetherian affine formal schemes over $\mathrm{Spf}(\mathcal{O})$

$$X_1 \xrightarrow{\cong} \mathrm{Spf}(\mathcal{O}[[I_K^{\mathrm{ab}}]]) \times \widehat{\mathbb{G}}_m$$

which at the level of $\mathrm{Spf}(A)$-points (for any π-adically complete $\mathcal{O}$-algebra A) is clearly described by the assignment

$$(\chi : W_K^{\mathrm{ab}} \to A^\times) \mapsto (\chi|_{I_K^{\mathrm{ab}}}, \chi(\sigma_K)).$$

Note that to define $\mathrm{Spf}(\mathcal{O}[[I_K^{\mathrm{ab}}]]$ we are taking the formal completion with respect to the usual topology of $\mathcal{O}[[I_K^{\mathrm{ab}}]]$ as completed group ring (which is not the π-adic topology!).

Given any discrete artinian quotient A of $\mathcal{O}[[I_K^{\mathrm{ab}}]]$, we can extend the given continuous surjection $\mathcal{O}[[I_K^{\mathrm{ab}}]] \twoheadrightarrow A$ to a continuous surjection $\mathcal{O}[[G_K^{\mathrm{ab}}]] \twoheadrightarrow A$ by sending $\sigma_K \mapsto 1$. This is equivalent to giving a Galois character $\chi_A : G_K \to A^\times$, which in turn gives rise to a rank one projective (and actually free) étale (φ,Γ)-module

$$M_A := ((W(\mathbb{C}_p^\flat)_A \otimes_A A(\chi_A))^{G_{K^{\mathrm{cyc}}}}$$

over $\mathbf{A}_{K,A}$ (cf. theorem 2 above), i.e. to a morphism $\mathrm{Spec}(A) \to \mathcal{X}_1$ (here clearly $A(\chi_A)$ denotes A viewed as G_K-module via χ_A).

Since $\mathrm{Spf}(\mathcal{O}[[I_K^{\mathrm{ab}}]])$ (as a sheaf over $Sch/\mathrm{Spec}(\mathcal{O})_{fppf}$) is the direct limit of $\mathrm{Spec}(A)$ for A running over the discrete artinian quotients of $\mathcal{O}[[I_K^{\mathrm{ab}}]]$, we automatically get a morphism $r : \mathrm{Spf}(\mathcal{O}[[I_K^{\mathrm{ab}}]]) \to \mathcal{X}_1$.

We also have a morphism $\widehat{\mathbb{G}}_m \to \mathcal{X}_1$ given by the universal unramified rank one étale (φ,Γ)-module $N_T \in \mathcal{X}_1(\widehat{\mathbb{G}}_m)$.

Such a module can be realized as

$$N_T := D_{K,T} \otimes_{W(k)\otimes_{\mathbb{Z}_p}\mathcal{O}\langle T,T^{-1}\rangle} \mathbf{A}_{K,\mathcal{O}\langle T,T^{-1}\rangle},$$

where $D_{K,T}$ is the étale φ-module free of rank one over $W(k) \otimes_{\mathbb{Z}_p} A$ afforded by the following lemma. We let φ act diagonally on N_T, while clearly the Γ-action takes place only on the second factor.

Note that the Frobenius φ on $W(k) \otimes_{\mathbb{Z}_p} A$ is given by A-linear extension of the unique ring automorphism of $W(k)$ such that $\varphi(x) \equiv x^p \mod p$.

Lemma 3. *Let A be an $\mathcal{O}$-algebra and let $a \in A^\times$. Then up to isomorphism there is a unique étale φ-module $(D_{K,a}, \varphi_a)$ of rank one over $W(k) \otimes_{\mathbb{Z}_p} A$ with the property that φ_a^f is given by the multiplication by $1 \otimes a$ on $D_{K,a}$.*

