BUILDING 6FF

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1. Six functor formalisms

All across mathematics we exploit the structure and properties of various cohomology theories of geometric objects:

$$\{\text{Geometric objects}\} \mapsto D(\mathbb{Z})/\operatorname{Sp}/\ldots, \quad X \mapsto H^*(X).$$

Examples are singular cohomology, Čech cohomology, coherent cohomology and étale cohomology.

In my joint work with Bastiaan Cnossen and Tobias Lenz I have developed higher categorical tools that allow us to construct the structure of six functors, in the sense of Grothendieck, on cohomology theories. My main goal is to explain what the six operations are, and what our result is. I will motivate six functors via there applications to proving Poincaré/Verdier duality. Then I will hopefully discuss a future goal which seems approachable via the methods we develop (this is slightly speculative). The first crucial observation is that cohomology is typically taken with coefficients:

Example 1.1.

(1) We may take singular cohomology of a CW-complex X with coefficients in a local system on X.
(2) More generally we can take Čech cohomology of a locally compact Hausdorff space X with coefficients in a sheaf of abelian groups on X.
(3) We can take coherent cohomology of a complex manifold or scheme X with coefficients in a coherent sheaf of abelian groups.

Date: April 22, 2026.

(4) We can take étale cohomology of a scheme with coefficients in an étale sheaf.

In fact cohomology is just the shadow of an assignment:

$$\{\text{Geometric objects}\} \mapsto \mathrm{Cat}_\infty, \quad X \mapsto \{D(X), \text{ a cat. of coeffs. for the cohomology of } X\}.$$

To make this formal suppose $\mathcal{C}$ is the ∞-category of "geometric objects" which has finite limits.

Definition 1.2. A *coefficient system* is a functor

$$D(-)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_\infty), \quad X \mapsto D(X), \quad f \mapsto f^*$$

such that

(1) $f^* \dashv f_*$ for all f,
(2) Each $D(X)$ is closed symmetric monoidal, $- \otimes \mathcal{F} \dashv \mathrm{Hom}(\mathcal{F}, -)$ for all $\mathcal{F} \in D(X)$.

We can slickly encode this by saying that D factors through $\mathrm{CAlg}(\mathrm{Cat}_L^{\otimes})$.

How do we recover cohomology?

Definition 1.3. Given $X \in \mathcal{C}$ and $\mathcal{F} \in D(X)$, we define

$$H^*(X, \mathcal{F}) = (X \to *)_*(\mathcal{F}).$$

More generally f_* computes "relative cohomology".

With this upgrade we can state Poincaré duality, in a "schematic" form:

Theorem 1.4 (Verdier Duality)**.** *Let $f\colon X \to Y$ be a map in $\mathcal{C}$ which is "proper and smooth of relative dimension d"*[1]*. Then there is an object $\omega_f \in D(X)$, called the* dualizing sheaf *such that*

$$f_* \dashv f^* \otimes \omega_f \quad (f_! \dashv f^* \otimes \omega_f)$$

and ω_f is locally of the form $\mathbb{1}[d]$.

[1] In the case of topological spaces this is a fiber bundle whose fibers are smooth manifolds of dimension d

Remark 1.5. Supposing this is the case for $f\colon X \to *$, we obtain an isomorphism

$$H^*(X, \omega_X) \simeq H^*(X)^\vee.$$

this is Poincaré duality.

The origins of 6FFs lie in trying to prove this sort of statement. Note that this theorem has a local assumption and local conclusion, and so we may hope to prove it "locally". But it also has a global assumption, which in fact twarts any such proof. In trying to remove the global conditions one encounters an issue, which is that f_* is typically not well-behaved and "local" enough. Let me explain what I mean.

Definition 1.6. Let $P \subset S$ be a geometric subcategory, meaning it is wide, closed under diagonals and pullbacks. Then we say P is D-proper if

(1) It satisfies basechange: for every pullback square

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & Y \end{array}$$

such that p (and therefore q) is in P, the Beck–Chevalley map filling the square of categories

$$\begin{array}{ccc} D(Y') & \xrightarrow{g^*} & D(X') \\ {\scriptstyle p_*}\downarrow & \nearrow_{BC_*} & \downarrow{\scriptstyle q_*} \\ D(Y) & \xrightarrow[f^*]{} & D(X) \end{array}$$

is an isomorphism. This tells us for example that cohomology is computed "fiber-wise".

