Contractions via moduli of sheaves

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Plan of the talk

Introduction

Moduli problems

- Classification questions
- Moduli functors
- Stacks

Main results

- Good moduli spaces
- Examples

Applications

Andres Fernandez Herrero

Motivating question

X – smooth projective variety over a field k

$j\colon Y \to X$ – closed subvariety

$\pi\colon Y \to B$ – surjective morphism

Question

When does there exist a proper algebraic space $\overline{X}$ with a surjective $f\colon X \to \overline{X}$ such that

- $f(Y) = B \hookrightarrow \overline{X}$, and
- $X \setminus Y \to \overline{X} \setminus B$ is an isomorphism?

Motivating question

X – smooth projective variety over a field k

$j\colon Y \to X$ – closed subvariety

$\pi\colon Y \to B$ – surjective morphism

Answers

- [Artin '70] Good understanding of the formal completion of $Y \subset X \rightsquigarrow$ conditions for existence of $\overline{X}$ as an algebraic space
- [Kollár, Mori '98] Contractions associated to ω_X-negative extremal faces in the cone of curves using a semiample line bundle on X

Our approach

Classifying

- Groups: finitely generated abelian; finite; Lie; reductive
- Lie algebras: semisimple
- Varieties: curves; K3 surfaces; homogeneous spaces; K-stable Fano
- Sheaves on varieties: vector bundles, coherent sheaves, Bridgeland stable complexes
- Representations: of Lie algebras, of quivers
- Objects in abelian categories

Families

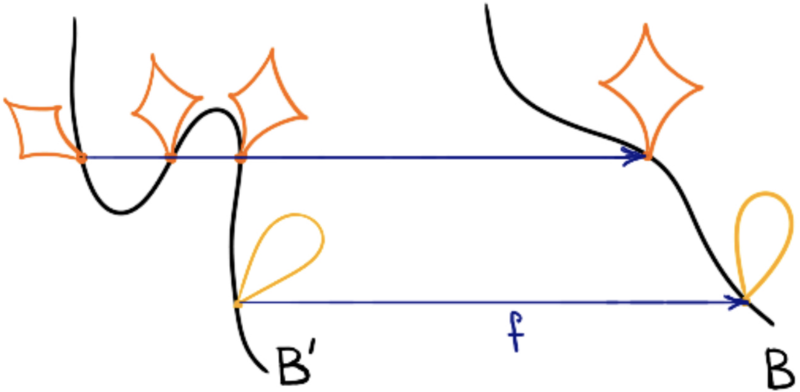
Question

How to say when two objects "differ slightly"?

Approach

Consider families

Pullback of families



Moduli functor

Definition

- *Moduli problem* = $\mathcal{M}\colon \mathcal{S}ch^{\mathrm{op}} \to \mathcal{S}et$ such that $\mathcal{M}(\operatorname{Spec} k)$ = set of isoclasses of objects
- *Fine moduli space* = $M \in \mathcal{S}ch + \mathcal{M} \cong \operatorname{Hom}(\_, M)$

Definition

Family = $F \in \mathcal{M}(B)$

Example 1: functor of points

X smooth projective variety over a field k

$T \in \mathcal{S}ch_k \rightsquigarrow \mathcal{M}(T) := \mathrm{Hom}(T, X)$

$k \subset K$ field extension $\rightsquigarrow \mathcal{M}(\operatorname{Spec} K) = X(K) = K$-points

X is the fine moduli space for $\mathcal{M}$

Example 2: functor of several points (aka Hilbert scheme of points)

X smooth projective variety over a field k

$\ell > 0$ natural number

$$T \in \mathcal{S}ch_k \rightsquigarrow$$
$$\mathcal{M}(T) := \left\{ Z \subset X \times T \;\middle|\; \begin{array}{l} Z \text{ is } T\text{-flat;} \\ \forall t \in T\colon Z_{|X \times t} \text{ length } \ell \text{ subscheme} \end{array} \right\}$$

Theorem

There is a fine moduli space of $\mathcal{M}$, denoted $\mathrm{Hilb}^{\ell}(X)$

Problem

Moduli functors are rarely representable

Solution 1

Throw out the non-schematic locus

Solution 2

Expand the notion of moduli functor

Why stacks?

