RECIPROCITY LAWS
FOR BALANCED DIAGONAL CLASSES

by

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Abstract. — This article constructs a 3-variable *balanced* diagonal class $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ in the cohomology of the Galois representation associated to a self-dual triple $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ of p-adic Hida families. Its first main result (Theorem A of Section 1.1) establishes an explicit reciprocity law relating $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ to the *unbalanced* Garrett–Rankin p-adic L-function attached to $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$. The class $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ arises from the p-adic interpolation of diagonal classes in the Bloch–Kato Selmer groups of the specialisations of $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ at balanced triples of classical weights. As a consequence, the value of $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ at a specialisation (f,g,h) of $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ at an unbalanced triple of classical weights is a p-adic limit of crystalline classes. Our second main result (Theorem B of Section 1.2) shows that the obstruction to the crystallinity of an appropriate derivative of $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ at (f,g,h) is encoded in the central critical value of the complex L-function of $f\otimes g\otimes h$.

To Bernadette Perrin-Riou on her 65th birthday

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1. Description and statement of results

The reciprocity laws alluded to in the title of this work concern the diagonal class arising in the cohomology of the big Galois representation attached to a self-dual triple of Hida p-adic families of cusp forms. Our construction of this class builds on the push-forward of a canonical generator of an invariant space of locally analytic functions along the diagonal morphism of a modular curve into the corresponding triple-product threefold. It constitutes a crucial step towards the proof of the main results of this paper and of those of our other contribution [**BSV20a**] to the present volume.

The specialisations of the diagonal class at triples of classical weights in the so-called *balanced* region, in which each weight is strictly smaller than the sum of the other two, give rise to cohomology classes admitting a similar description in terms of invariant theory which are closely related to diagonal cycles in Chow groups of Kuga–Sato varieties. As a consequence, the diagonal class belongs to a big Selmer group, called the balanced Selmer group, which interpolates in the geometric region of balanced weights the Bloch–Kato Selmer groups of the triple tensor product representations of the corresponding modular forms.

The first main result of this paper – Theorem A of Section 1.1 – pertains to the specialisation of the diagonal class to the three *unbalanced* regions where one weight is at least equal to the sum of the other two. The explicit reciprocity laws proved therein identify the image of the diagonal class by a branch of the Perrin-Riou big logarithm corresponding to the choice of unbalanced region as the 3-variable p-adic L-function interpolating the central critical values of the Garrett–Rankin complex L-functions attached to the triples of weights in that region.

Our second main result – Theorem B of Section 1.2 – proves that the specialisation of the diagonal class at an unbalanced point is crystalline at p if and only if the corresponding central critical value is zero. This criterion follows directly from the reciprocity law of Theorem A combined with Jacquet's conjecture proved by Harris–Kudla when the p-adic L-function for the corresponding unbalanced region does *not* have an exceptional zero in the sense of Mazur–Tate–Teitelbaum. The *exceptional cases* can only occur at unbalanced triples in which the modular form of dominant weight is multiplicative at p. These subtler cases require the proof of an exceptional zero formula for the 3-variable p-adic L-function, combined with an analysis of the derivatives of the Perrin-Riou logarithm at the unbalanced point and the costruction of an improved class.

Applications to the arithmetic of elliptic curves obtained from instances of the exceptional case constitute the object of the main results of our other contribution [**BSV20a**] to this volume, and represent one motivating feature of the present work. The Hida families considered in this setting respectively interpolate the weight-two modular form attached to an elliptic curve A over the rational numbers and two weight-one theta series associated to the same quadratic field K and subject to natural arithmetic conditions. In this setting, we establish a factorisation of the triple product p-adic L-function along the line $(k, 1, 1)$ as a product of two Hida–Rankin p-adic L-functions attached to A/K, which implies a relation between the fourth derivative at weights $(2,1,1)$ of the former p-adic L-function and the product of the second derivatives at $k=2$ of the latter. This translates into a formula for the Bloch–Kato logarithm of the specialisation of the diagonal class at $(2,1,1)$ as a product of formal group logarithms of Heegner points or Stark–Heegner points, depending respectively on whether K is imaginary quadratic or real quadratic. This result provides a bridge between the diagonal class arising from the geometry of higher dimensional varieties and the theory of rational points on elliptic curves, lending also some support to the conjecture on the rationality of Stark–Heegner points.

1.1. The three-variable reciprocity law. — Fix a prime $p \geqslant 5$, algebraic closures $\bar{\mathbf{Q}}$ and $\bar{\mathbf{Q}}_p$ of $\mathbf{Q}$ and $\mathbf{Q}_p$ respectively, and embeddings $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$ and $\bar{\mathbf{Q}} \hookrightarrow \mathbf{C}$. Let L be a finite extension of $\mathbf{Q}_p$ and let

$$\begin{aligned} \boldsymbol{f}^\sharp &= \sum_{n\geqslant 1} a_n(\boldsymbol{k}) \cdot q^n \in \mathcal{O}(U_{\boldsymbol{f}})[\![q]\!], \\ \boldsymbol{g}^\sharp &= \sum_{n\geqslant 1} b_n(\boldsymbol{l}) \cdot q^n \in \mathcal{O}(U_{\boldsymbol{g}})[\![q]\!] \\ \textit{and} \qquad \boldsymbol{h}^\sharp &= \sum_{n\geqslant 1} c_n(\boldsymbol{m}) \cdot q^n \in \mathcal{O}(U_{\boldsymbol{h}})[\![q]\!] \end{aligned}$$

be primitive, L-rational Hida p-adic families of modular forms of tame conductors $N_{\boldsymbol{f}}, N_{\boldsymbol{g}}$ and $N_{\boldsymbol{h}}$, centres k_o, l_o and m_o and tame characters $\chi_{\boldsymbol{f}}, \chi_{\boldsymbol{g}}$ and $\chi_{\boldsymbol{h}}$ respectively (cf. Section 5). Here $N_{\boldsymbol{f}}$ is a positive integer coprime to p, $U_{\boldsymbol{f}}$ is an L-rational open disc centred at $k_o \in \mathbf{Z}_{\geqslant 1}$ in the p-adic weight space $\mathcal{W}$, and $\mathcal{O}(U_{\boldsymbol{f}})$ is the ring of analytic functions on $U_{\boldsymbol{f}}$. For each k in $U_{\boldsymbol{f}}^{\mathrm{cl}} = \left\{k \in U_{\boldsymbol{f}} \cap \mathbf{Z}_{\geqslant 2} \mid k \equiv k_o \bmod 2(p-1)\right\}$ the weight-k specialisation $\boldsymbol{f}_k^\sharp = \sum_{n\geq 1} a_n(k) \cdot q^n \in L[\![q]\!] \cap S_k(N_{\boldsymbol{f}}p, \chi_{\boldsymbol{f}})$ is a p-stabilised newform of weight k, level $\Gamma_1(N_{\boldsymbol{f}}) \cap \Gamma_0(p)$ and character $\chi_{\boldsymbol{f}}$. In particular the p-th Fourier coefficient $a_p(\boldsymbol{k})$ is a unit in the ring $\Lambda_{\boldsymbol{f}}$ of functions $\alpha \in \mathcal{O}(U_{\boldsymbol{f}})$ satisfying $|\alpha(x)|_p \leqslant 1$ for all $x \in U_{\boldsymbol{f}}$. If $k > 2$ then $\boldsymbol{f}_k^\sharp$ is the ordinary p-stabilisation of a newform $f_k^\sharp$ in $S_k(N_{\boldsymbol{f}}, \chi_{\boldsymbol{f}})$. If $k=2$ then either $\boldsymbol{f}_2^\sharp = f_2^\sharp$ is new or it is the p-stabilisation of a newform $f_2^\sharp$ of level $N_{\boldsymbol{f}}$. A similar discussion applies to $\boldsymbol{g}^\sharp$ and $\boldsymbol{h}^\sharp$.

Let $(\boldsymbol{\xi}^\sharp, u_o)$ denote one of pairs $(\boldsymbol{f}^\sharp, k_o), (\boldsymbol{g}^\sharp, l_o)$ and $(\boldsymbol{h}^\sharp, m_o)$. If $u_o = 1$, then the weight-one specialisation $\boldsymbol{\xi}_1^\sharp$ of $\boldsymbol{\xi}^\sharp$ is a cuspidal-overconvergent (but not necessarily classical) ordinary modular form. Throughout the paper we make the following

Assumption 1.1. — *If $u_o = 1$, then $\boldsymbol{\xi}_1^\sharp$ is a p-stabilisation of a classical, cuspidal and p-regular newform of level $\Gamma_1(N_{\boldsymbol{\xi}})$, without real multiplication by a quadratic field in which p splits.*

A weight-one eigenform has *real multiplication* if it is equal to the theta series $\vartheta_\chi = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) \cdot q^{\mathbf{N}\mathfrak{a}}$ associated with a ray class character χ of a real quadratic field K, where $\mathfrak{a}$ runs over the non-zero ideals of $\mathcal{O}_K$ and $\mathbf{N}\mathfrak{a} = |\mathcal{O}_K/\mathfrak{a}|$. Moreover, a normalised weight-one eigenform $\xi = \sum_{n\geqslant 0} a_n(\xi) \cdot q^n$ of level $\Gamma_1(N_{\boldsymbol{\xi}})$ and character χ_ξ is said to be *p-regular* if its p-th Hecke polynomial $X^2 - a_p(\xi) \cdot X + \chi_\xi(p)$ is separable. We refer to Remarks 1.4 and to Section 5 below for explanations on the relevance of Assumption 1.1 for the main results of this paper.

Let N be the least common multiple of $N_{\boldsymbol{f}}, N_{\boldsymbol{g}}$ and $N_{\boldsymbol{h}}$. A *level-N test vector* for $(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$ is a triple $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ of Hida families of common tame level N, having $(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$ as associated triple of primitive families (cf. Section 5). For each k in $U_{\boldsymbol{f}}^{\mathrm{cl}}$ the weight-k specialisation $\boldsymbol{f}_k$ of $\boldsymbol{f}$ is an ordinary cusp form of weight k, level $\Gamma_1(N) \cap \Gamma_0(p)$ and character $\chi_{\boldsymbol{f}}$, which is an eigenvector for U_p and T_ℓ for all primes $\ell \nmid Np$, with the same eigenvalues as $\boldsymbol{f}_k^\sharp$. Similarly for $\boldsymbol{g}$ and $\boldsymbol{h}$. Fix a level-N test vector $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ for $(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$.

We make throughout this paper the following crucial *self-duality assumption.*

Assumption 1.2. — $\chi_{\boldsymbol{f}} \cdot \chi_{\boldsymbol{g}} \cdot \chi_{\boldsymbol{h}} = 1$.

Set $\Sigma = \tilde{U}_{\boldsymbol{f}}^{\mathrm{cl}} \times \tilde{U}_{\boldsymbol{g}}^{\mathrm{cl}} \times \tilde{U}_{\boldsymbol{h}}^{\mathrm{cl}}$, where $\tilde{U}_{\boldsymbol{f}}^{\mathrm{cl}} = U_{\boldsymbol{f}}^{\mathrm{cl}} \cup \{k_o\}$ (so that $\tilde{U}_{\boldsymbol{f}}^{\mathrm{cl}} = U_{\boldsymbol{f}}^{\mathrm{cl}}$ if $k_o \geqslant 2$), and $\tilde{U}_{\boldsymbol{g}}^{\mathrm{cl}}$ and $\tilde{U}_{\boldsymbol{h}}^{\mathrm{cl}}$ are defined similarly. Assumption 1.2 implies that $k + l + m$ is an even integer for all $w = (k, l, m)$ in $U_{\boldsymbol{f}}^{\mathrm{cl}} \times U_{\boldsymbol{g}}^{\mathrm{cl}} \times U_{\boldsymbol{h}}^{\mathrm{cl}}$, hence $c_w = (k + l + m - 2)/2$ is a positive integer. Let $\Sigma_{\boldsymbol{f}}$ be the set of w in Σ such that $k \geqslant l + m$, define similarly $\Sigma_{\boldsymbol{g}}$ and $\Sigma_{\boldsymbol{h}}$ and denote by Σ_{bal} the complement in Σ of the union of $\Sigma_{\boldsymbol{f}}$, $\Sigma_{\boldsymbol{g}}$ and $\Sigma_{\boldsymbol{h}}$. One calls Σ_{bal} the *balanced region.*

Denote by $\boldsymbol{\xi}$ one of the symbols $\boldsymbol{f}, \boldsymbol{g}$ and $\boldsymbol{h}$ and correspondingly by ξ one of f, g and h. Let $\mathscr{O}_{\boldsymbol{\xi}} = \Lambda_{\boldsymbol{\xi}}[1/p]$ be the space of bounded analytic functions on $U_{\boldsymbol{\xi}}$ and set $\mathscr{O}_{\boldsymbol{fgh}} = \mathscr{O}_{\boldsymbol{f}} \hat{\otimes}_L \mathscr{O}_{\boldsymbol{g}} \hat{\otimes}_L \mathscr{O}_{\boldsymbol{h}}$. Associated with $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ one has:

- *Garrett–Rankin square root p-adic L-functions* $\mathscr{L}_p^\xi(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ in $\mathscr{O}_{\boldsymbol{fgh}}$, interpolating the square roots of the central critical values $L(f_k^\sharp \otimes g_l^\sharp \otimes h_m^\sharp, c_w)$ of the complex Garrett–Rankin L-functions $L(f_k^\sharp \otimes g_l^\sharp \otimes h_m^\sharp, s)$ for classical triples $w = (k, l, m)$ in the region $\Sigma_{\boldsymbol{\xi}}$ (cf. Remark 1.8(1) and see Section 6 for details).
- An $\mathscr{O}_{\boldsymbol{fgh}}$-adic representation $V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ of $G_{\mathbf{Q}} = \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$, satisfying the following interpolation property (cf. Section 7.2). For each classical triple $w = (k, l, m)$ in Σ let $V(f_k^\sharp, g_l^\sharp, h_m^\sharp)$ be the central critical twist (i.e. the c_w-th Tate twist) of the tensor product of the Deligne representations of $f_k^\sharp, g_l^\sharp$ and $h_m^\sharp$. Then the base change $V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ of $V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ under evaluation at (k, l, m) on $\mathscr{O}_{\boldsymbol{fgh}}$ is isomorphic to $\bigoplus_{i=1}^a V(f_k^\sharp, g_l^\sharp, h_m^\sharp)$, for some integer $a \geqslant 1$ which is independent of $(k, l, m) \in \Sigma$ (cf. Section 7.2).
- A *balanced Selmer group* $H^1_{\mathrm{bal}}(\mathbf{Q}, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) \subset H^1(\mathbf{Q}, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$, which interpolates the Bloch–Kato Selmer groups $\mathrm{Sel}(\mathbf{Q}, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m))$ for all *balanced* triples $(k, l, m) \in \Sigma_{\mathrm{bal}}$ (cf. Section 7.2).
- Perrin-Riou big logarithms
$$\mathscr{L}_{\boldsymbol{\xi}} = \mathscr{L}og_{\boldsymbol{\xi}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) : H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) \longrightarrow \mathscr{O}_{\boldsymbol{fgh}},$$
satisfying the following interpolation properties. Say that $\boldsymbol{\xi} = \boldsymbol{f}$ to fix ideas. Then for all balanced triples $w = (k, l, m)$ in a subset of Σ_{bal} which is dense in $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}}$, and for all local balanced classes $\mathscr{Z}$ in $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$
$$\mathscr{L}_{\boldsymbol{f}}\big(\mathrm{res}_p(\mathscr{Z})\big)(w) = \mathscr{E}_{\boldsymbol{f}}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m) \cdot \log_p(\mathscr{Z}_w)\big(\eta^\alpha_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_l} \otimes \omega_{\boldsymbol{h}_m}\big).$$
Here $\mathscr{E}_{\boldsymbol{f}}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ is an explicit non-zero algebraic number, the class $\mathscr{Z}_w$ in $H^1_{\mathrm{fin}}(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m))$ is the specialisation of $\mathscr{Z}$ at w, $\log_p$ is the Bloch–Kato

logarithm and $\eta^{\alpha}_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_l} \otimes \omega_{\boldsymbol{h}_m}$ is the differential considered in Section 7.3, to which we refer for details.

According to a conjectural picture envisioned by Perrin-Riou the L-functions $\mathscr{L}_p^{\xi}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ should arise from a global balanced class via the logarithms $\mathscr{L}_{\boldsymbol{\xi}}$. Our first main result confirms this expectation.

Theorem A. — *There is a canonical class $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ in $H^1_{\mathrm{bal}}(\mathbf{Q}, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$ such that, for $\boldsymbol{\xi} = \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}$, one has*

$$\mathscr{L}_{\boldsymbol{\xi}}\big(\mathrm{res}_p\big(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})\big)\big) = \mathscr{L}_p^{\xi}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}).$$

Remarks 1.3. —

1. The equality displayed in Theorem A determines the class $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ only up to addition by an element in a suitable (conjecturally trivial) restricted Selmer group. Nonetheless Section 8.1 gives a geometric construction of a *canonical* three-variable balanced class $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ satisfying the conclusions of Theorem A.

2. Theorem 8.1 and Proposition 8.3 express the specialisation of $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ at a *balanced* triple $(k, l, m) \in \Sigma_{\mathrm{bal}}$ as an explicit multiple of a suitable Selmer *diagonal class* $\kappa(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m) \in \mathrm{Sel}(\mathbf{Q}, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m))$ associated in Section 3 with $(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ (cf. Proposition 3.2). The latter is in turn related to the values of $\mathscr{L}_p^{\xi}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ at (k, l, m) by an explicit reciprocity law (cf. Proposition 3.6). Theorem A then follows from analytic continuation.

3. Both the square-root p-adic L-function $\mathscr{L}_p^{\xi}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ and the big logarithm $\mathscr{L}_{\boldsymbol{\xi}} = \mathscr{L}og_{\boldsymbol{\xi}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ genuinely depend on the choice of the level-N test vector $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ for $(\boldsymbol{f}^{\sharp}, \boldsymbol{g}^{\sharp}, \boldsymbol{h}^{\sharp})$. On the other hand the big Galois representation $V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) = V_N(\boldsymbol{f}^{\sharp}, \boldsymbol{g}^{\sharp}, \boldsymbol{h}^{\sharp})$ and the balanced class
$$\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) = \kappa_N(\boldsymbol{f}^{\sharp}, \boldsymbol{g}^{\sharp}, \boldsymbol{h}^{\sharp})$$
depend on the test vector $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ only through its level N and the systems of eigenvalues defined by $(\boldsymbol{f}^{\sharp}, \boldsymbol{g}^{\sharp}, \boldsymbol{h}^{\sharp})$ (cf. Sections 5 and 8.1).

4. The construction of $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ given in Section 8.1 applies more generally to a triple $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ of (not necessarily ordinary) Coleman families. The p-adic L-function $\mathscr{L}_p^{\xi}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ has recently been constructed in [**AI21b**], and it is natural to wonder if one can generalise Theorem A to this setting.

Remark 1.4. — Let $(\boldsymbol{\xi}^{\sharp}, u_o)$ denote one of pairs $(\boldsymbol{f}^{\sharp}, k_o)$, $(\boldsymbol{g}^{\sharp}, l_o)$ and $(\boldsymbol{h}^{\sharp}, m_o)$. When $u_o = 1$, Assumption 1.1 guarantees that the big Galois representation $V(\boldsymbol{\xi})$ and its maximal $G_{\mathbf{Q}_p}$-unramified quotient $V(\boldsymbol{\xi})^-$ are free over $\mathscr{O}_{\boldsymbol{\xi}}$ (cf. Section 5 below for more details). It is likely that Theorem A can be proved without this assumption, at the cost of extending scalars to the fraction field of $\mathscr{O}_{\boldsymbol{fgh}}$ in the definition of $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ and in the statement of the explicit reciprocity law. On the other hand, the freeness of $V(\boldsymbol{\xi})$ and $V(\boldsymbol{\xi})^-$ are crucial in the proofs of Theorem B below and of the main result of our contribution [**BSV20a**].

Remark 1.5. — By using different methods, extending those of [**DR16**], the contribution of Darmon and Rotger [**DR20**] to this volume gives an alternate construction of the 3-variable diagonal class.

Remark 1.6. — The class $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ is constructed by interpolating diagonal classes in the Bloch-Kato Selmer groups $\mathrm{Sel}(\mathbf{Q}, V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m))$ for all triples $(k,l,m)\in\Sigma_{\mathrm{bal}}$. By using systems of étale sheaves attached to spaces of locally analytic functions and the big Abel–Jacobi map defined in equation (156), this geometric problem is reduced to the simpler one of constructing a canonical invariant in a space of locally analytic functions. This invariant element plays a central role in the construction, carried out in [**GS20**] (cf. also [**Hsi20**]), of a *balanced* triple-product p-adic L-function interpolating the square-roots of the central critical values $L(f_k^\sharp\otimes g_l^\sharp\otimes h_m^\sharp, c_w)$ for triples $w=(k,l,m)$ in the balanced region Σ_{bal}. We remark that a similar method can be applied in other settings, for example for the interpolation of generalised Heegner cycles. In this case, the relevant invariant function was instrumental for the definition in [**BD07**] of an anticyclotomic two-variable p-adic L-function. The resulting big Heegner class gives rise via an explicit reciprocity law to the p-adic L-functions considered in [**BDP13, AI21a**]. See also [**JLZ20**] for a related construction in the Heegner case.

1.2. Specialisations at unbalanced points. — Let $w_o=(k,l,m)$ be a classical triple in the unbalanced region $\Sigma_{\boldsymbol{f}}$. The following assumption will be in force in this section (cf. Remarks 1.8).

Assumption 1.7. — *The local sign $\varepsilon_\ell(f_k^\sharp,g_l^\sharp,h_m^\sharp)$ is equal to $+1$ for each rational prime ℓ.*

Theorem B stated below relates the specialisation of the big diagonal class $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ at w_o to the central value of the *complex* Garrett–Rankin L-function $L(f_k^\sharp\otimes g_k^\sharp\otimes h_k^\sharp,s)$. This relation is particularly intriguing and subtle when $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ has an *exceptional zero* at w_o in the sense of Mazur–Tate–Teitelbaum.

Let $\mathcal{H}_{\boldsymbol{g}}=\mathcal{H}_{\boldsymbol{g}}(w_o)$ be the *g-improving plane* in $U_{\boldsymbol{f}}\times U_{\boldsymbol{g}}\times U_{\boldsymbol{h}}$ defined by the equation

$$\boldsymbol{k}-\boldsymbol{l}+\boldsymbol{m}=k-l+m.$$

Let $\mathscr{O}_{\boldsymbol{gh}}=\mathscr{O}_{\boldsymbol{g}}\hat{\otimes}_L\mathscr{O}_{\boldsymbol{h}}$ and (shrinking $U_{\boldsymbol{g}}$ and $U_{\boldsymbol{h}}$ if necessary) let $\nu_{\boldsymbol{g}}:\mathscr{O}_{\boldsymbol{fgh}}\longrightarrow\mathscr{O}_{\boldsymbol{gh}}$ be the map sending $F(\boldsymbol{k},\boldsymbol{l},\boldsymbol{m})$ to its restriction $F(\boldsymbol{l}-\boldsymbol{m}+k+m-l,\boldsymbol{l},\boldsymbol{m})$ to $\mathcal{H}_{\boldsymbol{g}}$. Set $V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}}=V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})\otimes_{\nu_{\boldsymbol{g}}}\mathscr{O}_{\boldsymbol{gh}}$ and denote by

$$\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}}\in H^1(\mathbf{Q},V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}})$$

the image of $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ under the morphism induced in cohomology by $\nu_{\boldsymbol{g}}$. Define the analytic g-Euler factor

$$\mathcal{E}_g(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})=1-\frac{\bar{\chi}_{\boldsymbol{g}}(p)\cdot b_p(\boldsymbol{l})}{c_p(\boldsymbol{m})\cdot a_p(\boldsymbol{l}-\boldsymbol{m}+k+m-l)}\cdot p^{(k-l+m-2)/2}\in\mathscr{O}_{\boldsymbol{gh}}. \tag{1}$$

Section 9.3 proves the factorisation

$$\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}}=\mathcal{E}_g(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})\cdot\kappa_g^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \tag{2}$$

for a canonical *g-improved balanced diagonal class*

$$\kappa_g^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})\in H^1_{\mathrm{bal}}(\mathbf{Q}_p,V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}}).$$

This is not interesting nor surprising if $\mathcal{E}_g(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ does not vanish at w_o. On the other hand, if $\mathcal{E}_g(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)=0$ this implies that the specialisation of $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ at w_o vanishes independently of whether the complex L-function $L(f_k^\sharp\otimes g_l^\sharp\otimes h_m^\sharp,s)$ vanishes at the central point $s=c_{w_o}$. This phenomenon is the first source of exceptional zeros in the present setting. Since we are limiting our discussion to Hida families, the vanishing of $\mathcal{E}_g(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ at w_o is equivalent to the following conditions:

$$w_o=(2,1,1),\quad p\|c(\boldsymbol{f}_2),\quad p\nmid c(\boldsymbol{g}_1)\cdot c(\boldsymbol{h}_1)\ \text{ and }\ \chi_{\boldsymbol{h}}(p)\cdot a_p(2)\cdot b_p(1)=c_p(1), \tag{3}$$

where $c(\boldsymbol{f}_2),c(\boldsymbol{g}_1)$ and $c(\boldsymbol{h}_1)$ denote the conductors of $\boldsymbol{f}_2,\boldsymbol{g}_1$ and $\boldsymbol{h}_1$ respectively. In particular $\boldsymbol{g}_1$ and $\boldsymbol{h}_1$ are classical weight-one eigenforms.

The second source of exceptional zeros for $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ at w_o is of a different (non geometric) nature (cf. Section 9.2). It is related to the vanishing at w_o of the analytic f-*unbalanced* Euler factor

$$\mathcal{E}_f^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})=1-\frac{b_p(\boldsymbol{l})\cdot c_p(\boldsymbol{m})}{\bar{\chi}_{\boldsymbol{f}}(p)\cdot a_p(\boldsymbol{l}+\boldsymbol{m}+k-l-m)}p^{(k-l-m)/2}\in\mathscr{O}_{\boldsymbol{gh}}, \tag{4}$$

which on the f-*improving plane* in $U_{\boldsymbol{f}}\times U_{\boldsymbol{g}}\times U_{\boldsymbol{h}}$ defined by the equation

$$\boldsymbol{k}-\boldsymbol{l}-\boldsymbol{m}=k-l-m$$

interpolates a different Euler factor of $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$. In the present ordinary scenario, this vanishing is equivalent to the following conditions:

$$w_o=(2,1,1),\quad p\|c(\boldsymbol{f}_2),\quad p\nmid c(\boldsymbol{g}_1)\cdot c(\boldsymbol{h}_1)\ \text{ and }\ \chi_{\boldsymbol{f}}(p)\cdot b_p(1)\cdot c_p(1)=a_p(2). \tag{5}$$

We say that the unbalanced triple w_o in $\Sigma_{\boldsymbol{f}}$ is *exceptional* if the conditions displayed in Equation (3) or those displayed in Equation (5) are satisfied.

Remarks 1.8. —

1. Assumption 1.7 is in place to guarantee that for weights in the unbalanced region the Garrett–Rankin complex L-functions involved in the definition of the triple-product p-adic L-function have sign of the functional equation equal to $+1$, and that the corresponding central values can be described in terms of trilinear forms arising on $\mathrm{GL}_{2,\mathbf{Q}}$ (cf. [**HK91**]). On the other hand, Theorem A holds regardless of this assumption and does not exclude the possibility of vanishing of the diagonal class for sign reasons.

2. The exceptional zero condition (3) is symmetric in $\boldsymbol{g}$ and $\boldsymbol{h}$. Precisely, define $\mathcal{H}_{\boldsymbol{h}},V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_h},\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}}$ and $\mathcal{E}_h(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ by switching in the above definitions the roles of $\boldsymbol{g}$ of $\boldsymbol{h}$. Then

$$\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{h}}}=\mathcal{E}_h(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})\cdot\kappa_h^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$$

for a unique canonical h-*improved diagonal class* $\kappa_h^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ in the global Galois cohomology of $V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}}$.

3. The restriction of the class $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ to the plane $\mathcal{H}_{\boldsymbol{f}}$ also factors as the product of $\mathcal{E}_f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ and a canonical class $\kappa_f^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ in the Galois cohomology of $V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{f}}}$. This factorisation is uninteresting in the present setting, as the Euler factor $\mathcal{E}_f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ does not vanish at any classical point of the region $\Sigma_{\boldsymbol{f}}$.

4. Under Assumption 1.1, the exceptional zero conditions (3) and (5) are mutually exclusive. Indeed, if one of them holds, then the other is satisfied precisely if the form $g_1^\sharp$ (or equivalently $h_1^\sharp$) is p-irregular.

Define the *diagonal class*

$$\kappa^*(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m) \in H^1(\mathbf{Q}, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m))$$

by the following recipe. If the conditions stated in Equation (3) are *not* satisfied, then

$$\kappa^*(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m) = \kappa(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$$

is the specialisation of $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ at the classical triple $w_o = (k, l, m)$. If Equation (3) is satisfied, one defines

$$\kappa^*(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1) = \kappa_g^*(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1),$$

where the global class $\kappa_g^*(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1)$ is the specialisation of the g-improved diagonal class $\kappa_g^*(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ at $w_o = (2, 1, 1)$. (Note that $\kappa_h^*(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1) = -\kappa_g^*(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1)$.)

Theorem B. — *The diagonal class $\kappa^*(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ is crystalline at p if and only if the complex L-function $L(f_k^\sharp \otimes g_l^\sharp \otimes h_m^\sharp, s)$ vanishes at $s = \frac{k+l+m-2}{2}$.*

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2. Cohomology of modular curves

In a first reading of this paper it will be sufficient to get acquainted with the main definitions and notations of this section. The precise description of the various Hecke operators will be necessary for crucial computations in the arguments of later sections (see in particular Section 8). The exposition follows [**Kat04**, Section 2].

2.1. Modular curves. — Let $M \geqslant 1$ and $N \geqslant 1$ be positive integers such that $M + N \geqslant 5$. Denote by

$$Y(M, N) \longrightarrow \mathrm{Spec}(\mathbf{Z}[1/MN])$$

the scheme which represents the functor

$$S \longmapsto \big\{\text{isomorphism classes of } S\text{-triples } (E, P, Q)\big\},$$

where S is a $\mathbf{Z}[1/MN]$-scheme, E is an elliptic curve over S, and P and Q are sections of E over S such that $M \cdot P = 0$, $N \cdot Q = 0$ and the map $\mathbf{Z}/M\mathbf{Z} \times \mathbf{Z}/N\mathbf{Z} \to E$ which on (a, b) takes the value $a \cdot P + b \cdot Q$ is injective. More generally, for each rational prime $\ell \geqslant 1$, we consider as in [**Kat04**] the schemes

$$Y(M(\ell), N) \longrightarrow \mathbf{Z}[1/\ell MN] \quad \text{and} \quad Y(M, N(\ell)) \longrightarrow \mathbf{Z}[1/\ell MN].$$

The $\mathbf{Z}[1/\ell MN]$-scheme $Y(M(\ell), N)$ classifies 4-tuples (E, P, Q, C), where (E, P, Q) is as above and C is a cyclic subgroup of E of order ℓM which contains P and is complementary to Q (viz. the map $\mathbf{Z}/N\mathbf{Z} \times C \to E$ which sends (a, x) to $a \cdot Q + x$ is injective). Similarly $Y(M, N(\ell))$ classifies 4-tuples (E, P, Q, C) where C is a cyclic subgroup of order ℓN which contains Q and is complementary to P. Denote by

$$\begin{aligned} E(M, N) &\longrightarrow Y(M, N), \\ E(M(\ell), N) &\longrightarrow Y(M(\ell), N) \\ \text{and} \quad E(M, N(\ell)) &\longrightarrow Y(M, N(\ell)) \end{aligned}$$

the universal elliptic curves over $Y(M, N)$, $Y(M(\ell), N)$ and $Y(M, N(\ell))$ respectively.

Let $\mathbf{H} = \{z \in \mathbf{C} \mid \Im(z) > 0\}$ be the Poincaré upper half-plane and set

$$\Gamma(M, N) = \left\{\gamma \text{ in } \mathrm{SL}_2(\mathbf{Z}) \text{ such that } \gamma \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \bmod \left(\begin{smallmatrix} M & M \\ N & N \end{smallmatrix}\right)\right\}.$$

Then

$$Y(M, N)(\mathbf{C}) \cong (\mathbf{Z}/M\mathbf{Z})^* \times \Gamma(M, N)\backslash\mathbf{H}, \tag{6}$$

where the class of (a, z) in $(\mathbf{Z}/M\mathbf{Z})^* \times \mathbf{H}$ corresponds to the isomorphism class of the triple $(\mathbf{C}/\mathbf{Z} \oplus \mathbf{Z}z, az/M, 1/N)$. The Riemann surfaces $Y(M(\ell), N)(\mathbf{C})$ and $Y(M, N(\ell))(\mathbf{C})$ admit a similar complex uniformisation by $(\mathbf{Z}/M\mathbf{Z})^* \times \mathbf{H}$.

There is an isomorphism of $\mathbf{Z}[1/\ell MN]$-schemes

$$\varphi_\ell : Y(M, N(\ell)) \cong Y(M(\ell), N)$$

which on the 4-tuple $(E, P, Q, C)_{/S}$ in $Y(M, N(\ell))$ (for some $\mathbf{Z}[1/\ell MN]$-scheme S) takes the value

$$\varphi_\ell(E, P, Q, C) = \left(E/NC, P + NC, \ell^{-1}(Q) \cap C + NC, \left(\ell^{-1}(\mathbf{Z} \cdot P) + NC\right)/NC\right),$$

where $\ell^{-1}(\cdot)$ is the inverse image of $\cdot$ under multiplication by ℓ on E. On complex points (cf. Equation (6)) this is induced by the map $(\mathbf{Z}/M\mathbf{Z})^* \times \mathbf{H} \longrightarrow (\mathbf{Z}/M\mathbf{Z})^* \times \mathbf{H}$ which sends (a, z) to $(a, \ell \cdot z)$. If

$$\varphi_\ell^*(E(M(\ell), N)) \longrightarrow Y(M, N(\ell))$$

denotes the base change of $E(M(\ell), N) \to Y(M(\ell), N)$ under φ_ℓ, there is a natural degree-ℓ isogeny

$$\lambda_\ell : E(M, N(\ell)) \longrightarrow \varphi_\ell^*(E(M(\ell), N)).$$

When $M = 1$ one denotes by

$$Y_1(N) = Y(1, N) \quad \text{and} \quad Y_1(N, \ell) = Y(1, N(\ell)) \tag{7}$$

the affine modular curves over $\mathbf{Z}[1/N]$ and $\mathbf{Z}[1/N\ell]$ corresponding to the subgroups $\Gamma_1(N)$ and $\Gamma_1(N, \ell) = \Gamma_1(N) \cap \Gamma_0(\ell^{\mathrm{ord}_l(N)+1})$ of $\mathrm{SL}_2(\mathbf{Z})$ respectively. Similarly one writes

$$E_1(N) = E(1, N) \quad \text{and} \quad E_1(N, \ell) = E(1, N(\ell))$$

for the universal elliptic curves over $Y_1(N)$ and $Y_1(N, \ell)$ respectively.

2.2. Degeneracy maps. — Let M and N be as in the previous section, and let ℓ be a rational prime. Let

$$Y(M,N\ell) \xrightarrow{\mu_\ell} Y(M,N(\ell)) \xrightarrow{\nu_\ell} Y(M,N)$$
$$\text{and}\quad Y(M\ell,N) \xrightarrow{\check{\mu}_\ell} Y(M(\ell),N) \xrightarrow{\check{\nu}_\ell} Y(M,N)$$

be the natural degeneracy maps (e.g. $\mu_\ell(E,P,Q) = (E,P,\ell\cdot Q,\mathbf{Z}\cdot Q)$ and $\nu_\ell(E,P,Q,C) = (E,P,Q)$), and define

$$\mathrm{pr}_1 : Y(M,N\ell) \longrightarrow Y(M,N) \quad\text{and}\quad \mathrm{pr}_\ell : Y(M,N\ell) \longrightarrow Y(M,N)$$

by the formulae

$$\mathrm{pr}_1(E,P,Q) = (E,P,\ell\cdot Q) \quad\text{and}\quad \mathrm{pr}_\ell(E,P,Q) = (E/N\mathbf{Z}\cdot Q, P+N\mathbf{Z}\cdot Q, Q+N\mathbf{Z}\cdot Q).$$

Under the isomorphism (6) the map pr_1 (resp., pr_ℓ) is induced by the identity (resp., multiplication by ℓ) on the complex upper half-plane $\mathbf{H}$. Unwinding the definitions one easily checks the identities

$$\mathrm{pr}_1 = \nu_\ell \circ \mu_\ell \quad\text{and}\quad \mathrm{pr}_\ell = \check{\nu}_\ell \circ \varphi_\ell \circ \mu_\ell. \tag{8}$$

The degeneracy maps $\mu_\ell, \check{\mu}_\ell, \nu_\ell, \check{\nu}_\ell, \mathrm{pr}_1$ and pr_ℓ are finite étale morphisms of $\mathbf{Z}[1/MN\ell]$-schemes.

2.3. Relative Tate modules and Hecke operators. — Let N, M and ℓ be as in the previous section and let S be a $\mathbf{Z}[1/MN\ell p]$-scheme. For every $\mathbf{Z}[1/MN\ell p]$-scheme X write $X_S = X \times_{\mathbf{Z}[1/MN\ell p]} S$ and denote by $A = A_X$ either the locally constant sheaf $\mathbf{Z}/p^m\mathbf{Z}(j)$ or the locally constant p-adic sheaf (cf. [FK88, Definition 12.6]) $\mathbf{Z}_p(j)$ on $X_{\text{ét}}$, for fixed $m \geqslant 1$ and $j \in \mathbf{Z}$. Moreover fix an integer $r \geqslant 0$.

The previous sections yield the following commutative diagram, in which the smaller squares are cartesian.

(9)

$$\begin{tikzcd}
E(M,N)_S \arrow[d, "v_{M,N}"'] & E(M,N(\ell))_S \arrow[l] \arrow[r, "\lambda_\ell"] \arrow[d, "v_{M,N(\ell)}"'] & \varphi_\ell^*(E(M(\ell),N)_S) \arrow[r] \arrow[d] & E(M(\ell),N)_S \arrow[r] \arrow[d, "v_{M(\ell),N}"'] & E(M,N)_S \arrow[d, "v_{M,N}"] \\
Y(M,N)_S & Y(M,N(\ell))_S \arrow[l, "\nu_\ell"'] \arrow[r, equal] & Y(M,N(\ell))_S \arrow[r, "\varphi_\ell"] & Y(M(\ell),N)_S \arrow[r, "\check{\nu}_\ell"] & Y(M,N)_S
\end{tikzcd}$$

Here $v_{M,N}, v_{M(\ell),N}$ and $v_{M,N(\ell)}$ are the structural maps, one writes again ν_ℓ and $\check{\nu}_\ell$ (resp., λ_ℓ) for the base changes to S of the corresponding degeneracy maps (resp., isogeny), and the unlabelled maps are the natural projections.

If $Y(\cdot)_S$ denotes one of $Y(M,N)_S, Y(M(\ell),N)_S$ and $Y(M,N(\ell))_S$, set

$$\mathscr{T}_\cdot(A) = R^1 v_{\cdot *}\mathbf{Z}_p(1) \otimes_{\mathbf{Z}_p} A \quad\text{and}\quad \mathscr{T}_\cdot^*(A) = \mathrm{Hom}_A(\mathscr{T}_\cdot(A), A). \tag{10}$$

Here $R^q v_{\cdot *}$ is the q-th right derivative of $v_{\cdot *} : E(\cdot)_{\text{ét}} \longrightarrow Y(\cdot)_{\text{ét}}$ and one calls

$$\mathscr{T}_\cdot \overset{\text{def}}{=} \mathscr{T}_\cdot(\mathbf{Z}_p)$$

the *relative Tate module* of the universal elliptic curve $E(\cdot) \longrightarrow Y(\cdot)$. The perfect cup-product pairing

$$\mathscr{T}_\cdot \otimes_{\mathbf{Z}_p} \mathscr{T}_\cdot \longrightarrow R^2 v_{\cdot *}\mathbf{Z}_p(2)$$

and the relative trace $R^2 v_{\cdot *}\mathbf{Z}_p \cong \mathbf{Z}_p(-1)$ give the perfect relative Weil pairing

$$\langle \cdot, \cdot \rangle_{E(\cdot)_{p^\infty}} : \mathscr{T}_{\cdot} \otimes_{\mathbf{Z}_p} \mathscr{T}_{\cdot} \longrightarrow \mathbf{Z}_p(1), \tag{11}$$

under which one identifies $\mathscr{T}_{\cdot}(-1)$ with $\mathscr{T}_{\cdot}^* = \mathrm{Hom}_{\mathbf{Z}_p}(\mathscr{T}_{\cdot}, \mathbf{Z}_p)$. It is a consequence of the smooth base change theorem (cf. Corollary 4.2, Chapter IV of [Mil80]) that $\mathscr{T}_{\cdot}(A)$ and $\mathscr{T}_{\cdot}^*(A)$ are locally constant p-adic sheaves on $Y_1(N)_S$, of formation compatible with base changes along morphisms of $\mathbf{Z}[1/NM\ell p]$-schemes $S' \to S$. (This justifies the choice to suppress the dependence on S from the notations.) Define

$$\mathscr{L}_{\cdot,r}(A) = \mathrm{Tsym}^r_A \mathscr{T}_{\cdot}(A) \quad \text{and} \quad \mathscr{S}_{\cdot,r}(A) = \mathrm{Symm}^r_A \mathscr{T}_{\cdot}^*(A),$$

where for any finite free module M over a profinite $\mathbf{Z}_p$-algebra R one denotes by $\mathrm{Tsym}^r_R M$ the R-submodule of symmetric tensors in $M^{\otimes r}$ and by $\mathrm{Symm}^r_R M$ the maximal symmetric quotient of $M^{\otimes r}$.

Notation. — When $Y(\cdot)_S = Y(1,N)_S$ is the modular curve $Y_1(N)_S$ associated with the congruence subgroup $\Gamma_1(N)$, and the level N is clear from the context, we use the simplified notations

$$\mathscr{L}_r(A) = \mathscr{L}_{1,N,r}(A), \quad \mathscr{L}_r = \mathscr{L}_r(\mathbf{Z}_p), \quad \mathscr{S}_r(A) = \mathscr{S}_{1,N,r}(A) \text{ and } \mathscr{S}_r = \mathscr{S}_r(\mathbf{Z}_p). \tag{12}$$

If there is no risk of confusion, we use the same simplified notations to denote the étale sheaves $\mathscr{L}_{1,N(\ell),r}(A)$ and $\mathscr{S}_{1,N(\ell),r}(A)$ on the modular curve $Y(1,N(\ell))_S = Y_1(N,\ell)_S$ of level $\Gamma_1(N) \cap \Gamma_0(\ell^{\mathrm{ord}_l(N)+1})$ (cf. Equation (7)).

Throughout the rest of this section let $\mathscr{F}^r_{\cdot}$ denote either $\mathscr{L}_{\cdot,r}(A)$ or $\mathscr{S}_{\cdot,r}(A)$. According to the proper base change theorem [Mil80, Chapter VI, Corollary 2.3] and the diagram (9), associated with the finite étale morphisms ν_ℓ and $\check{\nu}_\ell$ one has natural isomorphisms

$$\nu_\ell^*\big(\mathscr{F}^r_{M,N}\big) \cong \mathscr{F}^r_{M,N(\ell)} \quad \text{and} \quad \check{\nu}_\ell^*\big(\mathscr{F}^r_{M,N}\big) \cong \mathscr{F}^r_{M(\ell),N}, \tag{13}$$

which induce pullbacks

$$\begin{tikzcd} & H^i_{\text{ét}}(Y(M,N)_S, \mathscr{F}^r_{M,N}) \arrow[dl, "\nu_\ell^*"'] \arrow[dr, "\check{\nu}_\ell^*"] & \\ H^i_{\text{ét}}(Y(M,N(\ell))_S, \mathscr{F}^r_{M,N(\ell)}) & & H^i_{\text{ét}}(Y(M(\ell),N)_S, \mathscr{F}^r_{M(\ell),N}) \end{tikzcd} \tag{14}$$

and traces (cf. [Mil80, Lemma 1.12, pag. 168])

$$\begin{tikzcd} & H^i_{\text{ét}}(Y(M,N)_S, \mathscr{F}^r_{M,N}) & \\ H^i_{\text{ét}}(Y(M,N(\ell))_S, \mathscr{F}^r_{M,N(\ell)}) \arrow[ur, "\nu_{\ell *}"] & & H^i_{\text{ét}}(Y(M(\ell),N)_S, \mathscr{F}^r_{M(\ell),N}) \arrow[ul, "\check{\nu}_{\ell *}"'] \end{tikzcd} \tag{15}$$

Similarly the (finite étale) isogeny λ_ℓ induces morphisms

$$\lambda_{\ell *} : \mathscr{F}^r_{M,N(\ell)} \longrightarrow \varphi_\ell^*\big(\mathscr{F}^r_{M(\ell),N}\big) \quad \text{and} \quad \lambda_\ell^* : \varphi_\ell^*\big(\mathscr{F}^r_{M(\ell),N}\big) \longrightarrow \mathscr{F}^r_{M,N(\ell)}. \tag{16}$$

More precisely, associated with the ℓ-isogeny λ_ℓ there is a trace $\lambda_{\ell *} \circ \lambda_\ell^* \longrightarrow \mathrm{id}$. As $v \circ \lambda_\ell = v_{M,N(\ell)}$, where $v : \varphi_\ell^*(E(M(\ell),N)_S) \to Y(M(\ell),N)_S$ is the first projection, it induces a map $v_{M,N(\ell)*} \circ \lambda_\ell^* \longrightarrow v_*$. Applying R^1 and using the natural isomorphisms

$\varphi_\ell^*(R^1 v_{M(\ell),N*}\mathbf{Z}_p(1)) \cong R^1 v_*\mathbf{Z}_p(1)$ and $\lambda_\ell^*\mathbf{Z}_p(1) \cong \mathbf{Z}_p(1)$, this in turn induces a morphism $R^1 v_{M,N(\ell)*}\mathbf{Z}_p(1) \longrightarrow \varphi_\ell^*\big(R^1 v_{M(\ell),N*}\mathbf{Z}_p(1)\big)$, and finally the push-forwards $\lambda_{\ell*}$ which appear in Equation (16). The pullbacks are defined similarly, after replacing the trace $\lambda_{\ell*}\circ\lambda_\ell^* \longrightarrow \mathrm{id}$ with the adjunction morphism $\mathrm{id} = \lambda_{\ell*}\circ\lambda_\ell^*$. Together with φ_ℓ the previous morphisms give a pushforward

$$\Phi_{\ell*} = \varphi_{\ell*}\circ\lambda_{\ell*} : H^i_{\text{ét}}(Y(M,N(\ell))_S, \mathscr{F}^r_{M,N(\ell)}) \longrightarrow H^i_{\text{ét}}(Y(M(\ell),N)_S, \mathscr{F}^r_{M(\ell),N}) \tag{17}$$

and a pullback

$$\Phi_\ell^* = \lambda_\ell^*\circ\varphi_\ell^* : H^i_{\text{ét}}(Y(M(\ell),N)_S, \mathscr{F}^r_{M(\ell),N}) \longrightarrow H^i_{\text{ét}}(Y(M,N(\ell))_S, \mathscr{F}^r_{M,N(\ell)}).$$

Define the *dual ℓ-th Hecke operator*

$$T'_\ell = \nu_{\ell*}\circ\Phi_\ell^*\circ\check{\nu}_\ell^* : H^i_{\text{ét}}(Y(M,N)_S, \mathscr{F}^r_{M,N}) \longrightarrow H^i_{\text{ét}}(Y(M,N)_S, \mathscr{F}^r_{M,N}).$$

We also consider the *ℓ-th Hecke operator*

$$T_\ell = \check{\nu}_{\ell*}\circ\Phi_{\ell*}\circ\nu_\ell^* : H^i_{\text{ét}}(Y(M,N)_S, \mathscr{F}^r_{M,N}) \longrightarrow H^i_{\text{ét}}(Y(M,N)_S, \mathscr{F}^r_{M,N}).$$

As customary, if the prime ℓ divides MN, we also denote by U_ℓ and U'_ℓ the Hecke operators T_ℓ and T'_ℓ respectively.

