Motives Seminar-WS 2025/26

The motives seminar this term will be giving an introduction to matrix factorizations, maximal Cohen-Macaulay modules and singularity categories. We hope this will form a foundation for further work in developing versions of these theories that incorporate a duality in the underlying categories.

Outline of talks

I (two lectures). Matrix factorizations, maximal Cohen-Macaulay modules and the singularity category.

i. Present section 6 of Eisenbud's paper [7], especially Theorem 6.1 (i), (ii). First give a discussion about Gorenstein rings and maximal Cohen-Macaulay modules. Show that for A a regular local ring and $x_1, \ldots, x_r$ a regular sequence, the ring $B := A/(x_1, \ldots, x_r)$ is Gorenstein. Introduce the category of matrix factorisations $\mathrm{MF}(A, x)$ and prove Theorem 6.1 (i),(i) and Corollary 6.3.

ii. Present Theorem 4.4.1 in the paper by Buchweitz [3]. Define the stable category of S-modules, the stable category of maximal CM modules $\underline{\mathrm{MCM}(S)}$ (for S Gorenstein) as a triangulated category, the stabilized derived category (singularity category) $\underline{D^b(S)}$ and the homotopy category of acyclic complexes of projective modules $\underline{\mathrm{APC}(S)}$. Prove Theorem 4.4.1: For S Gorenstein, there are equivalences $\Omega_0 : \underline{\mathrm{APC}(S)} \to \underline{\mathrm{MCM}(S)}$ and $\iota_S : \underline{\mathrm{MCM}(S)} \to \underline{D^b(S)}$.

iii. For $R = A/(x)$ a local hypersurface, show that the homotopy category of matrix factorizations $\underline{\mathrm{MF}(A, x)}$ has a natural triangulated structure, and that $\underline{MF(A, x)}$ is equivalent to $\underline{\mathrm{MCM}(A/(x))}$ as triangulated categories. You can use §2 in [6] as a source.

II. (1 lecture) CM modules on quadrics. Present the paper [5]. Concentrate on Theorems 2.1 and 2.2. Sketch the material in §1 without giving proofs.

III. (1 lecture) Knörrer periodicity. Present Knörrer's paper [10], §1-3. You can ignore the statements on the Auslander-Reiten quiver.

IV. (four lectures) Present Dyckerhoff's paper [6], sections 2-6. Lecture 1: give an overview of Toën's paper [13], concentrating on the definitions and statements of main results as needed for Dyckerhoff's paper, without proofs. Lecture 2: §2,3, just the local case, Lecture 3: §4. Lecture 4: §5,6.

V. (1 lecture) Real Knörrer periodicity. Present the paper by Brown [2]

VI (1 lecture) Real Knörrer periodicity. Present the paper by Spellmann-Young [12].

VII (? lectures) Further directions, time permitting. These could include looking at more modern work on singularity categories, such as the papers of Preygel [11], Toën-Vezzosi [14], Blanc-Robalo-Toën-Vezzosi [1], or the two papers of Hennion-Holstein-Robalo [8, 9].

References

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