Solution 2

Ask the moduli functor remember automorphisms $\rightsquigarrow$ stacks

Advantages

- Works when GIT cannot be applied,
  when we don't know enough about the structure
- Intrinsic, or "coordinate-free"

Definition

Definition

Groupoid = category, all morphisms are isomorphisms

$\mathcal{G}pd$ = 2-category of groupoids

Definition

Stack = $\mathcal{M}\colon \mathcal{S}ch_k^{\mathrm{op}} \to \mathcal{G}pd$ + gluing condition

Definition

$\mathcal{M}$ *algebraic* =

$\exists$ smooth surjection $U \to \mathcal{M}$ from some scheme U,

and $\mathcal{M} \to \mathcal{M} \times_k \mathcal{M}$ is representable

Example 0: $B\mathbb{G}_m$

k a field

$T \in \mathcal{S}ch_k \rightsquigarrow \mathcal{M}(T) := \{\mathbb{G}_m\text{-torsors over } T\}$

Morphisms in $\mathcal{M}(T)$ = isomorphisms

Definition

Denote this stack $\mathcal{M}$ by $B\mathbb{G}_m$

Example 1: ideal sheaves of one point

X smooth projective variety over a field k

$T \in \mathcal{S}ch_k \rightsquigarrow$

$$\mathcal{M}(T) := \left\{ \mathcal{I}_Z \text{ ideal sheaf} \;\middle|\; \begin{array}{l} Z \subset X \times T \text{ a } T\text{-flat subscheme;} \\ \forall t \in T\colon Z_{|X \times t} \text{ length } 1 \text{ subscheme} \end{array} \right\}$$

Morphisms in $\mathcal{M}(T)$ = isomorphisms of sheaves

Lemma

If $\dim X > 1$, then $\mathcal{M} \cong X \times B\mathbb{G}_m$

Example 2: ideal sheaves of several points

X smooth projective variety over a field k

$\ell > 0$ natural number

$T \in \mathcal{S}ch_k \rightsquigarrow$

$$\mathcal{M}(T) := \left\{ \mathcal{I}_Z \text{ ideal sheaf} \;\middle|\; \begin{array}{l} Z \subset X \times T \text{ a } T\text{-flat subscheme;} \\ \forall t \in T \colon Z_{|X \times t} \text{ length } \ell \text{ subscheme} \end{array} \right\}$$

Morphisms in $\mathcal{M}(T)$ = isomorphisms of sheaves

Lemma

If $\dim X > 1$, then $\mathcal{M} \cong \mathrm{Hilb}^{\ell}(X) \times B\mathbb{G}_m$

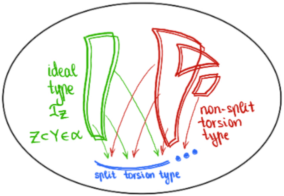
Example 3: stack $\mathcal{U}_{\{\mathbb{P}^1\},1}$

Setup

- $i : \mathbb{P}^1 \hookrightarrow X$ rigid smooth rational curve
  (i.e. $\{\mathbb{P}^1\} \subset \mathrm{Hilb}(X)$ is open and closed)
- $\mathcal{N}_{\mathbb{P}^1/X}$ splits into line bundles of degree ≤ 0

Points of $\mathcal{U}_{\{\mathbb{P}^1\},1}$ of two types:

- $F \cong \mathcal{I}_x$ for $x \in X$
- F that fits into SES
  $0 \to i_*(\mathcal{O}_{\mathbb{P}^1}(-1)) \to F \to \mathcal{I}_{\mathbb{P}^1} \to 0$



Good moduli spaces

Definition

Good moduli space (gms) for $\mathcal{M}$ = qcqs morphism $\varphi\colon \mathcal{M} \to M$ to an algebraic space M such that

- $\varphi_*\colon \operatorname{QCoh} \mathcal{M} \to \operatorname{QCoh} M$ exact
- $\mathcal{O}_M \to \varphi_* \mathcal{O}_{\mathcal{M}}$ isomorphism