For each profinite $\mathbf{Z}_p$-algebra R and each finite free R-module M, the evaluation map induces a perfect pairing

$$\mathrm{Tsym}^r_R M \otimes_R \mathrm{Symm}^r_R M^* \longrightarrow R,$$

where $M^* = \mathrm{Hom}_R(M, \mathbf{Z}_p)$. This defines a perfect pairing $\mathscr{L}_r \otimes_{\mathbf{Z}_p} \mathscr{S}_r \longrightarrow \mathbf{Z}_p$, hence a cup-product

$$\langle\cdot,\cdot\rangle_N : H^1_{\text{ét}}(Y_1(N)_{\bar{\mathbf{Q}}}, \mathscr{L}_r(1)) \otimes_{\mathbf{Z}_p} H^1_{\text{ét,c}}(Y_1(N)_{\bar{\mathbf{Q}}}, \mathscr{S}_r) \longrightarrow H^2_{\text{ét,c}}(Y_1(N)_{\bar{\mathbf{Q}}}, \mathbf{Z}_p(1)) \cong \mathbf{Z}_p, \tag{18}$$

which by Poincaré duality is perfect after inverting p. The Hecke operators $T^{\cdot}_\ell$ induce endomorphisms on the compactly supported cohomology $H^1_{\text{ét,c}}(Y_1(N)_{\bar{\mathbf{Q}}}, \mathscr{S}_r)$, and by construction T_ℓ and T'_ℓ (resp., T'_ℓ and T_ℓ) are adjoint to each other under $\langle\cdot,\cdot\rangle_N$. In addition, the Eichler–Shimura isomorphism (cf. Chapter 8 of [**Shi71**])

$$H^1_{\text{ét}}(Y_1(N)_{\bar{\mathbf{Q}}}, \mathscr{L}_r) \otimes_{\mathbf{Z}_p} \mathbf{C} \cong M_{r+2}(N, \mathbf{C}) \oplus \overline{S_{r+2}(N, \mathbf{C})} \tag{19}$$

(depending on a fixed embedding $\mathbf{Z}_p \hookrightarrow \mathbf{C}$) commutes with the action of the Hecke operators $T^{\cdot}_\ell$ on both sides.

After replacing the left hand square in the diagram (9) with the cartesian square

$$\begin{tikzcd} E(M,N\ell)_S \arrow[r] \arrow[d, "v_{M,N\ell}"'] & E(M,N(\ell))_S \arrow[d, "v_{M,N(\ell)}"] \\ Y(M,N\ell)_S \arrow[r, "\mu_\ell"] & Y(M,N(\ell))_S \end{tikzcd}$$

one defines as in Equations (14) and (15) the maps μ_ℓ^* and $\mu_{\ell*}$. For $\cdot = 1, \ell$ one can also define as above morphisms

$$H^i_{\text{ét}}(Y(M,N\ell)_S, \mathscr{F}^r_{M,N\ell}) \xrightarrow{\mathrm{pr}_{\cdot*}} H^i_{\text{ét}}(Y(M,N)_S, \mathscr{F}^r_{M,N}) \xrightarrow{\mathrm{pr}_{\cdot}^*} H^i_{\text{ét}}(Y(M,N\ell)_S, \mathscr{F}^r_{M,N\ell}), \tag{20}$$

which according to Equation (8) satisfy the identities

$$\mathrm{pr}_{1*} = \nu_{\ell*} \circ \mu_{\ell*}, \quad \mathrm{pr}_1^* = \mu_\ell^* \circ \nu_\ell^*, \quad \mathrm{pr}_{\ell*} = \check{\nu}_{\ell*} \circ \Phi_{\ell*} \circ \mu_{\ell*} \ \text{ and } \ \mathrm{pr}_\ell^* = \mu_\ell^* \circ \Phi_\ell^* \circ \check{\nu}_\ell^*. \tag{21}$$

As a consequence, if $\deg(\mu_\ell)$ denotes the degree of the finite morphism μ_ℓ, one has the relations

$$\deg(\mu_\ell) \cdot T_\ell = \mathrm{pr}_{\ell*} \circ \mathrm{pr}_1^* \quad \text{and} \quad \deg(\mu_\ell) \cdot T'_\ell = \mathrm{pr}_{1*} \circ \mathrm{pr}_\ell^*. \tag{22}$$

2.3.1. *Diamond and Atkin–Lehner operators.* — We recall here the geometric definition of the diamond and Atkin–Lehner operators on the cohomology groups $H^i_{\text{ét}}(Y(\cdot)_S, \mathscr{F}^r_\cdot)$ (where $\mathscr{F}^r_\cdot$ are the sheaves introduced in the previous section). For simplicity we limit the discussion to the modular curves $Y_1(\cdot)$ of level $\Gamma_1(\cdot)$, and denote by $\mathscr{F}_r$ the étale sheaf $\mathscr{F}^r_{1,\cdot}$ on $Y_1(\cdot)_S$.

For every unit d in $(\mathbf{Z}/N\mathbf{Z})^*$ the *diamond operator* $\langle d\rangle : Y_1(N)_S \to Y_1(N)_S$ is the automorphism of $Y_1(N)_S$ defined on the moduli problem by sending (E, P) to $(E, d \cdot P)$. Denote by $P_1(N)$ the universal point of order N of $E_1(N)_S$. The pair $(E_1(N)_S, d \cdot P_1(N))$ is an elliptic curve with $\Gamma_1(N)$-level structure over $Y_1(N)_S$, hence there exists a unique isomorphism $\langle d\rangle : E_1(N)_S \cong E_1(N)_S$ which makes the following diagram cartesian:

$$\begin{tikzcd} E_1(N)_S \arrow[r, "\langle d\rangle"] \arrow[d, "v_N"'] & E_1(N)_S \arrow[d, "v_N"] \\ Y_1(N)_S \arrow[r, "\langle d\rangle"] & Y_1(N)_S. \end{tikzcd}$$

This induces automorphisms $\langle d\rangle = \langle d\rangle^*$ and $\langle d\rangle' = \langle d\rangle_*$ of $H^i_{\text{ét}}(Y_1(N)_S, \mathscr{F}_r)$ which are inverse to each other.

Assume in the rest of this Section 2.3.1 that p does not divide N and that S is a scheme over $\mathbf{Z}[1/N, \mu_p]$. Set $\zeta_p = e^{2\pi i/p}$. For every elliptic curve E denote by E_p the kernel of multiplication by p and by $\langle \cdot, \cdot \rangle_{E_p} : E_p \times E_p \to \mu_p$ the Weil pairing. Since $p \nmid N$ the curve $Y_1(Np)$ classifies triples (E, P, Q), where E is an elliptic curve and P (resp., Q) is a point of exact order N (resp., p). (More precisely a pair (E, P_{Np}), where E is an elliptic curve over and P_{Np} is a section of exact order Np, corresponds in the above identification to the triple $(E, p \cdot P_{Np}, N \cdot P_{Np})$.) The *Atkin–Lehner operator* $w_p = w_{\zeta_p} : Y_1(Np)_S \cong Y_1(Np)_S$ is the automorphism of $Y_1(Np)_S$ defined by

$$w_p(E, P, Q) = (E/\mathbf{Z} \cdot Q, P + \mathbf{Z} \cdot Q, Q' + \mathbf{Z} \cdot Q),$$

where $Q' \in E_p$ is characterized by $\langle Q, Q'\rangle_{E_p} = \zeta_p$. There is a natural commutative diagram

$$\begin{tikzcd} E_1(Np)_S \arrow[r, "\check{w}_p"] \arrow[d, "v_{Np}"'] & w_p^*(E_1(Np))_S \arrow[r] \arrow[d] & E_1(Np)_S \arrow[d, "v_{Np}"] \\ Y_1(Np)_S \arrow[r, equal] & Y_1(Np)_S \arrow[r, "w_p"] & Y_1(Np)_S, \end{tikzcd}$$

in which the right-hand square is cartesian and $\check{w}_p$ is a degree-p isogeny. As in Equations (13)–(17), associated with the previous diagram one has a *Atkin–Lehner operator*

$$w_p : H^i_{\text{ét}}(Y_1(Np)_S, \mathscr{F}_r) \xrightarrow{w_p^*} H^i_{\text{ét}}(Y_1(Np)_S, w_p^*(\mathscr{F}_r)) \xrightarrow{\check{w}_p^*} H^i_{\text{ét}}(Y_1(Np)_S, \mathscr{F}_r)$$

and a *dual Atkin–Lehner operator*

$$w_p' : H^i_{\text{ét}}(Y_1(Np)_S, \mathscr{F}_r) \xrightarrow{\check{w}_{p*}} H^i_{\text{ét}}(Y_1(Np)_S, w_p^*(\mathscr{F}_r)) \xrightarrow{w_{p*}} H^i_{\text{ét}}(Y_1(Np)_S, \mathscr{F}_r).$$

More generally, let Q be a divisor of Np such that Q and Np/Q are coprime. After replacing the pair (p, N) with $(Q, Np/Q)$ in the previous construction, one defines the *Atkin–Lehner operators* w_Q on $H^1_{\text{ét}}(Y_1(Np)_S, \mathscr{F}_r)$.

2.4. Deligne representations. — Let

$$f = \sum_{n \geqslant 1} a_n(f) q^n \in S_k(N, \chi_f)$$

be a normalised cusp form of weight $k \geqslant 2$, level $\Gamma_1(N)$ and character χ_f. Set $N_o = N/p^{\mathrm{ord}_p(N)}$ and assume that f is an eigenvector for the Hecke operator T_ℓ for every prime $\ell \nmid N_o$. (In particular f is an eigenvector for U_p if p divides N.)

Let $L/\mathbf{Q}_p$ be a finite extension containing the Fourier coefficients of f. Define

$$H^1_{\text{ét}}(Y_1(N)_{\bar{\mathbf{Q}}}, \mathscr{L}_{k-2}(1))_L \twoheadrightarrow V(f) \tag{23}$$

to be the maximal L-quotient on which T_ℓ' and $\langle d \rangle' = \langle d \rangle_*$ act as multiplication by $a_\ell(f)$ and $\chi_f(d)$ respectively, for all $\ell \nmid N_o$ and $\langle d \rangle \in (\mathbf{Z}/N\mathbf{Z})^*$. If f is new of conductor N then $V(f)$ is the *dual* of the Deligne representation of f: for every prime $\ell \nmid Np$ an arithmetic Frobenius $\mathrm{Frob}_\ell \in G_{\mathbf{Q}}$ at ℓ acts on it with characteristic polynomial

$$\det\left(1 - \mathrm{Frob}_\ell | V(f) \cdot X\right) = 1 - a_\ell(f) \cdot X + \chi_f(\ell) \cdot \ell^{k-1} \cdot X^2.$$

In general $V(f) \cong \bigoplus_{i=1}^a V(f^{\mathrm{prim}})$ is (non-canonically) isomorphic to the direct sum of a finite number of copies of $V(f^{\mathrm{prim}})$, where f^{prim} is the primitive form (of conductor a divisor of N) associated with f. Dually let

$$V^*(f) \hookrightarrow H^1_{\text{ét,c}}(Y_1(N)_{\bar{\mathbf{Q}}}, \mathscr{S}_{k-2})_L$$

be the maximal L-submodule on which the Hecke operators T_ℓ and $\langle d \rangle = \langle d \rangle^*$ act as multiplication by $a_p(f)$ and $\chi_f(d)$ respectively, for every prime $\ell \nmid N_o$ and unit d modulo N. (Since f is cuspidal, one can replace the compactly supported cohomology $H^1_{\text{ét,c}}$ with the full cohomology $H^1_{\text{ét}}$ in the definition of $V^*(f)$.) If f is new of level N then $V^*(f)$ is the Deligne $G_{\mathbf{Q}}$-representation of f. In general $V^*(f) \cong \bigoplus_{i=1}^a V^*(f^{\mathrm{prim}})$ for a positive integer a.

Because (by construction) T_ℓ' and $\langle d \rangle^*$ are respectively the adjoints of T_ℓ and $\langle d \rangle_*$ under the morphism $\langle \cdot, \cdot \rangle_N$ defined in Equation (18), the latter induces a pairing

$$\langle \cdot, \cdot \rangle_f : V(f) \otimes_L V^*(f) \longrightarrow L, \tag{24}$$

which is perfect by Poincaré duality [Mil80, Chapter VI].

2.5. Comparison with de Rham cohomology. — Let A be a subring of $\mathbf{C}_p$. Write $v : E \to Y$ for one of the universal morphisms $v_{M,N}$ et cetera that as been previously introduced. Denote by

$$\mathscr{S}_{\mathrm{dR}} = \mathscr{S}_{\mathrm{dR}}(v) = \mathbf{R}^1 v_*\big(\mathscr{O}_E \longrightarrow \Omega^1_{E/Y}\big)$$

the relative de Rham cohomology of E/Y and for every $r \geqslant 0$ set

$$\mathscr{S}_{\mathrm{dR},r} = \mathrm{Symm}^r_{\mathscr{O}_Y} \mathscr{S}_{\mathrm{dR}}.$$

Let $\underline{\omega} = v_*\Omega^1_{E/Y}$ be the invertible sheaf of relative differentials on E/Y. The vector bundle $\mathscr{S}_{\mathrm{dR}}$ is equipped with the *Hodge filtration*

$$0 \longrightarrow \underline{\omega} \longrightarrow \mathscr{S}_{\mathrm{dR}} \longrightarrow \underline{\omega}^{-1} \longrightarrow 0$$

and with an integrable Gauß–Manin connection $\nabla : \mathscr{S}_{\mathrm{dR}} \to \mathscr{S}_{\mathrm{dR}} \otimes_{\mathscr{O}_Y} \Omega^1_{Y/K}$. For all $r \geqslant 0$ these give rise to the Hodge filtration

$$\underline{\omega}^r \hookrightarrow \cdots \hookrightarrow \underline{\omega} \otimes \mathscr{S}_{\mathrm{dR},r-1} \hookrightarrow \mathscr{S}_{\mathrm{dR},r} \tag{25}$$

and to an integrable connection on $\mathscr{S}_{\mathrm{dR},r}$, denoted again by ∇.

Set $\mathscr{L}_{\mathrm{dR}} = \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{S}_{\mathrm{dR}}, \mathscr{O}_Y)$ and $\mathscr{L}_{\mathrm{dR},r} = \mathrm{Tsym}^r_{\mathscr{O}_Y}\mathscr{L}_{\mathrm{dR}}$, equipped with the induced Hodge filtration and integrable connection (denoted again by ∇). If $\mathscr{F} = \mathscr{S}, \mathscr{L}$ define the de Rham cohomology groups

$$H^j_{\mathrm{dR}}(Y, \mathscr{F}_{\mathrm{dR},r}) = \mathbf{H}^j\big(Y, \mathscr{F}_{\mathrm{dR},r} \xrightarrow{\nabla} \mathscr{F}_{\mathrm{dR},r} \otimes_{\mathscr{O}_Y} \Omega^1_{Y/K}\big)$$

(where the complex $\mathscr{F}_{\mathrm{dR},r} \xrightarrow{\nabla} \mathscr{F}_{\mathrm{dR},r} \otimes_{\mathscr{O}_Y} \Omega^1_{Y/K}$ is concentrated in degrees zero and one). As in Section 2.3 one defines on $H^j_{\mathrm{dR}}(Y, \mathscr{F}_{\mathrm{dR},r})$ Hecke operators T_ℓ and T'_ℓ, for every prime ℓ (when $Y = Y(M,N)$), and diamond operators $\langle d \rangle$, for every unit d of $\mathbf{Z}/N\mathbf{Z}$ (when $Y = Y_1(N)$).

Taking $A = \mathbf{Q}_p$ the comparison theorem of Faltings–Tsuji [**Fal88, Tsu99**] (and the Leray spectral sequence for v_N, cf. the proof of [**BDP13**, Lemma 2.2]) gives a natural, Hecke equivariant isomorphism of filtered $\mathbf{Q}_p$-vector spaces

$$D_{\mathrm{dR}}\big(H^1_{\mathrm{\acute{e}t}}(Y_1(N)_{\bar{\mathbf{Q}}_p}, \mathscr{F}_r)_{\mathbf{Q}_p}\big) \cong H^1_{\mathrm{dR}}(Y_1(N)_{\mathbf{Q}_p}, \mathscr{F}_{\mathrm{dR},r}), \tag{26}$$

where $D_{\mathrm{dR}}(\cdot) = H^0(\mathbf{Q}_p, \cdot \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})$ with B_{dR} Fontaine's field of p-adic periods, and the filtration on the de Rham cohomology arises from the Hodge filtration on $\mathscr{F}_{\mathrm{dR}}$ (cf. Equation (25)). Denote by $M_{r+2}(N, \mathbf{Z})$ the $\mathbf{Z}$-module of modular forms of weight $r+2$, level $\Gamma_1(N)$ and integral Fourier coefficients, and set $M_{r+2}(N, R) = M_{r+2}(N, \mathbf{Z}) \otimes_{\mathbf{Z}} R$ for every ring R. It then follows that canonically

$$\mathrm{Fil}^i D_{\mathrm{dR}}\big(H^1_{\mathrm{\acute{e}t}}(Y_1(N)_{\bar{\mathbf{Q}}_p}, \mathscr{S}_r)_{\mathbf{Q}_p}\big) \otimes_{\mathbf{Q}} \mathbf{Q}(\mu_N) \cong M_{r+2}(N, \mathbf{Q}_p) \otimes_{\mathbf{Q}} \mathbf{Q}(\mu_N) \tag{27}$$

for every $1 \leqslant i \leqslant k-1$ (cf. [**BDP13**, Lemma 2.2]). Under the isomorphisms (26) and (27) the space $\mathrm{Fil}^1 H^1_{\mathrm{dR}}(Y_1(N)_{\mathbf{Q}}, \mathscr{S}_{\mathrm{dR},r})$ corresponds to the image of $M_{r+2}(N, \mathbf{Q})$ under the Atkin–Lehner operator w_N.

Let f and $L/\mathbf{Q}_p$ be as in the previous section and assume that L contains $\mathbf{Q}(\mu_N)$. Define

$$V^*_{\mathrm{dR}}(f) \hookrightarrow H^1_{\mathrm{dR}}(Y_1(N)_{\mathbf{Q}_p}, \mathscr{S}_{\mathrm{dR},k-2})_L$$

to be the maximal submodule on which T_ℓ and $\langle d\rangle_*$ act respectively as $a_\ell(f)$ and $\chi_f(d)$ for every prime $\ell \nmid N_o$ and every $d \in (\mathbf{Z}/N\mathbf{Z})^*$, and dually (cf. Section 2.4)

$$H^1_{\mathrm{dR}}(Y_1(N)_{\mathbf{Q}_p}, \mathscr{L}_{k-2}(1))_L \twoheadrightarrow V_{\mathrm{dR}}(f).$$

(Here $\mathscr{L}_{\mathrm{dR},r}(j) = \mathscr{L}_{\mathrm{dR},r}$ as flat sheaves and $\mathrm{Fil}^i\mathscr{L}_{\mathrm{dR},r}(j) = \mathrm{Fil}^{i+j}\mathscr{L}_{\mathrm{dR},r}$.) The comparison isomorphism (26) gives

$$D_{\mathrm{dR}}(V(f)) \cong V_{\mathrm{dR}}(f) \quad \text{and} \quad D_{\mathrm{dR}}(V^*(f)) \cong V^*_{\mathrm{dR}}(f), \tag{28}$$

and Equation (27) implies that they restrict to canonical isomorphisms

$$\mathrm{Fil}^0 V_{\mathrm{dR}}(f) \cong S_k(N,L)_{f^*} \quad \text{and} \quad \mathrm{Fil}^1 V^*_{\mathrm{dR}}(f) \cong S_k(N,L)_f. \tag{29}$$

Here $f^* = \sum_{n\geqslant 1} \bar{a}_n(f)\cdot q^n \in S_k(N,\bar{\chi}_f)$ is the dual of f and $S_k(N,L)_\cdot$ denotes the L-module of cusp forms in $S_k(N,L)$ which are eigenvectors for the Hecke operators T_ℓ and $\langle d\rangle$, with the same eigenvalues as $\cdot$, for all primes $\ell \nmid N_o$ and units d in $\mathbf{Z}/N\mathbf{Z}$. One denotes by

$$\omega_f \in \mathrm{Fil}^1 V^*_{\mathrm{dR}}(f) \tag{30}$$

the element corresponding to f under the second isomorphism in Equation (29).

The pairing (24) and the isomorphisms (28) induce a perfect duality

$$\langle\cdot,\cdot\rangle_f : V_{\mathrm{dR}}(f) \otimes_L V^*_{\mathrm{dR}}(f) \longrightarrow D_{\mathrm{dR}}(L) = L, \tag{31}$$

which together with the isomorphisms (29) gives rise to perfect pairings

$$\begin{aligned} &\langle\cdot,\cdot\rangle_f : S_k(N,L)_{f^*} \otimes_L V^*_{\mathrm{dR}}(f)/\mathrm{Fil}^1 \longrightarrow L \\ \text{and} \quad &\langle\cdot,\cdot\rangle_f : V_{\mathrm{dR}}(f)/\mathrm{Fil}^0 \otimes_L S_k(N,L)_f \longrightarrow L, \end{aligned} \tag{32}$$

under which we often identify $V^*_{\mathrm{dR}}(f)/\mathrm{Fil}^1$ with the L-linear dual of $S_k(N,L)_{f^*}$.

Denote by

$$f^w = w_N(f) = N^{k-1}\cdot(Nz)^{-k}\cdot f(-1/Nz) \tag{33}$$

the image of f under the Atkin–Lehner isomorphism

$$w_N : S_k(N,\chi_f) \cong S_k(N,\bar{\chi}_f)$$

and define

$$\eta_f \in V^*_{\mathrm{dR}}(f)/\mathrm{Fil}^1 \tag{34}$$

to be the element which represents the linear functional

$$J_f = \frac{(f^w,\cdot)_N}{(f^w,f^w)_N} : S_k(N,L)_{f^*} \longrightarrow L. \tag{35}$$

Here $(\mu,\nu)_N = \iint_{Y_1(N)_{\mathbf{C}}} \bar{\mu}(z)\nu(z)y^k\frac{dxdy}{y^2}$ (with $z = x+iy$) is the Petersson scalar product on $S_k(N,\mathbf{C})$. The *a priori* $\mathbf{C}$-valued functional J_f indeed takes values in L (cf. [**Hid85**, Proposition 4.5]).

Assume that $\mathrm{ord}_p(N) \leqslant 1$, that p does not divide the conductor of χ_f, and that $a_p(f)$ is a unit in $\mathscr{O}$. Then the $G_{\mathbf{Q}_p}$-representations $V^\cdot(f)$ are semistable, viz. $D_{\mathrm{dR}}(V^\cdot(f)) = D_{\mathrm{st}}(V^\cdot(f))$. It follows that $D_{\mathrm{dR}}(V^\cdot(f))$, hence $V^\cdot_{\mathrm{dR}}(f)$ by Equation

(28), are equipped with an L-linear Frobenius endomorphism φ. Enlarging L if necessary, let $\alpha_f \in \mathscr{O}^*$ be the unit root of the Hecke polynomial

$$h_{f,p} = X^2 - a_p(f) \cdot X + \chi_f(p)p^{k-1} = (X - \alpha_f) \cdot (X - \beta_f)$$

of f. As proved in [Sai97] the characteristic polynomial of the Frobenius endomorphism φ acting on $V^*_{\mathrm{dR}}(f)$ is a power of $h_{f,p}$, and

$$V^*_{\mathrm{dR}}(f) = \mathrm{Fil}^1 V^*_{\mathrm{dR}}(f) \oplus V^*_{\mathrm{dR}}(f)^{\varphi=\alpha_f}. \tag{36}$$

As a consequence η_f lifts uniquely to a differential

$$\eta_f^\alpha \in V^*_{\mathrm{dR}}(f)^{\varphi=\alpha_f}. \tag{37}$$

3. Diagonal classes

Notation. In this section $Y_1(N) = Y_1(N)_{\mathbf{Q}}$ denotes the modular curve of level $\Gamma_1(N) = \Gamma(1,N)$ over $\mathbf{Q}$ and $\mathscr{T} = \mathscr{T}_{1,N}$ denotes the relative Tate module of the universal elliptic curve $E_1(N) = E_1(N)_{\mathbf{Q}}$ (cf. Equation (10)).

Fix a geometric point $\eta = \eta_N : \mathrm{Spec}(\bar{\mathbf{Q}}) \to Y_1(N)$ and denote by $\mathcal{G}_N = \pi_1^{\text{ét}}(Y_1(N), \eta)$ the fundamental group of $Y_1(N)$ with base point η. Then the stalk $\mathscr{T}_\eta$ of $\mathscr{T}$ at η is a free $\mathbf{Z}_p$-module of rank two, equipped with a continuous action of $\mathcal{G}_N$. Choose an isomorphism of $\mathbf{Z}_p$-modules $\xi : \mathscr{T}_\eta \cong \mathbf{Z}_p \oplus \mathbf{Z}_p$ satisfying (cf. Equation (11))

$$\langle x, y \rangle_{E_{p^\infty}} = \xi(x) \wedge \xi(y) \tag{38}$$

for every $x, y \in \mathscr{T}_\eta$ (where one identifies $\bigwedge^2 \mathbf{Z}_p^2$ and $\mathbf{Z}_p$ via $(1,0) \wedge (0,1) = 1$) and denote by

$$\varrho_N : \mathcal{G}_N \longrightarrow \mathrm{Aut}_{\mathbf{Z}_p}(\mathscr{T}_\eta) \cong \mathrm{GL}_2(\mathbf{Z}_p)$$

the corresponding continuous group morphism. According to Proposition A I.8 of [FK88] the map which sends $\mathscr{F}$ to its stalk $\mathscr{F}_\eta$ gives an equivalence between the category of locally constant p-adic sheaves on $Y_1(N)_{\text{ét}}$ and that of p-adic representations of $\mathcal{G}_N$. Then restriction via ϱ_N allows to associate with every continuous representation of $\mathrm{GL}_2(\mathbf{Z}_p)$ into a free finite $\mathbf{Z}_p$-module M a smooth sheaf $M^{\text{ét}}$ on $Y_1(N)$ satisfying $M^{\text{ét}}_\eta = M$.

Let $S_i(A)$ be the set of two-variable homogeneous polynomials of degree i in $A[x_1, x_2]$, equipped with the action of $\mathrm{GL}_2(\mathbf{Z}_p)$ defined for every $g \in \mathrm{GL}_2(\mathbf{Z}_p)$ and $P(x_1, x_2) \in S_i(A)$ by

$$gP(x_1, x_2) = P\big((x_1, x_2) \cdot g\big),$$

and let $L_i(A)$ be the A-linear dual of $S_i(A)$, with $\mathrm{GL}_2(\mathbf{Z}_p)$-action defined by $g\mu(P(x_1, x_2)) = \mu(g^{-1}P(x_1, x_2))$ for every $g \in \mathrm{GL}_2(\mathbf{Z}_p)$, $\mu \in L_i(A)$ and $P(x_1, x_2) \in S_i(A)$. Then (as sheaves on $Y_1(N)_{\mathbf{Q}}$) one has (cf. Equation (12))

$$\mathscr{L}_i(A) = L_i(A)^{\text{ét}} \quad \text{and} \quad \mathscr{S}_i(A) = S_i(A)^{\text{ét}}. \tag{39}$$

In particular $\mathscr{T}_\eta$ is isomorphic to $L_1(\mathbf{Z}_p)$, hence $\mathbf{Z}_p(1)_\eta \cong \bigwedge^2 \mathscr{T}_\eta \cong \det^{-1}$, where $\det^j : \mathrm{GL}_2(\mathbf{Z}_p) \to \mathbf{Z}_p^*$ is defined by $\det^j(\cdot) = \det(\cdot)^j$ for $j \in \mathbf{Z}$. As a consequence, for every $j \in \mathbf{Z}$ and every p-adic representation M of $\mathrm{GL}_2(\mathbf{Z}_p)$:

$$H^0(\mathrm{GL}_2(\mathbf{Z}_p), M \otimes \det^{-j}) \hookrightarrow H^0(\mathcal{G}_N, M \otimes \det^{-j}) \cong H^0_{\text{ét}}(Y_1(N), M^{\text{ét}}(j)). \tag{40}$$

Let $\boldsymbol{r} = (r_1, r_2, r_3) \in \mathbf{N}^3$ be a triple of nonnegative integers satisfying the following assumption.

Assumption 3.1. — 1. $r_1 + r_2 + r_3 = 2 \cdot r$ *with* $r \in \mathbf{N}$.
2. *For every permutation* $\{i, j, k\}$ *of* $\{1, 2, 3\}$ *one has* $r_i + r_j \geqslant r_k$.

Let $S_{\boldsymbol{r}}$ denote the $\mathrm{GL}_2(\mathbf{Z}_p)$-representation $S_{r_1}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} S_{r_2}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} S_{r_3}(\mathbf{Z}_p)$, which we identify with the module of six-variable polynomials in $\mathbf{Z}_p[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}]$ which are homogeneous of degree r_1, r_2 and r_3 in the variables $\boldsymbol{x} = (x_1, x_2)$, $\boldsymbol{y} = (y_1, y_2)$ and $\boldsymbol{z} = (z_1, z_2)$ respectively. Following the Clebsch–Gordan decomposition of classical invariant theory, define (cf. Assumption 3.1)

$$\mathrm{Det}_N^{\boldsymbol{r}} = \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}^{r - r_3} \cdot \det \begin{pmatrix} x_1 & x_2 \\ z_1 & z_2 \end{pmatrix}^{r - r_2} \cdot \det \begin{pmatrix} y_1 & y_2 \\ z_1 & z_2 \end{pmatrix}^{r - r_1}, \tag{41}$$

which is a $\mathrm{GL}_2(\mathbf{Z}_p)$-invariant of $S_{\boldsymbol{r}} \otimes \det^{-r}$:

$$\mathrm{Det}_N^{\boldsymbol{r}} \in H^0(\mathrm{GL}_2(\mathbf{Z}_p), S_{\boldsymbol{r}} \otimes \det^{-r}).$$

After setting $\mathscr{S}_{\boldsymbol{r}} = \mathscr{S}_{r_1}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathscr{S}_{r_2}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathscr{S}_{r_3}(\mathbf{Z}_p)$, denote by

$$\mathtt{Det}_N^{\boldsymbol{r}} \in H^0_{\text{ét}}(Y_1(N), \mathscr{S}_{\boldsymbol{r}}(r)) \tag{42}$$

the class corresponding to $\mathrm{Det}_N^{\boldsymbol{r}}$ under the natural injection (40). Let

$$p_j : Y_1(N)^3 \to Y_1(N)$$

be the natural projections, let

$$\mathscr{S}_{[\boldsymbol{r}]} = p_1^* \mathscr{S}_{r_1}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} p_2^* \mathscr{S}_{r_2}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} p_3^* \mathscr{S}_{r_3}(\mathbf{Z}_p)$$

and set

$$\mathtt{W}_{N,\boldsymbol{r}} = H^3_{\text{ét}}(Y_1(N)^3_{\bar{\mathbf{Q}}}, \mathscr{S}_{[\boldsymbol{r}]})(r+2).$$

Since $Y_1(N)_{\bar{\mathbf{Q}}}$ is a smooth affine curve over $\bar{\mathbf{Q}}$ one has

$$H^4_{\text{ét}}(Y_1(N)^3_{\bar{\mathbf{Q}}}, \mathscr{S}_{[\boldsymbol{r}]}(r+2)) = 0,$$

hence the Hochschild–Serre spectral sequence

$$H^p(\mathbf{Q}, H^q_{\text{ét}}(Y^3_{\bar{\mathbf{Q}}}, \mathscr{S}_{[\boldsymbol{r}]}(r+2))) \Longrightarrow H^{p+q}_{\text{ét}}(Y_1(N)^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2))$$

defines a morphism

$$\mathtt{HS} : H^4_{\text{ét}}(Y_1(N)^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2)) \longrightarrow H^1(\mathbf{Q}, \mathtt{W}_{N,\boldsymbol{r}}).$$

Let $d : Y_1(N) \longrightarrow Y_1(N)^3$ be the diagonal embedding. As

$$E_1^{2r}(N) = E_1^{\boldsymbol{r}}(N) \times_{Y_1(N)^3} Y_1(N)$$

is isomorphic to the base change of $u_N^{\boldsymbol{r}} : E_1^{\boldsymbol{r}}(N) \to Y_1(N)^3$ under d, there is a natural isomorphism $d^* \mathscr{S}_{[\boldsymbol{r}]} \cong \mathscr{S}_{\boldsymbol{r}}$ of smooth sheaves on $Y_1(N)_{\text{ét}}$. The codimension-2 closed embedding d then gives a pushforward map

$$d_* : H^0_{\text{ét}}(Y_1(N), \mathscr{S}_{\boldsymbol{r}}(r)) \longrightarrow H^4_{\text{ét}}(Y_1(N)^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2)),$$

and one defines the *diagonal class* of level N and weights $\boldsymbol{r}+\mathbf{2}$:

$$\tilde{\kappa}_{N,\boldsymbol{r}} = \mathtt{HS}\circ d_*(\mathtt{Det}_N^{\boldsymbol{r}}) \in H^1(\mathbf{Q}, \mathtt{W}_{N,\boldsymbol{r}}) \tag{43}$$

as the image of $\mathtt{Det}_N^{\boldsymbol{r}}$ under the composition of d_* with $\mathtt{HS}$. Let $W_{N,\boldsymbol{r}} = \mathtt{W}_{N,\boldsymbol{r}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ and let $H^1_{\mathrm{geo}}(\mathbf{Q}, W_{N,\boldsymbol{r}})$ be the geometric Bloch–Kato Selmer group of $W_{N,\boldsymbol{r}}$ over $\mathbf{Q}$, viz. the module of classes in $H^1(\mathbf{Q}, W_{N,\boldsymbol{r}})$ which are unramified at every prime different from p, and whose restrictions at p belong to the geometric subspace

$$H^1_{\mathrm{geo}}(\mathbf{Q}_p, W_{N,\boldsymbol{r}}) = \ker\big(H^1(\mathbf{Q}_p, W_{N,\boldsymbol{r}}) \longrightarrow H^1(\mathbf{Q}_p, W_{N,\boldsymbol{r}} \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})\big)$$

(cf. [**BK90**, Section 3]). The results of [**NN16**] (cf. the proof of Theorem 5.9) yield the following crucial proposition.

Proposition 3.2. — *The class $\tilde{\kappa}_{N,\boldsymbol{r}}$ belongs to $H^1_{\mathrm{geo}}(\mathbf{Q}, W_{N,\boldsymbol{r}})$.*

The bilinear form $\det^* : L_i(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} L_i(\mathbf{Z}_p) \to \mathbf{Z}_p \otimes \det^{-i}$ defined by

$$\det{}^*(\mu \otimes \nu) = \mu \otimes \nu\big((x_1y_2 - x_2y_1)^i\big)$$

for all $\mu, \nu \in L_i(\mathbf{Z}_p)$ becomes perfect after extending scalars to $\mathbf{Q}_p$, hence induces an isomorphism of $\mathrm{GL}_2(\mathbf{Z}_p)$-modules

$$\mathtt{s}_i : S_i(\mathbf{Q}_p) = \mathrm{Hom}_{\mathbf{Q}_p}(L_i(\mathbf{Q}_p), \mathbf{Q}_p) \cong L_i(\mathbf{Q}_p) \otimes_{\mathbf{Z}_p} \det{}^i.$$

Under the equivalence $\cdot^{\text{ét}}$ this corresponds by Equation (39) to an isomorphism of sheaves

$$\mathtt{s}_i : \mathscr{S}_i(\mathbf{Q}_p) \cong \mathscr{L}_i(\mathbf{Q}_p) \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(-i). \tag{44}$$

Define the sheaves $\mathscr{L}_{\boldsymbol{r}}$ on $Y_1(N)$ and $\mathscr{L}_{[\boldsymbol{r}]}$ on $Y_1(N)^3$ as above, and set

$$\mathtt{V}_{N,\boldsymbol{r}} = H^3_{\text{ét}}(Y_1(N)^3_{\bar{\mathbf{Q}}}, \mathscr{L}_{[\boldsymbol{r}]})(2-r) \quad \text{and} \quad V_{N,\boldsymbol{r}} = \mathtt{V}_{N,\boldsymbol{r}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p. \tag{45}$$

The tensor product of the $\mathtt{s}_{r_j}$ gives an isomorphism $\mathtt{s}_{\boldsymbol{r}} : W_{N,\boldsymbol{r}} \cong V_{N,\boldsymbol{r}}$. Set

$$\kappa_{N,\boldsymbol{r}} = \mathtt{s}_{\boldsymbol{r}*}(\tilde{\kappa}_{N,\boldsymbol{r}}) \in H^1_{\mathrm{geo}}(\mathbf{Q}, V_{N,\boldsymbol{r}}). \tag{46}$$

Remarks 3.3. — 1. We strived to define diagonal classes with values in the representations $V_{N,\boldsymbol{r}}$, as the corresponding cohomology groups are those which are extensively studied in the literature (cf. Sections 4 and 5).

2. For every $0 \leqslant j \leqslant i$ denote by $[x_1, x_2]_j$ the projection of $x_1^{\otimes j} \otimes x_2^{\otimes i-j}$ in $S_i(\mathbf{Q}_p)$. Then $[x_1, x_2]_j$ is a $\mathbf{Q}_p$-basis of $S_i(\mathbf{Q}_p)$ and one writes $[x_1, x_2]_j^*$ for the dual basis of $L_i(\mathbf{Q}_p)$. A direct computation shows that $\mathtt{s}_i : S_i(\mathbf{Q}_p) \cong L_i(\mathbf{Q}_p)$ is given by the formula

$$(-1)^j \cdot \binom{i}{j} \cdot \mathtt{s}_i([x_1, x_2]_j) = [x_1, x_2]_j^*.$$

Set $k = r_1 + 2$, $l = r_2 + 2$ and $m = r_3 + 2$, and consider three cuspidal normalised modular forms

$$f = \sum_{n \geqslant 1} a_n(f) \cdot q^n \in S_k(N, \chi_f),$$
$$g = \sum_{n \geqslant 1} a_n(g) \cdot q^n \in S_l(N, \chi_g),$$
$$h = \sum_{n \geqslant 1} a_n(h) \cdot q^n \in S_m(N, \chi_h)$$

of level $\Gamma_1(N)$, weights k, l and m and characters χ_f, χ_g and χ_h. Assume in the rest of this section the following

Assumption 3.4. — 1. *The triple* (f, g, h) *is* self-dual, *that is* $\chi_f \cdot \chi_g \cdot \chi_h = 1$.
2. *The forms* f, g *and* h *are eigenvectors for the Hecke operators* T_ℓ, *for every* $\ell \nmid N$.
3. *If* p *divides* N *then* f, g *and* h *are eigenvectors for the Hecke operator* U_p.

Note that Assumption 3.4.1 implies Assumption 3.1.1, id est that $k + l + m$ is an even integer. Moreover, Assumption 3.1.2 states that the triple (k, l, m) is balanced (with the terminology introduced in Section 1.1). Set

$$V(f, g, h) = V(f) \otimes_L V(g) \otimes_L V(h)\big((4 - k - l - m)/2\big). \tag{47}$$

The Künneth decomposition and projection to the (f, g, h)-isotypic component give a morphism of $G_{\mathbf{Q}}$-modules

$$\mathrm{pr}_{fgh} : \mathbb{V}_{N,\boldsymbol{r}} \otimes_{\mathbf{Q}_p} L \twoheadrightarrow V(f, g, h) \tag{48}$$

and one defines the diagonal class associated to the triple (f, g, h) by

$$\kappa(f, g, h) = \mathrm{pr}_{fgh}(\kappa_{N,\boldsymbol{r}}) \in H^1_{\mathrm{geo}}(\mathbf{Q}, V(f, g, h)).$$

3.1. The explicit reciprocity law (*cf.* [BSV20b]). — Let $\boldsymbol{r}$ and (f, g, h) be as in the previous section. In particular $\boldsymbol{r}$ and (f, g, h) satisfy Assumption 3.1 and Assumption 3.4 respectively. In addition, assume in this section that $\mathrm{ord}_p(N) \leqslant 1$, that the conductors of χ_f, χ_g and χ_h are all coprime to p, and that the forms f, g and h are p-*ordinary* (viz. their p-th Fourier coefficients are p-adic units).

Lemma 3.5. — *For* $\bullet$ *in* $\{\mathrm{geo}, \mathrm{fin}, \mathrm{exp}\}$, *the Bloch–Kato local conditions*

$$H^1_\bullet(\mathbf{Q}_p, V(f, g, h)) \hookrightarrow H^1(\mathbf{Q}_p, V(f, g, h))$$

(cf. [BK90, Section 3]*) are all equal.*

Proof. — Set $w = (k, l, m)$. For $\xi = f, g, h$, denote by $\xi^\sharp$ the newform of conductor $N_\xi | N$ and weight $u = k, l, m$ associated to ξ, and set

$$V = V(f^\sharp) \otimes_L V(g^\sharp) \otimes_L V(h^\sharp)\big((4 - k - l - m)/2\big).$$

Since $V(\xi)$ is isomorphic to the direct sum of a finite number of copies of $V(\xi^\sharp)$ (cf. Section 2.4), it is sufficient to prove the statement after replacing $V(f, g, h)$ with V. Moreover, since V is isomorphic to its Kummer dual $V^* = \mathrm{Hom}_L(V, L(1))$, it is sufficient to prove that $H^1_{\exp}(\mathbf{Q}_p, V)$ equals $H^1_{\mathrm{fin}}(\mathbf{Q}_p, V)$ (cf. Proposition 3.8 of [BK90]). According to [BK90, Corollary 3.8.4], the quotient $H^1_{\mathrm{fin}}(\mathbf{Q}_p, V)/H^1_{\exp}(\mathbf{Q}_p, V)$ is isomorphic to $D/(\varphi-1)D$, where D is the crystalline module $D_{\mathrm{cris}}(V) = H^0(\mathbf{Q}_p, V \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}})$ associated with the restriction of V to $G_{\mathbf{Q}_p}$, and φ is the crystalline Frobenius acting on it. We are then reduced to prove the claim

$$D^{\varphi=1} = 0. \tag{49}$$

The assumptions $\mathrm{ord}_p(N) \leqslant 1$ and $p \nmid \mathrm{cond}(\chi_\xi)$ guarantee that $V(\xi^\sharp)|_{G_{\mathbf{Q}_p}}$ is semi-stable, hence so is $V|_{G_{\mathbf{Q}_p}}$. Denote by $D_{\mathrm{st}}(\xi^\sharp) = H^0(\mathbf{Q}_p, V(\xi^\sharp) \otimes_{\mathbf{Q}_p} B_{\mathrm{st}})$ and $D_{\mathrm{st}} = H^0(\mathbf{Q}_p, V \otimes_{\mathbf{Q}_p} B_{\mathrm{st}})$ the semi-stable Fontaine modules of $V(\xi^\sharp)|_{G_{\mathbf{Q}_p}}$ and $V|_{G_{\mathbf{Q}_p}}$ respectively. One has

$$D_{\mathrm{st}}(\xi^\sharp) = L \cdot \mathbf{a}_\xi \oplus L \cdot \mathbf{b}_\xi,$$

where $\mathbf{a}_\xi$ and $\mathbf{b}_\xi$ are φ-eigenvectors with eigenvalues $a_p(\xi^\sharp)^{-1}$ and $p^{1-u}\chi_\xi(p)^{-1}a_p(\xi^\sharp)$ respectively (cf. Section 2.5). Moreover the monodromy operator N_ξ on $D_{\mathrm{st}}(\xi^\sharp)$ is zero if $p \nmid N_\xi$, and satisfies $\mathrm{N}_\xi(\mathbf{a}_\xi) = \mathbf{b}_\xi$ and $\mathrm{N}_\xi(\mathbf{b}_\xi) = 0$ if $p \| N_\xi$. Consider the set $\mathcal{B}_w = \{\mathbf{a}_w^{\cdot}, \mathbf{b}_w^{\cdot} : \cdot = \emptyset, f, g, h\}$ of elements of

$$D_{\mathrm{st}} \cong D_{\mathrm{st}}(f^\sharp) \otimes_L D_{\mathrm{st}}(g^\sharp) \otimes_L D_{\mathrm{st}}(h^\sharp) \otimes_{\mathbf{Q}_p} D_{\mathrm{cris}}(\mathbf{Q}_p((4-k-l-m)/2))$$

defined by

$$\begin{aligned}
\mathbf{a}_w &= \mathbf{a}_f \otimes \mathbf{a}_g \otimes \mathbf{a}_h \otimes t^{(4-k-l-m)/2}, & \mathbf{a}_w^f &= \mathbf{b}_f \otimes \mathbf{a}_g \otimes \mathbf{a}_h \otimes t^{(4-k-l-m)/2},\\
\mathbf{b}_w^f &= \mathbf{a}_f \otimes \mathbf{b}_g \otimes \mathbf{b}_h \otimes t^{(4-k-l-m)/2}, & \mathbf{b}_w &= \mathbf{b}_f \otimes \mathbf{b}_g \otimes \mathbf{b}_h \otimes t^{(4-k-l-m)/2}
\end{aligned}$$

et cetera, where t is the canonical generator of $D_{\mathrm{cris}}(\mathbf{Q}_p(1))$. Then $\mathcal{B}_w$ is an L-basis of φ-eigenvectors of D_{st} with respective eigenvalues $\mathcal{E}_w = \{\alpha_w^{\cdot}, \beta_w^{\cdot} : \cdot = \emptyset, f, g, h\}$, where

$$\alpha_w = \frac{p^{c(w)-1}}{a_p(f^\sharp)a_p(g^\sharp)a_p(h^\sharp)}, \qquad \alpha_w^f = \frac{p^{c(w)-k} \cdot a_p(f^\sharp)}{\chi_f(p)a_p(g^\sharp)a_p(h^\sharp)},$$

α_w^g and α_w^h are defined similarly, and $\beta_w^{\cdot}$ is defined by the equality

$$p \cdot \alpha_w^{\cdot} \cdot \beta_w^{\cdot} = 1.$$

Since the forms f, g and h are ordinary and w is balanced, one has

$$\mathrm{ord}_p(\beta_w^{\cdot}) < 0 \leqslant \mathrm{ord}_p(\alpha_w^\xi) < \mathrm{ord}_p(\alpha_w)$$

for $\cdot = \emptyset, f, g, h$ and $\xi = f, g, h$. In particular the L-module $D_{\mathrm{st}}^{\varphi=1}$ (hence $D^{\varphi=1}$) is contained in the space generated by the eigenvectors $\mathbf{a}_w^\xi$ for $\xi = f, g, h$.

