Research seminar WS 2021-22 - talk 2

Irrationality of zeta values

References:

[1] Ball, Rivoal "Irrationalité d'une infinité de valeurs de la fonction zêta aux entiers impairs" Invent. Math. 146 (2001)

[2] Beukers "A note on the Irrationality of $\zeta(2)$ and $\zeta(3)$" Bull. London Math. Soc. (1979)

[F] Fischler "Irrationalité de valeurs de zêta" Séminaire Bourbaki, Astérisque no. 294 (2004)

[IK] Iwaniec, Kowalski "Analytic Number Theory", American Math. Soc., Colloquium Publ. vol. 53 (2004)

§ 1. Apéry - Beukers' proof of the irrationality of $\zeta(3)$

Theorem A: $\zeta(3) = \sum_{n=1}^{+\infty} \frac{1}{n^3} \notin \mathbb{Q}$ [2, thm. 2]

Lemma 1: Let $d_n := \operatorname{lcm}(1,2,\dots n)$ for $n \in \mathbb{Z}_{\geq 1}$. Then for n large enough $d_n < 3^n$.

proof: Fix $n \in \mathbb{Z}_{\geq 1}$ and let $p \in \mathbb{Z}_{\geq 1}$ be a prime with $0 < p \leq n$. Then $k := \operatorname{ord}_p(d_n)$ is the unique positive integer such that

$$p^k \leq n \underset{\neq}{<} p^{k+1} \implies k = \lfloor \log(n)/\log(p) \rfloor$$

Hence:

$$d_n = \prod_{\substack{0<p\leq n \\ \text{prime}}} p^{\lfloor \log(n)/\log(p) \rfloor} \leq \prod_{\substack{p \leq n \\ \text{prime}}} p^{\log(n)/\log(p)} = n^{\pi(n)}$$

where $\pi(n) = \#\{p \text{ prime}, p \leq n\}$

The Prime Number Theorem ([IK, § 2.1]) says that

$\pi(n) \sim \frac{\log(n)}{n}$ for $n \to +\infty$; hence:

$d_n \leq n^{\pi(n)} \sim e^n < 3^n$ for n large enough $\blacksquare$

Rmk: Actually $d_n \sim e^n$ for $n \to +\infty$

Lemma 2: Let $r, s \in \mathbb{Z}_{\geq 0}$.

(i) if $r > s$ then $I_{r,s} := \int_{(0,1)^2} \frac{-\log(xy)}{1-xy} x^r y^s \, dx dy \in \mathbb{Q}$

and $d_r^3 \cdot I_{r,s} \in \mathbb{Z}$

(ii) $I_r := \int_{(0,1)^2} \frac{-\log(xy)}{1-xy} x^r y^r \, dx dy = 2\left(\zeta(3) - \sum_{n=1}^{r} \frac{1}{n^3}\right)$

so $d_r^3 \cdot I_r \in \mathbb{Z}$ $\quad (I_0 = 2\zeta(3))$

Proof: Let $\sigma \in \mathbb{R}_{>0}$ and consider

$$J_{r,s}(\sigma) := \int_{(0,1)^2} \frac{x^{r+\sigma} y^{r+\sigma}}{1-xy} \, dx dy$$

writing $\frac{1}{1-xy} = \sum_{n=0}^{+\infty} (xy)^n$ one can compute:

$$J_{r,s}(\sigma) = \sum_{n=0}^{+\infty} \frac{1}{(n+r+\sigma+1)(n+s+\sigma+1)}$$

(i) Assume $r > s$ so that:

$$J_{r,s}(\sigma) = \sum_{n=0}^{+\infty} \frac{1}{r-s}\left(\frac{1}{n+s+\sigma+1} - \frac{1}{n+r+\sigma+1}\right) =$$

$$= \frac{1}{r-s}\left(\frac{1}{s+1+\sigma} + \dots + \frac{1}{r+\sigma}\right)$$