(2) It satisfies the projection formula: for every map $p\colon X \to Y$ in P and $\mathcal{F} \in D(X), \mathcal{G} \in D(Y)$, the map

$$p_*\mathcal{F} \otimes \mathcal{G} \to p_*(\mathcal{F} \otimes p^*\mathcal{G})$$

is an isomorphism.

(3) For all $p \in P$, p_* is a left adjoint (i.e. it commutes with colimits).

In examples, the "geometrically proper" maps are always D-proper. However the local models of smooth maps, $\mathbb{A}^n \times Y \to Y$ are almost never D-proper. Therefore it is not possible to run our dream argument which proves Verdier duality and computes ω_f locally.

2. Exceptional pushforwards

To solve this problem we follow Grothendieck's insight in the context of étale sheaves, which has been remarkably robust across math. It is as follow:

Idea 2.1. For a nice class of morphisms, we may find a "replacement" $f_!$ of f_* which comes equipped with:

(1) a natural transformation $f_! \to f_*$ which is an equivalence when f is proper
(2) and coherent *data* which witnesses $f_!$ as D-local, meaning there are natural isomorphisms witnessing:
   (a) exceptional basechange.
   (b) the projection formula,

$$f_!\mathcal{F} \otimes \mathcal{G} \to f_!(\mathcal{F} \otimes f^*\mathcal{G}).$$

Moreover we require $f_! \dashv f^!$. We call this the pair of an exceptional pushforward and pullback. Given this we can prove an analog of Verdier duality for the exceptional pushforward, and so deduce it for the usual pushforward when f is proper.

How do we construct this replacement? As mentioned, we want our six functor formalism to encode Verdier duality, which is the statement that when f is smooth,

$$f_! \dashv f^*(-) \otimes \omega_f.$$

This would give us a definition of $f_!$, namely just $f^*(-) \otimes \omega_f$, if we had a definition of the dualizing sheaf. This simplifies further if we assume that $f\colon X \to Y$ is of relative dimension zero, in which case the dualizing sheaf is trivial!

The upshot is that if $i \in I$ is a class of *étale* maps, then necessarily $f_! \coloneqq f_\sharp \dashv f^*$. In examples we often restrict further to a class $I \subset E$ of *open immersions*.

Definition 2.2. Let $I \subset S$ be another geometric subcategory. Then we say I is D-étale if:

(1) For all $i \in I$, i^* admits a left adjoint $i_\sharp$.

(2) It satisfies basechange: for every pullback square

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ {\scriptstyle j}\downarrow & \lrcorner & \downarrow{\scriptstyle i} \\ X & \xrightarrow[f]{} & Y \end{array}$$

such that i (and therefore j) is in I, the Beck–Chevalley map filling the square of categories

$$\begin{array}{ccc} D(Y') & \xrightarrow{g^*} & D(X') \\ {\scriptstyle i_\sharp}\downarrow & \overset{BC_\sharp}{\Swarrow} & \downarrow{\scriptstyle j_\sharp} \\ D(Y) & \xrightarrow[f^*]{} & D(X) \end{array}$$

is an isomorphism.

(3) It satisfies the projection formula: for every map $i\colon X \to Y$ in I and $\mathcal{F} \in D(X), \mathcal{G} \in D(Y)$, the map

$$i_\sharp(\mathcal{F} \otimes i^*\mathcal{G}) \to i_\sharp\mathcal{F} \otimes \mathcal{G}$$

is an isomorphism.

To define the exceptional pushforward in general we consider maps $f\colon X \to Y$ obtained as a composite

$$\begin{array}{ccccc} & & \bar{X} & & \\ & \overset{i}{\nearrow} & & \overset{p}{\searrow} & \\ X & & \xrightarrow[f]{\qquad\qquad} & & Y \end{array}$$

where $p \in P, i \in I$. This implies that we must have

$$f_! \simeq p_! i_! \coloneqq p_* i_\sharp.$$

In particular the exceptional pushforwards are "determined" by the decisions before. Also the locality data is "determined". In examples we obtain a proposed definition of the exceptional pushforward for many maps this way:

(a) Nagata compactification shows that any finite type seperated map of schemes admits such a factorization.
(b) Any map of locally compact Hausdorff spaces factros as above. One can take the universal factorization through the Stone–Ĉech compactification for example.

But the unaddressed question is: Does this actually give well-defined exceptional pushforwards? And how coherently can we actually make these definitions.