Define $\varepsilon_\xi \in \{0, 1\}$ to be 1 (resp., 0) if p divides (resp., does not divide) the conductor N_ξ of $\xi = f, g, h$, and set $\varepsilon_w = \varepsilon_f + \varepsilon_g + \varepsilon_h$. According to Theorems 4.5.17 (namely the Ramanujan–Petersson conjecture) and 4.6.17 of [Miy06] one has

$$|\alpha_w^\xi|_\infty = p^{(\varepsilon_w - 2\cdot\varepsilon_\xi - 1)/2}$$

for $\xi = f, g, h$, where $|\cdot|_\infty$ denotes the complex absolute value. As a consequence $D_{\mathrm{st}}^{\varphi=1}$ vanishes if $\varepsilon_w = 0$ or $\varepsilon_w = 2$. If $\varepsilon_w = 1$, say $\varepsilon_f = 1$, then $D_{\mathrm{st}}^{\varphi=1}$ is contained in $L \cdot \mathbf{a}_w^g \oplus L \cdot \mathbf{a}_w^h$. On the other hand, the monodromy operator N on D_{st} satisfies

$$\mathrm{N}(\mathbf{a}_w^g) = \mathbf{b}_w^h \quad \text{and} \quad \mathrm{N}(\mathbf{a}_w^h) = \mathbf{b}_w^g,$$

hence $D_{\mathrm{st}}^{\varphi=1,\mathrm{N}=0}$ vanishes in this case. Finally, if $\varepsilon_w = 3$, then

$$\mathrm{N}(\mathbf{a}_w^{\xi}) = \mathbf{b}_w^{\xi'} + \mathbf{b}_w^{\xi''}$$

for each permutation (ξ, ξ', ξ'') of (f, g, h), hence $D^{\varphi=1} = D_{\mathrm{st}}^{\varphi=1,\mathrm{N}=0} = 0$ also in this case, thus proving the claim (49). $\square$

It follows from the previous Lemma 3.5 that, upon setting

$$V_{\mathrm{dR}}(f,g,h) = V_{\mathrm{dR}}(f) \otimes_L V_{\mathrm{dR}}(g) \otimes_L V_{\mathrm{dR}}(h)\big((4-k-l-m)/2\big), \tag{50}$$

the Bloch–Kato exponential and the isomorphism (28) give an isomorphism

$$\exp_p : V_{\mathrm{dR}}(f,g,h)/\mathrm{Fil}^0 \cong H^1_{\mathrm{geo}}(\mathbf{Q}_p, V(f,g,h)).$$

Similarly for the dual representations define

$$V_{\mathrm{dR}}^*(f,g,h) = V_{\mathrm{dR}}^*(f) \otimes_L V_{\mathrm{dR}}^*(g) \otimes_L V_{\mathrm{dR}}^*(h)\big((k+l+m-2)/2\big). \tag{51}$$

Then the perfect dualities (31) (for f, g and h) yield a natural isomorphism

$$V_{\mathrm{dR}}(f,g,h)/\mathrm{Fil}^0 \cong \mathrm{Fil}^0 V_{\mathrm{dR}}^*(f,g,h)^{\vee},$$

where $\cdot^{\vee} = \mathrm{Hom}_L(\cdot, L)$. Its composition with $\exp_p^{-1}$ defines an isomorphism

$$\log_p : H^1_{\mathrm{geo}}(\mathbf{Q}_p, V(f,g,h)) \cong \mathrm{Fil}^0 V_{\mathrm{dR}}^*(f,g,h)^{\vee}. \tag{52}$$

For every global Selmer class κ in $H^1_{\mathrm{geo}}(\mathbf{Q}, V(f,g,h))$ one simply writes $\log_p(\kappa)$ as a shorthand for $\log_p(\mathrm{res}_p(\kappa))$.

Denote by $\omega_g \in \mathrm{Fil}^{l-1} V_{\mathrm{dR}}^*(g)$ and $\omega_h \in \mathrm{Fil}^{m-1} V_{\mathrm{dR}}^*(h)$ the differentials corresponding to g and h respectively under the isomorphism (29), and recall the class $\eta_f^{\alpha} \in V_{\mathrm{dR}}^*(f)^{\varphi=\alpha_f}$ defined in Equation (37). Since $\mathrm{Fil}^0 V_{\mathrm{dR}}^*(f)$ equals $V_{\mathrm{dR}}^*(f)$ and $l+m-2 \geqslant (k+l+m-2)/2$ by Assumption 3.1(2) one has

$$\eta_f^{\alpha} \otimes \omega_g \otimes \omega_h \in \mathrm{Fil}^0 V_{\mathrm{dR}}^*(f,g,h). \tag{53}$$

Assume in the rest of this section that p does not divide N. For every s in $\mathbf{Z}$ denote by

$$\mathbf{M}_s(N,L) \subset \mathbf{Z}_p[\![q]\!] \otimes_{\mathbf{Z}_p} L$$

the space of p-adic modular forms of weight s and level $\Gamma_1(N)$ defined over L. Let

$$\mathbf{S}_s(N,L) \subset q \cdot \mathscr{O}[\![q]\!] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

be the subspace of cuspidal p-adic modular forms. $\mathbf{M}_s(N,L)$ contains naturally the space $M_s(\Gamma_1(N,p),L)$ of classical modular forms of level $\Gamma_1(N,p) = \Gamma_1(N) \cap \Gamma_0(p)$ and q-expansion in $L[\![q]\!]$. It is equipped with the Hecke operators $U = U_p$ and $V = V_p$, which are described on q-expansions by

$$U\Big(\sum_{n\geqslant 0} a_n \cdot q^n\Big) = \sum_{n\geqslant 0} a_{np} \cdot q^n \quad \text{and} \quad V\Big(\sum_{n\geqslant 0} a_n \cdot q^n\Big) = \sum_{n\geqslant 0} a_n \cdot q^{pn}$$

respectively. Serre's derivative operator $d = q \cdot \frac{d}{dq}$ on $L[\![q]\!]$ restricts to a morphism

$$d : \mathbf{M}_s(N,L) \to \mathbf{M}_{s+2}(N,L).$$

For every $s \geqslant 2$ Hida defined in [**Hid85**] an *ordinary projector*

$$e_{\mathrm{ord}} : \mathbf{M}_s(N, L) \twoheadrightarrow M_s^{\mathrm{ord}}(\Gamma_1(N,p), L)$$

onto the space $M_s^{\mathrm{ord}}(\Gamma_1(N,p), L)$ of classical ordinary modular forms of level $\Gamma_1(N,p)$, which is a section of the natural inclusion $M_s^{\mathrm{ord}}(\Gamma_1(N,p), L) \hookrightarrow \mathbf{M}_s(N, L)$. Given $\xi \in S_l(\Gamma_1(N,p), L)$ and $\psi \in S_m(\Gamma_1(N,p), L)$ set

$$\Xi_k^{\mathrm{ord}}(\xi, \psi) = e_{\mathrm{ord}}\big(d^{(k-l-m)/2}\xi^{[p]} \times \psi\big) \in S_k^{\mathrm{ord}}(\Gamma_1(N,p), L),$$

where $\xi^{[p]}$ and $d^{(k-l-m)/2}\xi^{[p]}$ are defined as follows. Note first that $t = (k-l-m)/2$ is a negative integer by Assumption 3.1. The *p-depletion* $\xi^{[p]} \in \mathbf{S}_l(N,p)$ is defined by $\xi^{[p]} = (1 - VU)\xi$. If ξ has q-expansion $\sum_{n\geqslant 1} a_n(\xi)\cdot q^n$ then

$$\xi^{[p]} = \sum_{(n,p)=1} a_n(\xi)\cdot q^n,$$

hence the limit of p-adic modular forms

$$d^t\xi^{[p]} = \lim_{n\to\infty} d^{t+(p-1)p^n}\xi$$

defines a p-adic modular form of weight $l+2t$ such that $d^{-t}(d^t\xi^{[p]}) = \xi^{[p]}$, and $d^t\xi^{[p]}\times\psi$ belongs to $\mathbf{S}_k(N, L)$.

Let $\xi \in S_k(N, \chi_\xi, L)$ be a eigenvector for the Hecke operators T_ℓ, for all primes $\ell \nmid N$. Assume that ξ is p-ordinary, viz. $T_p(\xi) = a_p(\xi)\cdot\xi$ for a unit $a_p(\xi)$ in $\mathscr{O}^*$. Let α_ξ and β_ξ be the roots of the p-th Hecke polynomial $X^2 - a_p(\xi)\cdot X + \chi_\xi(p)p^{k-1}$ of ξ. Enlarging L if necessary, assume that α_ξ and β_ξ belong to L, and order them in such a way that $\alpha_f \in \mathscr{O}^*$ is a p-adic unit and $\beta_f \in p^{k-1}\cdot\mathscr{O}^*$. Then the *(ordinary) p-stabilisation* of ξ:

$$\xi_\alpha(q) = \xi(q) - \beta_\xi\cdot\xi(q^p) \in S_k^{\mathrm{ord}}(\Gamma_1(N,p), \chi_\xi) \tag{54}$$

is a normalised eigenvector for the Hecke operator T_ℓ, with the same eigenvalue as ξ, for every prime $\ell \nmid Np$, and is an eigenvector for U_p with eigenvalue α_ξ. Taking ξ to be one of f, g, h and $f^w = w_N(f)$ gives rise to the p-stabilised forms $f_\alpha, g_\alpha, h_\alpha$ and $f_\alpha^w = (f^w)_\alpha$ in $S_k(\Gamma_1(N,p), L)$. Define (cf. Sections 2.5 and 6)

$$\mathscr{L}_p^f(f_\alpha, g_\alpha, h_\alpha) = \frac{(f_\alpha^w, \Xi_k^{\mathrm{ord}}(g,h))_{Np}}{(f_\alpha^w, f_\alpha^w)_{Np}} \in L. \tag{55}$$

In [**BSV20b**] we proved the following *explicit reciprocity law*. Its proof uses the ideas and techniques introduced in [**BDP13, DR14, BDR15, KLZ20**]. In particular it relies on Besser's generalisation of Coleman's p-adic integration and the work of Bannai–Kings, Nekovář and Nizioł [**Nek04, Niz97, Niz01, Bes00, BK90**], which forces the assumption $p \nmid N$ in the statement.

Proposition 3.6 **([BSV20b]).** — *Assume that p does not divide N, and that the eigenforms f, g and h are p-ordinary. Then*

$$\log_p\big(\kappa(f,g,h)\big)(\eta_f^\alpha \otimes \omega_g \otimes \omega_h) = E(f,g,h)\cdot\mathscr{L}_p^f(f_\alpha, g_\alpha, h_\alpha),$$

where

$$E(f,g,h) = \frac{(-1)^{r-r_1}(r-r_1)!\left(1-\frac{\beta_f}{\alpha_f}\right)\left(1-\frac{\beta_f}{p\alpha_f}\right)}{\left(1-\frac{\beta_f\alpha_g\alpha_h}{p^{r+2}}\right)\left(1-\frac{\beta_f\alpha_g\beta_h}{p^{r+2}}\right)\left(1-\frac{\beta_f\beta_g\alpha_h}{p^{r+2}}\right)\left(1-\frac{\beta_f\beta_g\beta_h}{p^{r+2}}\right)}.$$

3.2. Comparison with Gross–Kudla–Schoen diagonal cycles. — This section elucidates the relation between the diagonal classes introduced above and the Gross–Kudla–Schoen diagonal cycles. It will not be used in the sequel of this paper.

Let the notations and assumptions be as in the previous section. In this section only we also assume $r_j \geqslant 1$ for $j = 1, 2, 3$. As in [DR14, Section 3.1] fix three subsets $A = \{a_1, \dots, a_{r_1}\}$, $B = \{b_1, \dots, b_{r_2}\}$ and $C = \{c_1, \dots, c_{r_3}\}$ of $\{1, \dots, r\}$ of cardinalities r_1, r_2 and r_3 respectively, such that $A \cap B \cap C = \emptyset$. This is possible by Assumption 3.1. For $1 \leqslant j \leqslant r$, let $p_j : E_1^r(N) = E_1(N) \times_{Y_1(N)} \cdots \times_{Y_1(N)} E_1(N) \longrightarrow E_1(N)$ be the projection from the r-fold fibered product of $E_1(N)$ over $Y_1(N)$ onto its j-th component. Define

$$\iota_{N,\boldsymbol{r}} = (p_A, p_B, p_C) : E_1^r(N) \longrightarrow E_1^{\boldsymbol{r}}(N) \overset{\text{def}}{=} E_1^{r_1}(N) \times_{\mathbf{Q}} E_1^{r_2}(N) \times_{\mathbf{Q}} E_1^{r_3}(N), \tag{56}$$

where $p_A = p_{a_1} \times \cdots \times p_{a_{r_1}} : E_1^r(N) \to E_1^{r_1}(N)$ and p_B and p_C are defined similarly. Then $\iota_{N,\boldsymbol{r}} = \iota_{N,(A,B,C)}$ is a closed immersion of relative dimension $\dim E_1^{\boldsymbol{r}}(N) - \dim E_1^r(N) = r+2$, and one defines the *generalised Gross–Kudla–Schoen diagonal cycle* of level N and weights $\boldsymbol{r}+\mathbf{2}$ (cf. Section 3 of [DR14]) as

$$\Delta_{N,\boldsymbol{r}} = \iota_{N,\boldsymbol{r}*}(E_1^r(N)) \in \mathrm{CH}^{r+2}(E_1^{\boldsymbol{r}}(N)), \tag{57}$$

where $\mathrm{CH}^j(\cdot)$ is the Chow group of codimension-j cycles in $\cdot$ modulo rational equivalence.

For $i \in \mathbf{N}$ denote by $\mathfrak{S}_i = \mu_2^i \rtimes \Sigma_i$ the semi-direct product of $\mu_2^i = \{\pm 1\}^i$ with the symmetric group Σ_i on i letters. The permutation action of Σ_i on $E_1^i(N)$ and the action of μ_2 on $E_1(N)$ induce an action of $\mathfrak{S}_i$ on $E_1^i(N)$. Define the character $\psi_i : \mathfrak{S}_i \to \{\pm 1\}$ by $\psi_i(s_1, \dots, s_i, \sigma) = \mathrm{sgn}(\sigma) \cdot s_1 \cdots s_i$, and set $\varepsilon_i = \frac{1}{2^i \cdot i!} \sum_{g \in \mathfrak{S}_i} \psi_i(g) \cdot g$. Then ε_i gives an idempotent in the ring $\mathrm{Corr}(E_1^i(N))_{\mathbf{Q}}$ of correspondences on $E_1^i(N)$ with rational coefficients. Set $\varepsilon_{\boldsymbol{r}} = \varepsilon_{r_1} \otimes \varepsilon_{r_2} \otimes \varepsilon_{r_3} \in \mathrm{Corr}(E_1^{\boldsymbol{r}}(N))_{\mathbf{Q}}$. The Lieberman trick (cf. the proof of Lemme 5.3 of [Del71]) shows that $\varepsilon_{\boldsymbol{r}}$ kills the cohomology group $H^j_{\text{ét}}(E_1^{\boldsymbol{r}}(N)_{\bar{\mathbf{Q}}}, \mathbf{Q}_p)$ for every $j \neq 2r+3$, hence the image

$$cl_{\text{ét}}\big(\varepsilon_{\boldsymbol{r}} \cdot \Delta_{N,\boldsymbol{r}}\big) \in H^{2r+4}_{\text{ét}}(E_1^{\boldsymbol{r}}(N), \mathbf{Q}_p(r+2))$$

of $\varepsilon_{\boldsymbol{r}} \cdot \Delta_{N,\boldsymbol{r}}$ under the cycle class map

$$cl_{\text{ét}} : \mathrm{CH}^{r+2}(E_1^{\boldsymbol{r}}(N))_{\mathbf{Q}} \to H^{2r+4}_{\text{ét}}(E_1^{\boldsymbol{r}}(N), \mathbf{Q}_p(r+2))$$

belongs to

$$\begin{aligned} &\mathrm{Fil}^0 H^{2r+4}_{\text{ét}}(E_1^{\boldsymbol{r}}(N), \mathbf{Q}_p(r+2)) \\ &\quad = \ker\left(H^{2r+4}_{\text{ét}}(E_1^{\boldsymbol{r}}(N), \mathbf{Q}_p(r+2)) \xrightarrow{\pi^*} H^{2r+3}_{\text{ét}}(E_1^{\boldsymbol{r}}(N)_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(r+2))\right), \end{aligned}$$

where $\pi : E_1^{\boldsymbol{r}}(N)_{\bar{\mathbf{Q}}} \to E_1^{\boldsymbol{r}}(N)$ is the projection. As a consequence one can consider the Abel–Jacobi image

$$\mathrm{AJ}_p^{\text{ét}}\big(\varepsilon_{\boldsymbol{r}} \cdot \Delta_{N,\boldsymbol{r}}\big) = \mathtt{HS} \circ cl_{\text{ét}}\big(\varepsilon_{\boldsymbol{r}} \cdot \Delta_{N,\boldsymbol{r}}\big) \in H^1(\mathbf{Q}, \varepsilon_{\boldsymbol{r}} \cdot H^{2r+3}_{\text{ét}}(E_1^{\boldsymbol{r}}(N)_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(r+2)))$$

of $\varepsilon_{\boldsymbol{r}} \cdot \Delta_{N,\boldsymbol{r}}$ under the composition of the cycle class map $cl_{\text{ét}}$ with the morphism

$$\mathtt{HS} : \mathrm{Fil}^0 H^{2r+4}_{\text{ét}}(E_1^{\boldsymbol{r}}(N), \mathbf{Q}_p(r+2)) \longrightarrow H^1(\mathbf{Q}, H^{2r+3}_{\text{ét}}(E_1^{\boldsymbol{r}}(N)_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(r+2))) \tag{58}$$

arising from the Hochschild–Serre spectral sequence. According to the Lieberman trick the Leray spectral sequence associated with the structural map $E_1^{\boldsymbol{r}}(N) \to Y_1(N)^3$ induces a natural isomorphism

$$\mathtt{L}_{\boldsymbol{r}} : \varepsilon_{\boldsymbol{r}} \cdot H^{2r+3}_{\text{ét}}(E_1^{\boldsymbol{r}}(N)_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(r+2)) \cong H^1_{\text{ét}}(Y_1(N)^3_{\bar{\mathbf{Q}}}, \mathscr{S}_{[\boldsymbol{r}]}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p(r+2) = W_{N,\boldsymbol{r}}. \tag{59}$$

Denote by

$$\mathtt{L}_{\boldsymbol{r}*} : H^1(\mathbf{Q}, \varepsilon_{\boldsymbol{r}} \cdot H^{2r+3}_{\text{ét}}(E_1^{\boldsymbol{r}}(N)_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(r+2))) \cong H^1(\mathbf{Q}, W_{N,\boldsymbol{r}})$$

the isomorphism induced in Galois cohomology by $\mathtt{L}_{\boldsymbol{r}}$.

Proposition 3.7. — *The image of* $\mathrm{AJ}_p^{ét}(\varepsilon_{\boldsymbol{r}} \cdot \Delta_{N,\boldsymbol{r}})$ *under the isomorphism* $\mathtt{L}_{\boldsymbol{r}*}$ *is equal (up to sign) to* $\tilde{\kappa}_{N,\boldsymbol{r}}$.

Proof. — To ease notation set $E^{\cdot} = E_1^{\cdot}(N)$, $Y = Y_1(N)$, $\iota_{\boldsymbol{r}} = \iota_{N,\boldsymbol{r}}$, and denote by $u^{\boldsymbol{r}} = u_N^{\boldsymbol{r}}$ the structural morphism

$$u_N^{r_1} \times_{\mathbf{Q}} u_N^{r_2} \times_{\mathbf{Q}} u_N^{r_3} : E_1^{\boldsymbol{r}}(N) \to Y_1(N)^3.$$

Let $\iota_r : E^r \to E^{2r}$ be the proper morphism defined by

$$\iota_r(P_1, \dots, P_r) = (\{P_{a_j}\}, \{P_{b_j}\}, \{P_{c_j}\}),$$

so that $\iota_{\boldsymbol{r}}$ is the composition of ι_r with the natural map $d_{\boldsymbol{r}} : E^{2r} \to E^{\boldsymbol{r}}$.

Define

$$\mathscr{R}^{2r} = R^{2r} u_*^{2r} \mathbf{Z}_p, \quad \mathscr{R}^{r} = R^{2r} u_*^{r} \mathbf{Z}_p \quad \text{and} \quad \mathscr{R}^{[\boldsymbol{r}]} = R^{2r} u_*^{\boldsymbol{r}} \mathbf{Z}_p.$$

Then ι_r induces relative pull-back and pushforward maps

$$\vartheta_r^* : \mathscr{R}^{2r}(r) \longrightarrow \mathbf{Z}_p \quad \text{and} \quad \vartheta_{r*} : \mathbf{Z}_p \longrightarrow \mathscr{R}^{2r}(r)$$

which are adjoint to each other under the perfect relative Poincaré duality

$$\mathscr{R}^{2r}(r) \otimes_{\mathbf{Z}_p} \mathscr{R}^{2r}(r) \longrightarrow R^{4r} u_*^{2r} \mathbf{Z}_p(2r) \cong \mathbf{Z}_p$$

induced by the cup-product pairing. (They induce on the stalks at a geometric point $y : \mathrm{Spec}(\bar{\mathbf{Q}}) \to Y$ the pull-back $H^{2r}_{\text{ét}}(E_y^{2r}, \mathbf{Z}_p(r)) \to H^{2r}_{\text{ét}}(E_y^{r}, \mathbf{Z}_p(r)) \cong \mathbf{Z}_p$ and push-forward $\mathbf{Z}_p = H^0_{\text{ét}}(E_y^r, \mathbf{Z}_p) \to H^{2r}_{\text{ét}}(E_y^{2r}, \mathbf{Z}_p(r))$ associated with $\iota_r \times_y \bar{\mathbf{Q}}$ respectively.) The Leray spectral sequences associated with the morphisms u^{2r} and $u^{\boldsymbol{r}}$ identify the $\mathbf{Q}_p$-linear extensions of $H^0_{\text{ét}}(Y, \mathscr{R}^{2r}(r))$ and $H^4_{\text{ét}}(Y^3, \mathscr{R}^{[\boldsymbol{r}]}(r+2))$ with direct summands of $H^{2r}_{\text{ét}}(E^{2r}, \mathbf{Q}_p(r))$ and $H^{2r+4}_{\text{ét}}(E^{\boldsymbol{r}}, \mathbf{Q}_p(r+2))$ respectively. (This is again a consequence of the Lieberman trick, cf. [**Del71**].) By the functoriality of the Leray spectral sequence, under these identifications ϑ_{r*} and d_* are compatible with the absolute push-forward maps attached to ι_r and $d_{\boldsymbol{r}}$, viz. the following diagram is commutative:

(60)
$$\begin{tikzcd}
\mathbf{Q}_p \arrow[r, "\vartheta_{r*}"] \arrow[d, equal] & H^0_{\text{ét}}(Y, \mathscr{R}^{2r}(r))_{\mathbf{Q}_p} \arrow[r, "d_*"] \arrow[d, "\text{Leray}"'] & H^4_{\text{ét}}(Y^3, \mathscr{R}^{[\boldsymbol{r}]}(r+2))_{\mathbf{Q}_p} \arrow[d, "\text{Leray}"'] \\
H^0_{\text{ét}}(E^r, \mathbf{Q}_p) \arrow[r, "\iota_{r*}"] & H^{2r}_{\text{ét}}(E^{2r}, \mathbf{Q}_p(r)) \arrow[r, "d_{\boldsymbol{r}*}"] & H^{2r+4}_{\text{ét}}(E^{\boldsymbol{r}}, \mathbf{Q}_p(r+2)).
\end{tikzcd}$$

On the other hand the compatibility of the cycle class

$$cl_{\text{ét}} : \mathrm{CH}^{r+2}(E^{\boldsymbol{r}})_{\mathbf{Q}} \to H^{2r+4}_{\text{ét}}(E^{\boldsymbol{r}}, \mathbf{Q}_p(r+2))$$

with proper push-forwards and the definition of the diagonal cycle $\Delta_{\boldsymbol{r}} = \Delta_{N,\boldsymbol{r}}$ yield the identities

$$cl_{\text{ét}}(\Delta_{\boldsymbol{r}}) = cl_{\text{ét}} \circ \iota_{\boldsymbol{r}*}(E^r) = \iota_{\boldsymbol{r}*}(1) = d_{\boldsymbol{r}*} \circ \iota_{r*}(1).$$

In addition, using again the functoriality of the Leray spectral sequences, one has the commutative diagram

$$\begin{tikzcd}
H^4_{\text{ét}}(Y^3, \mathscr{R}^{[\boldsymbol{r}]}(r+2))_{\mathbf{Q}_p} \arrow[r, two heads, "\mathtt{p}_{[\boldsymbol{r}]}"] \arrow[d, "\text{Leray}"'] & H^4_{\text{ét}}(Y^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2))_{\mathbf{Q}_p} \arrow[r, "\mathtt{HS}"] & H^1(\mathbf{Q}, W_{\boldsymbol{r}}) \arrow[d, "\text{Leray}"] \\
\mathrm{Fil}^0 H^{2r+4}_{\text{ét}}(E^{\boldsymbol{r}}, \mathbf{Q}_p(r+2)) \arrow[rr, "\varepsilon_{\boldsymbol{r}*} \circ \mathtt{HS}"] & & H^1(\mathbf{Q}, \varepsilon_{\boldsymbol{r}} \cdot H^{2r+3}_{\text{ét}}(E^{\boldsymbol{r}}_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(r+2))),
\end{tikzcd}$$

where $\mathtt{p}_{[\boldsymbol{r}]} : \mathscr{R}^{[\boldsymbol{r}]} \twoheadrightarrow \mathscr{S}_{[\boldsymbol{r}]}$ is the natural projection and $W_{\boldsymbol{r}} = W_{N,\boldsymbol{r}}$. Since $\varepsilon_{\boldsymbol{r}}$ acts as the identity on $\mathscr{S}_{[\boldsymbol{r}]}$, the previous three equations prove that (cf. Equation (59))

$$\mathtt{L}_{\boldsymbol{r}*}\big(\mathrm{AJ}^{\text{ét}}_p(\varepsilon_{\boldsymbol{r}} \cdot \Delta_{\boldsymbol{r}})\big) = \mathtt{HS} \circ \mathtt{p}_{[\boldsymbol{r}]} \circ d_* \circ \vartheta_{r*}(1).$$

After setting $\mathtt{Det}^{\boldsymbol{r}} = \mathtt{Det}^{\boldsymbol{r}}_N$, to conclude the proof of the proposition it is then sufficient to show that

$$\mathtt{Det}^{\boldsymbol{r}} = \mathtt{p}_{\boldsymbol{r}} \circ \vartheta_{r*}(1) \in H^0_{\text{ét}}(Y, \mathscr{S}_{\boldsymbol{r}}(r)), \tag{61}$$

where $\mathtt{p}_{\boldsymbol{r}} : \mathscr{R}^{2r}(r) \twoheadrightarrow \mathscr{S}_{\boldsymbol{r}}(r)$ is the natural projection. Let $S = S_1(\mathbf{Z}_p)$ be the standard representation of $\mathrm{GL}_2(\mathbf{Z}_p)$. Recall the geometric point $\eta : \mathrm{Spec}(\bar{\mathbf{Q}}) \to Y$ and the isomorphism $\xi : \mathscr{T}_\eta \cong S \otimes \det^{-1}$ fixed above (cf. Equations (39) and (44)). The $\mathrm{GL}_2(\mathbf{Z}_p)$-representation $\mathscr{R}^{2r}(r)_\eta$ contains $S^{\otimes 2r} \otimes \det^{-r}$ as a direct summand, and $\mathtt{p}_{\boldsymbol{r}} : \mathscr{R}^{2r}(r)_\eta \to \mathscr{S}_{\boldsymbol{r}}(r)_\eta = S_{\boldsymbol{r}} \otimes \det^{-r}$ is the composition of $\mathrm{pr} : \mathscr{R}^{2r}(r)_\eta \twoheadrightarrow S^{\otimes 2r} \otimes \det^{-r}$ and the natural projection $\mathrm{pr}_{\boldsymbol{r}} : S^{\otimes 2r} \otimes \det^{-r} \twoheadrightarrow S_{\boldsymbol{r}} \otimes \det^{-r}$. Let $\vartheta^o_{r*} : \mathbf{Z}_p \to \mathscr{R}^{2r}(r)$ be the relative push-forward associated (as above) with the morphism $E^r \to E^{2r}$ which sends the point $(P_1, \dots, P_r)$ to $(P_1, P_1, \dots, P_r, P_r)$. Then

$$\vartheta_{r*} = \sigma_{\boldsymbol{r}} \circ \vartheta^o_{r*}, \tag{62}$$

where $\sigma_{\boldsymbol{r}} = \sigma_{A,B,C}$ is any fixed permutation of $\{1, \dots, 2r\}$ satisfying

$$\sigma_{\boldsymbol{r}}(P_1, P_1, \dots, P_r, P_r) = (P_{a_1}, \dots, P_{a_{r_1}}, P_{b_1}, \dots, P_{b_{r_2}}, P_{c_1}, \dots, P_{c_{r_3}})$$

for every point $(P_1, \dots, P_r)$ of E^r. The image of 1 under the composition

$$\mathrm{pr} \circ \vartheta^o_{r*} : \mathbf{Z}_p = H^0_{\text{ét}}(E^r_\eta, \mathbf{Z}_p) \longrightarrow H^{2r}_{\text{ét}}(E^{2r}_\eta, \mathbf{Z}_p(r)) = \mathscr{R}^{2r}(r)_\eta \twoheadrightarrow S^{\otimes 2r} \otimes \det^{-r}$$

(where one writes again ϑ^o_{r*} for the morphism induced by ϑ^o_{r*} on the stalks at η) is equal to

$$F_r = \big(x \otimes y - y \otimes x\big)^{\otimes r},$$

where x and y give a $\mathbf{Z}_p$-basis of $S \subset \mathbf{Z}_p[x,y]$. It then follows by the definition of $\mathtt{Det}^{\boldsymbol{r}}$ (see Equation (42)) and Equation (62) that in order to prove the claim (61) is it sufficient to prove (setting $\mathrm{Det}^{\boldsymbol{r}} = \mathrm{Det}^{\boldsymbol{r}}_N$)

$$\mathrm{Det}^{\boldsymbol{r}} = \mathrm{pr}_{\boldsymbol{r}} \circ \sigma_{\boldsymbol{r}}(F_r). \tag{63}$$

The previous formula is easily verified if $r \leqslant 2$ or $\boldsymbol{r} = (2,2,2)$ (hence $r = 3$). Assume now $r \geqslant 3$ and $\boldsymbol{r} \neq (2,2,2)$. Then at least one of $|A \cap B|$, $|A \cap C|$ and $|B \cap C|$ is greater or equal than 2. Without loss of generality one can then assume $r_2 = \min\{r_1, r_2, r_3\}$ and that the sets A and C are of the form

$$A = \{1, r, a_3, \ldots, a_{r_1}\} \quad \text{ and } \quad C = \{c_1, \ldots, c_{r_3-2}, 1, r\}.$$

Let $\boldsymbol{s} = (r_1 - 2, r_2, r_3 - 2)$ and $s = r - 2$. Then $\boldsymbol{s}$ satisfies Assumption 3.1 and one can chose as above a permutation $\sigma_{\boldsymbol{s}} = \sigma_{A_o,B,C_o}$ of $\{1, \ldots, 2 \cdot (r-1)\}$ relative to $A_o = \{a_3, \ldots, a_{r_1 - 1}\}$, B and $C_o = \{c_1, \ldots, c_{r_3-2}\}$. Extend $\sigma_{\boldsymbol{s}}$ to a permutation (denoted by the same symbol) of $\{1, \ldots, 2r\}$ by $\sigma_{\boldsymbol{s}}(i) = i$ for $i = 1, 2, 2r-1, 2r$ Without loss of generality one can then assume that $\sigma_{\boldsymbol{r}} = \sigma_{A,B,C}$ is the composition of $\sigma_{\boldsymbol{s}}$ with the permutation $\sigma_{\boldsymbol{r}|\boldsymbol{s}}$ of $\{1, \ldots, 2r\}$ defined by $\sigma_{\boldsymbol{r}|\boldsymbol{s}}(2) = 2r - 1$ and $\sigma_{\boldsymbol{r}|\boldsymbol{s}}(i) = i$ for $i \neq 2, 2r-1$, hence by induction on r one has

$$\mathrm{pr}_{\boldsymbol{r}} \circ \sigma_{\boldsymbol{r}}(F_r) = \mathrm{pr}_{\boldsymbol{r}} \circ \sigma_{\boldsymbol{r}|\boldsymbol{s}}\big(F_1 \otimes \sigma_{\boldsymbol{s}}(F_s) \otimes F_1\big) = \det \begin{pmatrix} x_1 & x_2 \\ z_1 & z_2 \end{pmatrix}^2 \cdot \mathrm{Det}^{\boldsymbol{s}}.$$

Since $r - r_2 = s - s_2 + 2$ and $r - r_j = s - s_j$ for $j \neq 2$, this proves Equation (63), and with it the proposition. □

4. *Big* étale sheaves and Galois representations

Sections 4.1 and 4.2 collect the technical background entering the construction of the three-variable diagonal class of Theorem A. In particular they present a slight extension of the *overconvergent cohomology* theory developed by Ash–Stevens and Andreatta–Iovita–Stevens in [**AS08, AIS15**].

Notation. In this section N is a positive integer coprime with p. Set $\Gamma = \Gamma_1(N,p)$, let Y denote the affine modular curve $Y_1(N,p)$ of level Γ defined over $\mathbf{Z}[1/Np]$ and let $u : E \to Y$ be the universal elliptic curve $E_1(N,p)$. Denote by C_p the universal order-p cyclic subgroup $C_1(N,p)$ of $E_1(N,p)$.

4.1. Locally analytic functions and distributions. — Let L be a finite extension of $\mathbf{Q}_p$ with ring of integers $\mathscr{O}$ and maximal ideal $\mathfrak{m} = \pi \cdot \mathscr{O}$. Let $\mathcal{W}$ be the *weight space* over $\mathbf{Q}_p$, viz. the rigid analytic space over $\mathbf{Q}_p$ which parametrises the continuous characters of $\mathbf{Z}_p^*$. It is isomorphic to $p-1$ copies of the open unit disc, indexed by the powers ω^j of the Teichmüller character $\omega : \mathbf{F}_p^* \to \mathbf{Z}_p^*$. We identify $\mathbf{Z} \times \mathbf{Z}/(p-1)\mathbf{Z}$ with a subset of $\mathcal{W}(\mathbf{Q}_p)$ by sending the pair (n, a) to the character

$(n,a) : \mathbf{Z}_p^* \to \mathbf{Z}_p^*$ defined by $(n,a)(u\cdot\omega) = u^n \cdot \omega^a$ for every $u \in 1 + p\mathbf{Z}_p$ and $\omega \in \mathbf{F}_p^*$. Given $\kappa \in \mathcal{W}$ and $z \in \mathbf{Z}_p^*$ we often write z^κ for $\kappa(z)$.

Let $U \subset \mathcal{W}$ be a connected wide open disc defined over L. Write $U \cap \mathbf{Z}$ for the set of characters in $U(\mathbf{Q}_p)$ of the form (n, i_U) for some $n \in \mathbf{Z}$ with $n \,(\mathrm{mod}\; p-1) = i_U$, where $i_U \in \mathbf{Z}/(p-1)\mathbf{Z}$ satisfies $\kappa|_{\mathbf{F}_p^*} = \omega^{i_U}$ for every $\kappa \in U$. Denote by $\mathcal{O}(U)$ the ring of rigid analytic functions on U, and by $\Lambda_U \subset \mathcal{O}(U)$ the set of $a \in \mathcal{O}(U)$ such that $\mathrm{ord}_p(a(x)) \geqslant 0$ for every $x \in U$. The $\mathscr{O}$-algebra Λ_U is isomorphic to the power series ring $\mathscr{O}[\![T]\!]$. In particular it is a regular local ring, complete with respect to the topology defined by its maximal ideal $\mathfrak{m}_U \cong (\pi, T)$. Let

$$\kappa_U : \mathbf{Z}_p^* \longrightarrow \Lambda_U^*$$

be the character sending $z \in \mathbf{Z}_p^*$ to the analytic function $\kappa_U(z) \in \Lambda_U^*$ which on $t \in U$ takes the value

$$\kappa_U(z)(t) = z^{t-2}.$$

In what follows let (B,κ) denote either the pair (Λ_U, κ_U) or $(\mathscr{O}, r)$ for some $r \in \mathcal{W}(L)$, and write $\mathfrak{m}_B$ for the maximal ideal of B. For every nonnegative integer $m \geqslant 0$ let $LA_m(\mathbf{Z}_p, B)$ be the space of functions $\gamma : \mathbf{Z}_p \to B$ converging on balls of width m, viz. for every $[a] \in \mathbf{Z}/p^m\mathbf{Z}$ one has $\gamma(a + p^m z) = \sum_{n\geqslant 0} c_n(\gamma)\cdot z^n$ for a sequence $c_n(\gamma)$ in B which converges to zero in the $\mathfrak{m}_B$-adic topology. We always assume that U is contained in a connected affinoid domain in $\mathcal{W}$ and that the function sending z to $\kappa_U(1+pz)$ belongs to $LA_m(\mathbf{Z}_p, \Lambda_U)$. The latter condition is guaranteed by taking $m = m(U)$ big enough.

Define $\mathsf{T} = \mathbf{Z}_p^* \times \mathbf{Z}_p$ and $\mathsf{T}' = p\mathbf{Z}_p \times \mathbf{Z}_p^*$. Right multiplication on $\mathbf{Z}_p^2$ by the semi-group

$$\Sigma_0(p) = \begin{pmatrix} \mathbf{Z}_p^* & \mathbf{Z}_p \\ p\mathbf{Z}_p & \mathbf{Z}_p \end{pmatrix} \subset \mathrm{Mat}_{2\times 2}(\mathbf{Z}_p) \quad \Big(\text{resp.,} \quad \Sigma_0'(p) = \begin{pmatrix} \mathbf{Z}_p & \mathbf{Z}_p \\ p\mathbf{Z}_p & \mathbf{Z}_p^* \end{pmatrix} \subset \mathrm{Mat}_{2\times 2}(\mathbf{Z}_p)\Big)$$

preserves the subset T (resp., T'). In particular both T and T' are preserved by scalar multiplication by $\mathbf{Z}_p^*$ and right multiplication by the Iwahori subgroup

$$\Gamma_0(p\mathbf{Z}_p) = \Sigma_0(p) \cap \Sigma_0'(p)$$

of $\mathrm{GL}_2(\mathbf{Z}_p)$. Define

$$\mathcal{A}_{\kappa,m} = \Big\{ f : \mathsf{T} \longrightarrow B \;\Big|\; f(1,z) \in LA_m(\mathbf{Z}_p, B) \text{ and } f(a\cdot t) = \kappa(a)\cdot f(t) \text{ for every } a \in \mathbf{Z}_p^*,\ t \in \mathsf{T} \Big\}, \tag{64}$$

and similarly define $\mathcal{A}'_{\kappa,m}$ as the space of functions $f : \mathsf{T}' \to B$ such that $f(pz,1)$ belongs to $LA_m(\mathbf{Z}_p, B)$, and $f(a\cdot t) = \kappa(a)\cdot f(t)$ for all $a \in \mathbf{Z}_p^*$ and $t \in \mathsf{T}'$. Set

$$A^{\cdot}_{\kappa,m} = \mathcal{A}^{\cdot}_{\kappa,m} \otimes_{\mathscr{O}} L, \quad \mathcal{D}^{\cdot}_{\kappa,m} = \mathrm{Hom}_B(\mathcal{A}^{\cdot}_{\kappa,m}, B) \quad \text{and} \quad D^{\cdot}_{\kappa,m} = \mathcal{D}^{\cdot}_{\kappa,m} \otimes_{\mathscr{O}} L,$$

where the superscript $\cdot$ denotes either $\emptyset$ or $\prime$. We equip $\mathcal{A}^{\cdot}_{\kappa,m}$ with the $\mathfrak{m}_B$-adic topology and $\mathcal{D}^{\cdot}_{\kappa,m}$ with the weak-$*$ topology, viz. the weakest topology which makes the evaluation-at-f morphism continuous for every f in $\mathcal{A}^{\cdot}_{\kappa,m}$. The B-module $\mathcal{A}^{\cdot}_{\kappa,m}$ is preserved by the left action of $\Sigma^{\cdot}_0(p)$ on functions $f : \mathsf{T}^{\cdot} \to B$ given by $\gamma \cdot f(t) = f(t\cdot\gamma)$, for every $\gamma \in \Sigma^{\cdot}_0(p)$ and $t \in \mathsf{T}^{\cdot}$. This equips $\mathcal{A}^{\cdot}_{\kappa,m}$ with the structure of a

$B[\Sigma_0^{\cdot}(p)]$-module, and induce on $\mathcal{D}^{\cdot}_{\kappa,m}$ the structure of a right $B[\Sigma_0^{\cdot}(p)]$-module. If $(B,\kappa)=(\Lambda_U,\kappa_U)$ we write $\mathcal{A}^{\cdot}_{U,m}$ and $\mathcal{D}^{\cdot}_{U,m}$ as shorthands for $\mathcal{A}^{\cdot}_{\kappa_U,m}$ and $\mathcal{D}^{\cdot}_{\kappa_U,m}$.

Remark 4.1. — For any function $f:\mathsf{T}\to B$ define $f_o:\mathbf{Z}_p\to B$ by $f_o(z)=f(1,z)$. The map which to f associates f_o gives an isomorphism of B-modules between $\mathcal{A}_{\kappa,m}$ and $LA_m(\mathbf{Z}_p,B)$. This intertwines the action of $\Sigma_0(p)$ on $\mathcal{A}_{\kappa,m}$ with the one on $LA_m(\mathbf{Z}_p,B)$ given by

$$\sigma\cdot f_o(z)=(a+cz)^{\kappa}\cdot f_o\left(\frac{b+dz}{a+cz}\right),\quad\text{where}\quad \sigma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}.$$

The B-module $LA_m(\mathbf{Z}_p,B)$ is isomorphic to the product $\prod_{a=0}^{p^m-1}B[\![T]\!]^o$, where $B[\![T]\!]^o$ is the set of power series $\sum_{n\geqslant 0}b_n\cdot T^n$ in $B[\![T]\!]$ with $\lim_{n\to\infty}b_n=0$ in the $\mathfrak{m}_B$-adic topology. Under this isomorphism, for every $0\leqslant a\leqslant p^m-1$ and every $n\geqslant 0$, the power T^n in the a-th factor of $LA_m(\mathbf{Z}_p,B)$ corresponds to an element $f_{a,n}\in\mathcal{A}_{\kappa,m}$. Every $f\in\mathcal{A}_{\kappa,m}$ can be written uniquely as $f=\sum_{0\leqslant a\leqslant p^m-1,n\geqslant 0}b_{a,n}(f)\cdot f_{a,n}$ with $\lim_{n\to\infty}b_{a,n}(f)=0$ for every $0\leqslant a\leqslant p^m-1$. A similar discussion applies to $\mathcal{A}'_{\kappa,m}$.

4.1.1. *Hecke operators.* — Set $\Xi_0^{\cdot}(p)=\Sigma_0^{\cdot}(p)\cap\mathrm{GL}_2(\mathbf{Q}_p)$, and recall that Γ denotes the congruence subgroup $\Gamma_1(N)\cap\Gamma_0(p)$ of $\mathrm{SL}_2(\mathbf{Z})$. Let M be a *right* $\Xi_0^{\cdot}(p)$-module (e.g. $M=\mathcal{D}^{\cdot}_{\kappa,m}$). Given $\sigma\in\Xi_0^{\cdot}(p)$ one defines a Hecke operator

$$T_\sigma: H^j(\Gamma,M)\to H^j(\Gamma,M)$$

as follows (cf. [**AS86a**, Section 1.1]). Write $\Gamma\sigma\Gamma=\coprod_{i=1}^{n_\sigma}\Gamma\sigma_i$ with $\sigma_i\in\Xi_0^{\cdot}(p)$, and define $t_i:\Gamma\longrightarrow\Gamma$ by $\sigma_i\cdot\gamma=t_i(\gamma)\cdot\sigma_{i(\gamma)}$ (for some $1\leqslant i(\gamma)\leqslant n_\sigma$). If $\boldsymbol{\xi}\in H^j(\Gamma,M)$ is represented by the homogeneous j-cochain $\xi:\Gamma^{j+1}\longrightarrow M$ then $T_\sigma(\boldsymbol{\xi})=cl(\xi_\sigma)$, where $\xi_\sigma:\Gamma^{j+1}\longrightarrow M$ is defined by

$$\xi_\sigma(\gamma_0,\dots,\gamma_j)=\sum_{i=1}^{n_\sigma}\xi(t_i(\gamma_0),\dots,t_i(\gamma_j))\cdot\sigma_i.$$

For every prime ℓ denote by σ_ℓ (resp., σ'_ℓ) the diagonal matrix with diagonal $(1,\ell)$ (resp., $(\ell,1)$). If σ_ℓ (resp., σ'_ℓ) belongs to $\Xi_0^{\cdot}(p)$ set $T_\ell=T_{\sigma_\ell}$ (resp., $T'_\ell=T_{\sigma'_\ell}$). As usual one also writes $U^{\cdot}_\ell$ for $T^{\cdot}_\ell$ if ℓ divides Np. The previous discussion then equips $H^i(\Gamma,\mathcal{D}_{\kappa,m})$ (resp., $H^1(\Gamma,\mathcal{D}'_{\kappa,m})$) with the action of the p-th Hecke operator U_p (resp., p-th dual Hecke operator U'_p), as well as with the action of the Hecke operators T_ℓ and T'_ℓ for every prime $\ell\neq p$.