Also $\frac{d}{d\sigma} J_{r,s}(\sigma) = \int_{(0,1)^2} \frac{\log(xy)}{1-xy} x^{r+\sigma} y^{r+\sigma} \, dx dy$

$$\| $$

$$\frac{-1}{r-s}\left(\frac{1}{(s+1+\sigma)^2} + \dots + \frac{1}{(r+\sigma)^2}\right)$$

and taking limits for $\sigma \to 0^+$ we obtain

$$-I_{r,s} = \frac{-1}{r-s}\left(\frac{1}{(s+1)^2} + \quad + \frac{1}{r^2}\right) \in \mathbb{Q} \quad \text{with } d_r^3 \cdot I_{r,s} \in \mathbb{Z}$$

(ii) Assume $r=s$ so that

$$J_{r,r}(\sigma) = \sum_{n=0}^{+\infty} \frac{1}{(n+r+\sigma+1)^2} \quad \text{and}$$

$$-I_r = \frac{d}{d\sigma} J_{r,r}(\sigma)\Big|_{\sigma=0} = -2\sum_{n=0}^{+\infty} \frac{1}{(n+r+1)^3} =$$

$$= -2\left(\zeta(3) - \sum_{n=1}^{r} \frac{1}{n^3}\right) \quad \blacksquare$$

ASIDE: "Legendre type" polynomials

For $n \in \mathbb{Z}_{\geq 0}$ let $Q_n(T) := T^n(1-T)^n$, $P_n(T) := \frac{1}{n!}\frac{d^n}{dT^n} Q_n(T)$

Clearly $P_n(T) \in \mathbb{Z}[T]$ and more explicitly

$$P_n(T) = \sum_{k=0}^{n} \binom{n}{k}^2 (-)^k T^k (1-T)^{n-k}$$

<u>Exercise</u>: (i) Prove that $P_0(T) = 1$, $P_1(T) = 1-2T$ and that $\forall n \geq 1$

$$(n+1) P_{n+1}(T) = (2n+1) P_n(T)\cdot(1-2T) - n\cdot P_{n-1}(T)$$

(ii) Prove that $P_n(1-T) = (-)^n P_n(T)$ and then that

$$\int_0^1 P_n(T) P_m(T) = \begin{cases} 0 & \text{if } m \neq n \\ \frac{(2n)!}{n!} \sum_{k=0}^{n} \frac{(-)^k \binom{n}{k}}{n+k+1} & \text{if } m = n \end{cases}$$

(iii) Write $$\frac{1}{\sqrt{1+(4x-2)t+t^2}} = \sum_{n=0}^{+\infty} \widetilde{P}_n(x)\, t^n$$

as formal power series and prove that $\widetilde{P}_n(T) = P_n(T)$ $\forall n \in \mathbb{Z}_{\geq 0}$

Proof of Theorem A:

STEP 1: construction of linear forms $L_n(X_1, X_2) = l_{1,n} X_1 + l_{2,n} X_2 \in \mathbb{Z}[X_1, X_2]$

Let $$K_n := \int_{(0,1)^2} \frac{-\log(xy)}{1-xy} P_n(x) P_n(y)\, dx\, dy$$

By lemma 2 we know that

$$d_n^3 \cdot K_n = l_{1,n} + l_{2,n} \zeta(3) \quad \text{with } l_{i,n} \in \mathbb{Z} \quad i=1,2$$