Example

G (linearly reductive) acts on variety X,
$X \to Y$ good quotient $\Rightarrow [X/G] \to Y$ gms

Existence criterion [Alper–Halpern-Leistner–Heinloth '18-'23]

$\mathcal{M}$ algebraic, finite type, affine stabilisers, affine diagonal

Theorem

$\mathcal{M}$ has a separated gms $M \iff$
$\mathcal{M}$ is locally linearly reductive, Θ-reductive and S-complete

Theorem

M is proper $\iff \mathcal{M}$ satisfies existence of valuative criterion for properness

Examples

Example 0

$B\mathbb{G}_m \to \operatorname{Spec} k$ gms

Example 1: ideal sheaves of 1 point

$X \times B\mathbb{G}_m \to X$ gms

Example 2: ideal sheaves of ℓ points

$\mathrm{Hilb}^{\ell}(X) \times B\mathbb{G}_m \to \mathrm{Hilb}^{\ell}(X)$ gms

Examples 3 and 4

Theorem A [FH, M '25]

$\mathcal{U}_{\alpha,\ell}$ has a gms, denoted $\mathrm{Hilb}^{\ell}(X)_{\alpha}$

Example 4: stack $\mathcal{U}^{\circ}_{\{\mathbb{P}^1\},1} \subset \mathcal{U}_{\{\mathbb{P}^1\},1}$

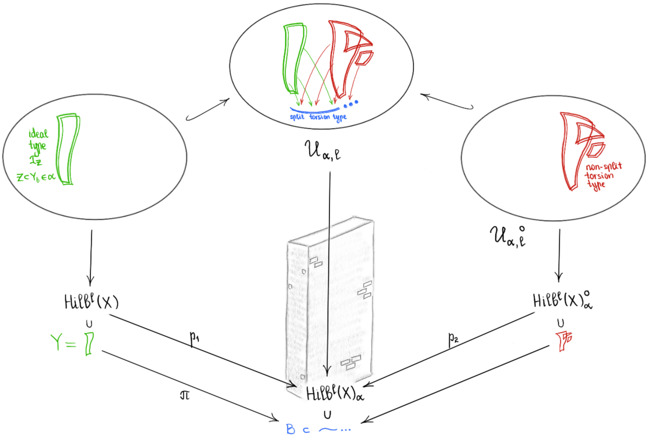
Take sheaves F of two types:

- $F \cong \mathcal{I}_x$ for $x \in X \setminus \mathbb{P}^1$
- fits into non-split SES $0 \to i_*(\mathcal{O}_{\mathbb{P}^1}(-1)) \to F \to \mathcal{I}_W \to 0$

Theorem B [FH, M '25]

$\mathcal{U}^{\circ}_{\alpha,\ell}$ has a gms, denoted $\mathrm{Hilb}^{\ell}(X)^{\circ}_{\alpha}$

Examples 3 and 4



Contracting rational curves

Known results

- [Reid '83] $\dim X = 3$,
  $\mathcal{N}_{\mathbb{P}^1/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ or $\mathcal{O} \oplus \mathcal{O}(-2))$
  $\Longrightarrow \mathbb{P}^1$ contracts
- [Jiménez '92] W – smooth curve of genus g – contracts
  $\Longleftrightarrow \exists$ l.b. $\mathcal{L}$ on a nbhd of W in X s.t.:
  $\deg(\mathcal{L}_{|W}) \leq g - 2$ and $H^1(\hat{X}, \hat{\mathcal{L}})$ is finite-dimensional

Corollary of Theorems A and B

Note: $\mathcal{N}_{\mathbb{P}^1/X}$ splits as $\bigoplus_i \mathcal{O}(n_i)$. If all $n_i \leq 0$, then:
$\mathbb{P}^1$ contracts in $\overline{X}$ and gets replaced by $\mathbb{P}H^1(\mathbb{P}^1, \mathcal{N}_{\mathbb{P}^1/X}(-1))$ in X°.