We pin down the precise condition under which this proposed definition is well-defined and completely coherent:

Definition 2.3. Given D and classes P and I of D-proper and D-étale maps, we say D satisfies (I, P)-interchange if every map in $I \cap P$ is truncated and for every pullback square

$$\begin{array}{ccc} X' & \xrightarrow{i} & Y' \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ X & \xrightarrow[j]{} & Y \end{array}$$

a canonical (Double Beck–Chevalley) map $\mathrm{BC}_{\sharp,*} \colon j_\sharp q_* \to p_* i_\sharp$ is an isomorphism.

Remark 2.4. Notice that for $f \in I \cap P$, we have two possible candidates of $f_!$, namely $f_\sharp$ and f_*. So for our "definition" of exceptional pushforwards to be well defined these must be canonically isomorphic. Potentially surprisingly, this is implied by the condition above! Here it is crucial that we assume that all maps in $I \cap P$ are truncated. This is essentially the story of (higher) semiadditivity, as first introduced by Hopkins–Lurie. A key step in our joint work was encapsulating the coherence of these isomorphisms, which we did by introducing the theory of *parametrized semiadditivity* and computing universal examples of such in parametrized category theory.

Given this additional condition, we prove it is possible to define the exceptional pushforwards and locality data completely coherently. To make this precise, I now define an $(\infty, 2)$-category encoding this data.

Definition 2.5. Let $(\mathcal{C}, I, P)$ be as before and define $E = P \circ I$. We define an $(\infty, 2)$-category $\mathbb{S}\mathrm{pan}(\mathcal{C}, E)_I^P$ with

(1) objects $X \in \mathcal{C}$,

(2) category of morphisms from X to Z given by

$$\left\{ \begin{array}{ccccc} & & Y & & \\ & \swarrow & \uparrow{\scriptstyle \in P} & \searrow{\scriptstyle \in E} & \\ X & \longleftarrow & W & \xrightarrow{\in E} & Z \\ & \nwarrow & \downarrow{\scriptstyle \in I} & \nearrow{\scriptstyle \in E} & \\ & & Y' & & \end{array} \right\}$$

(3) composition is given by pullback.

Taking product gives this the structure of a symmetric monoidal $(\infty, 2)$-category. Including as the backwards part of a span gives a functor

$$\mathcal{C}^{\mathrm{op}} \to \mathbb{S}\mathrm{pan}(\mathcal{C}, E)_I^P.$$

Theorem 2.6 (Factorization method)**.** *Let $(\mathcal{C}, I, P)$ be as before. Define $E = P \circ I$. Suppose P is D-proper, I is D-étale and D satisfies (I, P)-interchange. Then $D(-)$ extends uniquely to a lax monoidal functor*

$$D(-)\colon \mathbb{S}\mathrm{pan}(\mathcal{C}, E)_I^P \to \mathrm{Cat}_L^{\otimes}$$

such that $p_! = p_*$ and $i_! = i_\sharp$ for $p \in P, i \in I$.

Remark 2.7. We should comment on how D as a functor on the span category extends the coefficient system D. We note that the inclusion $\mathcal{C}^{\mathrm{op}}$ is symmetric monoidal, and in fact lax symmetric monoidal functors $D\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_L^{\otimes}$ is equivalent to functors $D\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{Cat}_L^{\otimes})$. Given the latter data, it is the composite

$$D(X) \times D(X) \xrightarrow{\boxtimes} D(X \times X) \xrightarrow{\Delta^*} D(X).$$

of the lax symmetric monoidal structure and pullback along the diagonal which encodes the tensor product on $D(X)$.

Remark 2.8. This theorem encodes all of the exceptional pushforwards, the locality data, as well as the transformations $f_! \to f_*$ for maps with proper diagonal, completely coherently. Let us unpack this a bit.

(1) As mentioned, the restriction to $\mathcal{C}^{\mathrm{op}}$ encodes the original coefficient system.

(2) The restriction to $E \subset \mathbb{S}\mathrm{pan}_{I,P}(\mathcal{C}, E)$ encodes the exceptional pushforwards:

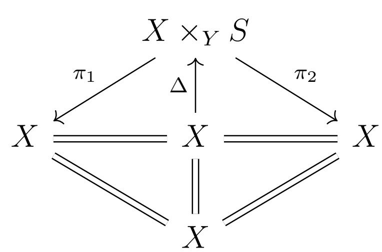
$$e_! = D(X = X \xrightarrow{e} Y).$$

(3) That D preserves composition encodes basechange. Given $f\colon Z \to Y$ and $e\colon X \to Y$ in $\mathcal{C}$ and E respectively, we may compute

$$f^* e_! = D(Y \xleftarrow{f} Z = Z) \circ D(X = X \xrightarrow{e} Y) \simeq D(X \xleftarrow{f'} X \times_Y Z \xrightarrow{e'} Z) \simeq e'_!(f')^*.$$