So we set $L_n(X_1, X_2) = l_{1,n} X_1 + l_{2,n} X_2$

STEP 2: "convergence", ie bound K_n

Notice that $$\frac{-\log(xy)}{1-xy} = \int_0^1 \frac{1}{1-(1-xy)z}\, dz \quad \text{for } \in (0,1)^2$$

$$\Rightarrow \quad K_n = \int_{(0,1)^3} \frac{P_n(x) P_n(y)}{1-(1-xy)z}\, dx\, dy\, dz = $$ $\rightarrow$ by parts in the variable x

$$= \int_{(0,1)^2} \left[\left(\frac{1}{n!} \frac{\partial^{n-1}}{\partial x^{n-1}} Q_n(x) \right) \frac{P_n(y)}{1-(1-xy)z} \Bigg|_{x=0}^{x=1} + \frac{1}{n!} \int_0^1 \left(\frac{\partial^{n-1}}{\partial x^{n-1}} Q_n(x) \right) \frac{P_n(y)\, yz}{(1-(1-xy)z)^2}\, dx \right] dy\, dz =$$

$$= \frac{k!}{n!} \int_{(0,1)^3} \left(\frac{\partial^{n-k}}{\partial x^{n-k}} Q_n(x) \right) \frac{P_n(y)\, (yz)^k}{(1-(1-xy)z)^{k+1}}\, dx\, dy\, dz =$$

$\downarrow$ after k steps $1 \leq k \leq n$

$$= \int_{(0,1)^3} \frac{(xyz)^n (1-x)^n P_n(y)}{(1-(1-xy)z)^{n+1}}\, dx\, dy\, dz = $$ $\rightarrow$ $w = \dfrac{1-z}{1-(1-xy)z}$

$$= \int_{(0,1)^3} \frac{(1-x)^n (1-w)^n P_n(y)}{1-(1-xy)w} \, dx\,dy\,dw = $$

again by parts n times in the variable y + rewrite $w = z$

$$= \int_{(0,1)^3} \frac{x^n(1-x)^n y^n (1-y)^n z^n (1-z)^n}{(1-(1-xy)z)^n} \frac{dx\,dy\,dz}{1-(1-xy)z}$$

Let $f(x,y,z) := \dfrac{x(1-x)y(1-y)z(1-z)}{1-(1-xy)z}$

Elementary calculus shows that the maximum of f in $[0,1]^3$ is attained at $x = y = \sqrt{2}-1 \quad z = \frac{\sqrt{2}}{2}$

$$\Rightarrow \quad 0 \leq f(x,y,z) \leq (\sqrt{2}-1)^4 \quad \text{if } (x,y,z) \in [0,1]^3$$

Hence:

lemma 2

$$0 < |d_n^3 k_n| \leq d_n^3 (\sqrt{2}-1)^{4n} \int_{(0,1)^3} \frac{1}{1-(1-xy)z} \, dx\,dy\,dz =$$

$$= 2\zeta(3) \cdot d_n^3 \cdot (\sqrt{2}-1)^{4n}$$

lemma 1

$$\Rightarrow \quad 0 < |l_{1,n} + l_{2,n}\zeta(3)| \leq 2\zeta(3)\, d_n^3 (\sqrt{2}-1)^{4n} <$$

$$< 2\zeta(3) \left(27(\sqrt{2}-1)^4\right)^n < 5 \cdot \left(4/5\right)^n \xrightarrow[n \to +\infty]{} 0$$

By a simple lemma proven in Talk 1 this shows that $\zeta(3) \notin \mathbb{Q}$ $\blacksquare$

§ 2. Ball-Rivoal theorem

Theorem B: For all $\varepsilon > 0$ $\exists k_\varepsilon \in \mathbb{Z}_{\geq 2}$ such that $\forall k \geq k_\varepsilon$ we have

$$\dim_{\mathbb{Q}} \langle 1, \zeta(3), \zeta(5), \ldots \zeta(2k-1) \rangle_{\mathbb{Q}} \geq \frac{1-\varepsilon}{1+\log 2} \cdot \log(2k-1)$$

[1, thm 1]

Rmk: Replacing $\frac{1-\varepsilon}{1+\log 2}$ by $1/3$ then actually

$$\dim_{\mathbb{Q}} \langle 1, \zeta(3), \zeta(5), \ldots \zeta(2k-1) \rangle_{\mathbb{Q}} \geq \frac{\log(2k-1)}{3} \quad \forall k \geq 2$$

but $$\frac{1-\varepsilon}{1+\log 2} \simeq 0,59\ldots > 1/3 \quad \text{for } \varepsilon \ll 1$$