Proof. Let V be a rank one free $W(k)\otimes_{\mathbb{Z}_p} A$-module with basis $\{v\}$. Assume that the required φ_a exists. In this case if $\varphi_a(v) = \lambda \cdot v$ for some $\lambda \in (W(k)\otimes_{\mathbb{Z}_p} A)^\times$, then the φ-semilinearity immediately implies that

$$\varphi_a^f(e) = \lambda \cdot \varphi(\lambda) \dots \varphi^{f-1}(\lambda) \cdot v$$

so we are essentially reduced to proving that there exists $\lambda \in (W(k)\otimes_{\mathbb{Z}_p} A)^\times$ such that

$$\lambda \cdot \varphi(\lambda) \dots \varphi^{f-1}(\lambda) = 1 \otimes a.$$

Recalling that by assumption $k \subseteq \mathbb{F}$, it is clear that A is naturally a $W(k)$-algebra and that there is an isomorphism of A-algebras

$$W(k)\otimes_{\mathbb{Z}_p} A \cong \underbrace{A \times A \times \dots \times A}_{f \text{ times}} \qquad \text{via the map} \qquad x \otimes 1 \mapsto (x, \varphi(x), \dots, \varphi^{f-1}(x))$$

In particular the existence of such a λ follows immediately. One can then check that if λ' satisfies the same property of λ, then $\lambda/\lambda' = \varphi(t)/t$ for some $t \in (W(k)\otimes_{\mathbb{Z}_p} A)^\times$, which (using the φ-semilinearity) translates into the fact that setting $\varphi_a(v) = \lambda \cdot v$ or $\varphi_a(v) = \lambda' \cdot v$ affords isomorphic étale φ-modules with A-coefficients. □

Going back to the general picture, we get a morphism $\mathcal{X}_1 \times \widehat{\mathbb{G}}_m \to \mathcal{X}_1$ which, at the level of $\mathrm{Spf}(A)$-points for A any π-adically complete $\mathcal{O}$-algebra, consists in the assignment $(M, a) \mapsto M \otimes_{\mathbf{A}_{K,A}} N_a$, where $M \in \mathcal{X}_1(A)$, $a \in A^\times = \widehat{\mathbb{G}}_m(A)$ and N_a the (φ,Γ)-module corresponding to the composition

$$\mathrm{Spf} A \xrightarrow{a} \widehat{\mathbb{G}}_m \xrightarrow{N_T} \mathcal{X}_1.$$

More concretely $N_a = D_{K,a} \otimes_{W(k)\otimes_{\mathbb{Z}_p} A} \mathbf{A}_{K,A}$ with diagonal action of φ and Γ-action on the second factor.

We finally define the morphism in (3) as the composition

$$X_1 \cong \mathrm{Spf}(\mathcal{O}[[I_K^{\mathrm{ab}}]]) \times \widehat{\mathbb{G}}_m \longrightarrow \mathcal{X}_1 \times \widehat{\mathbb{G}}_m \longrightarrow \mathcal{X}_1, \tag{5}$$

where the rightmost isomorphism corresponds to the choice of σ_K as explained above, the first arrow is given by $r \times 1_{\widehat{\mathbb{G}}_m}$ and the second arrow is the one that we have described in the previous paragraph.

Note that the morphism $X_1 \to \mathcal{X}_1$ that we have defined does not depend anymore on the choice of σ_K.

The relation between unramified Galois characters and the (φ,Γ)-modules N_a is elucidated by the following lemma.

Lemma 4. *Let A be a finite artinian $\mathcal{O}$-algebra and $a \in A^\times$. Then, under the equivalence of theorem 2, the (φ,Γ)-module N_a defined as above corresponds to the unramified Galois character sending the lifts of* Frob_k *to a.*

Proof. Cf. lemma 2.4 in [3]. □

Notation 5. If A is a π-adically complete $\mathcal{O}$-algebra and $(\delta : W_K \to A^\times) \in X_1(\mathrm{Spf}(A))$, we denote by $\mathbf{A}_{K,A}(\delta)$ the rank one étale (φ,Γ)-module associated to δ by the morphism $X_1 \to \mathcal{X}_1$ constructed above in (5).

3 Sketch of the proof of the main theorem

In this section we sketch the proof of the fact that the morphism $X_1 \to \mathcal{X}_1$ induces an isomorphism $[X_1 / \widehat{\mathbb{G}}_m] \cong \mathcal{X}_1$ as in (4).