(4) The lax monoidality encodes the projection formulas. $Y \xleftarrow{f} X = X$ makes X a Y-module in $\mathrm{Span}(\mathcal{C}, E)$. One proves that $X = X \xrightarrow{f} Y$ is a map of Y-modules in $\mathrm{Span}(\mathcal{C}, E)$. This data is transfered to Cat via the lax symmetric monoidal functor $D(-)$. This is equivalent to the projection formula!

(5) Suppose $f\colon X \to Y$ is a map with proper diagonal. Then D applied to the 2-morphism



gives a natural transformation $f^* f_! \simeq (\pi_2)_! \pi_1^* \to \mathrm{id}$. Adjoining over gives $f_! \to f_*$. Moreover, if f is proper then this map is the counit of an adjunction, witnessing $f_! = f_*$. The dual story holds for I.

To come full circle: Given this result, there are extremely streamlined approaches to prove Verdier duality. See work of Heyer–Mann, Zavyalov, also Kipp. They give the result in classical contexts such as étale cohomology, but also in more exotic contexts such as prismatic/syntomic cohomology.

3. One key idea in the proof

Let me make one conceptual jump in the proof explicit. First a small one, we note that all of the conditions involving the existence of right adjoints for $f_!$ and f^* and the closedness of $D(X)$ were snuck into our choice of coefficients $\mathrm{Cat}_L \subset \mathrm{Cat}$. These are all conditions. So really our main goal is to extend arbitrary functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ to *three functor formalisms*.

Now the big jump. We note that the conditions that

(1) P was a class of D-proper morphisms,
(2) I was a class of D-étale morphisms,
(3) and D satisfied (I, P)-interchange

We're simply the conditions that certain adjoints existed, and that certain canonical natural transformations were equivalences. Therefore we can consider the conditions above for functors $D\colon \mathcal{C}^{\mathrm{op}} \to \mathbb{E}$ into an arbitrary 2-category.

Definition 3.1. We say a functor $D\colon \mathcal{C}^{\mathrm{op}} \to \mathbb{E}$ is (I, P)-biadjointable if the conditions before hold. Explicitely,

(1) the image of maps in P admit right adjoints in $\mathbb{E}$ satisfying basechange,
(2) the image of maps in I admit left adjoint in $\mathbb{E}$ satisfying basechange,
(3) D satisfies (I, P)-interchange.

We prove that there is a universal $(\infty, 2)$-category $\mathbb{S}_{I,P}$ equipped with an (I, P)-biadjointable functor $\mathcal{C}^{\mathrm{op}} \to \mathbb{S}_{I,P}$. This is essentially clear, given we are just adding adjoints and inverting some two-morphisms. Therefore our theorem is actually an identification of the $(\infty, 2)$-category $\mathbb{S}_{I,P}$ with the span category above[2].

This perspective allows us to significantly reduce the necessary computations.

[2]In particular the extension theorem holds with values in an arbitrary $(\infty, 2)$-category, this is another benefit of our result.

4. Future perspective

You can interpret our result as telling you the precise structure given on a coefficient system which satisfies

(1) proper basechange
(2) étale basechange
(3) proper-étale interchange.

In particular you can construct the universal coefficient system, in a sort of motivic sense, which satisfies these properties. This recovers computations of various Mackey functors as "universal coefficients" with ambidexterity for étale maps. I hope we can push this perspective to compute universal cohomology theories which satisfy more properties. For example, it would be super cool if we could compute a universal 2-category where you add either

(1) excision
(2) $\mathbb{A}^1$-invariance

to the list of properties. You'd have to think about how to phrase that as the computation of universal 2-categories. I have some thoughts there, but no concrete suggestions.

A more concrete suggestion is the following. We may wonder what the universal coefficient system where you have (1),(2),(3) above, and also Verdier duality. One can show this is essentially the same as the theory having

(1) proper basechange
(2) smooth basechange
(3) proper-smooth interchange

Clearly this is a list of conditions where our methods immediately imply the existence of an initial 2-category equipped with a functor form $\mathcal{C}^{\mathrm{op}}$ satisfying these axioms. I have a guess for what the answer should be, and am hopeful we can prove it right by building on our methods.

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