Sketch of the proof of theorem B:

STEP 0: definition of $S_n(z)$

Let $a, r \in \mathbb{Z}_{\geq 0}$ with $a \geq 3$ $\quad 1 \leq r \leq a/2$; let $n \in \mathbb{Z}_{\geq 1}$ and set for $k \in \mathbb{Z}_{\geq 1}$

$$R_n^{a,r}(k) := R_n(k) := 2(k+n/2)(n!)^{a-2r} \frac{(k-rn)_{rn}(k+n+1)_{rn}}{(k)_{n+1}^{a}}$$

where $(\alpha)_m = \alpha \cdot (\alpha-1) \cdots (\alpha-m+1)$ for $m \in \mathbb{Z}_{\geq 1}$

is the Pochammer symbol

Set $$S_n^{a,r}(z) := S_n(z) := \sum_{k=1}^{+\infty} R_n(k) z^{-k}$$

Rmk: Since $a \geq 3$, one checks easily that $R_n(k) = O(1/k^2)$ for $k \to +\infty$

so that $S_n(z)$ converges absolutely for $z \in \mathbb{C}$ with $|z| \geq 1$

STEP 1: Properties of $S_n(1)$

Proposition 3: Assume that $a \in \mathbb{Z}_{\geq 3}$ is even. Then there exist $\hat{\ell}_{1,n}, \hat{\ell}_{2,n}, \ldots \hat{\ell}_{\frac{a}{2},n} \in \mathbb{Q}$ such that:

(i) $S_n(1) = \hat{\ell}_{1,n} + \hat{\ell}_{2,n}\zeta(3) + \cdots + \hat{\ell}_{\frac{a}{2},n}\zeta(a-1)$

(ii) for $s \in \{1, 2, \ldots \frac{a}{2}\}$ $\quad \limsup_{n \to +\infty} |\hat{\ell}_{s,n}|^{1/n} \leq 2^{a-2r}(2r+1)^{2r+1}$

(iii) for $s \in \{1, 2, \frac{a}{2}\}$ $\quad \ell_{s,n} := d_n^a \cdot \hat{\ell}_{s,n} \in \mathbb{Z}$

(iv) $\exists\, \varphi_{a,r} \in \mathbb{R}_{>0}$ s.t. $\lim_{n \to +\infty} |S_n(1)|^{1/n} = \varphi_{a,r} \leq \frac{2^{r+1}}{r}$

STEP 2 : Nesterenko's criterion for linear independence (~ talk 3)

Thm C : Let $\vartheta_1, \dots \vartheta_k \in \mathbb{R}$. For $n \geq 1$ let

$$L_n(X_1, \dots X_k) = l_{1,n} X_1 + \dots + l_{k,n} X_k \in \mathbb{Z}[X_1, \dots X_k]$$

Assume $\exists\, \alpha, \beta \in \mathbb{R}$ with $0 < \alpha < 1$, $\beta > 1$ such that:

- $\limsup_{n \to +\infty} |l_{s,n}|^{1/n} \leq \beta \quad \forall s \in \{1, \dots k\}$
- $\lim_{n \to +\infty} |L_n(\vartheta_1, \dots \vartheta_k)|^{1/n} = \alpha$

Then $\dim_{\mathbb{Q}} \langle \vartheta_1, \dots \vartheta_k \rangle_{\mathbb{Q}} \geq 1 - \frac{\log \alpha}{\log \beta}$

STEP 3 : mix the ingredients

Put: $\vartheta_1 = 1$, $\vartheta_s = \zeta(2s-1)$ $\quad s \in \{2, 3, \dots \frac{a}{2}\}$

$$L_n(X_1, \dots X_{\frac{a}{2}}) = l_{1,n} X_1 + \dots + l_{\frac{a}{2},n} X_{\frac{a}{2}}$$ (using $l_{s,n} \in \mathbb{Z}$ from prop. 3)