Let N be a *left* $\Xi_0^{\cdot}(p)$-module (e.g. $N=\mathcal{A}^{\cdot}_{\kappa,m}$) and let N^{op} denote the abelian group N equipped with the structure of right $\Xi_0^{\cdot}(p)^{-1}$-module by $n\cdot\tau=\tau^{-1}\cdot n$ for every $n\in N$ and $\tau\in\Xi_0^{\cdot}(p)^{-1}$. After identifying $H^i(\Gamma,N)$ and $H^i(\Gamma,N^{\mathrm{op}})$ define for every $\sigma\in\Xi_0^{\cdot}(p)$ the Hecke operator T_σ on $H^i(\Gamma,N)$ to be the Hecke operator $T_{\sigma^{-1}}$ on $H^i(\Gamma,N^{\mathrm{op}})$ defined in previous paragraph. This equips $H^i(\Gamma,\mathcal{A}_{\kappa,m})$ (resp., $H^i(\Gamma,\mathcal{A}'_{\kappa,m})$) with the action of the p-th Hecke operator $U_p=T_{\sigma_p}$ (resp., p-th dual Hecke operator $U'_p=T_{\sigma'_p}$), as well as with the action of the Hecke operators $T_\ell=T_{\sigma_\ell}$ and $T'_\ell=T_{\sigma'_\ell}$ for every prime ℓ different from p.

4.1.2. *Atkin–Lehner operators.* — Let Q be a positive divisor of Np, such that Q and Np/Q are coprime. Consider any matrix

$$\mathtt{w}_Q = \begin{pmatrix} Qa & b \\ Np & Qd \end{pmatrix} \in M_2(\mathbf{Z})$$

such that $\det(\mathtt{w}_Q) = Q$ and $d \equiv 1 \pmod{Np/Q}$. Such a matrix satisfies

$$\Gamma = \mathtt{w}_Q \cdot \Gamma \cdot \mathtt{w}_Q^{-1}. \tag{65}$$

If p divides Q, then right multiplication by $\mathtt{w}_Q$ on $\mathbf{Z}_p^2$ maps T onto T', hence induces a topological morphism of B-modules $w_Q : \mathcal{A}'_{\kappa,m} \longrightarrow \mathcal{A}_{\kappa,m}$. Together with conjugation by the inverse of $\mathtt{w}_Q$ on Γ (cf. Equation (65)), it yields a morphism of pairs $w_Q : (\Gamma, \mathcal{A}'_{\kappa,m}) \longrightarrow (\Gamma, \mathcal{A}_{\kappa,m})$, which in turn induces a morphism

$$w_Q : H^1(\Gamma, \mathcal{A}'_{\kappa,m}) \longrightarrow H^1(\Gamma, \mathcal{A}_{\kappa,m}). \tag{66}$$

A direct computation proves that, for each x in $H^1(\Gamma, \mathcal{A}'_{\kappa,m})$, one has

$$U_p \circ w_p(x) = w_p \circ U'_p \circ \langle p \rangle_N (x) \quad \text{and} \quad U_p \circ w_{Np}(x) = w_{Np} \circ U'_p(x),$$

where $\langle p \rangle_N = T_{\alpha_p}$ is the Hecke operator on $H^1(\Gamma, \mathcal{A}'_{\kappa,m})$ associated with any matrix α_p in $\mathrm{SL}_2(\mathbf{Z})$ of the form $\alpha_p = \left(\begin{smallmatrix} a & b \\ Npc & d \end{smallmatrix}\right)$ with $d \equiv 1 \pmod p$ and $d \equiv p \pmod N$. The dual of $w_Q : \mathcal{A}'_{\kappa,m} \longrightarrow \mathcal{A}_{\kappa,m}$ yields a map $w_Q : \mathcal{D}_{\kappa,m} \longrightarrow \mathcal{D}'_{\kappa,m}$, which together with conjugation by $\mathtt{w}_Q$ on Γ induces as above a morphism

$$w_Q : H^1(\Gamma, \mathcal{D}_{\kappa,m}) \longrightarrow H^1(\Gamma, \mathcal{D}'_{\kappa,m}). \tag{67}$$

For each y in $H^1(\Gamma, \mathcal{D}_{\kappa,m})$ one has

$$w_p \circ U_p(y) = U'_p \circ w_p \circ \langle p \rangle_N (y) \quad \text{and} \quad w_{Np} \circ U_p(y) = U'_p \circ w_{Np}(y). \tag{68}$$

If p does not divide Q, then $\mathtt{w}_Q$ belongs to $\Gamma_0(p\mathbf{Z}_p)$, and for $\cdot = \emptyset, \prime$ one defines

$$w_Q : H^1(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m}) \longrightarrow H^1(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m}) \ \text{ and } \ w_Q : H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m}) \longrightarrow H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m}) \tag{69}$$

to be the Hecke operators $T_{\mathtt{w}_Q}$ introduced in Section 4.1.1.

4.1.3. *Specialisations and comparison.* — Let $k = r + 2 \in U$ and let $\pi_k \in \Lambda_U$ be a uniformiser at $k - 2$ (hence π and π_k generate $\mathfrak{m}_U$). There are short exact sequences of $\Sigma^{\cdot}_0(p)$-modules (cf. [AIS15, Proposition 3.11])

$$\begin{aligned} &0 \longrightarrow \mathcal{A}^{\cdot}_{U,m} \xrightarrow{\pi_k} \mathcal{A}^{\cdot}_{U,m} \xrightarrow{\rho_k} \mathcal{A}^{\cdot}_{r,m} \longrightarrow 0; \\ &0 \longrightarrow \mathcal{D}^{\cdot}_{U,m} \xrightarrow{\pi_k} \mathcal{D}^{\cdot}_{U,m} \xrightarrow{\rho_k} \mathcal{D}^{\cdot}_{r,m} \longrightarrow 0. \end{aligned} \tag{70}$$

The morphisms ρ_k are defined by the formulae

$$\rho_k(f)(x, y) = f(x, y)(k) \quad \text{and} \quad \rho_k(\mu)(\gamma) = \mu(\gamma_U)(k)$$

for every $f \in \mathcal{A}^{\cdot}_{U,m}$, $(x, y) \in \mathsf{T}^{\cdot}$, $\mu \in \mathcal{D}^{\cdot}_{U,m}$, and $\gamma \in \mathcal{A}^{\cdot}_{r,m}$, where $\gamma_U(x, y) = \kappa_U(x) \cdot \gamma(1, y/x)$ if $\mathsf{T}^{\cdot} = \mathsf{T}$ and $\gamma_U(x, y) = \kappa_U(y) \cdot \gamma(x/y, 1)$ if $\mathsf{T}^{\cdot} = \mathsf{T}'$.

Let $r \in U \cap \mathbf{Z}_{\geqslant 0}$ be a nonnegative integer. Viewing two-variable polynomials as analytic functions on $\mathsf{T}^{\cdot}$ gives a natural map of $\Sigma^{\cdot}_0(p)$-modules $S_r(\mathscr{O}) \longrightarrow \mathcal{A}^{\cdot}_{r,m}$, and dually a morphism of $\Sigma_0^{\cdot}(p)$-modules $\mathcal{D}^{\cdot}_{r,m} \longrightarrow L_r(\mathscr{O})$. Together with the comparison isomorphisms between étale and Betti cohomology:

$$H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{S}_r(\mathscr{O})) \cong H^1(\Gamma, S_r(\mathscr{O})) \quad \text{and} \quad H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{L}_r(\mathscr{O})) \cong H^1(\Gamma, L_r(\mathscr{O})) \tag{71}$$

they induce *comparison* morphisms

$$H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{S}_r(\mathscr{O})) \longrightarrow H^1(\Gamma, \mathcal{A}^{\cdot}_{r,m}) \quad \text{and} \quad H^1(\Gamma, \mathcal{D}^{\cdot}_{r,m}) \longrightarrow H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{L}_r(\mathscr{O})). \tag{72}$$

The second isomorphism in Equation (71) is Hecke equivariant, hence so is the second morphism in Equation (72). On the other hand the first isomorphism in Equation (71) (resp., morphism in Equation (72)) intertwines the actions of the Hecke operators $U_p, T_\ell, U'_p, T'_\ell$ on the left hand side with those of Hecke operators $U'_p, T'_\ell, U_p, T_\ell$ respectively on the right hand side (whenever the latter are defined).

4.1.4. *Slope decompositions.* — Let $\mathscr{B}$ be a $\mathbf{Q}_p$-Banach algebra, let N be a module over $\mathscr{B}$, let $\boldsymbol{u}$ be a $\mathscr{B}$-linear endomorphism of N, and let $h \in \mathbf{Q}_{\geqslant 0}$. Following [**AS08**] one says that N admits a *slope* $\leqslant h$ *decomposition with respect to* $\boldsymbol{u}$ if there exists a (necessarily unique) direct sum decomposition

$$N = N^{\leqslant h} \oplus N^{>h}$$

into $\mathscr{B}[\boldsymbol{u}]$-modules such that the conditions 1–3 below are satisfied. One says that a polynomial $P(t)$ in $\mathscr{B}[t]$ has *slope* $\leqslant h$ if every edge of its Newton polygon has slope $\leqslant h$. Let $\mathscr{B}[t]^{\leqslant h}$ be the set of polynomials in $\mathscr{B}[t]$ of slope $\leqslant h$ and whose leading coefficient is a multiplicative unit. For every $P(t) \in \mathscr{B}[t]$ write $P^*(t) = t^{\deg(P)} \cdot P(1/t)$.

1. $N^{\leqslant h}$ is finitely generated over $\mathscr{B}$.
2. There exists $P(t) \in \mathscr{B}[t]^{\leqslant h}$ such that $P^*(\boldsymbol{u})$ kills $N^{\leqslant h}$.
3. For every $P(t) \in \mathscr{B}[t]^{\leqslant h}$ the endomorphism $P^*(\boldsymbol{u})$ of $N^{>h}$ is an isomorphism.

Let m and U be as in Section 4.1, let $k = r + 2 \in U(L)$, and let $h \in \mathbf{Q}_{\geqslant 0}$. Set

$$\mathcal{T}_r = \big\{(L, A_{r,m}, U_p), (L, A'_{r,m}, U'_p), (L, D_{r,m}, U_p), (L, D'_{r,m}, U'_p)\big\}$$

and

$$\mathcal{T}_U = \big\{(\mathscr{O}_U, A_{U,m}, U_p), (\mathscr{O}_U, A'_{U,m}, U'_p), (\mathscr{O}_U, D_{U,m}, U_p), (\mathscr{O}_U, D'_{U,m}, U'_p)\big\},$$

where $\mathscr{O}_U$ is a shorthand for $\Lambda_U[1/p]$. Recall that Λ_U is isomorphic to the power series ring $\mathscr{O}[\![T]\!]$, equipped with the topology defined by the maximal ideal $\mathfrak{m}_U \cong (\pi, T)$, hence $\mathscr{O}_U$ is isomorphic to the L-module $L[\![T]\!]^o$ of power series in $L[\![T]\!]$ with bounded Gauß norm. If s is a real number satisfying $0 < s < 1$, define $|\cdot|_s : L[\![T]\!]^o \longrightarrow \mathbf{R}_{\geqslant 0}$ by $|\sum_{n\geqslant 0} a_n \cdot T^n|_s = \sup_{n\geqslant 0} s^n \cdot |a_n|_p$. Then $|\cdot|_s$ is an L-Banach algebra norm on $L[\![T]\!]^o$, which is independent of s and induces the (π, T)-adic topology on $\mathscr{O}[\![T]\!]$. This corresponds to an L-Banach algebra norm on $\mathscr{O}_U$, which restricts to the $\mathfrak{m}_U$-adic topology on the $\mathscr{O}$-submodule Λ_U. The discussion on slope $\leqslant h$ decompositions then applies to each triple $(\mathscr{B}, M, \boldsymbol{u})$ in $\mathcal{T}_r \cup \mathcal{T}_U$. The following proposition is a consequence of the work of Coleman and Ash–Stevens [**Col97, AS08**] (see also [**AIS15**]).

Proposition 4.2. — *Let $(\mathscr{B}, M, \boldsymbol{u})$ be a triple in $\mathcal{T}_r \cup \mathcal{T}_U$. If $r \in U \cap \mathbf{Z}_{\geqslant 0}$, one also allows $(\mathscr{B}, M, \boldsymbol{u})$ to denote either $(L, S_r(L), U^{\cdot}_p)$ or $(L, L_r(L), U^{\cdot}_p)$, with $U^{\cdot}_p = U_p, U'_p$.*

1. *Up to shrinking U if necessary, the $\mathscr{B}$-module $H^1(\Gamma, M)$ admits a slope $\leqslant h$ decomposition with respect to $\boldsymbol{u}$. Moreover, for $\cdot = \emptyset, \prime$, the specialisation maps ρ_k defined in Equation (70) induce Hecke equivariant isomorphisms*

$$\rho_k : H^1(\Gamma, A^{\cdot}_{U,m})^{\leqslant h} \otimes_{\Lambda_U} \Lambda_U/\pi_k \cong H^1(\Gamma, A^{\cdot}_{r,m})^{\leqslant h}$$
$$\text{and} \quad \rho_k : H^1(\Gamma, D^{\cdot}_{U,m})^{\leqslant h} \otimes_{\Lambda_U} \Lambda_U/\pi_k \cong H^1(\Gamma, D^{\cdot}_{r,m})^{\leqslant h}.$$

2. *Assume that $r = (n, a) \in \mathbf{Z}_{\geqslant 0} \times \mathbf{Z}/(p-1)\mathbf{Z}$ with $n \equiv a \pmod{p-1}$ and $h < n+1$. Then (for $\cdot = \emptyset, \prime$) the natural maps $S_r(L) \longrightarrow A^{\cdot}_{r,m}$ and $D^{\cdot}_{r,m} \longrightarrow L_r(L)$ induce Hecke equivariant isomorphisms*

$$H^1(\Gamma, S_r(L))^{\leqslant^{\cdot} h} \cong H^1(\Gamma, A^{\cdot}_{r,m})^{\leqslant h} \quad \text{and} \quad H^1(\Gamma, D^{\cdot}_{r,m})^{\leqslant h} \cong H^1(\Gamma, L_r(L))^{\leqslant^{\cdot} h},$$

where the superscript $\leqslant^{\cdot} h$ in $H^1(\Gamma, -)^{\leqslant^{\cdot} h}$ refers to the slope decomposition with respect to the endomorphism $U_p^{\cdot}$.

Let r be a nonnegative integer and let $h \in \mathbf{Q}_{\geqslant 0}$ such that $h < r+1$. As the étale cohomology groups $H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{S}_r)_L$ and $H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{L}_r)_L$ are finite-dimensional over L, they admit slope $\leqslant^{\cdot} h$ decompositions with respect to $U_p^{\cdot}$. Part 2 of Proposition 4.2 then implies that the comparison maps defined in Equation (72) induce natural isomorphisms of L-modules (cf. the last lines of the previous section)

$$(73) \qquad H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{S}_r)_L^{\leqslant' h} \cong H^1(\Gamma, A_{r,m})^{\leqslant h} \quad \text{and} \quad H^1(\Gamma, D_{r,m})^{\leqslant h} \cong H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{L}_r)_L^{\leqslant h}.$$

One obtains similar isomorphisms after replacing $A_{r,m}$ and $D_{r,m}$ with $A'_{r,m}$ and $D'_{r,m}$ respectively.

4.2. Étale sheaves. — Let $\mathscr{T} = \mathscr{T}_{1(p),N}$ be the relative Tate module $R^1 u_* \mathbf{Z}_p(1)$ of E over Y (cf. Equation (10)). Fix a geometric point $\eta : \operatorname{Spec}(\bar{\mathbf{Q}}) \longrightarrow Y$ and denote by $\mathcal{G} = \mathcal{G}_{N,p}$ the fundamental group $\pi_1^{\text{ét}}(Y, \eta)$. Fix in addition an isomorphism $\xi : \mathscr{T}_\eta \cong \mathbf{Z}_p \oplus \mathbf{Z}_p$ of $\mathbf{Z}_p$-modules such that, for every $x, y \in \mathscr{T}_\eta$, one has

$$(74) \qquad \langle x, y \rangle_{E_{p^\infty}} = \xi(x) \wedge \xi(y) \quad \text{and} \quad \bar{\xi}(C_{p,\eta}) = \mathbf{F}_p \cdot (1, 0),$$

where $\langle \cdot, \cdot \rangle_{E_{p^\infty}}$ is the Weil pairing, $\bigwedge^2 \mathbf{Z}_p^2 = \mathbf{Z}_p$ via $(1,0) \wedge (0,1) = 1$, and $\bar{\xi} : E_{p,\eta} \cong \mathbf{F}_p \oplus \mathbf{F}_p$ is the reduction of ξ modulo p. The action of $\mathcal{G}$ on $\mathscr{T}_\eta$ and the isomorphism ξ give a continuous morphism $\varrho : \mathcal{G} \to \mathrm{GL}_2(\mathbf{Z}_p)$. Since the subgroup $C_{p,\eta}$ of $E_{p,\eta}$ is preserved by the action of $\mathcal{G}$, the second condition in Equation (74) implies that ϱ factors through a continuous morphism $\varrho : \mathcal{G} \longrightarrow \Gamma_0(p\mathbf{Z}_p)$. Let $\mathbf{S}_f(Y_{\text{ét}})$ be the category of locally constant constructible sheaves on $Y_{\text{ét}}$ with finite stalk of p-power order at η, and for every topological group G denote by $\mathbf{M}_f(G)$ the category of finite sets of p-power order, equipped with a continuous action of G. Taking the stalk at η defines an equivalence of categories $\cdot_\eta : \mathbf{S}_f(Y_{\text{ét}}) \cong \mathbf{M}_f(\mathcal{G})$, whose inverse $\cdot^{\text{ét}} : \mathbf{M}_f(\mathcal{G}) \cong \mathbf{S}_f(Y_{\text{ét}})$ restricts via ϱ to a functor $\cdot^{\text{ét}} : \mathbf{M}_f(\Gamma_0(p\mathbf{Z}_p)) \longrightarrow \mathbf{S}_f(Y_{\text{ét}})$. (Here both $\mathcal{G}$ and $\Gamma_0(p\mathbf{Z}_p)$ have the profinite topology.) Define $\mathbf{M}_{\text{cts}}(G)$ to be the category of G-modules M which are filtered unions $M = \bigcup_{i \in I} M_i$ with $M_i \in \mathbf{M}_f(G)$ for every $i \in I$, and $\mathbf{M}(G) \subset \mathbf{M}_{\text{cts}}(G)^{\mathbf{N}}$ to be the category of inverse systems of objects of $\mathbf{M}_{\text{cts}}(G)$. Define similarly $\mathbf{S}_{\text{cts}}(Y_{\text{ét}})$ and $\mathbf{S}(Y_{\text{ét}}) \subset \mathbf{S}_{\text{cts}}(Y_{\text{ét}})^{\mathbf{N}}$. If G denotes one of $\mathcal{G}$ and $\Gamma_0(p\mathbf{Z}_p)$, the functor $\cdot^{\text{ét}}$ extends to $\cdot^{\text{ét}} : \mathbf{M}(G) \longrightarrow \mathbf{S}(Y_{\text{ét}})$. Let $(M_i)_{i \in \mathbf{N}}$

be an inverse system of G-modules and let $M = \varprojlim M_i$. If the inverse system $(M_i)_i$ defining M is clear from the context, we say that M *belongs* to $\mathbf{M}(G)$ to mean that $(M_i)_i$ does. If this is the case we write $M^{\text{ét}}$ for $(M_i)_i^{\text{ét}}$.

More generally for every scheme S one defines the category $\mathbf{S}(S_{\text{ét}})$ as above. For every $\mathscr{F} = (\mathscr{F}_i)_{i\in\mathbf{N}} \in \mathbf{S}(S_{\text{ét}})$ set

$$H^j_{\text{ét}}(S,\mathscr{F}) = R^j\big(\lim_{\leftarrow i}\Gamma(S,\cdot)\big)(\mathscr{F}_i)_i \quad \text{and} \quad \mathtt{H}^j_{\text{ét}}(S,\mathscr{F}) = \lim_{\leftarrow i} H^j_{\text{ét}}(S,\mathscr{F}_i),$$

so that $(H^j_{\text{ét}}(S,\mathscr{F})$ is the continuous étale cohomology in the sense of [**Jan88**] and) there are short exact sequences

$$0 \longrightarrow R^1\lim_{\leftarrow i} H^{j-1}_{\text{ét}}(S,\mathscr{F}_i) \longrightarrow H^j_{\text{ét}}(S,\mathscr{F}) \longrightarrow \mathtt{H}^j_{\text{ét}}(S,\mathscr{F}) \longrightarrow 0. \tag{75}$$

One similarly defines compactly supported cohomology groups $H^j_{\text{ét,c}}(S,\mathscr{F})$ and $\mathtt{H}^j_{\text{ét,c}}(S,\mathscr{F})$ (cf. [**Jan88**]).

Let (B,κ) be as in Section 4.1. The modules $\mathcal{A}^{\cdot}_{\kappa,m}$ and $\mathcal{D}^{\cdot}_{\kappa,m}$ belong to $\mathbf{M}(\Gamma_0(p\mathbf{Z}_p))$:

$$\begin{aligned} \mathcal{D}^{\cdot}_{\kappa,m} &= \lim_{\leftarrow j} \mathcal{D}^{\cdot}_{\kappa,m}/\mathrm{Fil}^j\mathcal{D}^{\cdot}_{\kappa,m}, \\ \mathcal{A}^{\cdot}_{\kappa,m} &= \lim_{\leftarrow j} \mathcal{A}^{\cdot}_{\kappa,m}/\mathfrak{m}_B^j\mathcal{A}^{\cdot}_{\kappa,m} \\ \text{and} \quad \mathcal{A}^{\cdot}_{\kappa,m}/\mathfrak{m}_B^i\cdot\mathcal{A}^{\cdot}_{\kappa,m} &= \bigcup_{j\geqslant i}\mathrm{Fil}_{i,j}\mathcal{A}^{\cdot}_{\kappa,m}, \end{aligned}$$

where $(\mathrm{Fil}^j\mathcal{D}^{\cdot}_{\kappa,m})_{j\geqslant 0}$ is a decreasing filtration by $B[\Sigma^{\cdot}_0(p)]$-submodules on $\mathcal{D}^{\cdot}_{\kappa,m}$, such that $\mathcal{D}^{\cdot}_{\kappa,m}/\mathrm{Fil}^j$ is finite for every j, and where $(\mathrm{Fil}_{i,j}\mathcal{A}^{\cdot}_{\kappa,m})_{j\geqslant i}$ is an increasing filtration on $\mathcal{A}^{\cdot}_{\kappa,m}/\mathfrak{m}_B^i\cdot\mathcal{A}^{\cdot}_{\kappa,m}$ by $B[\Sigma^{\cdot}_0(p)]$-submodules of finite cardinality. Precisely one defines

$$\mathrm{Fil}^j\mathcal{D}^{\cdot}_{\kappa,m} = \Big\{\mu \in \mathcal{D}^{\cdot}_{\kappa,m} \;\Big|\; \mu(f_{a,n}) \in \mathfrak{m}_B^{j-n} \text{ for every } 0 \leqslant a \leqslant p^m-1 \text{ and } n \leqslant j\Big\}$$

(cf. [**AIS15**, Definition 3.9 and Proposition 3.10]) and

$$\mathrm{Fil}_{i,j}\mathcal{A}^{\cdot}_{\kappa,m} = \bigoplus_{0\leqslant a\leqslant p^m-1,\, n\leqslant j} B\cdot\big(f_{a,n} + \mathfrak{m}_B^i\big) \subset \mathcal{A}^{\cdot}_{\kappa,m}/\mathfrak{m}_B^i\cdot\mathcal{A}^{\cdot}_{\kappa,m},$$

where $(f_{a,n})_{0\leqslant a\leqslant p^m-1, n\geqslant 0}$ is the orthonormal basis of $\mathcal{A}^{\cdot}_{\kappa,m}$ defined in Remark 4.1. Denote by

$$\boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m} = \mathcal{A}^{\cdot\,\text{ét}}_{\kappa,m} \quad \text{and} \quad \boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m} = \mathcal{D}^{\cdot\,\text{ét}}_{\kappa,m}$$

the images of $\mathcal{A}^{\cdot}_{\kappa,m}$ and $\mathcal{D}^{\cdot}_{\kappa,m}$ respectively under $\cdot^{\text{ét}} : \mathbf{M}(\Gamma_0(p\mathbf{Z}_p)) \to \mathbf{S}(Y_{\text{ét}})$. For every $j \geqslant 0$ set

$$\begin{aligned} \mathcal{A}^{\cdot}_{\kappa,m,j} &= \mathcal{A}^{\cdot}_{\kappa,m}/\mathfrak{m}_B^j\cdot\mathcal{A}^{\cdot}_{\kappa,m}, \\ \mathcal{D}^{\cdot}_{\kappa,m,j} &= \mathcal{D}^{\cdot}_{\kappa,m}/\mathrm{Fil}^j, \\ \boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m,j} &= \mathcal{A}^{\cdot\,\text{ét}}_{\kappa,m,j} \\ \text{and} \quad \boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m,j} &= \mathcal{D}^{\cdot\,\text{ét}}_{\kappa,m,j}, \end{aligned}$$

so that $\boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m}$ is a shortened notation for the inverse system $(\boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m,j})_{j\in\mathbf{N}}$ and similarly $\boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m} = (\boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m,j})_{j\in\mathbf{N}}$. If S is a $\mathbf{Z}[1/Np]$-scheme one can define for every prime $\ell \nmid Np$

(resp., prime $\ell | Np$, unit $d \in (\mathbf{Z}/N\mathbf{Z})^*$) Hecke operators $T_\ell^{\cdot}$ (resp., $U_\ell^{\cdot}$, $\langle d \rangle$) acting on $H^i_{\text{ét}}(Y_S, \boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m,j})$ and $H^i_{\text{ét}}(Y_S, \boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m,j})$ (cf. Section 2.3 or [AIS15, Section 5]). We list below some of the basic properties satisfied by $\boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m}$ and $\boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m}$. Let S be a $\mathbf{Z}[1/Np]$-scheme and let $\chi : \mathbf{Z}_p^* \to B^*$ be a continuous character. Let $B/\mathfrak{m}_B^i(\chi) \in \mathbf{M}_f(\Gamma_0(p\mathbf{Z}_p))$ be a copy of $B/\mathfrak{m}_B^i$ equipped with the action of $\Gamma_0(p\mathbf{Z}_p)$ defined by $\gamma \cdot b = \chi(\det(\gamma)) \cdot b$, and let $B(\chi) = \lim_{\leftarrow i} B/\mathfrak{m}_B^i(\chi)$. If $\mathcal{C}^{\cdot}_{\kappa,m,\cdot}$ denotes either $\mathcal{A}^{\cdot}_{\kappa,m,\cdot}$ or $\mathcal{D}^{\cdot}_{\kappa,m,\cdot}$ define $\mathcal{C}^{\cdot}_{\kappa,m,\cdot}(\chi) = \mathcal{C}^{\cdot}_{\kappa,m,\cdot} \otimes_B B(\chi)$ and $\boldsymbol{\mathcal{C}}^{\cdot}_{\kappa,m}(\chi) = \mathcal{C}^{\cdot}_{\kappa,m}(\chi)^{\text{ét}} = \boldsymbol{\mathcal{C}}^{\cdot}_{\kappa,m} \otimes (B/\mathfrak{m}_B^i(\chi))^{\text{ét}}_{i \in \mathbf{N}}$. As usual, if $(B,\kappa) = (\Lambda_U, \kappa_U)$, one sets $\boldsymbol{\mathcal{C}}^{\cdot}_{U,m,\cdot} = \boldsymbol{\mathcal{C}}^{\cdot}_{\kappa_U,m,\cdot}$.

- For each $k = r+2 \in U(L)$, each $j \in \mathbf{N}$ and $\cdot = \emptyset, \prime$, the specialisation maps (70) induce morphisms
$$\rho_k : \boldsymbol{\mathcal{A}}^{\cdot}_{U,m,j}(\chi) \to \boldsymbol{\mathcal{A}}^{\cdot}_{r,m,j}(\chi) \quad \text{and} \quad \rho_k : \boldsymbol{\mathcal{D}}^{\cdot}_{U,m,j}(\chi) \to \boldsymbol{\mathcal{D}}^{\cdot}_{r,m,j}(\chi),$$
which in turn induce in cohomology *specialisation maps*
$$\begin{aligned} \rho_k &: H^1_{\text{ét}}(Y_S, \boldsymbol{\mathcal{A}}^{\cdot}_{U,m}(\chi)) \longrightarrow H^1_{\text{ét}}(Y_S, \boldsymbol{\mathcal{A}}^{\cdot}_{r,m}(\chi)) \\ \text{and} \quad \rho_k &: H^1_{\text{ét}}(Y_S, \boldsymbol{\mathcal{D}}^{\cdot}_{U,m}(\chi)) \longrightarrow H^1_{\text{ét}}(Y_S, \boldsymbol{\mathcal{D}}^{\cdot}_{r,m}(\chi)). \end{aligned} \tag{76}$$
- There are natural isomorphisms $H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m,j}) \cong H^1(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m,j})$, which induce isomorphisms (cf. Theorem 3.15 of [AIS15])
$$\begin{aligned} H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m}) &\cong \mathtt{H}^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m}) \cong H^1(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m}) \\ \text{and} \quad H^1_{\text{ét,c}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m}) &\cong \mathtt{H}^1_{\text{ét,c}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m}) \cong H^1_{\text{c}}(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m}) \end{aligned} \tag{77}$$
of B-modules compatible with the action of the Hecke operators and with the specialisation maps ρ_r. Here $H^j_{\text{c}}(\Gamma, \cdot) = H^{j-1}(\Gamma, I(\cdot))$ is defined to be the $(j-1)$-th cohomology group of Γ with values in the Γ-module
$$I(\cdot) = \mathrm{Hom}_{\mathbf{Z}}(\mathrm{Div}^0(\mathbf{P}^1(\mathbf{Q})), \cdot)$$
(cf. Proposition 4.2 of [AS86b]).
- There are natural maps $H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m,j}) \longrightarrow H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m,j})$, inducing an isomorphism of B-modules (cf. Lemma 4.3 below and the discussion preceding it)
$$\mathtt{H}^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m}) \cong H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m}) \tag{78}$$
compatible with the action of the Hecke operators and with the specialisation maps. In light of the exact sequence (75), the isomorphism (78) yields a Hecke equivariant short exact sequence of B-modules
$$0 \longrightarrow R^1\lim_{\leftarrow j} H^0(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m,j}) \longrightarrow H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m}) \longrightarrow H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m}) \longrightarrow 0. \tag{79}$$
- The B-modules $\mathtt{H}^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m})$ and $\mathtt{H}^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m})$ are equipped with natural continuous actions of $G_{\mathbf{Q}}$ which commute with the Hecke operators and the specialisation maps. Moreover as $G_{\mathbf{Q}}$-modules
$$\begin{aligned} \mathtt{H}^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m}(\chi)) &= \mathtt{H}^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m})(\chi_{\mathbf{Q}}) \\ \text{and} \quad \mathtt{H}^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m}(\chi)) &= \mathtt{H}^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m})(\chi_{\mathbf{Q}}), \end{aligned} \tag{80}$$

where $\chi_{\mathbf{Q}} = \chi \circ \chi_{\mathrm{cyc}}^{-1} : G_{\mathbf{Q}} \to B^*$ and $\chi_{\mathrm{cyc}} : G_{\mathbf{Q}} \to \mathbf{Z}_p^*$ is the p-adic cyclotomic character. A similar statement holds for the compactly supported cohomology $\mathtt{H}^1_{\text{ét},\mathrm{c}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{D}}^{\cdot}_{\kappa,m})$.

- We equip $H^1(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m}), H^1_{\mathrm{c}}(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m})$ and $H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m})$ with the structures of continuous $G_{\mathbf{Q}}$-modules via the isomorphisms (77) and (78) respectively. If $h \in \mathbf{Q}_{\geqslant 0}$ (and U is sufficiently small) the slope $\leqslant h$ submodules $H^1(\Gamma, D^{\cdot}_{\kappa,m})^{\leqslant h}$, $H^1_{\mathrm{c}}(\Gamma, D^{\cdot}_{\kappa,m})^{\leqslant h}$ and $H^1(\Gamma, A^{\cdot}_{\kappa,m})^{\leqslant h}$ of $H^1(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m})_{\mathbf{Q}_p}$, $H^1_{\mathrm{c}}(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m})_{\mathbf{Q}_p}$ and $H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m})_{\mathbf{Q}_p}$ respectively (cf. Proposition 4.2) are preserved by the action of $G_{\mathbf{Q}}$.
- Set $\boldsymbol{\Lambda}_{U,j} = (\Lambda_U/\mathfrak{m}^j)^{\text{ét}}$ and $\boldsymbol{\Lambda}_U = (\boldsymbol{\Lambda}_{U,j})_{j\in\mathbf{N}} \in \mathbf{S}(Y_{\text{ét}})$. There are canonical isomorphisms of Λ_U-modules

$$\mathrm{trace}_U : H^2_{\mathrm{c}}(\Gamma, \Lambda_U) \cong H^2_{\text{ét},\mathrm{c}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\Lambda}_U) \cong \Lambda_U. \tag{81}$$

The evaluation morphism $\mathcal{A}^{\cdot}_{U,m} \otimes_{\Lambda_U} \mathcal{D}^{\cdot}_{U,m} \longrightarrow \Lambda_U$ and the trace trace_U induce a cup-product

$$H^1(\Gamma, \mathcal{A}^{\cdot}_{U,m}) \otimes_{\Lambda_U} H^1_{\mathrm{c}}(\Gamma, \mathcal{D}^{\cdot}_{U,m}) \longrightarrow H^2_{\mathrm{c}}(\Gamma, \Lambda_U) \cong \Lambda_U,$$

under which the Hecke operator $U^{\cdot}_p$ acting on $H^1(\Gamma, \mathcal{A}^{\cdot}_{U,m})$ is adjoint to $U^{\cdot}_p$ acting on $H^1_{\mathrm{c}}(\Gamma, \mathcal{D}^{\cdot}_{U,m})$. This in turn induces for $h \in \mathbf{Q}_{\geqslant 0}$ (and U sufficiently small) morphisms of $\Lambda_U[1/p]$-modules

$$\xi^{\cdot}_{U,m} : H^1(\Gamma, A^{\cdot}_{U,m})^{\leqslant h} \longrightarrow \mathrm{Hom}_{\Lambda_U[1/p]}\big(H^1_{\mathrm{c}}(\Gamma, D^{\cdot}_{U,m})^{\leqslant h}, \Lambda_U[1/p]\big).$$

- Define $\det : \mathsf{T} \times \mathsf{T}' \longrightarrow \mathbf{Z}_p^*$ by $\det((x_1,x_2),(y_1,y_2)) = x_1y_2 - x_2y_1$, and denote by $\det_U : \mathsf{T} \times \mathsf{T}' \longrightarrow \Lambda_U^*$ the composition of det with $\kappa_U : \mathbf{Z}_p^* \longrightarrow \Lambda_U^*$. Evaluation at $\det_U$ defines a Γ-equivarint bilinear form $\mathcal{D}_{U,m} \otimes_{\Lambda_U} \mathcal{D}'_{U,m} \longrightarrow \Lambda_U$. Together with trace_U (cf. Equation (81)) this induces a cup-product pairing

$$\det{}^*_U : H^1(\Gamma, \mathcal{D}_{U,m}) \otimes_{\Lambda_U} H^1_{\mathrm{c}}(\Gamma, \mathcal{D}'_{U,m}) \longrightarrow H^2_{\mathrm{c}}(\Gamma, \Lambda_U) \cong \Lambda_U \tag{82}$$

under which the Hecke operators U_p and U'_p are adjoint to each other. For every $h \in \mathbf{Q}_{\geqslant 0}$ the (inverse of the) adjoint of $\det^*_U$ induces an isomorphism of $\Lambda_U[1/p]$-modules

$$\zeta'_{U,m} : \mathrm{Hom}_{\Lambda_U[1/p]}\big(H^1_{\mathrm{c}}(\Gamma, D'_{U,m})^{\leqslant h}, \Lambda_U[1/p]\big) \cong H^1(\Gamma, D_{U,m})^{\leqslant h}.$$

Similarly one defines an isomorphism

$$\zeta_{U,m} : \mathrm{Hom}_{\Lambda_U[1/p]}\big(H^1_{\mathrm{c}}(\Gamma, D_{U,m})^{\leqslant h}, \Lambda_U[1/p]\big) \cong H^1(\Gamma, D'_{U,m})^{\leqslant h}.$$

- Let $h \in \mathbf{Q}_{\geqslant 0}$. If U is sufficiently small the composition of $\zeta_{U,m}$ with $\xi_{U,m}$ gives a morphism of $G_{\mathbf{Q}}$-modules

$$\mathtt{s}_{U,h} : H^1(\Gamma, A_{U,m})^{\leqslant h}(\boldsymbol{\kappa}_U) \longrightarrow H^1(\Gamma, D'_{U,m})^{\leqslant h}, \tag{83}$$

where $\boldsymbol{\kappa}_U : G_{\mathbf{Q}} \longrightarrow \Lambda_U^*$ is defined by $\boldsymbol{\kappa}_U(g) = \kappa_U(\chi_{\mathrm{cyc}}(g))$ for every $g \in G_{\mathbf{Q}}$. For every integer $k = r+2$ in $U \cap \mathbf{Z}$ such that $h < k-1$, the following diagram of $L[G_{\mathbf{Q}}]$-modules commutes.

$$\begin{tikzcd} H^1(\Gamma, A_{U,m})^{\leqslant h}(\boldsymbol{\kappa}_U) \arrow[r, "\mathtt{s}_{U,h}"] \arrow[d, "\rho_k"'] & H^1(\Gamma, D'_{U,m})^{\leqslant h} \arrow[d, "\rho_k"] \\ H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{S}_r)_L^{\leqslant' h}(r) \arrow[r, "\mathtt{s}_r"] & H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{L}_r)_L^{\leqslant' h} \end{tikzcd} \tag{84}$$

By a slight abuse of notation, here one writes again ρ_k for the composition of the specialisation map $\rho_k : H^1(\Gamma, A_{U,m})^{\leqslant h} \to H^1(\Gamma, A_{r,m})^{\leqslant h}$ (resp., $\rho_k : H^1(\Gamma, D'_{U,m})^{\leqslant h} \to H^1(\Gamma, D'_{r,m})^{\leqslant h}$) with the comparison isomorphism $H^1(\Gamma, A_{r,m})^{\leqslant h} \cong H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{S}_r)^{\leqslant' h}$ (resp., $H^1(\Gamma, D'_{r,m})^{\leqslant h} \cong H^1(\Gamma, \mathscr{L}_r)_L^{\leqslant' h}$) defined in Equation (73). Similarly the composition of $\zeta'_{U,m}$ with $\xi'_{U,m}$ gives a morphism of $G_{\mathbf{Q}}$-modules

$$\mathtt{s}'_{U,h} : H^1(\Gamma, A'_{U,m})^{\leqslant h}(\boldsymbol{\kappa}_U) \longrightarrow H^1(\Gamma, D_{U,m})^{\leqslant h}$$

and the diagram of $G_{\mathbf{Q}}$-modules obtained by replacing $A_{U,m}, D'_{U,m}$ and $\mathtt{s}_{U,h}$ with $A'_{U,m}, D_{U,m}$ and $\mathtt{s}'_{U,h}$ respectively in Equation (84) commutes.

- The Atkin–Lehner operators w_p (resp., w_{Np}) defined in Equations (66) and (67) are $G_{\mathbf{Q}}$-equivariant (resp., $G_{\mathbf{Q}(\mu_N)}$-equivariant).

Due to the lack of a reference, we explain how to construct the crucial isomorphism (78). Let $\cdot$ denote either the empty symbol or $\prime$, and let $\mathrm{Fil}_{i,j}\boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m} = (\mathrm{Fil}_{i,j}\mathcal{A}^{\cdot}_{\kappa,m})^{\text{ét}}$ be the étale sheaf on Y associated with the finite $B/\mathfrak{m}^i B[\Gamma]$-module $\mathrm{Fil}_{i,j}\mathcal{A}^{\cdot}_{\kappa,m}$. The comparison isomorphisms between étale and Betti cohomology yields isomorphisms

$$\mathrm{comp}_{i,j} : H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathrm{Fil}_{i,j}\boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m}) \cong H^1(\Gamma, \mathrm{Fil}_{i,j}\mathcal{A}^{\cdot}_{\kappa,m}).$$

The étale cohomology of the affine scheme $Y_{\bar{\mathbf{Q}}}$ commutes with filtered direct limits. Moreover, since the group Γ is finitely generated, the cohomology functor $H^1(\Gamma, \cdot)$ commutes with filtered direct limits (cf. Exercises 1 and 4 on page 196 of [**Bro94**]). Taking the direct limit for $j \to \infty$ of the isomorphisms $\mathrm{comp}_{i,j}$ then gives isomorphisms of $B/\mathfrak{m}^i B$-modules

$$\mathrm{comp}_i : H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m,i}) \cong H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m,i}),$$

which in turn entail an isomorphism of B-modules

$$\mathrm{comp} : \lim_{\leftarrow i} H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m,i}) \cong \mathtt{H}^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}^{\cdot}_{\kappa,m}).$$

The sought for isomorphism (78) is defined as the composition of the comparison isomorphism comp and the natural map $H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m}) \longrightarrow \varprojlim_i H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m,i})$, which is an isomorphism by Lemma 4.3 below. The Hecke equivariance of the isomorphism (78) is proved precisely as in Sections 3.2 and 3.3 of [**AIS15**].

Lemma 4.3. — *The natural maps*

$$H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m}) \longrightarrow \varprojlim_i H^1(\Gamma, \mathcal{A}^{\cdot}_{\kappa,m,i})$$

are isomorphisms of B-modules.

Proof. — We adapt the proof of [**AIS15**, Lemma 3.13] to our setting. To ease notation, set $\mathcal{A}_i = \mathcal{A}^{\cdot}_{\kappa,m,i}$ and $\mathcal{A} = \mathcal{A}^{\cdot}_{\kappa,m}$. For each Γ-module M, let

$$C^\bullet(\Gamma, M) : 0 \longrightarrow M \xrightarrow{d^0} C^1(\Gamma, M) \xrightarrow{d^1} C^2(\Gamma, M) \longrightarrow \cdots$$

be the usual complex of inhomogeneous cochains computing the cohomology groups $H^j(\Gamma, M) = Z^j(\Gamma, M)/\mathrm{im}(d^{j-1})$, where $C^j(\Gamma, M)$ is the group of maps from Γ^j to M and $Z^j(\Gamma, M) = \ker(d^j)$. Denote by $d^\bullet$ (resp., $d_i^\bullet$) the differentials in $C^\bullet(\Gamma, \mathcal{A})$ (resp., $C^\bullet(\Gamma, \mathcal{A}_i)$), so that one has the following commutative diagram with exact rows. (Recall that by definition $\mathcal{A}_i$ is a shorthand for $\mathcal{A}/\mathfrak{m}_B^i \cdot \mathcal{A}$.)

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "d^0"] \arrow[d, equal] & Z^1(\Gamma, \mathcal{A}) \arrow[r] \arrow[d, "\zeta"'] & H^1(\Gamma, \mathcal{A}) \arrow[r] \arrow[d, "\vartheta"'] & 0 \\
\varprojlim_i \mathcal{A}_i \arrow[r, "(d_i^0)"] & \varprojlim_i Z^1(\Gamma, \mathcal{A}_i) \arrow[r, "\varepsilon"] & \varprojlim_i H^1(\Gamma, \mathcal{A}_i) &
\end{tikzcd}$$

To prove that ϑ is an isomorphism, it is then sufficient to show that ε is surjective and that ζ is an isomorphism. The cokernel of ε is contained in $R^1 \varprojlim_i (\mathcal{A}_i/H^0(\Gamma, \mathcal{A}_i))$, which vanishes since the maps $\mathcal{A}_{i+1}/H^0(\Gamma, \mathcal{A}_{i+1}) \longrightarrow \mathcal{A}_i/H^0(\Gamma, \mathcal{A}_i)$ are surjective. Moreover, as $\mathcal{A} = \varprojlim_i \mathcal{A}_i$, the natural map $C^\bullet(\Gamma, \mathcal{A}) \longrightarrow \varprojlim_i C^\bullet(\Gamma, \mathcal{A}_i)$ is an isomorphism, hence so is ζ by the left exactness of the inverse limit. □

4.3. The ordinary case. — This section explains the relations between the ordinary (id est slope $\leqslant 0$) parts of the modules $H^1(\Gamma, D^{\cdot}_{U,m})$ and the big ordinary Galois representations considered in [**Hid86, Oht95, Oht00**]. This will be particularly relevant for the study of the eigencurve in a neighbourhood of a classical weight-one eigenform (where the Eichler–Shimura isomorphism of [**AIS15**] does not apply).