Abel-Jacobi contractions

$\dim X = 2$

$\alpha \subset \mathrm{Hilb}(X)$ open and closed subfamily of some curves

$\mathcal{C} \to \alpha$ universal curve

Abel-Jacobi morphism $AJ_\ell \colon \mathrm{Hilb}^\ell(\mathcal{C}/\alpha) \to \overline{\mathrm{Pic}}(\mathcal{C}/\alpha)_{\chi(\mathcal{O})-\ell}$

$$\begin{array}{ccccc} & & \mathrm{Hilb}^\ell(\mathcal{C}/\alpha) & \longrightarrow & \mathrm{Hilb}^\ell(X) \\ & \swarrow^{AJ_\ell} & \downarrow & & \downarrow \\ \overline{\mathrm{Pic}}(\mathcal{C}/\alpha)_{\chi(\mathcal{O})-\ell} & \hookleftarrow & H_\ell & \hookrightarrow & \mathrm{Hilb}^\ell(X)_\alpha \end{array}$$

A Mukai flop

$X \to \mathbb{P}^1$ – elliptic K3 with two singular fibers X_a, X_b of type I_3
$X_a = C_1 \cup C_2 \cup C_3$ – cycle of three $\mathbb{P}^1$'s
$\alpha = \{C_1, C_2, C_3\}$ – isolated points in $\mathrm{Hilb}(X)$

$\ell \geq 2 \implies$ condition $(\dagger)$ is satisfied

Corollary of Theorems A and B

- $\mathrm{Hilb}^\ell(X) \to \mathrm{Hilb}^\ell(X)_\alpha$ is a contraction of $\bigsqcup \mathrm{Hilb}^\ell(C_i)$
- $\mathrm{Hilb}^\ell(X)^\circ_\alpha$ is a Mukai flop

$\mathrm{Hilb}^2(X)_\alpha$ is not...

- ...a scheme
- ...a Bridgeland moduli space (all such admit a strictly nef l.b.)

$\mathbb{P}^1$-fibration surgeries, $\ell = 1$

X – smooth projective variety
$Y \subset X$ – subvariety with a $\mathbb{P}^1$-fibration $Y \to B$ s.t.
B corresponds to an open and closed subscheme $\alpha \subset \mathrm{Hilb}(X)$
$\mathcal{N}_{Y/X}$ is fiberwise non-positive

Corollary of Theorems A and B

We have a surgery diagram

$$X \xrightarrow{p_1} M_1 \xleftarrow{p_2} M_1^\circ$$

DK conjecture

X – smooth projective variety

$X \supset Y \to B$ – $\mathbb{P}^1$-fibration

$B \longleftrightarrow \alpha \subset \mathrm{Hilb}(X)$

Assume M_1° is smooth

$\mathcal{N}_{Y/X}$ is fiberwise non-positive

$$X \xrightarrow{p_1} M_1 \xleftarrow{p_2} M_1^\circ$$

DK hypothesis [Kawamata '18, Conj. 1.2]

If $\forall b \in B$: $\omega_X . Y_b \leq 0$, then
$\exists$ an embedding of an admissible subcategory $D^b(M_1^\circ) \to D^b(X)$.

DK conjecture

Theorem [Bridgeland '01]

M – complex projective threefold with terminal singularities,
$X \to M_1 \leftarrow M_1^\circ$ – crepant resolutions.
Then $D^b(M_1^\circ) \cong D^b(X)$.

DK conjecture

Theorem [Bridgeland '01]

M – complex projective threefold with terminal singularities,
$X \to M_1 \leftarrow M_1^\circ$ – crepant resolutions.
Then $D^b(M_1^\circ) \cong D^b(X)$.

Theorem [FH, M '25]

Verified the DK hypothesis in the above setting.
Our assumptions $\implies \omega_X . Y_b = -2, -1, 0$.

- Case $\omega_X . Y_b = -2$: $M_1^\circ = \varnothing$
- Case $\omega_X . Y_b = -1$: $M_1^\circ =$ Fujiki-Nakano contraction
- Case $\omega_X . Y_b = 0$: $D^b(M_1^\circ) \cong D^b(X)$

Thank you!