Since the prescribed action of $\widehat{\mathbb{G}}_m$ on X_1 is the trivial one, it follows rather easily that there is an identification

$$[X_1 / \widehat{\mathbb{G}}_m] \cong [\mathrm{Spf}(\mathcal{O}) / \widehat{\mathbb{G}}_m] \times_{\mathrm{Spf}(\mathcal{O})} X_1$$

where at the level of A-points for A an $\mathcal{O}/(\pi^a)$-algebra for some $a \geq 1$, the identification of the groupoids $[X_1 / \widehat{\mathbb{G}}_m](A)$ and $[\mathrm{Spf}(\mathcal{O}) / \widehat{\mathbb{G}}_m](A) \times_{\mathrm{Spf}(\mathcal{O})(A)} X_1(A)$ is given as follows. An element $[X_1 / \widehat{\mathbb{G}}_m](A)$ corresponds to a diagram

$$\mathrm{Spec}(A) \leftarrow P \xrightarrow{u} X_1 \times_{\mathrm{Spf}(\mathcal{O})} \mathrm{Spec}(\mathcal{O}/(\pi^a))$$

with $P \to \mathrm{Spec}(A)$ a principal $\mathbb{G}_{m,\mathcal{O}/(\pi^a)}$-bundle and u a $\mathbb{G}_{m,\mathcal{O}/(\pi^a)}$-equivariant morphism. Since the action of $\mathbb{G}_{m,\mathrm{Spec}(\mathcal{O}/(\pi^a))}$ on $X_1 \times_{\mathrm{Spf}(\mathcal{O})} \mathrm{Spec}(\mathcal{O}/(\pi^a))$ is trivial, one can actually prove (after trivializing $P \to \mathrm{Spec}(A)$ and applying *fppf* descent), that there must exist a morphism $\mathrm{Spec}(A) \to X_1 \times_{\mathrm{Spf}(\mathcal{O})} \mathrm{Spec}(\mathcal{O}/(\pi^a))$ such that the following diagram commutes

$$\begin{array}{ccc} P & \xrightarrow{\quad u \quad} & X_1 \times_{\mathrm{Spf}(\mathcal{O})} \mathrm{Spec}(\mathcal{O}/(\pi^a)) \\ \downarrow & \overset{x}{\nearrow} & \\ \mathrm{Spec}(A) & & \end{array}$$

Then one associates to

$$\mathrm{Spec}(A) \leftarrow P \xrightarrow{u} X_1 \times_{\mathrm{Spf}(\mathcal{O})} \mathrm{Spec}(\mathcal{O}/(\pi^a))$$

the element $(L, x) \in [\mathrm{Spf}(\mathcal{O}) / \widehat{\mathbb{G}}_m](A) \times_{\mathrm{Spf}(\mathcal{O})(A)} X_1(A)$, where L is the invertible A-module corresponding to $P \to \mathrm{Spec}(A)$ and x is obtained as above (and postcomposing with the projection to X_1).

For A a π-complete $\mathcal{O}$-algebra, we have shown that the groupoid $[X_1 / \widehat{\mathbb{G}}_m](\mathrm{Spf}(A))$ is equivalent to the groupoid of (isomorphism classes of) pairs (L, δ) with L an invertible A-module and δ a continuous character $W_K \to A^\times$.

It is then not hard to see that the morphism $X_1 \to \mathcal{X}_1$ defined in the previous section factors as a morphism

$$[X_1 / \widehat{\mathbb{G}}_m] \to \mathcal{X}_1. \tag{6}$$

Explicitly, for A a π-adically complete $\mathcal{O}$-algebra and after the above identification, at the level of $\mathrm{Spf}(A)$-points our morphism (6) is given by the association

$$(L, \delta) \mapsto \mathbf{A}_{K,A}(\delta) \otimes_A L.$$

Let now M be any rank one étale (φ, Γ)-module with A-coefficients (A again a π-complete $\mathcal{O}$-algebra). Then one can prove (cf. [2], lemma 2.2.19 and proposition 2.2.12 for the last equality below) that

$$\mathrm{Aut}_{\mathbf{A}_{K,A},\varphi,\Gamma}(M) = \left((M \otimes_{\mathbf{A}_{K,A}} M^\vee)^{\varphi=1,\Gamma}\right)^\times = \left((\mathbf{A}_{K,A})^{\varphi=1,\Gamma})\right)^\times = A^\times. \tag{7}$$

Then, in order to prove that the morphism (6) is an isomorphism, we are reduced to prove the following.