Since $H^1(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m})$ is a profinite group (as $\mathcal{D}^{\cdot}_{\kappa,m}$ is), the limit $e^{\cdot}_{\mathrm{ord}} = \lim_{n\to\infty} U_p^{\cdot n!}$ defines an idempotent in the B-endomorphism ring of $H^1(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m})$. (Here as usual (B, κ) denotes either (Λ_U, κ_U) or $(\mathscr{O}, r)$ with r in $\mathcal{W}(L)$, and $\cdot$ denotes either the empty symbol or $\prime$.) Set

$$H^1(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m})^{\leqslant 0} = e^{\cdot}_{\mathrm{ord}} \cdot H^1(\Gamma, \mathcal{D}^{\cdot}_{\kappa,m}).$$

This is a finite Λ_B-module, which recasts $H^1(\Gamma, D^{\cdot}_{\kappa,m})^{\leqslant 0}$ after inverting p.

Following [**Hid86, Oht95**], define

$$\mathbf{T} = \varprojlim_{r} H^1_{\text{ét}}(Y_1(Np^r)_{\bar{\mathbf{Q}}}, \mathbf{Z}_p(1)),$$

where $r \in \mathbf{Z}_{\geqslant 1}$ and the transition maps are given by the traces pr_{1*} induced in cohomology by the degeneracy maps $\mathrm{pr}_1 : Y_1(Np^{r+1}) \longrightarrow Y_1(Np^r)$ introduced in Equation (8). As the maps pr_{1*} are Hecke-equivariant, the module $\mathbf{T}$ is equipped with the action of Hecke operators $T^{\cdot}_\ell$ (resp., $U^{\cdot}_\ell$), for each prime ℓ not dividing (resp., dividing) Np. Moreover, the action of $(\mathbf{Z}/p^r\mathbf{Z})^*$ on $H^1_{\text{ét}}(Y_1(Np^r)_{\bar{\mathbf{Q}}}, \mathbf{Z}_p(1))$ via diamond operators makes $\mathbf{T}$ a module over $\diamond = \mathbf{Z}_p[\![\mathbf{Z}_p^*]\!]$. Let

$$\mathbf{D}' = \mathrm{Hom}_{\mathbf{Z}_p}(\mathrm{Step}(\mathsf{T}'), \mathbf{Z}_p)$$

be the right $\Sigma_0'(p)$-module of measures on T', where $\mathrm{Step}(\mathsf{T}')$ is the set of $\mathbf{Z}_p$-valued step functions on T'. Section 4.1.1 equips $H^1(\Gamma, \mathbf{D}')$ with the action of Hecke operators U_p' and $T_\ell^{\cdot}$, for $\cdot = \emptyset, \prime$ and ℓ a rational prime different from p. A slight variant of Lemma 6.8 of [**GS93**] yields a Hecke-equivariant isomorphism of $\diamond$-modules

$$\mathbf{T} \cong H^1(\Gamma, \mathbf{D}'), \tag{85}$$

where the action of the Iwasawa algebra $\diamond$ on the right hand side arises from that of the group $\mathbf{Z}_p^* = \mathbf{Z}_p^* \cdot \left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right) \hookrightarrow \Sigma_0'(p)$ on $\mathbf{D}'$.

Each measure μ in $\mathbf{D}'$ extends to a Λ_U-linear morphism $\mu_U : \mathscr{C}(\mathsf{T}', \Lambda_U) \longrightarrow \Lambda_U$ on the space $\mathscr{C}(\mathsf{T}', \Lambda_U)$ of Λ_U-valued continuous functions on T'. The map sending μ to the restriction of μ_U to $\mathcal{A}_{U,m}' \hookrightarrow \mathscr{C}(\mathsf{T}', \Lambda_U)$ defines a morphism of $\Sigma_0'(p)$-modules

$$\mathbf{D}' \longrightarrow \mathcal{D}_{U,m}',$$

which in turn induces a Hecke-equivariant morphism of Λ_U-modules

$$H^1(\Gamma, \mathbf{D}') \otimes_\diamond \Lambda_U \longrightarrow H^1(\Gamma, \mathcal{D}_{U,m}'), \tag{86}$$

where Λ_U has the structure of $\diamond$-algebra arising from $\kappa_U : \mathbf{Z}_p^* \longrightarrow \Lambda_U^*$.

After setting

$$\mathbf{T}_U^{\leqslant 0} = e_{\mathrm{ord}}' \cdot \mathbf{T} \otimes_\diamond \Lambda_U,$$

the composition of the maps (85) and (86) yields an isomorphism of Λ_U-modules

$$\mathtt{Sh}_{U,m} : \mathbf{T}_U^{\leqslant 0} \cong H^1(\Gamma, \mathcal{D}_{U,m}')^{\leqslant 0}(1), \tag{87}$$

which is Hecke-equivariant and $G_{\mathbf{Q}}$-equivariant. In order to prove this, let r be a positive integer in U. Since $H^2(\Gamma, \cdot)$ vanishes for each Γ-module $\cdot$ of finite cardinality (and $\mathcal{D}_{U,m}'$ is profinite), evaluation at $k = r+2$ on Λ_U induces an isomorphism

$$H^1(\Gamma, \mathcal{D}_{U,m}')^{\leqslant 0} \otimes_{\Lambda_U} \Lambda_U/\pi_k \cong H^1(\Gamma, \mathcal{D}_{r,m}')^{\leqslant 0}. \tag{88}$$

Moreover, for each $j \geqslant 0$, the natural map $\mathcal{D}_{r,m}' \longrightarrow L_r(\mathscr{O})$ induces an isomorphism

$$H^j(\Gamma, \mathcal{D}_{r,m}')^{\leqslant 0} \cong H^j(\Gamma, L_r(\mathscr{O}))^{\leqslant' 0}, \tag{89}$$

which for $j = 1$ recasts the isomorphism displayed in Part 2 of Proposition 4.2 after inverting p. (Indeed a direct computation shows that $\left(\begin{smallmatrix} p & 0 \\ pNi & 1 \end{smallmatrix}\right) \in \Sigma_0'(p)$ maps the kernel $\mathcal{K}_{r,m}'$ of $\mathcal{D}_{r,m}' \longrightarrow L_r(\mathscr{O})$ into $p^{r+1} \cdot \mathcal{K}_{r,m}'$ for each $0 \leqslant i \leqslant p-1$, from which one deduces that the anti-ordinary projector e_{ord}' kills $H^j(\Gamma, \mathcal{K}_{r,m}')$ for each $j \geqslant 0$.) On the other hand, the inclusion $S_r(\mathbf{Z}_p) \hookrightarrow \mathscr{C}(\mathsf{T}', \mathbf{Z}_p)$ dualises to a specialisation map $\rho_k : \mathbf{D}' \longrightarrow L_r(\mathbf{Z}_p)$, and Hida's control theorem (cf. [**Hid86, Oht95**]) shows that the isomorphism (85) and ρ_k induce a Hecke-equivariant isomorphism

$$e_{\mathrm{ord}}' \cdot \mathbf{T} \otimes_\diamond \diamond/I_k \cong H^1(\Gamma, L_r(\mathbf{Z}_p))^{\leqslant' 0}, \tag{90}$$

where I_k is the ideal of $\diamond$ generated by $[1+p] - (1+p)^r$ and $[\mu] - \mu^r$, with μ a generator of $\mathbf{F}_p^*$ and $[\cdot] : \mathbf{Z}_p^* \longrightarrow \diamond^*$ the tautological map. It follows from Equations (88)–(90) that the base change of $\mathtt{Sh}_{U,m}$ along the projection $\Lambda_U \longrightarrow \Lambda_U/\pi_k$ is an isomorphism. Together with Nakayama's Lemma, this implies that $\mathtt{Sh}_{U,m}$ is surjective, and that $\ker(\mathtt{Sh}_{U,m}) \otimes_{\Lambda_U} \Lambda_U/\pi_k$ is a quotient of the π_k-torsion submodule of $H^1(\Gamma, \mathcal{D}_{U,m}')^{\leqslant 0}$. The latter is in turn a quotient of $H^0(\Gamma, \mathcal{D}_{r,m}')^{\leqslant 0}$, which vanishes by Equation (89).

Another application of Nakayama's Lemma then proves that $\mathtt{Sh}_{U,m}$ is injective, thus concluding the proof of the claim (87).

Set $\mathscr{O}_U = \Lambda_U[1/p]$ and denote by

$$\mathfrak{h}(U) = \mathfrak{h}(N,U) \hookrightarrow \mathrm{End}_{\Lambda_U}(H^1(\Gamma, \mathcal{D}'_{U,m})^{\leqslant 0})[1/p]$$

the Hecke algebra generated over $\mathscr{O}_U$ by the dual Hecke operators $(U'_q)_{q|Np}$, $(T'_\ell)_{\ell\nmid Np}$ and $(\langle d\rangle)_{d\in(\mathbf{Z}/N\mathbf{Z})^*}$ acting on $H^1(\Gamma, D'_{U,m})^{\leqslant 0}$. For each positive integer r and $\cdot = \emptyset, \prime$, let $h^{\cdot}(Np^r)$ be the ring generated by the Hecke operators $(U^{\cdot}_q)_{q|Np}$, $(T^{\cdot}_\ell)_{\ell\nmid Np}$ and $(\langle d\rangle)_{d\in(\mathbf{Z}/N\mathbf{Z})^*}$ acting on the space $M_2(Np^r)$ of complex modular forms of weight 2. Conjugation by the Atkin–Lehner isomorphism $w_{Np^r} \in \mathrm{Iso}_{\mathbf{C}}(M_2(Np^r))$ restricts to an isomorphism $h(Np^r) \cong h'(Np^r)$, sending U_q and T_ℓ to U'_q and T'_ℓ respectively. Set

$$(91)\qquad h^{\cdot}_{Np^\infty} = e^{\cdot}_{\mathrm{ord}} \cdot \lim_{\leftarrow r}(h^{\cdot}(Np^r)\otimes_{\mathbf{Z}}\mathbf{Z}_p) \quad\text{and}\quad h^{\cdot}_{Np^\infty}(U) = h^{\cdot}_{Np^\infty}\otimes_\diamond \mathscr{O}_U,$$

where the transition maps in the inverse limit defining h_{Np^∞} (resp., h'_{Np^∞}) are induced by the inclusions $M_2(Np^r) \subset M_2(Np^{r+1})$ (resp., the maps $M_2(Np^r) \hookrightarrow M_2(Np^{r+1})$ sending $f(z)$ to $f(pz)$). The Atkin–Lehner operators $(w_{Np^r})_{r\geqslant 1}$ induce an isomorphism of Λ_U-modules between $h_{Np^\infty}(U)$ and $h'_{Np^\infty}(N)$, and since $h(Np^r)$ acts faithfully on $H^1_{\mathrm{\acute{e}t}}(Y_1(Np^r)_{\bar{\mathbf{Q}}}, \mathbf{Z}_p(1))$ (cf. Equation (19)), the Shapiro isomorphism $\mathtt{Sh}_{U,m}$ defined in Equation (87) yields an isomorphisms of $\mathscr{O}_U$-modules

$$(92)\qquad h_{Np^\infty}(U) \cong \mathfrak{h}(N,U).$$

sending the Hecke operators T_ℓ and U_q to the corresponding duals T'_ℓ and U'_q.

Denote by $\mathcal{C} = \mathcal{C}(N) = \mathrm{Spf}(h_{Np^\infty})_{\mathbf{Q}_p}$ Berthelot's rigid fibre of the formal spectrum of h_{Np^∞} (cf. Section 7 of [dJ95]). The structural maps $\diamond \longrightarrow h_{Np^\infty}$ yield a finite and flat morphism $\kappa : \mathcal{C} \longrightarrow \mathcal{W}$, and Equation (92) gives an isomorphism of $\mathscr{O}_U$-modules

$$(93)\qquad \mathfrak{h}(U) \cong \mathscr{O}(\mathcal{C}\times_{\mathcal{W}} U)$$

mapping the dual Hecke operators T'_ℓ $(\ell\nmid Np)$ and U'_q $(q|Np)$ in the left hand side to the corresponding Hecke operators T_ℓ and U_q in the right hand side, where $\mathscr{O}(\cdot)$ denotes the ring of bounded analytic functions on $\cdot$.

Section 6 of [Pil13] gives an isomorphism between $\mathcal{C}$ and the ordinary locus $\mathscr{C}^{\mathrm{ord}} = \mathscr{C}^{\mathrm{ord}}(N)$ of the Buzzard–Coleman–Mazur eigencurve $\mathscr{C} = \mathscr{C}(N)$ of tame level N, mapping the Hecke operators in h_{Np^∞} to the corresponding Hecke operators in $\mathscr{O}(\mathscr{C}^{\mathrm{ord}})$. In light of Equation (93), this gives isomorphisms

$$(94)\qquad \mathfrak{h}(U) \cong \mathscr{O}(\mathscr{C}^{\mathrm{ord}}\times_{\mathcal{W}} U)$$

mapping the dual Hecke operators in the left hand side to the corresponding Hecke operators in the right hand side.

Remark 4.4. — If U is a sufficiently small open disc in $\mathcal{W}$ centred at an integer $k_o \geqslant 2$, and h is a non-negative rational number satisfying $h < k_o - 2$, then the overconvergent Eichler–Shimura isomorphism [AIS15, Theorem 1.3] implies that the isomorphism (94) holds after replacing $\mathscr{C}^{\mathrm{ord}}$ with the slope $\leqslant h$ locus of $\mathscr{C}$, and $\mathfrak{h}(U)$ with the Hecke algebra acting on the slope $\leqslant h$ subspace of $H^1(\Gamma, D'_{U,m})$. On the other hand, their result does not apply when U is centred at $k_o = 1$ (and $h = 0$), a crucial scenario for the applications of the main results of this paper to the arithmetic of elliptic curves (cf. [**BSV20a**]).

5. Hida families

As explained in Section 6 of [**AIS15**] (see also Section 6 of [**GS93**]), the *big* Galois representations associated to p-adic Coleman–Hida families (generically) appear as direct factors of the cohomology groups $H^1(\Gamma, \mathcal{D}_{U,m}^{\cdot})$. This section recalls these results, paying particular attention to the case (not covered in loc. cit.) where the open disc U is centred at weight 1 in $\mathcal{W}(\mathbf{Q}_p)$. To simplify the exposition we limit the discussion to Hida families. This suffices for the applications we have in mind (and requires no mention of the theory of (φ, Γ)-modules and trianguline representations).

Let M be a positive integer coprime to p, let $U \subset \mathcal{W}$ be an L-rational open disc centred at a positive integer $k_o \in \mathbf{Z}_{\geqslant 1}$, and let χ be a Dirichlet character modulo M. Let $\mathscr{O}_U = \Lambda_U[1/p]$ be the ring of bounded analytic functions on U, and let

$$U^{\mathrm{cl}} = \big\{ k \in U \cap \mathbf{Z} \;\big|\; k \geqslant 2 \text{ and } k \equiv k_o \bmod 2 \cdot (p-1) \big\}$$

be the set of *classical points of* U. An *$\mathscr{O}_U$-adic cusp form* of tame level M and tame character χ is a formal q-expansion

$$\boldsymbol{f} = \sum_{n \geqslant 1} a_n(\boldsymbol{f}; \boldsymbol{k}) \cdot q^n \in \mathscr{O}_U[\![q]\!]$$

such that, for each classical weight $k \in U^{\mathrm{cl}}$, the *weight-k specialisation*

$$\boldsymbol{f}_k = \sum_{n \geqslant 1} a_n(\boldsymbol{f}; k) \cdot q^n \in S_k^{\mathrm{ord}}(Mp, \chi)_L$$

is the q-expansion of a *p-ordinary* cusp form in $S_k^{\mathrm{ord}}(Mp, \chi)_L$. Here

$$S_k^{\mathrm{ord}}(Mp, \chi)_L = e_{\mathrm{ord}} \cdot S_k(Mp, \chi)_L,$$

where $e_{\mathrm{ord}} = \lim_{n \to \infty} U_p^{n!}$ is Hida's ordinary projector acting on the L-module $S_k(Mp, \chi)_L$ of cusp forms of weight k, level $\Gamma_1(M) \cap \Gamma_0(p)$, character χ and Fourier coefficients in $\bar{\mathbf{Q}} \cap L$ (under the fixed embedding $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$). Denote by $S_U^{\mathrm{ord}}(M, \chi)$ the $\mathscr{O}_U$-module of $\mathscr{O}_U$-adic cusp forms of tame level M and character χ. It is equipped with the action of Hecke operators T_ℓ, for primes $\ell \nmid Mp$, and U_ℓ, for primes $\ell | Mp$, which are compatible with the usual Hecke operators on $S_k^{\mathrm{ord}}(Mp, \chi)$ for each $k \in U^{\mathrm{cl}}$. A cusp form $\boldsymbol{f}$ in $S_U^{\mathrm{ord}}(M, \chi)$ is *normalised* if $a_1(\boldsymbol{f}; \boldsymbol{k})$ is the constant function with value one on U. A *(L-rational) Hida family* of tame level M, tame character χ and centre $k_o \in \mathbf{Z}_{\geqslant 1}$ is an $\mathscr{O}_U$-adic cusp form $\boldsymbol{f} \in S_U^{\mathrm{ord}}(M, \chi)$, for some U as above, which is an eigenvector for the Hecke operators U_p and T_ℓ, for each prime $\ell \nmid Mp$ (equivalently such that, for each $k \in U^{\mathrm{cl}}$, the weight-k specialisation $\boldsymbol{f}_k$ is an eigenvector for the Hecke operators U_p and T_ℓ, for all primes $\ell \nmid Mp$.) A normalised Hida family $\boldsymbol{f} \in S_U^{\mathrm{ord}}(M, \chi)$ is *new* (or *primitive*) of tame level M if the conductor of the eigenform $\boldsymbol{f}_k$ is equal to M for all $k > 2$ in U^{cl}. To each Hida family $\boldsymbol{f} \in S_U^{\mathrm{ord}}(M, \chi)$ is associated a unique pair $(M_{\boldsymbol{f}}, \boldsymbol{f}^\sharp)$, where $M_{\boldsymbol{f}}$ is a positive divisor of M and $\boldsymbol{f}^\sharp = \sum_{n \geqslant 1} a_n(\boldsymbol{k}) \cdot q^n$ in $S_U^{\mathrm{ord}}(M_{\boldsymbol{f}}, \chi)$ is a new Hida family of tame level $M_{\boldsymbol{f}}$

such that $U_p(\boldsymbol{f}) = a_p(\boldsymbol{k}) \cdot \boldsymbol{f}$ and $T_\ell(\boldsymbol{f}) = a_\ell(\boldsymbol{k}) \cdot \boldsymbol{f}$ for all primes $\ell \nmid M$. We call $M_{\boldsymbol{f}}$ the *conductor of* $\boldsymbol{f}$ and $\boldsymbol{f}^\sharp$ the *primitive Hida family associated with* $\boldsymbol{f}$. Moreover, we denote by

$$S_U^{\mathrm{ord}}(M, \chi_{\boldsymbol{f}})[\boldsymbol{f}^\sharp] \hookrightarrow S_U^{\mathrm{ord}}(M, \chi_{\boldsymbol{f}})$$

the $\mathscr{O}_U$-module of Hida families in $S_U^{\mathrm{ord}}(M, \chi_{\boldsymbol{f}})$ having $\boldsymbol{f}^\sharp$ as associated primitive Hida family. A *level-N test vector* for $\boldsymbol{f}^\sharp$ is an element of $S_U^{\mathrm{ord}}(M, \chi_{\boldsymbol{f}})[\boldsymbol{f}^\sharp]$ of the form

$$\boldsymbol{f} = \sum_{0<d|M/M_{\boldsymbol{f}}} r_d \cdot \boldsymbol{f}^\sharp(q^d), \tag{95}$$

for analytic functions $(r_d)_d$ in $\mathscr{O}_U$ without common zeros in U.

Fix in the rest of this section a positive divisor $N_{\boldsymbol{f}}$ of N and a normalised eigenform

$$\boldsymbol{f}_{k_o}^\sharp = \sum_{n \geqslant 1} a_n \cdot q^n \in M_{k_o}(\Gamma_1(N_{\boldsymbol{f}}) \cap \Gamma_0(p), \chi_{\boldsymbol{f}})_L$$

of weight $k_o \geqslant 1$, level $N_{\boldsymbol{f}} p$, character $\chi_{\boldsymbol{f}} : (\mathbf{Z}/N_{\boldsymbol{f}}\mathbf{Z})^* \longrightarrow L^*$ and Fourier coefficients in L, satisfying the following (cf. Assumption 1.1)

Assumption 5.1. — *One of the following statements* 1–2 *holds true.*

1. *The form $\boldsymbol{f}_{k_o}^\sharp$ is cuspidal of weight $k_o \geqslant 2$, p-ordinary (id est a_p is a p-adic unit under the fixed embedding $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$) and its conductor is divisible by $N_{\boldsymbol{f}}$.*
2. *The form $\boldsymbol{f}_{k_o}^\sharp$ is a p-stabilisation of a cuspidal and p-regular weight-one newform of level $N_{\boldsymbol{f}}$, without real multiplication by a quadratic field in which p splits.*

The previous assumption guarantees that the eigencurve $\kappa : \mathscr{C}(N_{\boldsymbol{f}}) \longrightarrow \mathcal{W}$ (cf. Section 4.3) is étale at (the L-rational point corresponding to) $\boldsymbol{f}_{k_o}^\sharp$. In case 5.1(1) (resp., case 5.1(2)) this follows from Corollary 1.4 of [**Hid86**] and Section 6 of [**Pil13**] (resp., Theorems 1.1 and 7.2 of [**BD16**]). As a consequence, there exists an open connected disc $U_{\boldsymbol{f}}$ in $\mathcal{W}_L$ centred at k_o, and a section $U_{\boldsymbol{f}} \hookrightarrow \mathscr{C}(N_{\boldsymbol{f}}) \otimes_{\mathbf{Q}_p} L$ of $\kappa \otimes_{\mathbf{Q}_p} L$ mapping $U_{\boldsymbol{f}}$ isomorphically onto an open admissible neighbourhood of $\boldsymbol{f}_{k_o}^\sharp$. In light of Equation (94), this yields an idempotent $e_{\boldsymbol{f}^\sharp}$ in the Hecke algebra (cf. Section 4.3)

$$\mathcal{H} \stackrel{\mathrm{def}}{=} \mathfrak{h}(N_{\boldsymbol{f}}, U_{\boldsymbol{f}}),$$

and an isomorphism of $\mathscr{O}_{U_{\boldsymbol{f}}}$-algebras between $e_{\boldsymbol{f}^\sharp} \cdot \mathcal{H}$ and $\mathscr{O}_{U_{\boldsymbol{f}}}$. Let

$$\boldsymbol{\varphi} : \mathcal{H} \longrightarrow \mathscr{O}_{U_{\boldsymbol{f}}} \tag{96}$$

be the composition of this isomorphism with the projection onto $e_{\boldsymbol{f}^\sharp} \cdot \mathcal{H}$.

For each positive integer n, denote by $\Delta_n' \subset \Sigma_0'(p) \cap M_2(\mathbf{Z})$ the set of integral matrices $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ satisfying $\det(\alpha) = n$, $d \equiv 1 \mod N$, $p \nmid d$ and $c \equiv 0 \mod Np$. Define $T_n' = \sum_{\alpha \in \Delta_n'} T_\alpha$, where T_α is the endomorphism of $H^1(\Gamma_1(N_{\boldsymbol{f}}) \cap \Gamma_0(p), \mathcal{D}'_{U_{\boldsymbol{f}},m})^{\leqslant 0}$ introduced in Section 4.1.1 (and $m = m(U_{\boldsymbol{f}})$ is sufficiently large). The dual Hecke operator T_n' belongs to $\mathcal{H}$ (cf. [**Shi71**, Chapter 3]), and after setting

$$a_n(\boldsymbol{k}) = a_n(\boldsymbol{f}^\sharp, \boldsymbol{k}) = \boldsymbol{\varphi}(T_n'),$$

the formal q-expansion

$$\boldsymbol{f}^\sharp = \sum_{n \geqslant 1} a_n(\boldsymbol{k}) \cdot q^n \in \mathscr{O}_{U_{\boldsymbol{f}}}[\![q]\!]$$

is the (unique) cuspidal primitive Hida family in $S^{\mathrm{ord}}_{U_{\boldsymbol{f}}}(N_{\boldsymbol{f}},\chi_{\boldsymbol{f}})$ of tame level $N_{\boldsymbol{f}}$ and character $\chi_{\boldsymbol{f}}$ specialising to $\boldsymbol{f}^\sharp_{k_o}$ at k_o. For each positive integer n, it is an eigenvector for the Hecke operator T_n with eigenvalue $a_n(\boldsymbol{k})$.

The rest of this section summarises the main result from Hida theory needed in the sequel of the paper. Fix a level-N test vector

$$\boldsymbol{f} \in S^{\mathrm{ord}}_{U_{\boldsymbol{f}}}(N,\chi_{\boldsymbol{f}})[\boldsymbol{f}^\sharp]$$

for $\boldsymbol{f}^\sharp$. To ease notation, set $\Lambda_{\boldsymbol{f}} = \Lambda_{U_{\boldsymbol{f}}}$, $\mathscr{O}_{\boldsymbol{f}} = \mathscr{O}_{U_{\boldsymbol{f}}}$, $\mathcal{D}^{\cdot}_{\boldsymbol{f},m} = \mathcal{D}^{\cdot}_{U_{\boldsymbol{f}},m}$ and $D^{\cdot}_{\boldsymbol{f},m} = D^{\cdot}_{U_{\boldsymbol{f}},m}$ (where as usual $\cdot$ denotes either the empty symbol or $\prime$). Denote by $\boldsymbol{k}-k_o$ a uniformiser at k_o in $\Lambda_{\boldsymbol{f}}$, so that $\mathscr{O}_{\boldsymbol{f}}$ is a module of power series in $L[\![\boldsymbol{k}-k_o]\!]$ which converge for any $\boldsymbol{k}$ in $U_{\boldsymbol{f}}$. One has $\kappa_{U_{\boldsymbol{f}}}(t) = \omega(t)^{k_o-2}\cdot\langle t\rangle^{\boldsymbol{k}-2}$ for all $t\in\mathbf{Z}_p^*$, and

$$\boldsymbol{\kappa}_{U_{\boldsymbol{f}}} = \omega_{\mathrm{cyc}}^{k_o-2}\cdot\kappa_{\mathrm{cyc}}^{\boldsymbol{k}-2} : G_{\mathbf{Q}} \longrightarrow \Lambda_{\boldsymbol{f}}^*. \tag{97}$$

Here ω_{cyc} and κ_{cyc} denote the composition of the p-adic cyclotomic character

$$\chi_{\mathrm{cyc}} : G_{\mathbf{Q}} \longrightarrow \mathbf{Z}_p^*$$

with the projections $\omega : \mathbf{Z}_p^* \longrightarrow \mathbf{F}_p^*$ and $\langle\cdot\rangle : \mathbf{Z}_p \longrightarrow 1+p\mathbf{Z}_p$ respectively, and $\kappa_{\mathrm{cyc}}^{\boldsymbol{k}-2}$ is the $\Lambda_{\boldsymbol{f}}^*$-valued character which on $g\in G_{\mathbf{Q}}$ takes the value $\kappa_{\mathrm{cyc}}(g)^{\boldsymbol{k}-2}$.

- For every classical weight $k>2$ in $U_{\boldsymbol{f}}^{\mathrm{cl}}$ the weight-k specialisation $\boldsymbol{f}_k$ is old at p. Indeed $\boldsymbol{f}_k = f_\alpha$ is the ordinary p-stabilisation of an eigenform $f=f_k$ in $S_k(N,\chi_{\boldsymbol{f}})$ (cf. Equation (54)), hence $a_p(k)=\alpha_f$ is the unit root of
$$X^2 - \frac{a_p(f)}{a_1(f)}\cdot X + \chi_{\boldsymbol{f}}(p)p^{k-1} = (X-\alpha_f)\cdot(X-\beta_f).$$
(We refer the reader to [**Hid86**] for more details.)
- To ease notation, set
$$\boldsymbol{\mathcal{V}} = H^1(\Gamma_1(N_{\boldsymbol{f}})\cap\Gamma_0(p), D'_{\boldsymbol{f},m})^{\leqslant 0}(1) \quad\text{and}\quad \mathcal{H} = \mathfrak{h}(N_{\boldsymbol{f}},U_{\boldsymbol{f}}).$$
According to the main results of [**Oht00**] and the isomorphism (92), there is a short exact sequence of $\mathcal{H}[G_{\mathbf{Q}_p}]$-modules
$$0\longrightarrow\boldsymbol{\mathcal{V}}^+\longrightarrow\boldsymbol{\mathcal{V}}\longrightarrow\boldsymbol{\mathcal{V}}^-\longrightarrow 0, \tag{98}$$
where $\boldsymbol{\mathcal{V}}^\pm$ are finite free $\mathscr{O}_{\boldsymbol{f}}$-modules. The $G_{\mathbf{Q}_p}$-module $\boldsymbol{\mathcal{V}}^-$ is the maximal unramified $\mathscr{O}_{\boldsymbol{f}}$-quotient of $\boldsymbol{\mathcal{V}}$, and an arithmetic Frobenius acts on it as multiplication by the p-th Fourier coefficient $a_p(k)$ of $\boldsymbol{f}^\sharp$. Moreover, there are canonical isomorphisms of $\mathcal{H}$-modules $\boldsymbol{\mathcal{V}}^+\cong\mathcal{H}_{\mathrm{par}}$ and $\boldsymbol{\mathcal{V}}^-\cong\mathrm{Hom}_{\mathscr{O}_U}(\mathcal{H},\mathscr{O}_U)$, where $\mathcal{H}_{\mathrm{par}}$ is the quotient of $\mathcal{H}$ acting faithfully on the parabolic subspace $H^1_{\mathrm{par}}(\Gamma_{\boldsymbol{f}},D'_{\boldsymbol{f},m})^{\leqslant 0}(1)$ of the cohomology group $\boldsymbol{\mathcal{V}}$.

  Applying the idempotent $e_{\boldsymbol{f}^\sharp}$ (defined before Equation (96)) to the short exact sequence (98) gives a short exact sequence of $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}_p}]$-modules
$$0\longrightarrow V(\boldsymbol{f}^\sharp)^+\longrightarrow V(\boldsymbol{f}^\sharp)\longrightarrow V(\boldsymbol{f}^\sharp)^-\longrightarrow 0, \tag{99}$$
where (for $\cdot$ equal to one of the symbols $\emptyset,+$ and $-$)
$$V(\boldsymbol{f}^\sharp)^{\cdot} = e_{\boldsymbol{f}^\sharp}\cdot\boldsymbol{\mathcal{V}}^{\cdot}$$

is a free $\mathscr{O}_{\boldsymbol{f}}$-direct summand of $\boldsymbol{\mathcal{V}}^{\cdot}$.

- The $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}}]$-module $V(\boldsymbol{f}^{\sharp})$ has rank two over $\mathscr{O}_{\boldsymbol{f}}$, and is unramified outside $N_{\boldsymbol{f}}p$. For every prime ℓ not dividing $N_{\boldsymbol{f}}p$, the characteristic polynomial of an arithmetic Frobenius Frob_{ℓ} in $G_{\mathbf{Q}}$ at ℓ is given by (cf. Equation (106) below)

$$\det\bigl(1-\mathrm{Frob}_{\ell}|V(\boldsymbol{f}^{\sharp})\cdot X\bigr)=1-a_{\ell}(\boldsymbol{k})\cdot X+\chi_{\boldsymbol{f}}(\ell)\cdot\kappa_{U_{\boldsymbol{f}}}(\ell)\cdot\ell\cdot X^{2}.$$

In particular the determinant of $V(\boldsymbol{f}^{\sharp})$ is given by (cf. Equation (97))

$$\det_{\mathscr{O}_{\boldsymbol{f}}}V(\boldsymbol{f})=\chi_{\boldsymbol{f}}\cdot\chi_{\mathrm{cyc}}\cdot\boldsymbol{\kappa}_{U_{\boldsymbol{f}}}=\chi_{\boldsymbol{f}}\cdot\omega_{\mathrm{cyc}}^{k_{o}-1}\cdot\kappa_{\mathrm{cyc}}^{\boldsymbol{k}-1}. \tag{100}$$

As the arithmetic Frobenius $\mathrm{Frob}_{p}\in G_{\mathbf{Q}_{p}}$ acts on $\boldsymbol{\mathcal{V}}^{-}$ as multiplication by $a_{p}(\boldsymbol{k})$, one deduces isomorphisms of $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}_{p}}]$-modules

$$V(\boldsymbol{f}^{\sharp})^{+}\cong\mathscr{O}_{\boldsymbol{f}}\bigl(1+\boldsymbol{\kappa}_{U_{\boldsymbol{f}}}+\chi_{\boldsymbol{f}}-\breve{a}_{p}(\boldsymbol{k})\bigr)\quad\text{and}\quad V(\boldsymbol{f}^{\sharp})^{-}\cong\mathscr{O}_{\boldsymbol{f}}(\breve{a}_{p}(\boldsymbol{k})), \tag{101}$$

where for every $a\in\Lambda_{\boldsymbol{f}}^{*}$ one writes $\breve{a}:G_{\mathbf{Q}_{p}}\to\Lambda_{\boldsymbol{f}}^{*}$ for the continuous unramified character satisfying $\breve{a}(\mathrm{Frob}_{p})=a$.

- Recall the level-N test vector $\boldsymbol{f}$ for $\boldsymbol{f}^{\sharp}$ fixed above, and define

$$H^{1}(\Gamma,D'_{\boldsymbol{f},m})^{\leqslant 0}(1)\twoheadrightarrow V(\boldsymbol{f})$$

to be the maximal $\mathscr{O}_{\boldsymbol{f}}$-quotient of $H^{1}_{\cdot}(\Gamma,D'_{\boldsymbol{f},m})^{\leqslant 0}(1)$ on which the dual Hecke operators T'_{ℓ}, U'_{p}, and $\langle d\rangle'$ act respectively as multiplication by $a_{\ell}(\boldsymbol{k})$, $a_{p}(\boldsymbol{k})$ and $\chi_{\boldsymbol{f}}(d)$, for each prime ℓ not dividing Np and each unit d in $(\mathbf{Z}/N\mathbf{Z})^{*}$. This is equal to the $G_{\mathbf{Q}}$-modules $V(\boldsymbol{f}^{\sharp})=e_{\boldsymbol{f}^{\sharp}}\cdot\boldsymbol{\mathcal{V}}$ introduced above when $N=N_{\boldsymbol{f}}$ and $\boldsymbol{f}=\boldsymbol{f}^{\sharp}$. In general, the $\mathscr{O}_{U}[G_{\mathbf{Q}}]$-module $V(\boldsymbol{f})$ is (non-canonically) isomorphic to the direct sum of a finite number of copies of $V(\boldsymbol{f}^{\sharp})$. In particular, $V(\boldsymbol{f})$ is a free $\mathscr{O}_{U}$-module, and there is a short exact sequence of $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}_{p}}]$-modules

$$0\longrightarrow V(\boldsymbol{f})^{+}\longrightarrow V(\boldsymbol{f})\longrightarrow V(\boldsymbol{f})^{-}\longrightarrow 0 \tag{102}$$

with $V(\boldsymbol{f})^{\pm}$ free of finite rank over $\mathscr{O}_{\boldsymbol{f}}$, and $V(\boldsymbol{f})\twoheadrightarrow V(\boldsymbol{f})^{-}$ the maximal unramified $\mathscr{O}_{\boldsymbol{f}}$-quotient of $V(\boldsymbol{f})$.

Dually, define

$$V^{*}(\boldsymbol{f})\hookrightarrow H^{1}_{\mathrm{c}}(\Gamma,D_{\boldsymbol{f},m})^{\leqslant 0}(-\boldsymbol{\kappa}_{U_{\boldsymbol{f}}})$$

be the maximal $\mathscr{O}_{\boldsymbol{f}}$-submodule of $H^{1}_{\mathrm{c}}(\Gamma,D_{\boldsymbol{f},m})^{\leqslant 0}(-\boldsymbol{\kappa}_{U_{\boldsymbol{f}}})$ on which the Hecke operators T_{ℓ}, U_{p} and $\langle d\rangle$ act respectively as multiplication by $a_{\ell}(\boldsymbol{k})$, $a_{p}(\boldsymbol{k})$ and $\chi_{\boldsymbol{f}}(d)$, for every prime $\ell\nmid Np$ and every unit d in $(\mathbf{Z}/N\mathbf{Z})^{*}$. Then $V^{*}(\boldsymbol{f})$ is an $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}}]$-direct summand of $H^{1}_{\mathrm{c}}(\Gamma,D_{\boldsymbol{f},m})^{\leqslant 0}(-\boldsymbol{\kappa}_{U_{\boldsymbol{f}}})$, isomorphic to the $\mathscr{O}_{\boldsymbol{f}}$-dual of $V(\boldsymbol{f})$. Indeed the bilinear form $\det^{*}_{U_{\boldsymbol{f}}}$ defined in Equation (82) induces a perfect pairing of $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}}]$-modules (cf. [Oht00] and Section 4.3)

$$\langle\cdot,\cdot\rangle_{\boldsymbol{f}}:V(\boldsymbol{f})\otimes_{\mathscr{O}_{\boldsymbol{f}}}V^{*}(\boldsymbol{f})\longrightarrow\mathscr{O}_{\boldsymbol{f}}. \tag{103}$$

Let $V^{*}(\boldsymbol{f})^{+}\hookrightarrow V^{*}(\boldsymbol{f})$ be the maximal unramified submodule of the restriction of $V^{*}(\boldsymbol{f})$ to $G_{\mathbf{Q}_{p}}$, and let $V^{*}(\boldsymbol{f})^{-}$ be the quotient of $V^{*}(\boldsymbol{f})$ by $V^{*}(\boldsymbol{f})^{+}$. There is then a short exact sequence of $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}_{p}}]$-modules

$$0\longrightarrow V^{*}(\boldsymbol{f})^{+}\longrightarrow V^{*}(\boldsymbol{f})\longrightarrow V^{*}(\boldsymbol{f})^{-}\longrightarrow 0,$$

and the bilinear form $\langle\cdot,\cdot\rangle_{\boldsymbol{f}}$ induces perfect, $G_{\mathbf{Q}_p}$-equivariant pairings

$$\langle\cdot,\cdot\rangle_{\boldsymbol{f}} : V(\boldsymbol{f})^{\pm} \otimes_{\mathscr{O}_{\boldsymbol{f}}} V^*(\boldsymbol{f})^{\mp} \longrightarrow \mathscr{O}_{\boldsymbol{f}}. \tag{104}$$

Because $H^1_{\cdot}(\Gamma, D^{\cdot}_{\boldsymbol{f},m})^{\leqslant 0}$ is an $\mathscr{O}_{\boldsymbol{f}}$-direct summand of $H^1_{\cdot}(\Gamma, D^{\cdot}_{\boldsymbol{f},m})$, there are natural $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}}]$-projections

$$\mathrm{pr}_{\boldsymbol{f}} : H^1(\Gamma, D'_{\boldsymbol{f},m})(1) \twoheadrightarrow V(\boldsymbol{f}) \quad \text{and} \quad \mathrm{pr}^*_{\boldsymbol{f}} : H^1_{\mathrm{c}}(\Gamma, D_{\boldsymbol{f},m})(-\boldsymbol{\kappa}_{U_{\boldsymbol{f}}}) \twoheadrightarrow V^*(\boldsymbol{f}). \tag{105}$$

- For all classical points k in $U^{\mathrm{cl}}_{\boldsymbol{f}}$ the specialisation map ρ_k in the right column of Equation (84) gives rise to an isomorphism of $L[G_{\mathbf{Q}}]$-modules

$$\rho_k : V(\boldsymbol{f}) \otimes_{\Lambda_{\boldsymbol{f}}} \Lambda_{\boldsymbol{f}}/(\pi_k) \cong H^1_{\text{ét}}(Y_1(N,p)_{\bar{\mathbf{Q}}}, \mathscr{L}_{k-2}(1))_{\boldsymbol{f}^*_k} \cong V(\boldsymbol{f}_k). \tag{106}$$

Here

$$H^1_{\text{ét}}(Y_1(N,p)_{\bar{\mathbf{Q}}}, \mathscr{L}_{k-2}(1))_L \longrightarrow H^1_{\text{ét}}(Y_1(N,p)_{\bar{\mathbf{Q}}}, \mathscr{L}_{k-2}(1))_{\boldsymbol{f}^*_k}$$

is the maximal quotient on which T'_ℓ, U'_p and $\langle d\rangle'$ act respectively as multiplication by $a_\ell(k), a_p(k)$ and $\chi_{\boldsymbol{f}}(p)$ for any prime $\ell \nmid Np$ and any unit d in $(\mathbf{Z}/Np\mathbf{Z})^*$. If $t : Y_1(Np) \to Y_1(N,p)$ is the natural projection (viz. the one induced by the identity on $\mathbf{H}$ under (6)), the second isomorphism in Equation (106) is the one induced by the pull-back

$$t^* : H^1_{\text{ét}}(Y_1(N,p)_{\bar{\mathbf{Q}}}, \mathscr{L}_{k-2}(1)) \longrightarrow H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{L}_{k-2}(1)).$$

If $k_o = 1$, so that $\boldsymbol{f}_1 = \sum_{n\geqslant 1} a_n(1)\cdot q^n$ is a classical, cuspidal weight-one Hecke eigenform (cf. Assumption 5.1), then the weight-one specialisation

$$V(\boldsymbol{f}^\sharp_1) = V(\boldsymbol{f}^\sharp) \otimes_{\Lambda_{\boldsymbol{f}}} \Lambda_{\boldsymbol{f}}/(\pi_1)$$

of $V(\boldsymbol{f}^\sharp)$ yields a canonical model of the dual of the Deligne–Serre representation attached to $\boldsymbol{f}^\sharp_1$. More generally, if $\boldsymbol{f}_1$ is classical, set $V(\boldsymbol{f}_1) = V(\boldsymbol{f}) \otimes_{\Lambda_{\boldsymbol{f}}} \Lambda_{\boldsymbol{f}}/\pi_1$ (which is non-canonically isomorphism to the direct sum of a finite number of $V(\boldsymbol{f}^\sharp_1)$.) In order to have coherent notation and terminology, we still denote by

$$\rho_1 : V(\boldsymbol{f}) \otimes_{\Lambda_{\boldsymbol{f}}} \Lambda_{\boldsymbol{f}}/(\pi_1) \longrightarrow V(\boldsymbol{f}_1) \tag{107}$$

the identity map, and refer to it as the specialisation map at weight one.

Similarly for each classical weight k in $U^{\mathrm{cl}}_{\boldsymbol{f}}$ there are natural isomorphisms of $L[G_{\mathbf{Q}_p}]$-modules

$$\rho_k : V^*(\boldsymbol{f}) \otimes_{\Lambda_{\boldsymbol{f}}} \Lambda_{\boldsymbol{f}}/(\pi_k) \cong V^*(\boldsymbol{f}_k) \tag{108}$$

(cf. the discussion following Equation (84)). Moreover for each $x \in V(\boldsymbol{f})$ and $y \in V^*(\boldsymbol{f})$ one has

$$\langle x,y\rangle_{\boldsymbol{f}}(k) = \langle \rho_k(x), \rho_k(y)\rangle_{\boldsymbol{f}_k}, \tag{109}$$

where $\langle\cdot,\cdot\rangle_{\boldsymbol{f}_k}$ is the perfect bilinear form defined in Equation (24).