$\rotatebox{90}{$\in$}$ $\mathbb{Z}[X_1, \dots, X_{\frac{a}{2}}]$

We will assume a is large enough and set $r := \left\lfloor \frac{a}{\log^2 a} \right\rfloor$

so that $r^r = O\left(e^{\frac{a}{\log a}}\right)$ and $\frac{r^r}{c^a} \xrightarrow[a \to +\infty]{} 0 \quad \forall c \in \mathbb{R}, c > 1$

Then using (ii) and (iv) in prop. 3 (and the fact that $d_n \sim e^n$ for $n \to +\infty$)
to take:

$\beta = (2e)^a \cdot \delta_\beta > 1 \qquad \alpha = (e/r)^a \cdot \delta_\alpha < 1$ (for a large $r > 3$)

with $\frac{\log(\delta_*)}{a} \xrightarrow[a \to +\infty]{} 0$ for $* \in \{\alpha, \beta\}$

then by thm C we know that:

$$\dim_{\mathbb{Q}} \langle 1, \zeta(3), \dots, \zeta(a-1) \rangle_{\mathbb{Q}} \geq 1 - \frac{\log \alpha}{\log \beta}$$

But:

$$1-\frac{\log\alpha}{\log\beta}=\frac{a(1+\log 2-1+\log 2)+\log\delta_\beta-\log\delta_\alpha}{a(1+\log 2)+\log\delta_\beta}\geq$$

$\uparrow$ $\log(2 2)\geq\log\left(\frac{a-1}{\log^2 a}\right)$

$$\geq\frac{\log(a-1)}{1+\log 2}\left[\frac{1-\overbrace{\frac{2\log\log a}{\log(a-1)}+\frac{\log(\delta_\beta/\delta_\alpha)}{a\cdot\log(a-1)}}^{\varepsilon_1\ll 1 \text{ if } a \text{ large}}}{1+\underbrace{\frac{\log(\delta_\beta)}{a(1+\log 2)}}_{\varepsilon_2\ll 1 \text{ if } a \text{ large}}}\right]\geq\frac{\log(a-1)}{1+\log 2}\cdot(1-\varepsilon)$$

if $a\gg 1$ $\blacksquare$

- Sketch of proof of (i) in prop. 3 (for the proof of (ii), (iii), (iv) we refer to [7, § 2.3]

We define polylogarithms:

$$\mathrm{Li}_s(z):=\sum_{k=1}^{+\infty}\frac{z^k}{k^s}\qquad \text{for } s\in\mathbb{C},\quad z\in\mathbb{C},\ |z|<1\ \text{(a priori)}$$

$$\mathrm{Li}_0(z)=\frac{z}{1-z},\qquad \mathrm{Li}_1(z)=-\ln(1-z)$$

If $s\in\mathbb{Z}_{\geq 2}$ then $\mathrm{Li}_s(z)$ can be analytically continued to an entire function and clearly $\mathrm{Li}_s(1)=\zeta(s)$

<u>IDEA</u>: Show that $S_n(z)=\sum_{s=1}^{a}P_s(z)\,\mathrm{Li}_s(1/z)+P_0(z)$

with $P_s(z)\in\mathbb{Q}[z]$ for $s\in\{0,1,\dots a\}$ (a priori for $|z|>1$)

We write:

$$R_n(k)=\sum_{j=0}^{n}\sum_{s=1}^{a}\frac{c_{js}}{(k+j)^s}\qquad c_{j,s}\in\mathbb{Q}$$

where one can check that $c_{js}=\frac{1}{(a-s)!}\frac{d^{a-s}}{dX^{a-s}}\left(R_n(X)(X+j)^a\right)\Big|_{X=-j}$