Proposition 6. *Let A be a π-adically complete $\mathcal{O}$-algebra and let $M \in \mathcal{X}_1(\mathrm{Spf}(A))$ a rank one étale (φ, Γ)-module with A-coefficients. Then there exists a unique continuous character $\delta : W_K \to A^\times$ and a unique (up to isomorphism) invertible A-modules L, such that $M \cong \mathbf{A}_{K,A}(\delta) \otimes_A L$ as (φ, Γ)-modules.*

Sketch of the proof. We first prove the uniqueness part (which corresponds to proving, together with the equalities in (7), that $[X_1 / \widehat{\mathbb{G}}_m] \to \mathcal{X}_1$ is a monomorphism). Since (as we have already recalled) $\mathbf{A}_{K,A}^{\varphi=1} = A$, one sees immediately that if $M \cong \mathbf{A}_{K,A}(\delta) \otimes_A L$, then necessarily

$$L \cong \left(M \otimes_{\mathbf{A}_{K,A}} \mathbf{A}_{K,A}(\delta^{-1})\right)^{\varphi=1}.$$

After passing to an $fppf$ cover of A trivializing L, we are reduced to prove that if $\mathbf{A}_{K,A}(\delta) \cong \mathbf{A}_{K,A}(\delta')$, then $\delta = \delta'$. Equivalently, we have to prove that the natural map

$$X_1 \times_{\mathcal{X}_1} X_1 \longrightarrow X_1 \times_{\mathrm{Spf}(\mathcal{O})} X_1$$

factors as

$$\begin{array}{ccc} X_1 \times_{\mathcal{X}_1} X_1 & \xrightarrow{\quad (\dagger) \quad} & X_1 \times_{\mathrm{Spf}(\mathcal{O})} X_1 \\ & \searrow \qquad \nearrow_{\Delta} & \\ & X_1 & \end{array}$$

where Δ is the diagonal morphism.

Thanks to [2], lemma 7.1.14, it is enough to check the above factorization at the level of A-points, for A any finite local artinian $\mathcal{O}$-algebra A. But for such an A the claimed factorization follows immediately from the equivalence of categories of theorem 2 and the fact that the map $X_1 \to \mathcal{X}_1$ respects such an equivalence.

To be precise, in order to apply the aforementioned lemma one has to check that $(\dagger)$ and the structure morphism $X_1 \times_{\mathrm{Spf}(\mathcal{O})} X_1 \to \mathrm{Spec}(\mathcal{O})$ are limit preserving on objects. One checks easily that $X_1 \times_{\mathrm{Spf}(\mathcal{O})} X_1$ is limit preserving, while for $(\dagger)$ one observes that this map is a base change of the diagonal $\mathcal{X}_1 \to \mathcal{X}_1 \times_{\mathrm{Spf}(\mathcal{O})} \mathcal{X}_1$, which is affine and of finite presentation, hence limit preserving on objects.

For the existence part of the statement we limit ourselves to discuss the strategy applied in [3]. There the author proceeds as follows:

- prove that it is enough to assume that A is reduced $\mathbb{F}$-algebra of finite type, i.e. that it is enough to prove that the induced morphism between the reduced substacks $[X_1 / \widehat{\mathbb{G}}_m]_{red} \to \mathcal{X}_{1,red}$ is an isomorphism;
- observe that there is an isomorphism of stacks over $\mathbb{F}$

$$\coprod_{\delta} [\mathbb{G}_{m,\mathbb{F}} / \mathbb{G}_{m,\mathbb{F}}] \cong [X_1 / \widehat{\mathbb{G}}_m]_{red}$$

  where δ runs over the *Serre weights* $\delta : I_K \to \mathbb{F}^\times$ (extended to characters of W_K sending Frobenii to 1);
- show that for every δ as above, the induced map $[\mathbb{G}_{m,\mathbb{F}} / \mathbb{G}_{m,\mathbb{F}}] \to \mathcal{X}_{1,red}$ indexed by δ is a closed immersion of stacks over $\mathbb{F}$;
- conclude that, since every continuous character $G_K \to \overline{\mathbb{F}}_p^\times$ is an unramified twist of one of the Serre weights δ, the closed immersion $X_1 \hookrightarrow \mathcal{X}_1$ is indeed an isomorphism (note that here we are implicitly using the equivalence of theorem 2 for $A = \overline{\mathbb{F}}_p$).