- For each k in $U^{\mathrm{cl}}_{\boldsymbol{f}}$ and $\cdot = \emptyset, *$, one has short exact sequences of $L[G_{\mathbf{Q}_p}]$-modules

$$0 \longrightarrow V^{\cdot}(\boldsymbol{f}_k)^+ \longrightarrow V^{\cdot}(\boldsymbol{f}_k) \longrightarrow V^{\cdot}(\boldsymbol{f}_k)^- \longrightarrow 0, \tag{110}$$

where $V(\boldsymbol{f}_k)^-$ is the maximal $G_{\mathbf{Q}_p}$-unramified L-quotient of $V(\boldsymbol{f}_k)$, and $V^*(\boldsymbol{f}_k)^+$ is the maximal $G_{\mathbf{Q}_p}$-unramified L-submodule of $V^*(\boldsymbol{f}_k)$. The specialisation maps (106) and (108) induce isomorphisms

$$\rho_k : V^{\cdot}(\boldsymbol{f})^{\pm} \otimes_k L \cong V^{\cdot}(\boldsymbol{f}_k)^{\pm}. \tag{111}$$

According to Equation (101) the inertia subgroup $I_{\mathbf{Q}_p}$ of $G_{\mathbf{Q}_p}$ acts on $V(\boldsymbol{f}_k)^+$ via $\chi_{\mathrm{cyc}}^{k-1}$, and trivially on $V(\boldsymbol{f}_k)^-$. If $k \geqslant 2$, applying $D_{\mathrm{dR}}(\cdot)$ to the previous exact sequence and using Equation (28) gives natural isomorphisms

$$D_{\mathrm{cris}}(V(\boldsymbol{f}_k)^+) \cong V_{\mathrm{dR}}(\boldsymbol{f}_k)/\mathrm{Fil}^0 \quad \text{and} \quad \mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{f}_k) \cong D_{\mathrm{cris}}(V(\boldsymbol{f}_k)^-). \tag{112}$$

Similarly $I_{\mathbf{Q}_p}$ acts trivially on $V^*(\boldsymbol{f}_k)^+$ and via $\chi_{\mathrm{cyc}}^{1-k}$ on $V^*(\boldsymbol{f}_k)^-$, hence Equations (28) and (110) give

$$D_{\mathrm{cris}}(V^*(\boldsymbol{f}_k)^+) \cong V^*_{\mathrm{dR}}(\boldsymbol{f}_k)/\mathrm{Fil}^1 \quad \text{and} \quad \mathrm{Fil}^1 V^*_{\mathrm{dR}}(\boldsymbol{f}_k) \cong D_{\mathrm{cris}}(V^*(\boldsymbol{f}_k)^-). \tag{113}$$

- The Atkin–Lehner operator w_{Np} introduced in Equation (67) induces an isomorphism of $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}(\zeta_N)}]$-modules (cf. Equation (68))

$$w_{Np} : H^1(\Gamma, D_{U,m})^{\leqslant 0} \cong H^1(\Gamma, D'_{U,m})^{\leqslant 0},$$

intertwining the action of the dual Hecke operators U'_p, T'_ℓ and $\langle d\rangle$ on the left hand side with that of the Hecke operators U_p, T_ℓ and $\langle d\rangle^{-1}$ on the right hand side, for each prime ℓ not dividing Np and each unit d modulo N. Since the form $\boldsymbol{f}^\sharp_{k_o}$ is cuspidal, it induces Galois equivariant isomorphisms

$$w^{\cdot}_{Np} : V^*(\boldsymbol{f})^{\cdot}(1 + \boldsymbol{\kappa}_{U_{\boldsymbol{f}}} + \chi_{\boldsymbol{f}}) \cong V(\boldsymbol{f})^{\cdot}, \tag{114}$$

for $\cdot$ equal to one of the symbols $\emptyset, +$ and $-$.
- Set

$$D^*(\boldsymbol{f})^- = \big(\mathtt{V}^*(\boldsymbol{f})^-(1 + \boldsymbol{\kappa}_{U_{\boldsymbol{f}}} + \chi_{\boldsymbol{f}})\hat{\otimes}_{\mathbf{Z}_p} \hat{\mathbf{Z}}_p^{\mathrm{nr}}\big)^{G_{\mathbf{Q}_p}}[1/p], \tag{115}$$

where $\mathtt{V}^*(\boldsymbol{f})^-$ is a $G_{\mathbf{Q}_p}$-stable $\Lambda_{\boldsymbol{f}}$-lattice in $V^*(\boldsymbol{f})^-$, and $\hat{\mathbf{Z}}_p^{\mathrm{nr}}$ is the ring of integers of the p-adic completion $\hat{\mathbf{Q}}_p^{\mathrm{nr}}$ of the maximal unramified extension of $\mathbf{Q}_p$. (Note that $V^*(\boldsymbol{f})^-(1 + \boldsymbol{\kappa}_{U_{\boldsymbol{f}}} + \chi_{\boldsymbol{f}})$ is an unramified $G_{\mathbf{Q}_p}$-module, cf. Equations (101) and (104).) It is a free finite $\mathscr{O}_{\boldsymbol{f}}$-module (of rank one if $\boldsymbol{f} = \boldsymbol{f}^\sharp$ is primitive). For each classical point k in $U_{\boldsymbol{f}}^{\mathrm{cl}}$, the isomorphism (111) and the second isomorphism in Equation (113) induce a specialisation isomorphism

$$\rho_k : D^*(\boldsymbol{f})^- \otimes_k L \cong \Big(V^*(\boldsymbol{f}_k)^-(k - 1 + \chi_{\boldsymbol{f}}) \otimes_{\mathbf{Q}_p} \hat{\mathbf{Q}}_p^{\mathrm{nr}}\Big)^{G_{\mathbf{Q}_p}} \cong \mathrm{Fil}^1 V^*_{\mathrm{dR}}(\boldsymbol{f}_k). \tag{116}$$

As $V^*(\boldsymbol{f}_k)^-(k-1)$ is unramified, in the previous equation one identifies the middle term with the tensor product of $D_{\mathrm{cris}}(V^*(\boldsymbol{f}_k)^-)$, $D_{\mathrm{cris}}(\mathbf{Q}_p(k-1))$ and $D_{\mathrm{cris}}(L(\chi_{\boldsymbol{f}}))$. The second isomorphism then arises from Equation (113), the canonical isomorphism $D_{\mathrm{cris}}(\mathbf{Q}_p(k-1)) \cong \mathbf{Q}_p$, and the isomorphism between $D_{\mathrm{cris}}(L(\chi_{\boldsymbol{f}}))$ and L sending the Gauß sum $\sum_{a \in (\mathbf{Z}/c(\chi_{\boldsymbol{f}})\mathbf{Z})^*} \check{\chi}_{\boldsymbol{f}}(a) \otimes \zeta^a_{c(\chi_{\boldsymbol{f}})}$ of the primitive character $\check{\chi}_{\boldsymbol{f}}$ attached to $\chi_{\boldsymbol{f}}$ to the identity, where $c(\chi_{\boldsymbol{f}})$ is the conductor of $\chi_{\boldsymbol{f}}$ and $\zeta_{c(\chi_{\boldsymbol{f}})}$ is a primitive $c(\chi_{\boldsymbol{f}})$-th root of unity.

In light of the isomorphisms (87) and (114), the main result of [**Oht00**] and Theorem 9.5.2 of [**KLZ17**] yield an *Eichler–Shimura isomorphism*

$$\mathtt{ES}_{\boldsymbol{f}}^{-} : D^*(\boldsymbol{f})^{-} \cong S_{U_{\boldsymbol{f}}}^{\mathrm{ord}}(N, \chi_{\boldsymbol{f}})[\boldsymbol{f}^\sharp], \tag{117}$$

whose base change along evaluation at a classical point $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$ is equal to the composition of the specialisation isomorphism (116) with the isomorphism $\mathrm{Fil}^1 V_{\mathrm{dR}}^*(\boldsymbol{f}_k) \cong S_k(Np, L)_{\boldsymbol{f}_k}$ defined in Equation (27). One defines

$$\omega_{\boldsymbol{f}} \in D^*(\boldsymbol{f})^{-} \tag{118}$$

to be the image of the Hida family $\boldsymbol{f}$ under the inverse of $\mathtt{ES}_{\boldsymbol{f}}^{-}$, so that

$$\rho_k(\omega_{\boldsymbol{f}}) = \omega_{\boldsymbol{f}_k} \tag{119}$$

for each classical point k in $U_{\boldsymbol{f}}^{\mathrm{cl}}$ (cf. Equation (30)). (When $k_o \geqslant 2$, the overconvergent Eichler–Shimura isomorphism proved in [**AIS15**] extends these results to Coleman families of slope at most $k_o - 2$.)

- Set

$$D^*(\boldsymbol{f})^{+} = \left(\mathtt{V}^*(\boldsymbol{f})^{+} \hat{\otimes}_{\mathbf{Z}_p} \hat{\mathbf{Z}}_p^{\mathrm{nr}}\right)^{G_{\mathbf{Q}_p}}[1/p], \tag{120}$$

where $\mathtt{V}^*(\boldsymbol{f})^{+}$ is a $G_{\mathbf{Q}_p}$-stable $\Lambda_{\boldsymbol{f}}$-lattice in $V^*(\boldsymbol{f})^{+}$. The perfect duality $\langle \cdot, \cdot \rangle_{\boldsymbol{f}}$ (cf. Equation (104)), the Atkin–Lehner isomorphism w_{Np}^{+} (cf. Equation (114)) and the Eichler–Shimura isomorphism $\mathtt{ES}_{\boldsymbol{f}}^{-}$ give rise to an isomorphism

$$\mathtt{ES}_{\boldsymbol{f}}^{+} : D^*(\boldsymbol{f})^{+} \cong \mathrm{Hom}_{\mathscr{O}_{\boldsymbol{f}}}(S_{U_{\boldsymbol{f}}}^{\mathrm{ord}}(N, \chi_f)[\boldsymbol{f}^\sharp], \mathscr{O}_{\boldsymbol{f}}),$$

whose base change along evaluation at $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$ on $\mathscr{O}_{\boldsymbol{f}}$ equals the composition of the specialisation isomorphism

$$\rho_k : D^*(\boldsymbol{f})^{+} \otimes_k L \cong \left(V^*(\boldsymbol{f}_k)^{+} \otimes_{\mathbf{Q}_p} \hat{\mathbf{Q}}_p^{\mathrm{nr}}\right)^{G_{\mathbf{Q}_p}} \cong V_{\mathrm{dR}}^*(\boldsymbol{f}_k)/\mathrm{Fil}^1 \tag{121}$$

arising from Equations (111) and (113), and the isomorphism

$$V_{\mathrm{dR}}^*(\boldsymbol{f}_k)/\mathrm{Fil}^1 \cong \mathrm{Hom}_L(S_k(Np, L)_{\boldsymbol{f}_k^*}, L) \cong \mathrm{Hom}_L(S_k(Np, L)_{\boldsymbol{f}_k}, L),$$

where the first map is the adjoint of the perfect duality $\langle \cdot, \cdot \rangle_{\boldsymbol{f}_k}$ defined in Equation (32) (cf. Equation (109)), and the second is the dual of

$$(-1)^{k_o - 2} \cdot w_{Np} : S_k(Np, L)_{\boldsymbol{f}_k} \cong S_k(Np, L)_{\boldsymbol{f}_k^*}.$$

We claim that (shrinking $U_{\boldsymbol{f}}$ if necessary) there exists

$$\eta_{\boldsymbol{f}} \in D^*(\boldsymbol{f})^{+} \tag{122}$$

such that, for each classical point k in $U_{\boldsymbol{f}}^{\mathrm{cl}}$, one has (cf. Equation (34))

$$\rho_k(\eta_{\boldsymbol{f}}) = (p-1) a_p(k) \cdot \eta_{\boldsymbol{f}_k}. \tag{123}$$

Indeed, write $\boldsymbol{f} = \sum_d r_d \cdot \boldsymbol{f}^\sharp(q^d)$, with functions $(r_d)_{d|(N/N_{\boldsymbol{f}})}$ in $\mathscr{O}_{\boldsymbol{f}}$ without common zeros. For each positive divisor d of $N/N_{\boldsymbol{f}}$, the $\mathbf{Q}$-rational morphism $v_d : Y_1(N,p)_{\mathbf{Q}} \longrightarrow Y_1(N_{\boldsymbol{f}}, p)_{\mathbf{Q}}$ arising from multiplication by d on $\mathbf{H}$ (cf. Equation (6)) induces a $G_{\mathbf{Q}}$-equivariant morphism $v_{d*} : V^*(\boldsymbol{f}) \longrightarrow V^*(\boldsymbol{f}^\sharp)$ (cf.

Equation (77)), which in turn induces $v_{d*} : D^*(\boldsymbol{f})^- \longrightarrow D^*(\boldsymbol{f}^\sharp)^-$. Under the Eichler–Shimura isomorphism $\mathtt{ES}_{\boldsymbol{f}}^-$, the latter gives rise to a map

$$v_{d_*} : S^{\mathrm{ord}}_{U_{\boldsymbol{f}}}(N, \chi_{\boldsymbol{f}})[\boldsymbol{f}^\sharp] \longrightarrow S^{\mathrm{ord}}_{U_{\boldsymbol{f}}}(N_{\boldsymbol{f}}, \chi_{\boldsymbol{f}})[\boldsymbol{f}^\sharp] = \mathscr{O}_{\boldsymbol{f}} \cdot \boldsymbol{f}^\sharp.$$

Set $\mathrm{Trace}_{\boldsymbol{f}} = \sum_d r_d \cdot v_{d*}$, and define the big differential $\check{\eta}_{\boldsymbol{f}} \in D^*(\boldsymbol{f})^+$ to be the image under the inverse of $\mathtt{ES}_{\boldsymbol{f}}^+$ of the linear form sending the Hida family $\boldsymbol{f}'$ in $S^{\mathrm{ord}}_{U_{\boldsymbol{f}}}(N, \chi_{\boldsymbol{f}})[\boldsymbol{f}^\sharp]$ to the first Fourier coefficient of $\mathrm{Trace}_{\boldsymbol{f}}(\boldsymbol{f}')$:

$$\mathtt{ES}_{\boldsymbol{f}}^+(\check{\eta}_{\boldsymbol{f}})(\boldsymbol{f}') = a_1(\mathrm{Trace}_{\boldsymbol{f}}(\boldsymbol{f}')).$$

It follows from the definitions and Equation (109) that

$$\rho_k(\check{\eta}_{\boldsymbol{f}}) = (-1)^{k_o-2} \cdot \frac{(\boldsymbol{f}_k, \boldsymbol{f}_k)_{Np}}{(\boldsymbol{f}_k^\sharp, \boldsymbol{f}_k^\sharp)_{N_{\boldsymbol{f}}p}} \cdot \eta_{\boldsymbol{f}_k}$$

for each classical point k in $U_{\boldsymbol{f}}^{\mathrm{cl}}$. As explained in the proof of Lemma 2.19 of [**DR14**], the elements $(-1)^{k_o-2} \cdot \frac{(\boldsymbol{f}_k, \boldsymbol{f}_k)_{Np}}{(\boldsymbol{f}_k^\sharp, \boldsymbol{f}_k^\sharp)_{N_{\boldsymbol{f}}p}}$ are interpolated by an analytic function $\mathscr{E}_{\boldsymbol{f}}$ on $U_{\boldsymbol{f}}$, which does not vanish at k_o (as $\boldsymbol{f}_{k_o}$ is non-zero by the definition of level-N test vector for $\boldsymbol{f}^\sharp$). Shrinking $U_{\boldsymbol{f}}$ if necessary, one can then assume that $\mathscr{E}_{\boldsymbol{f}}$ is a unit in $\mathscr{O}_{\boldsymbol{f}}$, and define the sought-for $\mathscr{O}_{\boldsymbol{f}}$-adic differential $\eta_{\boldsymbol{f}}$ to be $(p-1) \cdot \mathscr{E}_{\boldsymbol{f}}^{-1} \cdot a_p(\boldsymbol{k})$ times $\check{\eta}_{\boldsymbol{f}}$.

- Similarly as in Equations (115) and (120), for $\cdot = \pm$, define the $\mathscr{O}_{\boldsymbol{f}}$-module

$$D(\boldsymbol{f})^\cdot = \big(\mathtt{V}(\boldsymbol{f})^\cdot(\boldsymbol{\nu}^\cdot)\hat{\otimes}_{\mathbf{Z}_p}\hat{\mathbf{Z}}_p^{\mathrm{nr}}\big)^{G_{\mathbf{Q}_p}}[1/p], \tag{124}$$

where $\mathtt{V}(\boldsymbol{f})^\cdot$ is a $G_{\mathbf{Q}_p}$-stable $\mathscr{O}_{\boldsymbol{f}}$-lattice in $V(\boldsymbol{f})^\cdot$, $\boldsymbol{\nu}^-$ is the trivial character and $\boldsymbol{\nu}^+ = -1 - \boldsymbol{\kappa}_{U_{\boldsymbol{f}}}$ (so that the twist of $V(\boldsymbol{f})^\cdot$ by $\boldsymbol{\nu}^\cdot$ is unramified, cf. Equation (101)). The pairings $\langle \cdot, \cdot \rangle_{\boldsymbol{f}}$ defined in Equation (104) and the isomorphism $D_{\mathrm{cris}}(L(\chi_{\boldsymbol{f}})) \cong L$ sending the Gauß sum $G(\chi_{\boldsymbol{f}})$ to the identity induce perfect dualities of $\mathscr{O}_{\boldsymbol{f}}$-modules (denoted again by the same symbols)

$$\langle \cdot, \cdot \rangle_{\boldsymbol{f}} : D(\boldsymbol{f})^\pm \otimes_{\mathscr{O}_{\boldsymbol{f}}} D^*(\boldsymbol{f})^\mp \longrightarrow \mathscr{O}_{\boldsymbol{f}}. \tag{125}$$

Similarly as in Equations (116) and (121), for each classical point $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$, the specialisation maps (111) and the isomorphisms (112) give rise to specialisation isomorphisms of L-modules

$$\rho_k : D(\boldsymbol{f})^+ \otimes_k L \cong V_{\mathrm{dR}}(\boldsymbol{f}_k)/\mathrm{Fil}^0 \quad \text{and} \quad \rho_k : D(\boldsymbol{f})^- \otimes_k L \cong \mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{f}_k). \tag{126}$$

Under the isomorphisms (116), (121) and (126), the base change of (125) along evaluation at k on $\mathscr{O}_{\boldsymbol{f}}$ is compatible with the perfect duality (31).

- If $k_o = 1$, the representations $V(\boldsymbol{f}_1)$ and $V^*(\boldsymbol{f}_1)$ are Artin representations unramified at p. After setting $V^\cdot(\boldsymbol{f}_1)^\pm = V^\cdot(\boldsymbol{f})^\pm \otimes_1 L$ (for $\cdot = \emptyset, *$), one has a decomposition of $G_{\mathbf{Q}_p}$-modules

$$V^\cdot(\boldsymbol{f}_1) \cong V^\cdot(\boldsymbol{f}_1)^+ \oplus V^\cdot(\boldsymbol{f}_1)^-.$$

Indeed, according to Assumption 5.1(2) one has

$$V(\boldsymbol{f}_1)^+ = V(\boldsymbol{f}_1)^{\mathrm{Frob}_p = \beta_{\boldsymbol{f}_1}} \quad \text{and} \quad V(\boldsymbol{f}_1)^- = V(\boldsymbol{f}_1)^{\mathrm{Frob}_p = \alpha_{\boldsymbol{f}_1}},$$

where Frob_p is an arithmetic Frobenius, $\alpha_{\boldsymbol{f}_1} = a_p(1)$ and $\alpha_{\boldsymbol{f}_1} \cdot \beta_{\boldsymbol{f}_1} = \chi_{\boldsymbol{f}}(p)$.

In order to have a uniform notation (cf. Equation (112)), if $k_o = 1$ one sets $V_{\mathrm{dR}}^{\cdot}(\boldsymbol{f}_1) = D_{\mathrm{cris}}(V^{\cdot}(\boldsymbol{f}_1))$ and defines

$$V_{\mathrm{dR}}(\boldsymbol{f}_1)/\mathrm{Fil}^0 = D_{\mathrm{cris}}(V(\boldsymbol{f}_1)^+) \quad \text{and} \quad \mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{f}_1) = D_{\mathrm{cris}}(V(\boldsymbol{f}_1)^-). \tag{127}$$

Similarly set $\mathrm{Fil}^1 V_{\mathrm{dR}}^*(\boldsymbol{f}_1) = D_{\mathrm{cris}}(V^*(\boldsymbol{f}_1)^-)$ and $V_{\mathrm{dR}}^*(\boldsymbol{f}_1)/\mathrm{Fil}^1 = D_{\mathrm{cris}}(V^*(\boldsymbol{f}_1)^+)$. The pairing (103) then induces a perfect and $G_{\mathbf{Q}}$-equivariant duality

$$V(\boldsymbol{f}_1) \otimes_L V^*(\boldsymbol{f}_1) \longrightarrow L,$$

under which $V(\boldsymbol{f}_1)^+$ is the orthogonal complement of $V^*(\boldsymbol{f}_1)^+$. This in turn induces on the crystalline Dieudonné modules a perfect pairing

$$\langle \cdot, \cdot \rangle_{\boldsymbol{f}_1} : V_{\mathrm{dR}}(\boldsymbol{f}_1) \otimes_L V_{\mathrm{dR}}^*(\boldsymbol{f}_1) \longrightarrow L, \tag{128}$$

which identifies $\mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{f}_1)$ and $V_{\mathrm{dR}}(\boldsymbol{f}_1)/\mathrm{Fil}^0$ with the duals of $V_{\mathrm{dR}}^*(\boldsymbol{f}_1)/\mathrm{Fil}^1$ and $\mathrm{Fil}^1 V_{\mathrm{dR}}^*(\boldsymbol{f}_1)$ respectively. One finally defines

$$\omega_{\boldsymbol{f}_1} = \rho_1(\omega_{\boldsymbol{f}}) \in \mathrm{Fil}^1 V_{\mathrm{dR}}^*(\boldsymbol{f}_1) \quad \text{and} \quad \eta_{\boldsymbol{f}_1} = \rho_1(\eta_{\boldsymbol{f}}) \in V_{\mathrm{dR}}^*(\boldsymbol{f}_1)/\mathrm{Fil}^1 \tag{129}$$

as the specialisations of $\omega_{\boldsymbol{f}}$ and $\eta_{\boldsymbol{f}}$ respectively at weight one.

6. Garrett–Rankin p-adic L-functions

Fix three primitive L-rational Hida families

$$\begin{aligned} \boldsymbol{f}^\sharp &= \sum_{n \geqslant 1} a_n(\boldsymbol{k}) \cdot q^n \in S_{U_{\boldsymbol{f}}}^{\mathrm{ord}}(N_{\boldsymbol{f}}, \chi_{\boldsymbol{f}}), \\ \boldsymbol{g}^\sharp &= \sum_{n \geqslant 1} b_n(\boldsymbol{l}) \cdot q^n \in S_{U_{\boldsymbol{g}}}^{\mathrm{ord}}(N_{\boldsymbol{g}}, \chi_{\boldsymbol{g}}) \\ \text{and} \quad \boldsymbol{h}^\sharp &= \sum_{n \geqslant 1} c_n(\boldsymbol{m}) \cdot q^n \in S_{U_{\boldsymbol{h}}}^{\mathrm{ord}}(N_{\boldsymbol{h}}, \chi_{\boldsymbol{h}}). \end{aligned}$$

Let N be the least common multiple of $N_{\boldsymbol{f}}, N_{\boldsymbol{g}}$ and $N_{\boldsymbol{h}}$, and let

$$\boldsymbol{f} \in S_{U_{\boldsymbol{f}}}^{\mathrm{ord}}(N, \chi_{\boldsymbol{f}}), \quad \boldsymbol{g} \in S_{U_{\boldsymbol{g}}}^{\mathrm{ord}}(N, \chi_{\boldsymbol{g}}) \text{ and } \boldsymbol{h} \in S_{U_{\boldsymbol{h}}}^{\mathrm{ord}}(N, \chi_{\boldsymbol{h}})$$

be Hida families with associated primitive forms $\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp$ and $\boldsymbol{h}^\sharp$ respectively. Suppose that Assumption 1.2 holds true, namely $\chi_{\boldsymbol{f}} \cdot \chi_{\boldsymbol{g}} \cdot \chi_{\boldsymbol{h}}$ is the trivial character modulo N. Denote by $\Sigma_{\boldsymbol{f}}^{\mathrm{gen}}$ the set of classical triples $w = (k, l, m)$ in $\Sigma_{\boldsymbol{f}}$ such that p does not divide the conductor of $\boldsymbol{f}_k, \boldsymbol{g}_l$ and $\boldsymbol{h}_m$.

For any $w \in \Sigma_{\boldsymbol{f}}^{\mathrm{gen}}$ one has $\boldsymbol{f}_k = (f_k)_\alpha, \boldsymbol{g}_l = (g_l)_\alpha$ and $\boldsymbol{h}_m = (h_m)_\alpha$ for (unique) p-ordinary eigenforms f_k, g_l and h_m of common level N (cf. Equation (54)). Similarly $\boldsymbol{f}_k^\sharp, \boldsymbol{g}_l^\sharp$ and $\boldsymbol{h}_m^\sharp$ are the ordinary p-stabilisations of newforms $f_k^\sharp, g_l^\sharp$ and $h_m^\sharp$ of levels $N_{\boldsymbol{f}}$, $N_{\boldsymbol{g}}$ and $N_{\boldsymbol{h}}$ respectively.

Lemma 6.1. — *There exists a Hida family $w_N(\boldsymbol{f})$ in $S_{U_{\boldsymbol{f}}}^{\mathrm{ord}}(N, \bar{\chi}_{\boldsymbol{f}})$ such that, for any $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$ with p not dividing the conductor of $\boldsymbol{f}_k$, the weight-k specialisation $w_N(\boldsymbol{f})_k$ is the ordinary p-stabilisation of $f_k^w = w_N(f_k)$.*

Proof. — A direct computation (see Proposition 1.5 of [**AL78**]) shows that

$$w_N \circ \mathrm{pr}_p^* = \langle (p,1)\rangle \cdot \mathrm{pr}_p^* \circ w_N \quad \text{and} \quad w_N \circ \mathrm{pr}_1^* = \mathrm{pr}_1^* \circ w_N$$

as morphisms from $H^1_{\mathrm{dR}}(Y_1(N)_{\mathbf{Q}_p}, \mathscr{S}_{\mathrm{dR},k-2})_L$ to $H^1_{\mathrm{dR}}(Y_1(Np)_{\mathbf{Q}_p}, \mathscr{S}_{\mathrm{dR},k-2})_L$, where $\langle (p,1)\rangle$ is the diamond operator associated with $(p,1)$ under the identification $\mathbf{Z}/Np\mathbf{Z} = \mathbf{Z}/N\mathbf{Z} \times \mathbf{F}_p$. As a consequence

$$(f_k^w)_\alpha = \left(\mathrm{pr}_1^* \circ w_N - \frac{\bar{\chi}_f(p)\beta_{f_k}}{p^{k-1}} \cdot \mathrm{pr}_p^* \circ w_N\right) f_k \tag{130}$$
$$= w_N \circ \left(\mathrm{pr}_1^* - \frac{\beta_{f_k}}{p^{k-1}} \cdot \mathrm{pr}_p^*\right) f_k = w_N(\boldsymbol{f}_k).$$

The lemma follows from the previous equation and [**KLZ17**, Proposition 10.1.2], namely the existence of a morphism $w_N : S^{\mathrm{ord}}_{U_{\boldsymbol{f}}}(N, \chi_{\boldsymbol{f}}) \to S^{\mathrm{ord}}_{U_{\boldsymbol{f}}}(N, \bar{\chi}_{\boldsymbol{f}})$ which specialises to the Atkin–Lehner operator w_N on the ordinary part of $S_k(\Gamma_1(N,p), \chi_{\boldsymbol{f}})$ for each classical weight k in $U_{\boldsymbol{f}}^{\mathrm{cl}}$ (cf. Equations (69) and (117)). □

According to the previous lemma and the results of [**HT01, DR14, Hid85**] Hida's method (cf. [**Hid85**]) can be applied to construct a *square-root Garrett–Rankin p-adic L-function*

$$\mathscr{L}_p^f(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \in \mathscr{O}_{\boldsymbol{fgh}}$$

such that, for each classical triple $w = (k,l,m)$ in $\Sigma_{\boldsymbol{f}}^{\mathrm{gen}}$, one has

$$\mathscr{L}_p^f(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})(w) = \mathscr{L}_p^f(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m), \tag{131}$$

where $\mathscr{L}_p^f(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ is the p-adic period associated in Equation (55) to the p-stabilisation of the triple (f_k, g_l, h_m).

Remark 6.2. — The p-adic L-function $\mathscr{L}_p^f(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ slightly differs from the one denoted by the same symbol in [**DR14**]. Precisely our $\mathscr{L}_p^f(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ is equal to their $\mathscr{L}_p^f(w_N(\boldsymbol{f}^*), \boldsymbol{g}, \boldsymbol{h})$, where $\boldsymbol{f}^*$ is the Hida family which specialises to the dual of $\boldsymbol{f}_k$ for each k in $U_{\boldsymbol{f}}^{\mathrm{cl}}$.

6.1. Test vectors and special value formulae. — In this section assume the following hypotheses (cf. [**Hsi20**]).

Assumption 6.3. —

1. *There is a triple (k,l,m) in Σ such that the local sign $\varepsilon_q(\boldsymbol{f}_k^\sharp, \boldsymbol{g}_l^\sharp, \boldsymbol{h}_m^\sharp)$ is equal to $+1$ for all primes $q|N$.*
2. *The greatest common divisor of $N_{\boldsymbol{f}}, N_{\boldsymbol{g}}$ and $N_{\boldsymbol{h}}$ is squarefree.*
3. *There is a classical point k in $U_{\boldsymbol{f}}^{\mathrm{cl}}$ such that $V(f_k^\sharp)$ is residually irreducible and p-distinguished.*

Under these assumptions, Section 3.5 of [**Hsi20**] implies the existence of an explicit level-N test vector $(\boldsymbol{f}^\star, \boldsymbol{g}^\star, \boldsymbol{h}^\star)$ for $(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$ such that the *Garrett–Rankin triple product p-adic L-function*

$$L_p(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp) = \mathscr{L}_p^f(\boldsymbol{f}^\star, \boldsymbol{g}^\star, \boldsymbol{h}^\star)^2$$

satisfies the following interpolation property (see Theorem A of loc. cit.). For all $w = (k, l, m)$ in $\Sigma_{\boldsymbol{f}}^{\mathrm{gen}}$

(132)
$$L_p(\boldsymbol{f}_k^\sharp, \boldsymbol{g}_l^\sharp, \boldsymbol{h}_m^\sharp) = \frac{\Gamma(k,l,m)}{2^{\alpha(k,l,m)}} \cdot \frac{\mathcal{E}(\boldsymbol{f}_k^\sharp, \boldsymbol{g}_l^\sharp, \boldsymbol{h}_m^\sharp)^2}{\mathcal{E}_0(\boldsymbol{f}_k^\sharp)^2 \cdot \mathcal{E}_1(\boldsymbol{f}_k^\sharp)^2} \cdot \prod_{q|N} \mathrm{Loc}_q \cdot \frac{L(f_k^\sharp \otimes g_l^\sharp \otimes h_m^\sharp, \frac{k+l+m-2}{2})}{\pi^{2(k-2)} \cdot (f_k^\sharp, f_k^\sharp)_{N_f}^2},$$

where the notations are as follows.

- $\alpha(\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}) \in \mathscr{O}_{\boldsymbol{fgh}}$ is a linear form in the variables $\boldsymbol{k}, \boldsymbol{l}$ and $\boldsymbol{m}$ and

(133)
$$\Gamma(k,m,l) = \big((k+l+m-4)/2\big)! \cdot \big((k+l-m-2)/2\big)! \cdot \big((k+m-l-2)/2\big)! \cdot \big((k-l-m)/2\big)!.$$

- Set $c_w = (k+l+m-2)/2$, $\alpha_k = a_p(k)$, $\beta_k = \chi_{\boldsymbol{f}}(p)p^{k-1}/\alpha_k$, $\alpha_l = b_p(l)$ et cetera. Then

(134)
$$\mathcal{E}(\boldsymbol{f}_k^\sharp, \boldsymbol{g}_l^\sharp, \boldsymbol{h}_m^\sharp) = \left(1 - \frac{\beta_k\alpha_l\alpha_m}{p^{c_w}}\right)\left(1 - \frac{\beta_k\beta_l\alpha_m}{p^{c_w}}\right)\left(1 - \frac{\beta_k\alpha_l\beta_m}{p^{c_w}}\right)\left(1 - \frac{\beta_k\beta_l\beta_m}{p^{c_w}}\right),$$

$$\mathcal{E}_0(\boldsymbol{f}_k^\sharp) = 1 - \frac{\beta_k}{\alpha_k} \quad \text{and} \quad \mathcal{E}_1(\boldsymbol{f}_k^\sharp) = 1 - \frac{\beta_k}{p \cdot \alpha_k}. \tag{135}$$

- For each rational prime q dividing N, Loc_q is an explicit non-zero rational number, independent of w.
- Let $\pi(f_k^\sharp), \pi(g_l^\sharp)$ and $\pi(h_m^\sharp)$ be the cuspidal automorphic representations of GL_2 attached to $f_k^\sharp, g_l^\sharp$ and $h_m^\sharp$ respectively, and set $\Pi_x = \pi(f_k^\sharp) \otimes \pi(g_l^\sharp) \otimes \pi(h_m^\sharp)$. Then
$$L(f_k^\sharp \otimes g_l^\sharp \otimes h_m^\sharp, s) = L(\Pi_w, s + (3-k-l-m)/2).$$
Thanks to the results of Garrett and Harris–Kudla [**Gar87, HK91**] one knows that $L(f_k^\sharp \otimes g_l^\sharp \otimes h_m^\sharp, s)$ admits an analytic continuation to all of $\mathbf{C}$ and satisfies a functional equation with global epsilon factor $\varepsilon(\Pi_x, 1/2)$ equal to $+1$ relating its values at s and $k+l+m-2-s$.

This is proved by Hsieh in Theorem A of [**Hsi20**] relying on the special value formulae of Garrett, Harris–Kudla and Ichino [**Gar87, HK91, Ich08**].

7. Selmer groups and big logarithms

Let $(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$ and $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ be as in Section 6.

7.1. A four-variable big logarithm.

— Let (cf. Section 5, in particular Equations (97), (102) and (101))
$$M(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f = V(\boldsymbol{f})^- \hat{\otimes}_L V(\boldsymbol{g})^+ \hat{\otimes}_L V(\boldsymbol{h})^+ \left(\omega_{\mathrm{cyc}}^{2-l_o-m_o} \cdot \kappa_{\mathrm{cyc}}^{2-\boldsymbol{l}-\boldsymbol{m}}\right).$$
This is a free $\mathscr{O}_{\boldsymbol{fgh}}$-module on which $G_{\mathbf{Q}_p}$ acts via the *unramified* character
$$\Psi : G_{\mathbf{Q}_p} \twoheadrightarrow G_{\mathbf{Q}_p}^{\mathrm{ur}} \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}^*$$

defined by

$$\Psi(\mathrm{Frob}_p) = \frac{\chi_{\boldsymbol{g}}\chi_{\boldsymbol{h}}(p)\cdot a_p(\boldsymbol{k})}{b_p(\boldsymbol{l})\cdot c_p(\boldsymbol{m})} \tag{136}$$

(cf. Equation (101)). Let $\mathscr{O}_{\mathrm{cyc}} \subset \mathbf{Q}_p[\![\boldsymbol{j}-j_o]\!]$ be the ring of bounded analytic functions on an open disc U_{cyc} centred at $j_o = (k_o - l_o - m_o)/2$, and let $\kappa_{\mathrm{cyc}}^{-\boldsymbol{j}} : G_{\mathbf{Q}} \to \mathscr{O}_{\mathrm{cyc}}^*$ be defined by $\kappa_{\mathrm{cyc}}^{-\boldsymbol{j}}(g) = \exp_p(-\boldsymbol{j}\cdot\log_p(\chi_{\mathrm{cyc}}(g)))$. Denote by $\bar{\mathscr{O}}_{\boldsymbol{fgh}}$ the tensor product $\mathscr{O}_{\boldsymbol{fgh}}\hat{\otimes}_{\mathbf{Q}_p}\mathscr{O}_{\mathrm{cyc}}$ and define the $\bar{\mathscr{O}}_{\boldsymbol{fgh}}[G_{\mathbf{Q}_p}]$-module

$$\bar{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f = M(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f\hat{\otimes}_{\mathbf{Q}_p}\mathscr{O}_{\mathrm{cyc}}\left(\omega_{\mathrm{cyc}}^{-j_o}\cdot\kappa_{\mathrm{cyc}}^{-\boldsymbol{j}}\right). \tag{137}$$

Denote by $\mathcal{Z} = \mathcal{Z}_{\boldsymbol{fgh}}$ the set of integers such that $j \equiv j_o \pmod{p-1}$ and set $\bar{\Sigma} = \Sigma\times\mathcal{Z}$. For all $w = (k,l,m)\in\Sigma$ let $\Psi_w : G_{\mathbf{Q}_p} \to L^*$ be the composition of Ψ with evaluation at w on $\mathscr{O}_{\boldsymbol{fgh}}$ and define $M(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f = M(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f\otimes_w L$ as the base change of $M(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ under evaluation at x on $\mathscr{O}_{\boldsymbol{fgh}}$, which is isomorphic to $L(\Psi_w)^a$ for some positive integer $a \geqslant 1$. If $x = (w,j)\in\bar{\Sigma}$ then evaluation at x on $\bar{\mathscr{O}}_{\boldsymbol{fgh}}$ induces a natural isomorphism of $L[G_{\mathbf{Q}_p}]$-modules

$$\rho_x : \bar{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f\otimes_x L \cong M(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f(-j).$$

If

$$\Lambda_{\boldsymbol{fgh}} = \Lambda_{\boldsymbol{f}}\hat{\otimes}_{\mathscr{O}}\Lambda_{\boldsymbol{g}}\hat{\otimes}_{\mathscr{O}}\Lambda_{\boldsymbol{h}}$$

then

$$M(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f = \mathtt{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f[1/p]$$

for a $\Lambda_{\boldsymbol{fgh}}[G_{\mathbf{Q}_p}]$-module $\mathtt{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f$, free of finite rank over $\Lambda_{\boldsymbol{fgh}}$. Let $\hat{\mathbf{Z}}_p^{\mathrm{nr}} = W(\bar{\mathbf{F}}_p)$ be the ring of Witt vectors of an algebraic closure of $\mathbf{F}_p$ and define

$$D(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f = \left(\mathtt{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f\hat{\otimes}_{\mathbf{Z}_p}\hat{\mathbf{Z}}_p^{\mathrm{nr}}\right)^{G_{\mathbf{Q}_p}}[1/p]$$

and

$$\bar{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f = D(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f\hat{\otimes}_{\mathbf{Q}_p}\mathscr{O}_{\mathrm{cyc}}.$$

(Note that $D(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f$ is naturally isomorphic to $D(\boldsymbol{f})^-\hat{\otimes}_L D(\boldsymbol{g})^+\hat{\otimes}_L D(\boldsymbol{h})^+$, cf. Equation (124).) As $\mathtt{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f$ is unramified and free over $\Lambda_{\boldsymbol{fgh}}$, $D(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f$ is a free $\mathscr{O}_{\boldsymbol{fgh}}$-module of the same rank as $M(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f$. For all classical triples $w = (k,l,m)$ in Σ the specialisation maps (106) induce a natural isomorphism

$$\rho_w : D(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f\otimes_w L \cong D_{\mathrm{cris}}(M(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f)$$

Let t_o denote Fontaine's p-adic analogue of $2\pi i$, which depends on a fixed choice of a compatible sequence ζ_{p^∞} of p^n-th roots of unit for $n \geqslant 0$. The element $t = t_o^{-1}\otimes\zeta_{p^\infty}$ is a canonical generator of $D_{\mathrm{cris}}(\mathbf{Q}_p(1))$, and gives rise to a generator t^i of $D_{\mathrm{cris}}(\mathbf{Q}_p(i))$ for each $i\in\mathbf{Z}$. For any $x = (w,j)$ in $\bar{\Sigma}$ define the isomorphism

$$\rho_x : \bar{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f\otimes_x L \cong D_{\mathrm{cris}}\big(M(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f(-j)\big). \tag{138}$$

by the formulae $\rho_x(\alpha\hat{\otimes}\beta) = \beta(j)\cdot\rho_w(\alpha)\otimes t^{-j}$, for each $\alpha\in D(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f$ and $\beta\in\mathscr{O}_{\mathrm{cyc}}$.

If $j<0$ then the Bloch–Kato exponential map gives an isomorphism

$$\exp_x : D_{\mathrm{cris}}\big(M(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f(-j)\big) \cong H^1(\mathbf{Q}_p, M(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f(-j)),$$

and one writes $\log_x$ for its inverse. If $j \geqslant 0$ denote by

$$\exp_x^* : H^1(\mathbf{Q}_p, M(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)_f(-j)) \longrightarrow D_{\mathrm{cris}}(M(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)_f(-j))$$

the Bloch–Kato dual exponential map. The following proposition is a consequence of the work of Ochiai [Och03] and Loeffler–Zerbes [LZ14], which extends previous work of Coleman–Perrin-Riou [Col79, PR94] (see also Theorem 8.2.3 of [KLZ17]).

Proposition 7.1. — *There exists a unique morphism of $\bar{\mathscr{O}}_{\boldsymbol{fgh}}$-modules*

$$\bar{\mathcal{L}}_{\boldsymbol{f}} : H^1(\mathbf{Q}_p, \bar{M}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f) \longrightarrow \bar{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f$$

such that for any $x = (w, j)$ in $\bar{\Sigma}$ with $\Psi_w(\mathrm{Frob}_p) \neq p^{1+j}$ and any $\mathscr{Z}$ in $H^1(\mathbf{Q}_p, \bar{M}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f)$ one has

$$\bar{\mathcal{L}}_{\boldsymbol{f}}(\mathscr{Z})_x = \left(1 - \frac{p^j}{\Psi_w(\mathrm{Frob}_p)}\right)\left(1 - \frac{\Psi_w(\mathrm{Frob}_p)}{p^{1+j}}\right)^{-1} \cdot \begin{cases} \frac{(-1)^{j+1}}{(-j-1)!} \log_x(\mathscr{Z}_x) & \text{if} \quad j < 0 \\ j! \exp_x^*(\mathscr{Z}_x) & \text{if} \quad j \geqslant 0 \end{cases},$$

where $\bar{\mathcal{L}}_{\boldsymbol{f}}(\mathscr{Z})_x$ and $\mathscr{Z}_x$ are shorthands for $\rho_x \circ \bar{\mathcal{L}}_{\boldsymbol{f}}(\mathscr{Z})$ and $\rho_{x*}(\mathscr{Z})$ respectively.

7.1.1. *$\mathscr{O}_{\boldsymbol{fgh}}$-adic differentials.* — Recall the $\mathscr{O}_{\boldsymbol{f}}$-modules $D^*(\boldsymbol{f})^{\pm}$ (resp., $D(\boldsymbol{f})^{\pm}$) introduced in Equations (115) and (120) (resp., Equation (124)), and define similarly $D^*(\boldsymbol{\xi})^{\pm}$ and $D(\boldsymbol{\xi})^{\pm}$ for $\boldsymbol{\xi} = \boldsymbol{g}, \boldsymbol{h}$. Then (cf. Section 7.1)

$$\bar{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f = D(\boldsymbol{f})^- \hat{\otimes}_L D(\boldsymbol{g})^+ \hat{\otimes}_L D(\boldsymbol{h})^+ \hat{\otimes}_{\mathbf{Q}_p} \mathscr{O}_{\mathrm{cyc}},$$

and one defines dually

$$\bar{D}^*(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f = D^*(\boldsymbol{f})^+ \hat{\otimes}_L D^*(\boldsymbol{g})^- \hat{\otimes}_L D^*(\boldsymbol{h})^- \hat{\otimes}_{\mathbf{Q}_p} \mathscr{O}_{\mathrm{cyc}},$$

so that the perfect dualities $\langle \cdot, \cdot \rangle_{\boldsymbol{\xi}}$, for $\boldsymbol{\xi} = \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}$ (cf. Equation (125)) yield a pairing

$$\langle \cdot, \cdot \rangle_{\boldsymbol{fgh}} : \bar{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f \otimes_{\bar{\mathscr{O}}_{\boldsymbol{fgh}}} \bar{D}^*(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f \longrightarrow \bar{\mathscr{O}}_{\boldsymbol{fgh}}. \tag{139}$$

Moreover, identifying $D_{\mathrm{cris}}(\mathbf{Q}_p(i)) = \mathbf{Q}_p \cdot t^i$ with $\mathbf{Q}_p$ ($i \in \mathbf{Z}$), the isomorphisms (116), (121), (126) (and their analogues for $\boldsymbol{g}$ and $\boldsymbol{h}$) give specialisation isomorphisms

$$\rho_x : \bar{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f \otimes_x L \cong \mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{f}_k) \otimes_L V_{\mathrm{dR}}(\boldsymbol{g}_l)/\mathrm{Fil}^0 \otimes_L V_{\mathrm{dR}}(\boldsymbol{h}_m)/\mathrm{Fil}^0 \tag{140}$$

and

$$\rho_x : \bar{D}^*(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f \otimes_x L \cong V_{\mathrm{dR}}^*(\boldsymbol{f}_k)/\mathrm{Fil}^1 \otimes_L \mathrm{Fil}^1 V_{\mathrm{dR}}^*(\boldsymbol{g}_l) \otimes_L \mathrm{Fil}^1 V_{\mathrm{dR}}^*(\boldsymbol{h}_m), \tag{141}$$

for each classical 4-tuple $x = (k, l, m, j)$ in $\bar{\Sigma}$ with $k, l, m \geqslant 2$.

Define the $\bar{\mathscr{O}}_{\boldsymbol{fgh}}$*-adic differential* (cf. Equations (118) and (122))

$$\eta_{\boldsymbol{f}} \omega_{\boldsymbol{g}} \omega_{\boldsymbol{h}} = \eta_{\boldsymbol{f}} \otimes \omega_{\boldsymbol{g}} \otimes \omega_{\boldsymbol{h}} \otimes 1 \in \bar{D}^*(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f. \tag{142}$$

According to Equation (119), Equation (123), and the discussion following Equation (126), for each $x = (k, l, m, j) \in \bar{\Sigma}$ with $k, l, m \geqslant 2$ and each $\boldsymbol{\mu}$ in $\bar{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f$ one has

$$\langle \boldsymbol{\mu}, \eta_{\boldsymbol{f}} \omega_{\boldsymbol{g}} \omega_{\boldsymbol{h}} \rangle_{\boldsymbol{fgh}}(x) = (p-1) a_p(k) \cdot \left\langle \rho_x(\boldsymbol{\mu}), \eta_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_l} \otimes \omega_{\boldsymbol{h}_m} \right\rangle_{\boldsymbol{f}_k \boldsymbol{g}_l \boldsymbol{h}_m}, \tag{143}$$

where $\langle \cdot, \cdot \rangle_{\boldsymbol{f}_k \boldsymbol{g}_l \boldsymbol{h}_m}$ is the product of the perfect dualities $\langle \cdot, \cdot \rangle_{\xi}$ introduced in Equation (32), for ξ equal to $\boldsymbol{f}_k$, $\boldsymbol{g}_l$ and $\boldsymbol{h}_m$.

Define the *four-variable $\boldsymbol{f}$-big logarithm*

$$\bar{\mathscr{L}}_{\boldsymbol{f}} = \bar{\mathscr{L}}og(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \stackrel{\text{def}}{=} \left\langle \bar{\mathcal{L}}_{\boldsymbol{f}}(\cdot), \eta_{\boldsymbol{f}}\omega_{\boldsymbol{g}}\omega_{\boldsymbol{h}} \right\rangle_{\boldsymbol{fgh}} : H^1(\mathbf{Q}_p, \bar{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f) \longrightarrow \bar{\mathscr{O}}_{\boldsymbol{fgh}} \tag{144}$$

to be the composition of $\bar{\mathcal{L}}_{\boldsymbol{f}}$ with the linear form $\langle \cdot, \eta_{\boldsymbol{f}}\omega_{\boldsymbol{g}}\omega_{\boldsymbol{h}}\rangle_{\boldsymbol{fgh}}$ on $\bar{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f$.

Mutatis mutandis the previous constructions apply after replacing $\boldsymbol{f}$ with $\boldsymbol{a} = \boldsymbol{g}, \boldsymbol{h}$. One obtains four-variable *$\boldsymbol{a}$-big logarithms* $\bar{\mathscr{L}}_{\boldsymbol{a}} : H^1(\mathbf{Q}_p, \bar{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_a) \longrightarrow \bar{\mathscr{O}}_{\boldsymbol{fgh}}$.

7.1.1.1. *Weight-one specialisations.* — With the notations introduced in the last part of Section 5 (cf. Equations (127)–(129)), the isomorphisms (140) and (141) and the definition of the pairing $\langle\cdot,\cdot\rangle_{\boldsymbol{f}_k\boldsymbol{g}_l\boldsymbol{h}_m}$ extend to all classical 4-tuples $x = (k,l,m,j)$ in $\bar{\Sigma}$, independently on whether the weights k, l and m are geometric or not (id est equal to 1). Moreover, if $k \geqslant 2$, Equation (143) still holds.

7.2. The balanced Selmer group. — Define the continuous character

$$\Xi_{\boldsymbol{fgh}} : G_{\mathbf{Q}} \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}^*$$

by the formula

$$\Xi_{\boldsymbol{fgh}}(g) = \omega_{\mathrm{cyc}}(g)^{(4-k_o-l_o-m_o)/2} \cdot \kappa_{\mathrm{cyc}}(g)^{(4-\boldsymbol{k}-\boldsymbol{l}-\boldsymbol{m})/2},$$

for every g in $G_{\mathbf{Q}}$, and the $\mathscr{O}_{\boldsymbol{fgh}}[G_{\mathbf{Q}}]$-representation

$$V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = V(\boldsymbol{f})\hat{\otimes}_L V(\boldsymbol{g})\hat{\otimes}_L V(\boldsymbol{h}) \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \Xi_{\boldsymbol{fgh}}.$$

Equations (103) and (114) imply that $V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ is Kummer self-dual: the product of the perfect dualities $[\cdot,\cdot]_{\boldsymbol{\xi}} : V(\boldsymbol{\xi}) \otimes_{\mathscr{O}_{\boldsymbol{\xi}}} V(\boldsymbol{\xi}) \longrightarrow \mathscr{O}_{\boldsymbol{\xi}}(1 + \boldsymbol{\kappa}_{U_{\boldsymbol{\xi}}} + \chi_{\boldsymbol{\xi}})$ defined by $[x,y]_{\boldsymbol{\xi}} = \langle x, w_{Np}^{-1}(y)\rangle_{\boldsymbol{\xi}}$ yields a perfect, skew-symmetric duality (cf. Assumption 1.2)

$$[\cdot,\cdot]_{\boldsymbol{fgh}} : V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}(1),$$

whose adjoint identifies $V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ with its own Kummer dual. Moreover, for all $w = (k,l,m)$ in Σ the specialisation maps (106) induce isomorphisms

$$\rho_w : V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \otimes_w L \cong V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m) \tag{145}$$

(cf. Equation (47)), where $\cdot \otimes_w L$ denotes the base change under evaluation at w.

Define a decreasing filtration $\mathscr{F}^{\cdot}V(\boldsymbol{f})$ on $V(\boldsymbol{f})$ by $\mathscr{F}^jV(\boldsymbol{f}) = V(\boldsymbol{f})$ for every $j \leqslant 0$, $\mathscr{F}^1V(\boldsymbol{f}) = V(\boldsymbol{f})^+$ and $\mathscr{F}^jV(\boldsymbol{f}) = 0$ for $j \geqslant 2$, and similarly $\mathscr{F}^{\cdot}V(\boldsymbol{g})$ and $\mathscr{F}^{\cdot}V(\boldsymbol{h})$. Let $\mathscr{F}^{\cdot}V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ be the tensor product filtration:

$$\mathscr{F}^nV(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = \left[\sum_{p+q+r=n} \mathscr{F}^pV(\boldsymbol{f})\hat{\otimes}_L\mathscr{F}^qV(\boldsymbol{g})\hat{\otimes}_L\mathscr{F}^rV(\boldsymbol{h})\right] \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \Xi_{\boldsymbol{fgh}}.$$

This is a decreasing filtration of $V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ by $\mathscr{O}_{\boldsymbol{fgh}}[G_{\mathbf{Q}_p}]$-submodules, satisfying $\mathscr{F}^4V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = 0$ and $\mathscr{F}^0V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$. The annihilator of $\mathscr{F}^iV(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ under the duality $[\cdot,\cdot]_{\boldsymbol{fgh}}$ is equal to $\mathscr{F}^{4-i}V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$, hence the adjoint of $[\cdot,\cdot]_{\boldsymbol{fgh}}$ induces isomorphisms of $\mathscr{O}_{\boldsymbol{fgh}}[G_{\mathbf{Q}_p}]$-modules

$$\mathrm{gr}^iV(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \cong \mathrm{Hom}_{\mathscr{O}_{\boldsymbol{fgh}}}(\mathrm{gr}^{3-i}V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}), \mathscr{O}_{\boldsymbol{fgh}}(1)) \tag{146}$$

(where $\mathrm{gr}^iV(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = \mathscr{F}^iV(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})/\mathscr{F}^{i+1}$). If one sets

$$V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f = V(\boldsymbol{f})^-\hat{\otimes}_L V(\boldsymbol{g})^+\hat{\otimes}_L V(\boldsymbol{h})^+ \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \Xi_{\boldsymbol{fgh}},$$

and defines similarly $V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_g$ and $V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_h$, then

$$\mathrm{gr}^2 V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f \oplus V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_g \oplus V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_h \tag{147}$$

as $\mathscr{O}_{\boldsymbol{fgh}}[G_{\mathbf{Q}_p}]$-modules. It follows form Equation (146) and the definitions that the inertia subgroup $I_{\mathbf{Q}_p(\mu_p)}$ of the absolute Galois group of $\mathbf{Q}(\mu_p)$ acts on $\mathrm{gr}^3 V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ and $\mathrm{gr}^0 V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ via the characters $\kappa_{\mathrm{cyc}}^{(\boldsymbol{k}+\boldsymbol{l}+\boldsymbol{m}-2)/2}$ and $\kappa_{\mathrm{cyc}}^{(4-\boldsymbol{k}-\boldsymbol{l}-\boldsymbol{m})/2}$ respectively. In addition, Equations (146) and (147) show that $\mathrm{gr}^2 V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ and $\mathrm{gr}^1 V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ are isomorphic respectively to the direct sum of a finite number of copies of

$$\kappa_{\mathrm{cyc}}^{\frac{\boldsymbol{l}+\boldsymbol{m}-\boldsymbol{k}}{2}} \oplus \kappa_{\mathrm{cyc}}^{\frac{\boldsymbol{l}+\boldsymbol{k}-\boldsymbol{m}}{2}} \oplus \kappa_{\mathrm{cyc}}^{\frac{\boldsymbol{k}+\boldsymbol{m}-\boldsymbol{l}}{2}} \quad \text{and} \quad \kappa_{\mathrm{cyc}}^{\frac{\boldsymbol{k}-\boldsymbol{l}-\boldsymbol{m}+2}{2}} \oplus \kappa_{\mathrm{cyc}}^{\frac{\boldsymbol{m}-\boldsymbol{l}-\boldsymbol{k}+2}{2}} \oplus \kappa_{\mathrm{cyc}}^{\frac{\boldsymbol{l}-\boldsymbol{k}-\boldsymbol{m}+2}{2}}$$

as $I_{\mathbf{Q}(\mu_p)}$-modules (where $\kappa_{\mathrm{cyc}}^{\bullet} = \mathscr{O}_{\boldsymbol{fgh}}(\kappa_{\mathrm{cyc}}^{\bullet})$). In particular, for each $i \in \mathbf{Z}$ one has

$$H^0(\mathbf{Q}_p, \mathrm{gr}^i V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) = 0. \tag{148}$$

Define the *balanced local condition*

$$H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) = H^1(\mathbf{Q}_p, \mathscr{F}^2 V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})).$$

In light of Equation (148), the morphism induced on the first $G_{\mathbf{Q}_p}$-cohomology groups by the inclusion $\mathscr{F}^2 V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \hookrightarrow V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ is injective, hence we can, and will, identify the balanced local condition with a submodule of $H^1(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))$, namely

$$H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) = \mathrm{Im}\big(H^1(\mathbf{Q}_p, \mathscr{F}^2 V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) \longrightarrow H^1(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))\big).$$

For $\cdot = f, g, h$, one denotes by $p_\cdot$ both the natural $G_{\mathbf{Q}_p}$-equivariant projection

$$p_\cdot : \mathscr{F}^2 V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \twoheadrightarrow V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_\cdot$$

arising from Equation (147) and the morphism

$$p_\cdot : H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) \longrightarrow H^1(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_\cdot)$$

it induces in cohomology.

For all morphisms of L-algebras $\varphi : \mathscr{O}_{\boldsymbol{fgh}} \longrightarrow \mathscr{O}_\varphi$, set

$$V_\varphi(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_\cdot = V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_\cdot \otimes_\varphi \mathscr{O}_\varphi \quad \text{and} \quad \mathscr{F}^\cdot V_\varphi(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = \mathscr{F}^\cdot V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \otimes_\varphi \mathscr{O}_\varphi,$$

denote again by by $p_\cdot : V_\varphi(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \twoheadrightarrow V_\varphi(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_\cdot$ the natural projections, and define

$$H^1_{\mathrm{bal}}(\mathbf{Q}_p, V_\varphi(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) = \mathrm{Im}\big(H^1(\mathbf{Q}_p, \mathscr{F}^2 V_\varphi(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) \longrightarrow H^1(\mathbf{Q}_p, V_\varphi(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))\big).$$

If $w = (k, l, m)$ is a triple in Σ and φ is evaluation at w, we identify $V_\varphi(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ with $V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ under the specialisation isomorphism ρ_w (cf. Equation (145)).

One has the following crucial lemma.

Lemma 7.2. — *If $w = (k, m, l) \in \Sigma_{\mathrm{bal}}$ is a* balanced *classical triple, then*

$$H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)) = H^1_{\mathrm{fin}}(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)), \tag{149}$$

where $H^1_{\mathrm{fin}}(\mathbf{Q}_p, \cdot)$ is the Bloch–Kato finite local condition (cf. Lemma 3.5). As a consequence, the Bloch–Kato exponential map gives an isomorphism

$$\exp_p : V_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)/\mathrm{Fil}^0 \cong H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)).$$

Proof. — Set $V = V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$, and consider the exact sequence of $G_{\mathbf{Q}_p}$-modules

$$0 \longrightarrow \mathscr{F}^2V \longrightarrow V \longrightarrow V/\mathscr{F}^2 \longrightarrow 0.$$

The discussion preceding Equation (148) shows that $\mathscr{F}^2V$ has Hodge–Tate weights

$$\frac{k+l+m-2}{2}, \quad \frac{k+l-m}{2}, \quad \frac{k+m-l}{2} \quad \text{and} \quad \frac{l+m-k}{2},$$

while $V/\mathscr{F}^2$ has Hodge–Tate weights

$$\frac{k-l-m+2}{2}, \quad \frac{l-k-m+2}{2}, \quad \frac{m-k-l+2}{2} \quad \text{and} \quad \frac{4-k-l-m}{2}.$$

Since w is a balanced classical triple, it follows that all the Hodge–Tate weights of $\mathscr{F}^2V$ (resp., $V/\mathscr{F}^2$) are positive (resp., non-positive), hence

$$\mathrm{tg}_{\mathrm{dR}}(\mathscr{F}^2V) = D_{\mathrm{dR}}(\mathscr{F}^2V) \quad \text{and} \quad \mathrm{Fil}^0 D_{\mathrm{dR}}(V/\mathscr{F}^2) = D_{\mathrm{dR}}(V/\mathscr{F}^2) \tag{150}$$

(where $\mathrm{tg}_{\mathrm{dR}}(\cdot) = D_{\mathrm{dR}}(\cdot)/\mathrm{Fil}^0$). The second equality implies that $H^1_{\mathrm{exp}}(\mathbf{Q}_p, V/\mathscr{F}^2)$ vanishes (cf. Corollary 3.8.4 of [BK90]), and since $\mathscr{F}^2V$ is isomorphic to the Kummer dual of $V/\mathscr{F}^2$, this in turn implies $H^1(\mathbf{Q}_p, \mathscr{F}^2V) = H^1_{\mathrm{geo}}(\mathbf{Q}_p, \mathscr{F}^2V)$ (cf. Proposition 3.8 of [BK90]). As $H^1_{\mathrm{fin}}(\mathbf{Q}_p, V) = H^1_{\mathrm{geo}}(\mathbf{Q}_p, V)$ by Lemma 3.5, one deduces that $H^1_{\mathrm{fin}}(\mathbf{Q}_p, V)$ contains the balanced subspace $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V)$. On the other hand, Equation (150) shows that the inclusion $\mathscr{F}^2V \hookrightarrow V$ induces an isomorphism between the tangent space of $\mathscr{F}^2V$ and that of V. It follows that $H^1_{\mathrm{exp}}(\mathbf{Q}_p, V)$ is contained in the image of $H^1_{\mathrm{exp}}(\mathbf{Q}_p, \mathscr{F}^2V)$, hence a fortiori in the balanced subspace $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V)$. Since $H^1_{\mathrm{exp}}(\mathbf{Q}_p, V) = H^1_{\mathrm{fin}}(\mathbf{Q}_p, V)$ by Lemma 3.5, this concludes the proof of the first statement. The second statement follows from the first and Lemma 3.5. □

7.3. The three-variable big logarithms. — Let $w = (k, l, m)$ be a classical triple in Σ. If $w \in \Sigma_{\mathrm{bal}}$ is *balanced*, then the differential $\eta^\alpha_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_l} \otimes \omega_{\boldsymbol{h}_m}$ belongs to $\mathrm{Fil}^0 V^*_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ by Equation (53). In this case denote by

$$\log_p : H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)) \cong V_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)/\mathrm{Fil}^0$$

the inverse of the Bloch–Kato exponential (cf. Lemma 7.2), and define

$$\log_p(\cdot)_f = \log_p(\cdot)\big(\eta^\alpha_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_l} \otimes \omega_{\boldsymbol{h}_m}\big) : H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)) \longrightarrow L$$

to be the composition of $\log_p$ with evaluation on $\eta^\alpha_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_l} \otimes \omega_{\boldsymbol{h}_m}$. Here one identifies $V_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)/\mathrm{Fil}^0$ with the dual of $\mathrm{Fil}^0 V^*_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ under the product of the perfect dualities $\langle \cdot, \cdot \rangle_{\boldsymbol{\xi}_u}$ introduced in Equation (31), for $\boldsymbol{\xi}_u = \boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m$.

If w belongs to $\Sigma_{\boldsymbol{f}}$ denote by

$$\exp^*_p : H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)) \longrightarrow \mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$$

the Bloch–Kato dual exponential map, and by

$$\exp^*_p(\cdot)_f = \exp^*_p(\cdot)\big(\eta^\alpha_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_l} \otimes \omega_{\boldsymbol{h}_m}\big) : H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)) \longrightarrow L$$

its composition with evaluation at $\eta^\alpha_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_l} \otimes \omega_{\boldsymbol{h}_m}$. As above, here one identifies $\mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ with a subspace of the dual of $V^*_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ under the tensor product of the pairings $\langle \cdot, \cdot \rangle_{\boldsymbol{\xi}_u}$ defined in (31) and (128). (If either l or m is equal to

1, the definitions of $V_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ and $V^*_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ given in Equations (50) and (51) are understood in light of the conventions of Section 5, cf. Equation (127).)

To ease notation set $\alpha_k = a_p(k), \beta_k = \chi_{\boldsymbol{f}}(p)p^{k-1}/\alpha_k, \alpha_l = b_p(l)$ et cetera. Recall that for each classical triple $w = (k, l, m)$ in Σ one writes $c_w = (k + l + m - 2)/2$ (which belongs to $\mathbf{N}$ by Assumption 1.2).

Proposition 7.3. — *There is a unique morphism of $\mathscr{O}_{\boldsymbol{fgh}}$-modules*

$$\mathscr{L}_{\boldsymbol{f}} = \mathscr{L}og(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) : H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}$$

such that, for all $w = (k, l, m) \in \Sigma$ with $\alpha_k\beta_l\beta_m \neq p^{c_w}$ and $\mathfrak{Z} \in H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$

$$\mathscr{L}_{\boldsymbol{f}}(\mathfrak{Z})(w) = (p-1)\alpha_k \cdot \frac{\left(1 - \frac{\beta_k\alpha_l\alpha_m}{p^{c_w}}\right)}{\left(1 - \frac{\alpha_k\beta_l\beta_m}{p^{c_w}}\right)} \cdot \begin{cases} \frac{(-1)^{c_w-k}}{(c_w-k)!} \log_p(\mathfrak{Z}_w)_f & \text{if} \quad w \in \Sigma_{\mathrm{bal}} \\ (k - c_w - 1)! \exp^*_p(\mathfrak{Z}_w)_f & \text{if} \quad w \in \Sigma_{\boldsymbol{f}} \end{cases},$$

where $\mathfrak{Z}_w = \rho_{w*}(\mathfrak{Z})$. Moreover $\mathscr{L}_{\boldsymbol{f}}$ factors through

$$p_{f*} : H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) \to H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f).$$

Proof. — Set $\bar{M}_f = \bar{M}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f$, $V = V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ and $V_f = V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f$. Let

$$\vartheta : \bar{\mathscr{O}}_{\boldsymbol{fgh}} \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}$$

be the surjective morphism of L-algebras which sends the analytic function $F(\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \boldsymbol{j})$ to its restriction $F(\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, (\boldsymbol{k} - \boldsymbol{l} - \boldsymbol{m})/2)$ to the hyperplane defined by the equation $2\boldsymbol{j} = \boldsymbol{k} - \boldsymbol{l} - \boldsymbol{m}$. (Here we implicitly shrink the discs $U_{\boldsymbol{f}}, U_{\boldsymbol{g}}$ and $U_{\boldsymbol{h}}$ if necessary, in order to guarantee that $(\boldsymbol{k} - \boldsymbol{l} - \boldsymbol{m})/2$ takes values in the disc U_{cyc} fixed in Section 7.1.) Unwinding the definitions one finds that ϑ induces an isomorphism of $\mathscr{O}_{\boldsymbol{fgh}}[G_{\mathbf{Q}_p}]$-modules (denoted by the same symbol)

$$\vartheta : \bar{M}_f \otimes_\vartheta \mathscr{O}_{\boldsymbol{fgh}} \cong V_f. \tag{151}$$

We claim that this map entails an isomorphism

$$\vartheta_* : H^1(\mathbf{Q}_p, \bar{M}_f) \otimes_\vartheta \mathscr{O}_{\boldsymbol{fgh}} \cong H^1(\mathbf{Q}_p, V_f). \tag{152}$$

Granting this, one can define $\mathscr{L}_{\boldsymbol{f}}$ by the composition

$$\begin{aligned} \mathscr{L}_{\boldsymbol{f}} : H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) &\xrightarrow{p_{f*}} H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f) \\ &\xrightarrow{\vartheta_*^{-1}} H^1(\mathbf{Q}_p, \bar{M}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f) \otimes_\vartheta \mathscr{O}_{\boldsymbol{fgh}} \xrightarrow{\bar{\mathscr{L}}_{\boldsymbol{f}} \otimes \mathrm{id}} \mathscr{O}_{\boldsymbol{fgh}}, \end{aligned}$$

where $\bar{\mathscr{L}}_{\boldsymbol{f}}$ is the four-variable $\boldsymbol{f}$-big logarithm defined in Equation (144). Unravelling the definitions, one checks that the interpolation property satisfied by $\mathscr{L}_{\boldsymbol{f}}$ is a direct consequence of Proposition 7.1. It then remains to prove the claim (152).

As $\bar{M}_f$ is a free module over the domain $\bar{\mathscr{O}}_{\boldsymbol{fgh}}$, the claim (152) is equivalent to the vanishing of the $(2\boldsymbol{j} - \boldsymbol{k} + \boldsymbol{l} + \boldsymbol{m})$-torsion submodule of $H^2(\mathbf{Q}_p, \bar{M}_f)$. Set

$$\bar{\Lambda} = \Lambda_{\boldsymbol{fgh}} \hat{\otimes}_{\mathbf{Z}_p} \Lambda_{\mathrm{cyc}},$$

where Λ_{cyc} is the $\mathbf{Z}_p$-module of functions in $\mathscr{O}_{\mathrm{cyc}}$ bounded by one. The $\mathscr{O}$-algebra $\bar{\Lambda}_{\boldsymbol{fgh}}$ is isomorphic to a power series ring in four variables with coefficients in $\mathscr{O}$. In particular, it is a regular local complete Noetherian ring with finite residue field (hence a UFD). Write $\bar{M}_f = \bar{\mathtt{M}}_f[1/p]$ for a $\bar{\Lambda}[G_{\mathbf{Q}_p}]$-module $\bar{\mathtt{M}}_f$ free of finite rank over

$\bar{\Lambda}$. For every discrete or compact $\bar{\Lambda}$-module $\cdot$ write $\mathscr{D}(\cdot) = \mathrm{Hom}_{\mathrm{cont}}(\cdot, \mathbf{Q}_p/\mathbf{Z}_p)$ for its Pontrjagin dual. According to the local Tate duality and the Pontrjagin duality

$$H^2(\mathbf{Q}_p, \bar{M}_f)[2\boldsymbol{j} - \boldsymbol{k} + \boldsymbol{l} + \boldsymbol{m}] = \mathscr{D}\Big(\mathscr{D}(\bar{\mathtt{M}}_f(-1))^{G_{\mathbf{Q}_p}}/(2\boldsymbol{j} - \boldsymbol{k} + \boldsymbol{l} + \boldsymbol{m})\Big)[1/p]. \tag{153}$$

Let Frob_p be the arithmetic Frobenius in $G_p^{\mathrm{nr}} = \mathrm{Gal}(\mathbf{Q}_p^{\mathrm{nr}}/\mathbf{Q}_p)$ and let γ be a topological generator of $G_p^{\mathrm{tr}} = \mathrm{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p)$ (recall that p is odd). By construction (after identifying $G_{\mathbf{Q}_p}^{\mathrm{ab}}$ with the product of G_p^{nr} and G_p^{tr}) Frob_p acts on $\bar{\mathtt{M}}_f$ as multiplication by $\Psi_{\boldsymbol{o}} = \Psi(\mathrm{Frob}_p)$ and γ acts on $\bar{\mathtt{M}}_f(-1)$ as multiplication by the inverse of $\Gamma_{\boldsymbol{o}} = \omega_o^{1+j_o} \cdot \gamma_o^{1+\boldsymbol{j}}$, where $\omega_o = \omega_{\mathrm{cyc}}(\gamma)$ and $\gamma_o = \kappa_{\mathrm{cyc}}(\gamma)$. This yields

$$\mathscr{D}(\bar{\mathtt{M}}_f(-1))^{G_{\mathbf{Q}_p}}/(2\boldsymbol{j} - \boldsymbol{k} + \boldsymbol{l} + \boldsymbol{m}) \cong \bigoplus_{i=0}^{a} \mathscr{D}\bigg(\frac{\bar{\Lambda}}{(\Psi_{\boldsymbol{o}} - 1, \Gamma_{\boldsymbol{o}} - 1)}[2\boldsymbol{j} - \boldsymbol{k} + \boldsymbol{l} + \boldsymbol{m}]\bigg)$$

for some positive integer a (cf. Equation (137)). We prove that the module

$$\frac{\bar{\Lambda}}{(\Psi_{\boldsymbol{o}} - 1, \Gamma_{\boldsymbol{o}} - 1)}[2\boldsymbol{j} - \boldsymbol{k} + \boldsymbol{l} + \boldsymbol{m}]$$

is killed by a power of p, which together with Equation (153) proves the claim (152). If $j_o \neq -1$, the function $\Gamma_{\boldsymbol{o}} - 1$ is a unit in $\Lambda_{\mathrm{cyc}}[1/p]$, hence $\bar{\Lambda}/(\Psi_{\boldsymbol{o}} - 1, \Gamma_{\boldsymbol{o}} - 1)$ is killed by a power of p. Assume then $j_o = -1$ and let $F = F(\boldsymbol{w}, \boldsymbol{j})$ be an element of $\bar{\Lambda}$ whose image in $\bar{\Lambda}/(\Psi_{\boldsymbol{o}} - 1, \Gamma_{\boldsymbol{o}} - 1)$ is killed by $2\boldsymbol{j} - \boldsymbol{k} + \boldsymbol{l} + \boldsymbol{m}$. This implies that

$$(\boldsymbol{l} + \boldsymbol{m} - \boldsymbol{k} - 2) \cdot F(\boldsymbol{w}, -1) = (\Psi_o(\boldsymbol{w}) - 1) \cdot G(\boldsymbol{w})$$

for some $G(\boldsymbol{w})$ in $\Lambda_{\boldsymbol{fgh}}$. As $j_o = -1$ there is a classical triple $w = (k, l, m) \in \Sigma$ such that $l + m - k - 2 = 0$ and such that p does not divide the conductor of $\boldsymbol{f}_k, \boldsymbol{g}_l$ and $\boldsymbol{h}_m$. According to the Ramanujan–Petersson conjecture the inverse of $\Psi_o(w)$ has complex absolute value $\sqrt{p}$ for every such w (cf. Equation (136)). As a consequence $\boldsymbol{l} + \boldsymbol{m} - \boldsymbol{k} - 2$ is not an irreducible factor of $\Psi_{\boldsymbol{o}} - 1$, hence the latter divides $F(\boldsymbol{w}, -1)$ by the previous equation. This proves that F belongs to the ideal generated by $\Psi_{\boldsymbol{o}} - 1$ and $\boldsymbol{j} + 1$. As $(\Gamma_{\boldsymbol{o}} - 1)/(1 + \boldsymbol{j})$ is a unit in $\Lambda_{\mathrm{cyc}}[1/p]$, it follows that $p^{N(\gamma_o)} \cdot F$ maps to zero in $\bar{\Lambda}/(\Psi_{\boldsymbol{o}} - 1, \Gamma_{\boldsymbol{o}} - 1)$ for a non-negative integer $N(\gamma_o)$ independent of F, as was to be shown. $\square$

We call $\mathscr{L}_{\boldsymbol{f}}$ the *three variable $\boldsymbol{f}$-big logarithm.* Mutatis mutandis, for $\boldsymbol{a} = \boldsymbol{g}, \boldsymbol{h}$ one defines *$\boldsymbol{a}$-big logarithms*

$$\mathscr{L}_{\boldsymbol{a}} : H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) \longrightarrow \mathscr{O}_{\boldsymbol{fgh}},$$

which factor through $p_{a*} : H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) \longrightarrow H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_a)$ and satisfy similar interpolation properties.

8. Proof of Theorem A

This section proves Theorem A stated in the Introduction.

8.1. Construction of $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$. — Fix a nonnegative integer $\imath \geqslant 1$, which will be made sufficiently large below. For $\boldsymbol{\xi}=\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}$ and $\cdot=\emptyset,\prime$ set $\mathcal{A}^{\cdot}_{\boldsymbol{\xi}}=\mathcal{A}^{\cdot}_{U_{\boldsymbol{\xi}},\imath}$, $\boldsymbol{\mathcal{A}}^{\cdot}_{\boldsymbol{\xi}}=\boldsymbol{\mathcal{A}}^{\cdot}_{U_{\boldsymbol{\xi}},\imath}$ $\mathcal{D}^{\cdot}_{\boldsymbol{\xi}}=\mathcal{D}^{\cdot}_{U_{\boldsymbol{f}},\imath}$ and $\boldsymbol{\mathcal{D}}^{\cdot}_{\boldsymbol{\xi}}=\boldsymbol{\mathcal{D}}^{\cdot}_{U_{\boldsymbol{\xi}},\imath}$ (cf. Section 4 for the relevant definitions). Similarly, for any $u\in U_{\boldsymbol{\xi}}\cap\mathbf{Z}$, set $\mathcal{A}^{\cdot}_{u}=\mathcal{A}^{\cdot}_{u,\imath}$, $\mathcal{D}^{\cdot}_{u}=\mathcal{D}^{\cdot}_{u,\imath}$, $\boldsymbol{\mathcal{A}}^{\cdot}_{u}=\boldsymbol{\mathcal{A}}^{\cdot}_{u,\imath}$ and $\boldsymbol{\mathcal{D}}^{\cdot}_{u}=\boldsymbol{\mathcal{D}}^{\cdot}_{u,\imath}$.

Set

$$(\mathsf{T}\times\mathsf{T})_0=\big\{(t_1,t_2)\in\mathsf{T}\times\mathsf{T}\mid \det(t_1,t_2)\in\mathbf{Z}_p^*\big\},$$

where $\det((x_1,x_2),(y_1,y_2))=x_1y_2-x_2y_1$. Let $(\mathsf{T}\times\mathsf{T})^0$ be the complement of $(\mathsf{T}\times\mathsf{T})_0$ in $\mathsf{T}\times\mathsf{T}$. Note that $(\mathsf{T}\times\mathsf{T})_0$ and $(\mathsf{T}\times\mathsf{T})^0$ are open compact subsets of $\mathsf{T}\times\mathsf{T}$, preserved by the diagonal action of $\Gamma_0(p\mathbf{Z}_p)$. Identify $\mathcal{A}_{\boldsymbol{g}}\hat{\otimes}\mathcal{A}_{\boldsymbol{h}}=\mathcal{A}_{\boldsymbol{g}}\hat{\otimes}_{\mathscr{O}}\mathcal{A}_{\boldsymbol{h}}$ with a space of locally analytic functions on $\mathsf{T}\times\mathsf{T}$, homogeneous of weights $\kappa_{\boldsymbol{g}}=\kappa_{U_{\boldsymbol{g}}}$ and $\kappa_{\boldsymbol{h}}=\kappa_{U_{\boldsymbol{h}}}$ in the first and second variable respectively. The orthonormal basis of $\mathcal{A}_{\boldsymbol{g}}\hat{\otimes}\mathcal{A}_{\boldsymbol{h}}$ arising from Remark 4.1 gives a decomposition of $\Gamma_0(p\mathbf{Z}_p)$-modules

$$\mathcal{A}_{\boldsymbol{g}}\hat{\otimes}\mathcal{A}_{\boldsymbol{h}}=(\mathcal{A}_{\boldsymbol{g}}\hat{\otimes}\mathcal{A}_{\boldsymbol{h}})_0\oplus(\mathcal{A}_{\boldsymbol{g}}\hat{\otimes}\mathcal{A}_{\boldsymbol{h}})^0,$$

where $(\mathcal{A}_{\boldsymbol{g}}\hat{\otimes}\mathcal{A}_{\boldsymbol{h}})_0$ and $(\mathcal{A}_{\boldsymbol{g}}\hat{\otimes}\mathcal{A}_{\boldsymbol{h}})^0$ consist in locally analytic functions supported on $(\mathsf{T}\times\mathsf{T})_0$ and $(\mathsf{T}\times\mathsf{T})^0$ respectively. Let $\Lambda_{\boldsymbol{fgh}}=\Lambda_{\boldsymbol{f}}\hat{\otimes}_{\mathscr{O}}\Lambda_{\boldsymbol{g}}\hat{\otimes}_{\mathscr{O}}\Lambda_{\boldsymbol{h}}$ and define the characters $\kappa^*_{\boldsymbol{f}}:\mathbf{Z}_p^*\to\Lambda^*_{\boldsymbol{fgh}}$ and $\kappa^*_{\boldsymbol{fgh}}:\mathbf{Z}_p^*\to\Lambda^*_{\boldsymbol{fgh}}$ by

$$\kappa^*_{\boldsymbol{f}}(u)=\omega(u)^{(l_o+m_o-k_o-2)/2}\cdot\langle u\rangle^{(\boldsymbol{l}+\boldsymbol{m}-\boldsymbol{k}-2)/2}$$
$$\text{and}\qquad \kappa^*_{\boldsymbol{fgh}}(u)=\omega(u)^{(k_o+l_o+m_o-6)/2}\cdot\langle u\rangle^{(\boldsymbol{k}+\boldsymbol{l}+\boldsymbol{m}-6)/2}$$

for every $u=\omega(u)\cdot\langle u\rangle$ in $\mathbf{Z}_p^*=\mathbf{F}_p^*\times 1+p\mathbf{Z}_p$. (Recall by the discussion preceding Equation (97) that $\kappa_{\boldsymbol{f}}(u)$ is equal to $\omega(u)^{k_o-2}\cdot\langle u\rangle^{\boldsymbol{k}-2}$, and similarly for $\kappa_{\boldsymbol{g}}$ and $\kappa_{\boldsymbol{h}}$.) Here one uses Assumption 1.2, which guarantees that the quantity $k_o+l_o+m_o$ is an even integer. Define similarly $\kappa^*_{\boldsymbol{g}}$ and $\kappa^*_{\boldsymbol{h}}$, so that $\kappa^*_{\boldsymbol{fgh}}=\kappa^*_{\boldsymbol{f}}+\kappa^*_{\boldsymbol{g}}+\kappa^*_{\boldsymbol{h}}$ (again with additive notation). After noting that $\det:\mathbf{Z}_p^2\times\mathbf{Z}_p^2\to\mathbf{Z}_p$ maps $\mathsf{T}'\times\mathsf{T}$ to $\mathbf{Z}_p^*$, let

$$\mathbf{Det}=\mathbf{Det}^{\boldsymbol{fgh}}_{N,p}:\mathsf{T}'\times\mathsf{T}\times\mathsf{T}\longrightarrow\Lambda_{\boldsymbol{fgh}}$$

be the function which vanishes identically on $\mathsf{T}'\times(\mathsf{T}\times\mathsf{T})^0$ and on an element $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ in $\mathsf{T}'\times(\mathsf{T}\times\mathsf{T})_0$ takes the value

$$\mathbf{Det}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})=\det(\boldsymbol{x},\boldsymbol{y})^{\kappa^*_{\boldsymbol{h}}}\cdot\det(\boldsymbol{x},\boldsymbol{z})^{\kappa^*_{\boldsymbol{g}}}\cdot\det(\boldsymbol{y},\boldsymbol{z})^{\kappa^*_{\boldsymbol{f}}}.$$

Because $\kappa^*_{\boldsymbol{g}}+\kappa^*_{\boldsymbol{h}}=\kappa_{\boldsymbol{f}}$, one has $\mathbf{Det}(u\cdot\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})=\kappa_{\boldsymbol{f}}(u)\cdot\mathbf{Det}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$ for every $u\in\mathbf{Z}_p^*$, hence for $\imath$ big enough $\mathbf{Det}(\boldsymbol{x},\boldsymbol{y}_o,\boldsymbol{z}_o)$ belongs to $\mathcal{A}'_{\boldsymbol{f}}$ for every $(\boldsymbol{y}_o,\boldsymbol{z}_o)\in\mathsf{T}\times\mathsf{T}$. Similarly $\mathbf{Det}(\boldsymbol{x}_o,\boldsymbol{y},\boldsymbol{z}_o)\in\mathcal{A}_{\boldsymbol{g}}$ and $\mathbf{Det}(\boldsymbol{x}_o,\boldsymbol{y}_o,\boldsymbol{z})\in\mathcal{A}_{\boldsymbol{h}}$ for every $(\boldsymbol{x}_o,\boldsymbol{z}_o)\in\mathsf{T}'\times\mathsf{T}$ and $(\boldsymbol{x}_o,\boldsymbol{y}_o)\in\mathsf{T}'\times\mathsf{T}$ respectively. Moreover

$$\mathbf{Det}(\boldsymbol{x}\cdot\gamma,\boldsymbol{y}\cdot\gamma,\boldsymbol{z}\cdot\gamma)=\det(\gamma)^{\kappa^*_{\boldsymbol{fgh}}}\cdot\mathbf{Det}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$$

for every $\gamma\in\Gamma_0(p\mathbf{Z}_p)$. As a consequence $\mathbf{Det}$ can be identified with an element of $\mathcal{A}'_{\boldsymbol{f}}\hat{\otimes}\mathcal{A}_{\boldsymbol{g}}\hat{\otimes}\mathcal{A}_{\boldsymbol{h}}(-\kappa^*_{\boldsymbol{fgh}})$, which is invariant under the diagonal action of $\Gamma_0(p\mathbf{Z}_p)$ (cf. Section 4.2). Since the $\Gamma_0(p\mathbf{Z}_p)$-representation $\mathcal{A}'_{\boldsymbol{f}}\hat{\otimes}\mathcal{A}_{\boldsymbol{g}}\hat{\otimes}\mathcal{A}_{\boldsymbol{h}}$ corresponds to the pro-sheaf $\boldsymbol{\mathcal{A}}'_{\boldsymbol{f}}\otimes\boldsymbol{\mathcal{A}}_{\boldsymbol{g}}\otimes\boldsymbol{\mathcal{A}}_{\boldsymbol{h}}$ on $Y=Y_1(N,p)$ under the functor $\cdot^{\text{ét}}$ (cf. loco citato) this yields

$$\mathbf{Det}^{\boldsymbol{fgh}}_{N,p}\in H^0_{\text{ét}}(Y,\boldsymbol{\mathcal{A}}'_{\boldsymbol{f}}\otimes\boldsymbol{\mathcal{A}}_{\boldsymbol{g}}\otimes\boldsymbol{\mathcal{A}}_{\boldsymbol{h}}(-\kappa^*_{\boldsymbol{fgh}})).\tag{154}$$

Let $\Gamma = \Gamma_1(N,p)$ and let $d : Y \longrightarrow Y^3$ be the diagonal embedding. Define

$$\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = \frac{1}{a_p(\boldsymbol{k})} \cdot \kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})^o \in H^1(\mathbf{Q}, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})), \tag{155}$$

where

$$\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})^o = \mathrm{AJ}_{\text{ét}}^{\boldsymbol{fgh}}\big(\mathbf{Det}_{N,p}^{\boldsymbol{fgh}}\big)$$

is the image of the *big* invariant $\mathbf{Det}_{N,p}^{\boldsymbol{fgh}}$ under the *big Abel–Jacobi map* $\mathrm{AJ}_{\text{ét}}^{\boldsymbol{fgh}}$ defined by the following composition.

$$\begin{aligned}
H^0_{\text{ét}}(Y, \boldsymbol{\mathcal{A}}'_{\boldsymbol{f}} \otimes \boldsymbol{\mathcal{A}}_{\boldsymbol{g}} \otimes \boldsymbol{\mathcal{A}}_{\boldsymbol{h}}(-\kappa^*_{\boldsymbol{fgh}})) &\xrightarrow{d_*} H^4_{\text{ét}}(Y^3, \boldsymbol{\mathcal{A}}'_{\boldsymbol{f}} \boxtimes \boldsymbol{\mathcal{A}}_{\boldsymbol{g}} \boxtimes \boldsymbol{\mathcal{A}}_{\boldsymbol{h}}(-\kappa^*_{\boldsymbol{fgh}}) \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(2)) \\
&\xrightarrow{\mathtt{HS}} H^1\big(\mathbf{Q}, H^3_{\text{ét}}\big(Y^3_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}'_{\boldsymbol{f}} \boxtimes \boldsymbol{\mathcal{A}}_{\boldsymbol{g}} \boxtimes \boldsymbol{\mathcal{A}}_{\boldsymbol{h}}\big)(2+\boldsymbol{\kappa}^*_{\boldsymbol{fgh}})\big) \\
&\xrightarrow{\mathtt{K}} H^1\big(\mathbf{Q}, H^1(\Gamma, A'_{\boldsymbol{f}})\hat{\otimes}_L H^1(\Gamma, A_{\boldsymbol{g}})\hat{\otimes}_L H^1(\Gamma, A_{\boldsymbol{h}})(2+\boldsymbol{\kappa}^*_{\boldsymbol{fgh}})\big) \\
&\xrightarrow{(w_p\otimes\mathrm{id}\otimes\mathrm{id})_*} H^1\big(\mathbf{Q}, H^1(\Gamma, A_{\boldsymbol{f}})\hat{\otimes}_L H^1(\Gamma, A_{\boldsymbol{g}})\hat{\otimes}_L H^1(\Gamma, A_{\boldsymbol{h}})(2+\boldsymbol{\kappa}^*_{\boldsymbol{fgh}})\big) \\
&\xrightarrow{\mathtt{s}_{\boldsymbol{fgh}*}} H^1\big(\mathbf{Q}, H^1(\Gamma, D'_{\boldsymbol{f}})^{\leqslant 0}\hat{\otimes}_L H^1(\Gamma, D'_{\boldsymbol{g}})^{\leqslant 0}\hat{\otimes}_L H^1(\Gamma, D'_{\boldsymbol{h}})^{\leqslant 0}(2-\boldsymbol{\kappa}^*_{\boldsymbol{fgh}})\big) \\
&\xrightarrow{\mathrm{pr}_{\boldsymbol{fgh}*}} H^1\big(\mathbf{Q}, V(\boldsymbol{f})\hat{\otimes}_L V(\boldsymbol{g})\hat{\otimes}_L V(\boldsymbol{h})(-1-\boldsymbol{\kappa}^*_{\boldsymbol{fgh}})\big) = H^1(\mathbf{Q}, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})).
\end{aligned} \tag{156}$$

Here $\boldsymbol{\kappa}^*_{\boldsymbol{fgh}} : G_{\mathbf{Q}} \to \Lambda^*_{\boldsymbol{fgh}}$ denotes the composition of $\kappa^*_{\boldsymbol{fgh}}$ with the p-adic cyclotomic character χ_{cyc}. The first arrow is the push-forward by the diagonal embedding d. The morphism HS arises from the Hochschild–Serre spectral sequence and Equation (80). (Note that $H^4_{\text{ét}}(Y^3_{\bar{\mathbf{Q}}}, \mathscr{F})$ vanishes for every pro-sheaf $\mathscr{F} \in \mathbf{S}(Y^3_{\text{ét}})$, as follows easily from Equation (75) and [**Mil80**, Chapter VI, Theorem 7.2].) The map K comes from the Künneth decomposition and the projection in Equation (79). The morphism $(w_p \otimes \mathrm{id} \otimes \mathrm{id})_*$ is the one induced in cohomology by the $G_{\mathbf{Q}}$-equivariant Atkin–Lehner operator $w_p : H^1(\Gamma, \mathcal{A}'_{\boldsymbol{f}}) \to H^1(\Gamma, \mathcal{A}_{\boldsymbol{f}})$ (cf. Sections 4.1.2 and 4.2). The penultimate arrow $\mathtt{s}_{\boldsymbol{fgh}*}$ is induced by the tensor product of the morphisms of $G_{\mathbf{Q}}$-modules

$$H^1(\Gamma, A_{\boldsymbol{a}}) \twoheadrightarrow H^1(\Gamma, A_{\boldsymbol{a}})^{\leqslant 0} \xrightarrow{\mathtt{s}_{\boldsymbol{a}}} H^1(\Gamma, D'_{\boldsymbol{a}})^{\leqslant 0}(-\boldsymbol{\kappa}_{U_{\boldsymbol{a}}})$$

for $\boldsymbol{a} = \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}$, where the first map is the projection to the slope $\leqslant 0$ part and $\mathtt{s}_{\boldsymbol{a}} = \mathtt{s}_{U_{\boldsymbol{a}},0}$ is defined in Equation (83). Finally $\mathrm{pr}_{\boldsymbol{fgh}}$ denotes the tensor product of the $G_{\mathbf{Q}}$-equivariant projections $\mathrm{pr}_{\boldsymbol{a}}$ defined in Equation (105).

8.2. Balanced specialisations of $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$. — Let $w = (k,l,m) \in \Sigma_{\text{bal}}$ be a *balanced* triple of classical weights, let $\boldsymbol{r} = (k-2, l-2, m-2) = w - \mathbf{2}$, and let $r = (r_1 + r_2 + r_3)/2$. Recall the diagonal classes

$$\tilde{\kappa}_{Np,\boldsymbol{r}} \in H^1_{\text{geo}}(\mathbf{Q}, W_{Np,\boldsymbol{r}}) \quad \text{and} \quad \kappa_{Np,\boldsymbol{r}} = \mathtt{s}_{\boldsymbol{r}*}(\tilde{\kappa}_{Np,\boldsymbol{r}}) \in H^1_{\text{geo}}(\mathbf{Q}, V_{Np,\boldsymbol{r}})$$

introduced in Equations (43) and (46), and define the *twisted diagonal class*

$$\kappa^\dagger(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m) = \mathrm{pr}_{\boldsymbol{f}_k\boldsymbol{g}_l\boldsymbol{h}_m*}\big(\mathtt{s}_{\boldsymbol{r}*}\left((w'_p \otimes \mathrm{id} \otimes \mathrm{id})_*\big(\tilde{\kappa}_{Np,\boldsymbol{r}}\big)\right)\big) \in H^1_{\text{geo}}(\mathbf{Q}, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)). \tag{157}$$

Here $\mathrm{pr}_{\boldsymbol{f}_k\boldsymbol{g}_l\boldsymbol{h}_m}$ is the projection defined in Equation (48) and

$$(w'_p \otimes \mathrm{id} \otimes \mathrm{id})_* : H^1(\mathbf{Q}(\mu_p), W_{Np,\boldsymbol{r}}) \longrightarrow H^1(\mathbf{Q}(\mu_p), W_{Np,\boldsymbol{r}})$$

is the map induced by the dual Atkin–Lehner operator

$$w_p' : H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{S}_{r_1}) \cong H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{S}_{r_1})$$

(cf. Section 2.3.1) and the Künneth decomposition on $W_{Np,\boldsymbol{r}}$. A priori the class $\kappa^\dagger(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ then lives in the geometric subgroup of $H^1(\mathbf{Q}(\mu_p), V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m))$. On the other hand the forms $\boldsymbol{f}_k$, $\boldsymbol{g}_l$ and $\boldsymbol{h}_m$ have level $\Gamma_1(N,p) = \Gamma_1(N) \cap \Gamma_0(p)$, hence the cohomology class $\tilde{\kappa}_{Np,\boldsymbol{r}}$ is in the image of the map induced in $G_{\mathbf{Q}}$-cohomology by the pull-back $H^3_{\text{ét}}(Y_1(N,p)^3_{\bar{\mathbf{Q}}}, \mathscr{S}_{[\boldsymbol{r}]})(c_w) \longrightarrow H^3_{\text{ét}}(Y_1(Np)^3_{\bar{\mathbf{Q}}}, \mathscr{S}_{[\boldsymbol{r}]})(c_w) = W_{Np,\boldsymbol{r}}$. Because the Atkin–Lehner operator w_p' acting on $H^1_{\text{ét}}(Y_1(N,p)_{\bar{\mathbf{Q}}}, \mathscr{S}_{k-2})$ is $G_{\mathbf{Q}}$-equivariant, this implies that $\kappa^\dagger(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ is fixed by the action of the Galois group of $\mathbf{Q}(\mu_p)$ over $\mathbf{Q}$, hence can naturally be viewed as a geometric class in $H^1(\mathbf{Q}, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m))$.