□

Remark 7. The proof of theorem 1 given by Emerton and Gee in [2] (chapter 7) consists in developing a suitable theoretical machinery in order to check minimal hypothesis on the morphism $X_1 \to \mathcal{X}_1$ to deduce that it induces an isomorphism $[X_1 / \widehat{\mathbb{G}}_m] \cong \mathcal{X}_1$. In particular most of the actual proof reduces to showing that the morphism $X_1 \to \mathcal{X}_1$ is representable by algebraic stacks (or equivalently by algebraic spaces). When $K/\mathbb{Q}_p$ is unramified, this boils down to proving that

$$\mathcal{Y}^a_{1,h,s} := X_1 \times_{\mathcal{X}_1} \mathcal{X}^a_{1,h,s}$$

is an algebraic stack for all $a \geq 1$, $h \geq 0$, $s \geq 0$ (with $\mathcal{X}^a_{d,h,s}$ defined as in the previous talk). The idea is to prove that $\mathcal{Y}^a_{1,h,s}$ is a closed subsheaf of $\mathrm{Spec}\left((\mathcal{O}/\pi^a)[I_K^{\mathrm{ab}}/U]\right) \times \mathbb{G}_{m,\mathcal{O}/(\pi^a)}$ for some open subgroup $U = U_{a,h,s}$ of I_K^{ab} (depending only on a, h, s). Again one checks this statement on A-points, for A a finite local artinian $\mathcal{O}/(\pi^a)$-algebra and the result essentially follows from the fact that for fixed height h and s large enough, there is a unique possible semilinear action of $\langle \gamma^{p^s} \rangle$ on projective weak Wach Modules $\mathcal{M} \in \mathcal{W}^a_{1,h,s}(A)$ (for $\gamma \in \Gamma$ a topological generator).

Remark 8. According to [2], remark 7.2.19, the fact that $\mathcal{X}_1$ can be described as the moduli stack of 1-dimensional continuous representations of the Weil group W_K does not generalize to $\mathcal{X}_d$ for $d \geq 2$. In other words, it does not seem possible to realize $\mathcal{X}_d$ as the moduli space of representation of a group in a way that is compatible with the fact that the closed points of such a moduli space have to be interpreted as representations of G_K. One can still try to define a morphism from the moduli stack, say $\mathcal{V}_d$, of continuous d-dimensional representation of WD_K to $\mathcal{X}_d$. Recall that in the previous talks we have defined the group WD_K via a pullback diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_K & \longrightarrow & G_K & \longrightarrow & G_K^+ & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & P_K & \longrightarrow & WD_K & \longrightarrow & \mathbb{Z}[1/p] \rtimes \mathbb{Z} & \longrightarrow & 1 \end{array}$$

where P_K is the wild inertia subgroup and one has to choose a lift of Frobenius and a lift of a topological generator of the tame inertia to get the subgroup $\mathbb{Z}[1/p] \rtimes \mathbb{Z}$ of the tame Galois group G_K^+. One can try to adapt the proof of [2] for the rank one case to this setting, but the hypothetical morphism $\mathcal{V}_d \to \mathcal{X}_d$ will not be representable by algebraic spaces. The crucial failure is given by the fact that the ramification groups (playing the role of the $U = U_{a,h,s}$ of the previous remark) will not be open in the wild inertia P_K in the case $d \geq 2$, i.e. when we cannot pass to the abelianized version of Galois theory.

4 Determinant character of étale (φ, Γ)-modules

In this section we follow [1], section 2.6. Let A be a π-complete $\mathcal{O}$-algebra and pick $M \in \mathcal{X}_d(\mathrm{Spf}(A))$ a rank d projective étale (φ, Γ)-module with A-coefficients. Then $\wedge^d M$ is a rank 1 projective étale (φ, Γ)-module with A-coefficients and can apply theorem 1 (or more precisely proposition 6) to associated to it a unique character

$$\det(M) : W_K \to A^\times.$$

Clearly if A is a finite artinian $\mathcal{O}$-algebra, then the equivalence of theorem 2 realizes $\det(M)$ as the determinant of the G_K-representation $T_A(M)$ (essentially because the equivalence of theorem 2 also respects tensor structures on both sides).

References

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