With the notations already introduced one has the following

Theorem 8.1. — *For each balanced triple $w = (k,l,m)$ in Σ_{bal} one has*

$$(p-1)\alpha_{\boldsymbol{f}_k} \cdot \rho_w\big(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})\big) = \left(1 - \frac{\alpha_{\boldsymbol{f}_k}\beta_{\boldsymbol{g}_l}\beta_{\boldsymbol{h}_m}}{p^{r+2}}\right) \cdot \kappa^\dagger(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m).$$

Before giving the proof of Theorem 8.1 we deduce the following

Corollary 8.2. — *$\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ lies in the balanced Selmer group $H^1_{\text{bal}}(\mathbf{Q}, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$.*

Proof. — By definition one has to prove that the class

$$\text{res}_{\mathscr{F},p}(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) \in H^1(\mathbf{Q}, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})/\mathscr{F}^2 V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$$

is zero, where $\text{res}_{\mathscr{F},p}$ is the composition of the restriction at p and the map induced by $V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \twoheadrightarrow V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})/\mathscr{F}^2$. According to Proposition 3.2 for every balanced triple $w = (k,l,m)$ in Σ_{bal} one has

$$\text{res}_p\big(\kappa^\dagger(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)\big) \in H^1_{\text{geo}}(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)).$$

Let Σ^o_{bal} be the set of (k,l,m) in Σ_{bal} such that p does not divide the conductors of $\boldsymbol{f}_k, \boldsymbol{g}_l$ and $\boldsymbol{h}_m$. One has

$$H^1_{\text{geo}}(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_k, \boldsymbol{h}_m)) = \ker\big(H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)) \to H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)/\mathscr{F}^2)\big)$$

and

$$\alpha_{\boldsymbol{f}_k}\beta_{\boldsymbol{g}_l}\beta_{\boldsymbol{h}_m} \neq p^{r+2}$$

for all $w = (k,l,m)$ in Σ^o_{bal} (by the Ramunajan–Petersson conjecture). The previous two equations and Theorem 8.1 imply that the class $\text{res}_{\mathscr{F},p}(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$ specialises to zero in $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)/\mathscr{F}^2)$ at every w in Σ^o_{bal}. Because Σ^o_{bal} is dense in $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}}$, to conclude the proof it is then sufficient to show that $H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})/\mathscr{F}^2)$ is $\mathscr{O}_{\boldsymbol{fgh}}$-torsion free (hence a submodule of a reflexive $\mathscr{O}_{\boldsymbol{fgh}}$-module), which implies that $\bigcap_{w \in \Sigma^o_{\text{bal}}}(\boldsymbol{k}-k, \boldsymbol{l}-l, \boldsymbol{m}-m) \cdot H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})/\mathscr{F}^2) = 0$. This is a consequence of the following claim. If $\wp \in \mathscr{O}_{\boldsymbol{fgh}}$ is irreducible and one sets $\mathscr{O}_\wp = \mathscr{O}_{\boldsymbol{fgh}}/(\wp)$, then

$$H^0(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})/\mathscr{F}^2 \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \mathscr{O}_\wp) = 0. \tag{158}$$

The rest of the proof is then devoted to the proof of this claim.

Section 7.2 shows that there is a short exact sequence of $G_{\mathbf{Q}_p(\mu_p)}$-modules

$$\mathscr{O}_\wp(\theta_{\boldsymbol{f}})^{\oplus a} \oplus \mathscr{O}_\wp(\theta_{\boldsymbol{g}})^{\oplus a} \oplus \mathscr{O}_\wp(\theta_{\boldsymbol{h}})^{\oplus a} \hookrightarrow V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})/\mathscr{F}^2 \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \mathscr{O}_\wp \twoheadrightarrow \mathscr{O}_\wp(\theta_{\boldsymbol{fgh}})^{\oplus a},$$

where a is a positive integer and the characters $\theta_\cdot : G_{\mathbf{Q}_p(\mu_p)} \to \mathscr{O}_\wp^*$ are defined by

$$\begin{aligned}\theta_{\boldsymbol{fgh}} &= \kappa_{\mathrm{cyc}}^{(4-\boldsymbol{k}-\boldsymbol{l}-\boldsymbol{m})/2} \cdot \check{a}_p(\boldsymbol{k}) \cdot \check{b}_p(\boldsymbol{l}) \cdot \check{c}_p(\boldsymbol{m}),\\ \theta_{\boldsymbol{f}} &= \kappa_{\mathrm{cyc}}^{(\boldsymbol{k}-\boldsymbol{l}-\boldsymbol{m}+2)/2} \cdot \chi_{\boldsymbol{f}} \cdot \check{b}_p(\boldsymbol{l}) \cdot \check{c}_p(\boldsymbol{m}) \cdot \check{a}_p(\boldsymbol{k})^{-1}\end{aligned}$$

and similarly for $\theta_{\boldsymbol{g}}$ and $\theta_{\boldsymbol{h}}$. Set $\wp_{\boldsymbol{fgh}} = 4-\boldsymbol{k}-\boldsymbol{l}-\boldsymbol{m}$, set $\wp_{\boldsymbol{f}} = \boldsymbol{k}-\boldsymbol{l}-\boldsymbol{m}+2$ and define similarly $\wp_{\boldsymbol{g}}$ and $\wp_{\boldsymbol{h}}$. Denote by $\wp_{\boldsymbol{a}}$ and $\theta_{\boldsymbol{a}}$ one of $\wp_\cdot$ and $\theta_\cdot$ respectively. If $\wp \cdot \mathscr{O}_{\boldsymbol{fgh}}$ is different from one of the ideals $\wp_{\boldsymbol{a}} \cdot \mathscr{O}_{\boldsymbol{fgh}}$, then $H^0(I_{\mathbf{Q}_p(\mu_p)}, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})/\mathscr{F}^2 \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \mathscr{O}_\wp)$ is trivial and (158) holds true. Assume now $\wp = u \cdot \wp_{\boldsymbol{a}}$ for a unit u in $\mathscr{O}_{\boldsymbol{fgh}}$, so that $\theta_{\boldsymbol{a}}$ is an unramified character of $G_{\mathbf{Q}_p(\mu_p)}$. According to the Ramanujan–Petersson conjecture one has

$$|\theta_{\boldsymbol{a}}(\mathrm{Frob}_p)(w)| = \sqrt{p}$$

for all $w \in \Sigma_{\mathrm{bal}}^o \cap V(\wp)$ (where $|\cdot|$ is the complex absolute value and $V(\wp)$ is the zero locus of $\wp$). Shrinking the discs $U_\cdot$ if necessary, we can assume that $\Sigma_{\mathrm{bal}}^o \cap V(\wp)$ is non-empty (otherwise $\wp$ would be a unit). The previous equation then implies that the characters $\theta_\cdot$ are non-trivial and (158) follows. □

Proof of Theorem 8.1. — According to [Mil06, Section II.7] for every $n, i \geqslant 1$ there is a trace isomorphism

$$\mathrm{Trace}_{Y^n} : H^{2n+3}_{\mathrm{\acute{e}t,c}}(Y^n, \mathscr{O}/\mathfrak{m}^i(n+1)) \cong \mathscr{O}/\mathfrak{m}^i.$$

(See Chapter II, Section 2 of loc. cit. for the definition of $H^\cdot_{\mathrm{\acute{e}t,c}}(Y^n,\cdot)$, denoted $H^\cdot_c(Y^n,\cdot)$ there.) For all finite smooth sheaves $\mathscr{F}$ of $\mathscr{O}/\mathfrak{m}^i$-modules on $Y^n_{\mathrm{\acute{e}t}}$, Trace_{Y^n} and the cup-product define perfect pairings

$$(159)\qquad (\cdot,\cdot)_{Y^n} = \mathrm{Trace}_{Y^n} \circ \cup : H^j_{\mathrm{\acute{e}t}}(Y^n, \mathscr{F}) \otimes_L H^{2n+3-j}_{\mathrm{\acute{e}t,c}}(Y^n, \mathscr{G}(n+1)) \longrightarrow \mathscr{O}/\mathfrak{m}^i,$$

where $\mathscr{G}$ is the dual of $\mathscr{F}$ (cf. Chapter II, Corollary 7.7 of [Mil06]). Denote by $\mathscr{F}_u^\cdot$ in $\mathbf{S}_f(Y_{\mathrm{\acute{e}t}})$ the sheaf associated to $\mathrm{Fil}_{i,j}\mathcal{A}^\cdot_{u,\imath}$ for $u \geqslant 0$ and fixed $j \geqslant i \geqslant 0$, and by $\mathscr{G}_u^\cdot$ the $\mathscr{O}/\mathfrak{m}^i$-dual of $\mathscr{F}_u^\cdot$. One has a Hecke equivariant diagram of adjoint morphisms, where the Hecke operators are defined by constructions similar to those of Section 2.3.

(160)
$$\begin{array}{ccccc} H^0_{\mathrm{\acute{e}t}}(Y, \mathscr{F}'_{r_1} \otimes \mathscr{F}_{r_2} \otimes \mathscr{F}_{r_3}(r)) & \times & H^5_{\mathrm{\acute{e}t,c}}(Y, \mathscr{G}'_{r_1} \otimes \mathscr{G}_{r_2} \otimes \mathscr{G}_{r_3}(2-r)) & \xrightarrow{(\cdot,\cdot)_Y} & \mathscr{O}/\mathfrak{m}^i \\ \Big\downarrow{\scriptstyle d_*} & & \Big\uparrow{\scriptstyle d^*} & & \Big\| \\ H^4_{\mathrm{\acute{e}t}}(Y^3, \mathscr{F}'_{r_1} \boxtimes \mathscr{F}_{r_2} \boxtimes \mathscr{F}_{r_3}(r+2)) & \times & H^5_{\mathrm{\acute{e}t,c}}(Y^3, \mathscr{G}'_{r_1} \boxtimes \mathscr{G}_{r_2} \boxtimes \mathscr{G}_{r_3}(2-r)) & \xrightarrow{(\cdot,\cdot)_{Y^3}} & \mathscr{O}/\mathfrak{m}^i \end{array}$$

Let $\mathcal{A}^\cdot_\cdot$ and $\boldsymbol{\mathcal{A}}^\cdot_\cdot$ be shorthands for $\mathcal{A}^\cdot_{\cdot,\imath}$ and $\boldsymbol{\mathcal{A}}^\cdot_{\cdot,\imath}$ respectively. Similarly as above, the orthonormal basis of $\mathcal{A}_u \hat{\otimes} \mathcal{A}_v$ arising from Remark 4.1 gives a decomposition of $\Gamma_0(p\mathbf{Z}_p)$-modules

$$\mathcal{A}_u \hat{\otimes} \mathcal{A}_v = (\mathcal{A}_u \hat{\otimes} \mathcal{A}_v)_0 \oplus (\mathcal{A}_u \hat{\otimes} \mathcal{A}_v)^0,$$

where $(\mathcal{A}_r\hat{\otimes}\mathcal{A}_s)_0$ (resp., $(\mathcal{A}_r\hat{\otimes}\mathcal{A}_s)^0$) can be identified with a space of locally analytic functions on $\mathsf{T}\times\mathsf{T}$ supported on $(\mathsf{T}\times\mathsf{T})_0$ (resp., $(\mathsf{T}\times\mathsf{T})^0$). This in turn induces similar decompositions

$$\mathscr{F}_u\otimes\mathscr{F}_v=(\mathscr{F}_u\otimes\mathscr{F}_v)_0\oplus(\mathscr{F}_u\otimes\mathscr{F}_v)^0 \quad\text{and}\quad \mathscr{G}_u\otimes\mathscr{G}_v=(\mathscr{G}_u\otimes\mathscr{G}_v)_0\oplus(\mathscr{G}_u\otimes\mathscr{G}_v)^0.$$

Let $t:Y_1(Np)\to Y_1(N,p)=Y$ be the natural projection. To ease notations, let $\mathtt{Det}\in H^0_{\text{ét}}(Y,\boldsymbol{\mathcal{A}}'_{r_1}\otimes\boldsymbol{\mathcal{A}}_{r_2}\otimes\boldsymbol{\mathcal{A}}_{r_3}(r))$ denote the image of $\mathtt{Det}^{\boldsymbol{r}}_{Np}$ under the composition of the push-forward t_* with the natural map

$$H^0_{\text{ét}}(Y,\mathscr{S}_{r_1}\otimes\mathscr{S}_{r_2}\otimes\mathscr{S}_{r_3}(r))\longrightarrow H^0_{\text{ét}}(Y,\boldsymbol{\mathcal{A}}'_{r_1}\otimes\boldsymbol{\mathcal{A}}_{r_2}\otimes\boldsymbol{\mathcal{A}}_{r_3}(r)).$$

For $j=j(i)$ large enough, let $\mathtt{D}=\mathtt{D}^{\boldsymbol{r}}_{i,j}\in H^0_{\text{ét}}(Y,\mathscr{F}'_{r_1}\otimes\mathscr{F}_{r_2}\otimes\mathscr{F}_{r_3}(r))$ be a representative of $\mathtt{Det}\,(\mathrm{mod}\ \mathfrak{m}^i)$ (cf. Section 4.2), and let $\mathtt{D}_0=\mathtt{D}^{\boldsymbol{r}}_{i,j,0}$ be its projection to the cohomology group $H^0_{\text{ét}}(Y,\mathscr{F}'_{r_1}\otimes(\mathscr{F}_{r_2}\otimes\mathscr{F}_{r_3})_0(r))$. By construction

$$(p-1)\cdot\rho_w(\mathbf{Det})=\lim_{\leftarrow i}\mathtt{D}^{\boldsymbol{r}}_{i,j,0}. \tag{161}$$

For all z in $H^5_{\text{ét},c}(Y^3,\mathscr{G}'_{r_1}\boxtimes\mathscr{G}_{r_2}\boxtimes\mathscr{G}_{r_3}(2-r))$ one has the equalities (cf. Equation (160))

$$\begin{aligned}\big(d_*(\mathtt{D}-\mathtt{D}_0),1\otimes U_p^{\otimes 2}(z)\big)_{Y^3}&=\big(\mathtt{D}-\mathtt{D}_0,d^*\big(1\otimes U_p^{\otimes 2}(z)\big)\big)_Y\\&=\big(\mathtt{D}-\mathtt{D}_0,\delta^*\big(1\otimes\delta^*\big(1\otimes U_p^{\otimes 2}(z)\big)\big)\big)_Y\\&=(\mathtt{D},\delta^*(1\otimes U_p(1\otimes\delta^*(z))))_Y\\&=p^{r-r_1}\cdot\big(\mathtt{D},\delta^*\big(U'_p\otimes 1\,(1\otimes\delta^*(z))\big)\big)_Y\\&=p^{r-r_1}\cdot\big(\mathtt{D},d^*\big(U'_p\otimes 1\otimes 1(z)\big)\big)_Y\\&=p^{r-r_1}\cdot\big(d_*(\mathtt{D}),U'_p\otimes 1\otimes 1(z)\big)_{Y^3},\end{aligned} \tag{162}$$

where $\delta:Y\to Y^2$ is the diagonal embedding. To justify the third equality one notes that

$$1\otimes\delta^*\circ 1\otimes U_p^{\otimes 2}-1\otimes U_p\circ 1\otimes\delta^*$$

(resp., $1\otimes U_p\circ 1\otimes\delta^*$) takes values in the submodule $H^5_{\text{ét},c}(Y,\mathscr{G}'_{r_1}\otimes(\mathscr{G}_{r_2}\otimes\mathscr{G}_{r_3})_0(2-r))$ (resp., in $H^5_{\text{ét},c}(Y,\mathscr{G}'_{r_1}\otimes(\mathscr{G}_{r_2}\otimes\mathscr{G}_{r_3})^0(2-r))$), and that $H^5_{\text{ét},c}(Y,\mathscr{G}'_{r_1}\otimes(\mathscr{G}_{r_2}\otimes\mathscr{G}_{r_3})_0(2-r))$ is orthogonal to $H^0_{\text{ét},c}(Y,\mathscr{F}'_{r_1}\otimes(\mathscr{F}_{r_2}\otimes\mathscr{F}_{r_3})^0(r))$. (Compare with the proof of Proposition 5.4 of [**GS20**].)

All the other equalities in Equation (162) but the fourth are standard. To prove the remaining equality, let $\pi:Y\to\operatorname{Spec}(\mathbf{Z}[1/Np])$ and $\boldsymbol{\pi}=\pi\times\pi:Y^2\to\operatorname{Spec}(\mathbf{Z}[1/Np])$ be the structural maps. Let $R\pi_!$ and $R\boldsymbol{\pi}_!$ be the δ-functors associated in [**FK88**, Chapter I, Definition 8.6] with the compactifiable maps π and $\boldsymbol{\pi}$, so that by definition $H^q_{\text{ét},c}(Y,\cdot)=H^q_{\text{ét},c}(\mathbf{Z}[1/Np],R\pi_!\cdot)$ and $H^q_{\text{ét},c}(Y^2,\cdot)=H^q_{\text{ét},c}(\mathbf{Z}[1/Np],R\boldsymbol{\pi}_!\cdot)$ for any $q\geqslant 0$ (cf. Section II.7 of [**Mil06**]). If $\mathscr{G}$ denotes the étale sheaf $\mathscr{G}'_{r_1}\boxtimes(\mathscr{G}_{r_2}\otimes\mathscr{G}_{r_3})(2-r)$ on Y^2, one can lift the Hecke operators $1\otimes U_p$ and $U'_p\otimes 1$ on $H^{\cdot}_{\text{ét},c}(Y^2,\mathscr{G})$ to morphisms (denoted by the same symbols) $R\boldsymbol{\pi}_!\mathscr{G}\longrightarrow R\boldsymbol{\pi}_!\mathscr{G}$ (cf. Section 2.3). The diagonal embedding $\delta^*:Y\longrightarrow Y^2$, the morphism of sheaves

$$\beta:\delta^*\mathscr{G}=\mathscr{G}'_{r_1}\otimes\mathscr{G}_{r_2}\otimes\mathscr{G}_{r_3}\,(2-r)\longrightarrow\mathscr{O}/\mathfrak{m}^i(2)$$

defined by the cup product with $\mathtt{D}$, and the trace morphism

$$\mathrm{tr}_Y : R\pi_! \mathscr{O}/\mathfrak{m}^i(2) \longrightarrow \mathscr{O}/\mathfrak{m}^i[-2]$$

(see the discussion preceding Theorem 7.6 in [Mil06, Chapter II, Section 7]) induce a map $\vartheta = \mathrm{tr}_Y \circ \beta \circ \delta^* : R\boldsymbol{\pi}_! \mathscr{G} \longrightarrow \mathscr{O}/\mathfrak{m}^i[-2]$. In order to prove the forth equality in Equation (162) it is then sufficient to prove that the composition $\Xi = \vartheta \circ 1 \otimes U_p$ agrees with $\Psi = \bar{\chi}_{\boldsymbol{f}}(p) p^{r-r_1} \cdot \vartheta \circ U_p' \otimes 1$. By using the Künneth isomorphism

$$R\boldsymbol{\pi}_! \mathscr{G} \cong R\pi_! \mathscr{G}_{r_1}' \otimes^{\mathbf{L}}_{\mathscr{O}} R\pi_!(\mathscr{G}_{r_2} \otimes \mathscr{G}_{r_3}(2-r)),$$

the sought for equality $\Xi = \Psi$ follows from the same formal computation as in the proof of Proposition 2.9 of [GS20].

Since the operators $1 \otimes U_p^{\otimes 2}$ and $U_p' \otimes 1 \otimes 1$ acting on $H^5_{\text{ét},c}(Y^3, \mathscr{G}_{r_1}' \boxtimes \mathscr{G}_{r_2} \boxtimes \mathscr{G}_{r_3}(2-r))$ are the adjoints under $(\cdot,\cdot)_{Y^3}$ of the operators $1 \otimes U_p^{\otimes 2}$ and $U_p' \otimes 1 \otimes 1$ acting on $H^4_{\text{ét}}(Y^3, \mathscr{F}_{r_1}' \boxtimes \mathscr{F}_{r_2} \boxtimes \mathscr{F}_{r_3}(r+2))$, and since $(\cdot,\cdot)_{Y^3}$ is perfect, Equation (162) yields

$$(1 \otimes U_p \otimes U_p) \circ d_*(\mathtt{D} - \mathtt{D}_0) = p^{r-r_1} \cdot (U_p' \otimes 1 \otimes 1) \circ d_*(\mathtt{D}).$$

In light of Equation (161), this implies

$$\begin{aligned}(163) \qquad & (p-1) \cdot (1 \otimes U_p \otimes U_p) \circ \mathtt{K} \circ \mathtt{HS} \circ d_* \circ \rho_w(\mathbf{Det}) \\ & \qquad = \big(1 \otimes U_p \otimes U_p - p^{r-r_1} \cdot U_p' \otimes 1 \otimes 1\big) \circ \mathtt{K} \circ \mathtt{HS} \circ d_*(\mathtt{Det})\end{aligned}$$

in $H^1_{\text{ét}}\big(\mathbf{Q}, H^1(\Gamma, A_{r_1}') \hat{\otimes}_L H^1(\Gamma, A_{r_2}) \hat{\otimes}_L H^1(\Gamma, A_{r_3})(r+2)\big)$, where $A_u^{\cdot}$ is a shorthand for $A_{u,\imath}^{\cdot}$, and the morphisms $\mathtt{K}, \mathtt{HS}$ and d_* are defined as in Equation (156), after replacing the big étale sheaf $\boldsymbol{\mathcal{A}}_{\boldsymbol{f}}' \otimes \boldsymbol{\mathcal{A}}_{\boldsymbol{g}} \otimes \boldsymbol{\mathcal{A}}_{\boldsymbol{h}}$ with $\boldsymbol{\mathcal{A}}_{r_1}' \otimes \boldsymbol{\mathcal{A}}_{r_2} \otimes \boldsymbol{\mathcal{A}}_{r_3}$. To ease notations write $\heartsuit$ (resp., $\spadesuit$) for the left (resp., right) hand side of Equation (163).

For each nonnegative integer u and $\mathscr{F}_u = \mathscr{S}_u, \mathscr{L}_u$, let

$$H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{F}_u)_o \hookrightarrow H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{F}_u)_L$$

be the L-direct summand on which the diamond operator $\langle d \rangle$ acts trivially for each integer d coprime to p and congruent to one modulo N, so that the pull-back t^* yields an isomorphism between $H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{F}_u)_L$ and $H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{F}_u)^o$, with inverse $\frac{1}{p-1}$ times the push-forward t_*. For $\cdot = \emptyset, \prime$ denote by

$$c_u^{\cdot} : H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{S}_u)_o \longrightarrow H^1(\Gamma, A_u^{\cdot})$$

the composition of t_* with the comparison morphism introduced in Equation (72). By construction

$$(c_{r_1}' \hat{\otimes} c_{r_2} \hat{\otimes} c_{r_3})_* \circ \mathtt{K}\,(\tilde{\kappa}_{Np,\boldsymbol{r}}) = \mathtt{K} \circ \mathtt{HS} \circ d_*(\mathtt{Det})$$

(where the morphism $\mathtt{K}$ which appear in the left hand side refers to the Künneth decomposition of $W_{Np,\boldsymbol{r}} = H^3_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{S}_{[\boldsymbol{r}]})(r+2)$), hence

$$\spadesuit = (c_{r_1}' \hat{\otimes} c_{r_2} \hat{\otimes} c_{r_3})_* \circ \big(1 \otimes U_p' \otimes U_p' - p^{r-r_1} \cdot U_p \otimes 1 \otimes 1\big) \circ \mathtt{K}\,(\tilde{\kappa}_{Np,\boldsymbol{r}})$$

(cf. the discussion following Equation (72)). Since $w_p \circ c_u' = c_u \circ w_p'$, where w_p' is the Atkin–Lehner operator defined in Section 2.3.1 and w_p is the one defined in

Equation (66), and since $w_p'U_p = \langle p\rangle_N\, U_p' w_p'$ as endomorphisms of $H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{S}_u)$, one deduces

(164) $$w_{p,f*}(\spadesuit) = c_{\boldsymbol{r}*} \circ \left(1 \otimes U_p' \otimes U_p' - p^{r-r_1} \cdot \langle p\rangle_N\, U_p' \otimes 1 \otimes 1\right) \circ w'_{p,f*} \circ \mathtt{K}\,(\tilde{\kappa}_{Np,\boldsymbol{r}}),$$

where $w_{p,f} = w_p \otimes \mathrm{id} \otimes \mathrm{id}$, $w'_{p,f} = w'_p \otimes \mathrm{id} \otimes \mathrm{id}$ and $c_{\boldsymbol{r}} = c_{r_1} \hat{\otimes} c_{r_2} \hat{\otimes} c_{r_3}$.

Taking $h = 0$ and replacing A_U and D'_U with A_u and D'_u (for $u \in \mathbf{N}$) respectively in the definition of the map $\mathtt{s}_{U,h}$ (cf. Equation (83)) yields a $G_{\mathbf{Q}}$-equivariant morphism

$$\mathtt{s}_{u,0} : H^1(\Gamma, A_u)^{\leqslant 0}(u) \longrightarrow H^1(\Gamma, D'_u)^{\leqslant' 0},$$

which intertwines the action of U_p on the source with that of U'_p on the target. If

$$\mathrm{comp}_u : H^1(\Gamma, D'_u)^{\leqslant' 0} \longrightarrow H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{L}_u)^{\leqslant' 0}_o$$

denotes the composition of $t^* : H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{L}_u)_L \longrightarrow H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{L}_u)_o$ with the comparison isomorphism defined in Equation (73), then (cf. Equation (44))

(165) $$\mathrm{comp}_u \circ \mathtt{s}_{u,0} \circ c_u = \frac{1}{p-1} \cdot \mathtt{s}_{u*}$$

as maps from $H^1_{\text{ét}}(Y_1(Np), \mathscr{S}_u)^{\leqslant' 0}_L(u)$ to $H^1(\Gamma, \mathscr{L}_u)^{\leqslant' 0}_L$. Set $\mathtt{s}_{\boldsymbol{r},0} = \mathtt{s}_{r_1,0} \otimes \mathtt{s}_{r_2,0} \otimes \mathtt{s}_{r_3,0}$ and $\mathrm{comp}_{\boldsymbol{r}} = \mathrm{comp}_{r_1} \otimes \mathrm{comp}_{r_2} \otimes \mathrm{comp}_{r_3}$. It then follows from Equation (164) and the definition of the twisted diagonal class $\kappa^\dagger(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ that the equality

(166)
$$\mathrm{pr}_{\boldsymbol{f}_k\boldsymbol{g}_l\boldsymbol{h}_m*} \circ \mathrm{comp}_{\boldsymbol{r}*} \circ \mathtt{s}_{\boldsymbol{r},0*} \circ w_{p,f*}(\spadesuit) = \frac{\alpha_{\boldsymbol{g}_l}\alpha_{\boldsymbol{h}_m}}{p-1}\left(1 - \frac{\bar{\chi}_{\boldsymbol{f}}(p)p^{r-r_1}\alpha_{\boldsymbol{f}_k}}{\alpha_{\boldsymbol{g}_l}\alpha_{\boldsymbol{h}_m}}\right) \cdot \kappa^\dagger(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$$

holds in $H^1_{\text{ét}}(\mathbf{Z}[1/Np], V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m))$. (Here $\mathrm{pr}_{\boldsymbol{f}_k\boldsymbol{g}_l\boldsymbol{h}_m}$ is the tensor product of the projections $\mathrm{pr}_\cdot$ defined in Equation (23), for $\cdot$ equal to $\boldsymbol{f}_k, \boldsymbol{g}_l$ and $\boldsymbol{h}_m$.)

By construction, one has

$$\mathtt{K} \circ \mathtt{HS} \circ d_* \circ \rho_w(\mathbf{Det}) = \rho_w \circ \mathtt{K} \circ \mathtt{HS} \circ d_*(\mathbf{Det}),$$

where the maps $\mathtt{K}, \mathtt{HS}$ and d_* which appear in the right hand side are the ones introduced in Equation (156). Since the maps ρ_w and $\mathrm{comp}_{\boldsymbol{r}}$ are Hecke-equivariant, and since $\mathtt{s}_{u,0}$ intertwines the action of U_p on $H^1(\Gamma, A_u)^{\leqslant 0}$ with that of U'_p on $H^1(\Gamma, D'_u)^{\leqslant' 0}$ (for each nonnegative integer u), it follows that

(167) $$\diamondsuit = (p-1)\alpha_{\boldsymbol{g}_l}\alpha_{\boldsymbol{h}_m} \cdot \mathrm{pr}_{\boldsymbol{f}_k\boldsymbol{g}_l\boldsymbol{h}_m} \circ \mathrm{comp}_{\boldsymbol{r}*} \circ \mathtt{s}_{\boldsymbol{r},0*} \circ w_{p,f*} \circ \rho_w \circ \mathtt{K} \circ \mathtt{HS} \circ d_*(\mathbf{Det}),$$

where one defines

$$\diamondsuit = \mathrm{pr}_{\boldsymbol{f}_k\boldsymbol{g}_l\boldsymbol{h}_m*} \circ \mathrm{comp}_{\boldsymbol{r}*} \circ \mathtt{s}_{\boldsymbol{r},0*} \circ w_{p,f*}(\heartsuit).$$

One has $w_{p,f*} \circ \rho_w = \rho_w \circ w_{p,f*}$. Moreover the diagram (84) and Equation (165) yield

$$\mathrm{comp}_u \circ \mathtt{s}_{u,0} \circ \rho_{u+2} = \frac{1}{p-1} \cdot \mathtt{s}_{u*} \circ c_u^{-1} \circ \rho_{u+2} = \frac{1}{p-1} \cdot \mathrm{comp}_u \circ \rho_{u+2} \circ \mathtt{s}_{U_{\boldsymbol{\xi}},0}$$

as morphisms from $H^1(\Gamma, A_{\boldsymbol{\xi}})^{\leqslant 0}(\kappa_{\boldsymbol{\xi}}) \longrightarrow H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{L}_u)^{\leqslant' 0}_o$, for $(\boldsymbol{\xi}, u)$ equal to one of the pairs $(\boldsymbol{f}, k-2)$, $(\boldsymbol{g}, l-2)$ and $(\boldsymbol{h}, m-2)$, (cf. the discussion following the diagram (84)). (With a slight abuse of notation, in the previous equation one writes c_u^{-1} for the inverse of the isomorphism between $H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{S}_u)^{\leqslant' 0}_o$ and $H^1(\Gamma, A_u)^{\leqslant 0}$ induced by c_u.) Finally, with the notations introduced in Equations (105) and (106), one has the following equality of $G_{\mathbf{Q}}$-equivariant maps from $H^1(\Gamma, D'_{\boldsymbol{\xi}})^{\leqslant 0}(1)$ to $V(\boldsymbol{f}_k)$:

$$\mathrm{pr}_{\boldsymbol{\xi}_u} \circ \mathrm{comp}_u \circ \rho_{u+2} = \rho_{u+2} \circ \mathrm{pr}_{\boldsymbol{\xi}}.$$

It then follows from Equation (167) and the definitions of ($\diamondsuit$ and) $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})^o$ that

$$\mathrm{pr}_{\boldsymbol{f}_k \boldsymbol{g}_l \boldsymbol{h}_m *} \circ \mathrm{comp}_{\boldsymbol{r}*} \circ \mathbf{s}_{\boldsymbol{r},0*} \circ w_{p,f*}(\heartsuit) = \alpha_{\boldsymbol{g}_l} \alpha_{\boldsymbol{h}_m} \cdot \rho_w\big(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})^o\big). \tag{168}$$

As $\chi_{\boldsymbol{f}} \chi_{\boldsymbol{g}} \chi_{\boldsymbol{h}} = 1$ by Assumption 1.2, and by definition $\alpha_{\boldsymbol{g}_l} \beta_{\boldsymbol{g}_l} = \chi_{\boldsymbol{g}}(p) p^{r_2+1}$, $\alpha_{\boldsymbol{h}_m} \beta_{\boldsymbol{h}_m} = \chi_{\boldsymbol{h}}(p) p^{r_3+1}$ and $2r = r_1 + r_2 + r_3$, the theorem follows from Equations (163), (166) and (168). (Recall that $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})^o = a_p(\boldsymbol{k}) \cdot \kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$.)

□

8.3. p-stabilisation of diagonal classes. — Write in this section

$$Y_1(M) = Y_1(M)_{\mathbf{Q}},$$

for every integer $M \geqslant 3$. Recall the degeneracy maps $\mathrm{pr}_i : Y_1(Np) \to Y_1(N)$, for $i = 1, p$, defined in Section 2.2.

Let $w \in \Sigma_{\mathrm{bal}}$ and $\boldsymbol{r} = w - \mathbf{2}$ be as in the previous section. Assume $k, l, m \geqslant 3$ and that p does not divide the conductors of $\boldsymbol{f}_k, \boldsymbol{g}_l$ and $\boldsymbol{h}_m$. As in Section 6 let $f = f_k$ (resp., $g = g_l$ and $h = h_m$) be the cusp form of weight k (resp., l, m), level $\Gamma_1(N)$ and character $\chi_{\boldsymbol{f}}$ (resp., $\chi_{\boldsymbol{g}}$, $\chi_{\boldsymbol{h}}$) whose ordinary p-stabilisation is $\boldsymbol{f}_k$ (resp., $\boldsymbol{g}_l$, $\boldsymbol{h}_m$). It is an eigenvector for the Hecke operator T_ℓ, with the same eigenvalue as $\boldsymbol{f}_k$ (resp., $\boldsymbol{g}_l$, $\boldsymbol{h}_m$), for every prime $\ell \nmid Np$, and an eigenvector for T_p with eigenvalue $a_p(f) = \alpha_{\boldsymbol{f}_k} + \beta_{\boldsymbol{f}_k}$ (resp., $a_p(g) = \alpha_{\boldsymbol{g}_l} + \beta_{\boldsymbol{g}_l}$, $a_p(h) = \alpha_{\boldsymbol{h}_m} + \beta_{\boldsymbol{h}_m}$). Assume without loss of generality that $\beta_{\boldsymbol{a}}$ belongs to L for $\boldsymbol{a} = \boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m$, and denote by

$$\Pi^{\alpha}_{\boldsymbol{r}*} : V_{Np,\boldsymbol{r}} \otimes_{\mathbf{Q}_p} L \longrightarrow V_{N,\boldsymbol{r}} \otimes_{\mathbf{Q}_p} L$$

the morphism (cf. Equations (20) and (45))

$$\Pi^{\alpha}_{\boldsymbol{r}*} = \left(\mathrm{pr}_{1*} - \frac{\beta_{\boldsymbol{f}_k}}{p^{k-1}} \cdot \mathrm{pr}_{p*}\right) \otimes \left(\mathrm{pr}_{1*} - \frac{\beta_{\boldsymbol{g}_l}}{p^{l-1}} \cdot \mathrm{pr}_{p*}\right) \otimes \left(\mathrm{pr}_{1*} - \frac{\beta_{\boldsymbol{h}_m}}{p^{m-1}} \cdot \mathrm{pr}_{p*}\right). \tag{169}$$

A direct computation shows that the composition $\mathrm{pr}_{fgh} \circ \Pi^{\alpha}_{\boldsymbol{r}*}$ factors through the projection $\mathrm{pr}_{\boldsymbol{f}_k \boldsymbol{g}_l \boldsymbol{h}_m}$, hence $\Pi^{\alpha}_{\boldsymbol{r}*}$ induces a morphism

$$\Pi^{\alpha}_{f_k g_l h_m *} : V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m) \longrightarrow V(f_k, g_l, h_m)$$

of $L[G_{\mathbf{Q}}]$-modules, which is indeed an isomorphism (see Equation (48) for the definition of the projections pr_{fgh} and $\mathrm{pr}_{\boldsymbol{f}_k \boldsymbol{g}_l \boldsymbol{h}_m}$). Note that $\boldsymbol{r} = (r_1, r_2, r_3)$ and (f_k, g_l, h_m) satisfy Assumption 3.1 and Assumption 3.4 respectively, hence the class $\kappa(f_k, g_l, h_m)$ in $H^1(\mathbf{Q}, V(f_k, g_l, h_m))$ is defined. Denote again by

$$\Pi^{\alpha}_{f_k g_l h_m *} : H^1(\mathbf{Q}, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)) \to H^1(\mathbf{Q}, V(f_k, g_l, h_m))$$

the morphism induced in Galois cohomology by $\Pi^{\alpha}_{f_k g_l h_m *}$.

Proposition 8.3. — *Assume $k, l, m \geqslant 3$ and that p does not divide the conductors of $\boldsymbol{f}_k, \boldsymbol{g}_l$ and $\boldsymbol{h}_m$. Then*

$$\Pi^{\alpha}_{f_k g_l h_m *}\big(\kappa^{\dagger}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)\big)$$

is equal to

$$(p-1)\alpha_{\boldsymbol{f}_k}\left(1 - \frac{\beta_{\boldsymbol{f}_k}\alpha_{\boldsymbol{g}_l}\beta_{\boldsymbol{h}_m}}{p^{r+2}}\right)\left(1 - \frac{\beta_{\boldsymbol{f}_k}\beta_{\boldsymbol{g}_l}\alpha_{\boldsymbol{h}_m}}{p^{r+2}}\right)\left(1 - \frac{\beta_{\boldsymbol{f}_k}\beta_{\boldsymbol{g}_l}\beta_{\boldsymbol{h}_m}}{p^{r+2}}\right) \cdot \kappa(f_k, g_l, h_m).$$

Proof. — Fix a geometric point $\eta : \mathrm{Spec}(\mathbf{C}) \to Y(1, N(p))$, corresponding to the class of z in $\mathbf{H}$ under the isomorphism (6). With a slight abuse of notation denote again by η the complex point $\nu_p \circ \eta : \mathrm{Spec}(\mathbf{C}) \to Y(1, N)$, and by $\check{\eta}$ both the complex points $\varphi_p \circ \eta : \mathrm{Spec}(\mathbf{C}) \to Y(1(p), N)$ and $\check{\nu}_p \circ \varphi_p \circ \eta : \mathrm{Spec}(\mathbf{C}) \to Y(1, N)$. Then η and $\check{\eta}$ correspond respectively to the classes of z and $p \cdot z$ under the analytic isomorphisms (6). With the notations of Section 2.3 (see in particular the diagram (9)) write

$$\mathscr{T}_{(p)} = R^1 v_{1,N(p)*}\mathbf{Z}_p(1), \quad \mathscr{T}^{(p)} = R^1 v_{1(p),N*}\mathbf{Z}_p(1) \ \text{ and } \ \mathscr{T} = R^1 v_{1,N*}\mathbf{Z}_p(1)$$

for the relative Tate modules of $E(1, N(p)) \to Y(1, N(p))$, $E(1(p), N) \to Y(1(p), N)$ and $E(1, N) \to Y(1, N)$ respectively (cf. Section 2.3). There are then natural isomorphisms

$$\mathscr{T}_{(p),\eta} \cong \mathbf{Z}_p \oplus \mathbf{Z}_p \cdot z \cong \mathscr{T}_{\eta} \quad \text{and} \quad \mathscr{T}^{(p)}_{\check{\eta}} \cong \mathbf{Z}_p \oplus \mathbf{Z}_p \cdot pz \cong \mathscr{T}_{\check{\eta}}. \tag{170}$$

Here the subscripts η and $\check{\eta}$ denote the stalks at η and $\check{\eta}$ respectively, and for each ω in $\mathbf{H}$ one writes

$$\mathbf{Z}_p \oplus \mathbf{Z}_p \cdot \omega = H_1(\mathbf{C}/\Lambda_\omega, \mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}_p$$

for the p-adic completion of the integral homology of the complex elliptic curve $\mathbf{C}/\Lambda_\omega$, where $\Lambda_\omega = \mathbf{Z} \oplus \mathbf{Z} \cdot \omega$. As in Sections 3 and 4.2, after identifying $\mathscr{T}_{(p),\eta}$ with $\mathbf{Z}_p \oplus \mathbf{Z}_p$ under the $\mathbf{Z}_p$-basis $\{1, z\}$, the natural action of the étale fundamental group $\mathcal{G}_{(p)} = \pi_1^{\text{ét}}(Y(1, N(p)), \eta)$ (resp., $\mathcal{G}^{(p)} = \pi_1^{\text{ét}}(Y(1(p), N), \check{\eta})$) on $\mathscr{T}_{(p),\eta}$ (resp., $\mathscr{T}^{(p)}_{\check{\eta}}$) gives a continuous representation $\varrho_{(p)} : \mathcal{G}_{(p)} \to \Gamma(1, N(p)) \otimes_{\mathbf{Z}} \mathbf{Z}_p \hookrightarrow \mathrm{GL}_2(\mathbf{Z}_p)$ ($\varrho^{(p)} : \mathcal{G}^{(p)} \to \Gamma(1(p), N) \otimes_{\mathbf{Z}} \mathbf{Z}_p \hookrightarrow \mathrm{GL}_2(\mathbf{Z}_p)$), where $\Gamma(1, N(p))$ (resp., $\Gamma(1(p), N)$) is the subgroup of matrices in $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbf{Z})$ with $c \equiv 0$, $d \equiv 1 \pmod N$ and $c \equiv 0 \pmod p$ (resp., $b \equiv 0 \pmod p$). For each $i \geqslant 0$ set

$$\mathscr{S}_{(p),i} = \mathrm{Symm}^i_{\mathbf{Z}_p} \mathscr{T}_{(p)}(-1) \quad \text{and} \quad \mathscr{S}^{(p)}_i = \mathrm{Symm}^i_{\mathbf{Z}_p} \mathscr{T}^{(p)}(-1),$$

where as in Section 2.3 the Tate twists $\mathscr{T}_{(p)}(-1)$ and $\mathscr{T}^{(p)}(-1)$ are identified with the duals of $\mathscr{T}_{(p)}$ and $\mathscr{T}^{(p)}$ under the Weil pairings on $E(1, N(p))$ and $E(1(p), N)$ respectively. Then the stalks of $\mathscr{S}_{(p),i}$ and $\mathscr{S}^{(p)}_i$ at η and $\check{\eta}$, viewed as representations of $\mathcal{G}_{(p)}$ and $\mathcal{G}^{(p)}$ respectively, correspond via $\varrho_{(p)}$ and $\varrho^{(p)}$ to the $\Gamma(1, N(p))$-module $S_i = S_i(\mathbf{Z}_p)$ and the $\Gamma(1(p), N)$-module S_i (cf. Section 3). As a consequence, for each

$j \geqslant 0$ and $u \in \mathbf{Z}$ there is a natural inclusion (cf. Section 4.2)

$$\begin{tikzcd} H^0(\Gamma(1,N(p)), S_i \otimes \det^{-u}) \arrow[r] & H^0(\mathcal{G}_{(p)}, S_i \otimes \det^{-u}) \arrow[d, equal] \\ & H^0_{\text{ét}}(Y(1,N(p)), \mathscr{S}_{(p),i} \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(u)), \end{tikzcd} \tag{171}$$

and an isomorphism

$$H^j_{\text{ét}}(Y(1,N(p))_{\bar{\mathbf{Q}}}, \mathscr{S}_{(p),i}) \cong H^j(\Gamma(1,N(p)), S_i),$$

and similarly for the data $(\Gamma(1(p),N), \mathcal{G}^{(p)}, \mathscr{S}_i^{(p)})$ in place of $(\Gamma(1,N(p)), \mathcal{G}_{(p)}, \mathscr{S}_{(p),i})$. As already explained in Section 3, there are similar isomorphisms after replacing $\varrho_{(p)}$ with the representations $\varrho : \mathcal{G} \to \mathrm{GL}_2(\mathbf{Z}_p)$ (resp., $\check{\varrho} : \check{\mathcal{G}} \to \mathrm{GL}_2(\mathbf{Z}_p)$) arising from the action of $\mathcal{G} = \pi_1^{\text{ét}}(Y(1,N), \eta)$ (resp., $\check{\mathcal{G}} = \pi_1^{\text{ét}}(Y(1,N), \check{\eta})$) on the stalk at η (resp., $\check{\eta}$) of $\mathscr{S}_i = \mathscr{S}_i(\mathbf{Z}_p)$. Under these isomorphisms, the maps

$$\begin{aligned} \lambda^i_{p*} &= (\lambda^i_{p*})_{\check{\eta}} : S_i \cong (\mathscr{S}_{(p),i})_\eta \longrightarrow (\mathscr{S}_i^{(p)})_{\check{\eta}} \cong S_i \\ \text{and} \quad \lambda^{i*}_p &= (\lambda^{i*}_p)_\eta : S_i \cong (\mathscr{S}_i^{(p)})_{\check{\eta}} \longrightarrow (\mathscr{S}_{(p),i})_\eta \cong S_i \end{aligned} \tag{172}$$

induced respectively on the stalks at $\check{\eta}$ and η by the morphisms (16) are given by

$$\lambda^i_{p*}(P) = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \cdot P \quad \text{and} \quad \lambda^{i*}_p(P) = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \cdot P, \tag{173}$$

for P in S_i. Indeed the base change $\lambda_{\check{\eta}} : \mathbf{C}/\Lambda_z = E(1,N(p)) \times_\eta \mathbf{C} \longrightarrow E(1(p),N) \times_{\check{\eta}} \mathbf{C} \cong \mathbf{C}/\Lambda_{pz}$ of the p-isogeny λ_p along $\check{\eta}$ is induced by multiplication by p on $\mathbf{C}$, hence the map $\lambda_{\check{\eta}*} : \mathscr{T}^{(p)} \longrightarrow \mathscr{T}_{(p)}$ it induces on the Tate modules is represented by $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$, once one identifies $\mathscr{T}_{(p)}$ and $\mathscr{T}^{(p)}$ with $\mathbf{Z}_p^2$ under the $\mathbf{Z}_p$-bases $\{1, z\}$ and $\{1, pz\}$ (cf. Equation (170)). Because the dual isogeny $\lambda'_{\check{\eta}}$ of $\lambda_{\check{\eta}}$ is the map $\mathbf{C}/\Lambda_{pz} \to \mathbf{C}/\Lambda_z$ induced by the identity on $\mathbf{C}$, and $\lambda_{\check{\eta}*}$ and $\lambda'_{\check{\eta}*}$ are adjoint to each other under the Weil pairings on $\mathbf{C}/\Lambda_z$ and $\mathbf{C}/\Lambda_{pz}$, Equation (173) follows.

After this preliminary discussion, we divide the proof into three steps. For each triple i, j, k of elements of $\{1, p\}$ write

$$\mathrm{pr}_{ijk*} = \mathrm{pr}_{i*} \otimes \mathrm{pr}_{j*} \otimes \mathrm{pr}_{k*} : Z_{Np,\boldsymbol{r}}(n) \to Z_{Np,\boldsymbol{r}}(n),$$

for $n \in \mathbf{Z}$ and $Z = V$ or $Z = W$, and denote by the same symbol the map they induce in $G_{\mathbf{Q}}$-cohomology. For any curve X over $\mathbf{Q}$ write $d : X \longrightarrow X^3$ for the diagonal embedding.

Step 1. One has the identities in $H^1(\mathbf{Q}, V_{N,\boldsymbol{r}}(r+2))$:

$$\mathrm{pr}_{111*}(\kappa_{Np,\boldsymbol{r}}) = (p^2-1) \cdot \kappa_{N,\boldsymbol{r}} \quad \text{and} \quad \mathrm{pr}_{ppp*}(\kappa_{Np,\boldsymbol{r}}) = (p^2-1)p^r \cdot \kappa_{N,\boldsymbol{r}}. \tag{174}$$

As the element $\mathrm{Det}^{\boldsymbol{r}} = \mathrm{Det}^{\boldsymbol{r}}_N$ is invariant under $\mathrm{GL}_2(\mathbf{Z}_p)$, it defines under the inclusion (171) an element $\mathtt{Det}^{\boldsymbol{r}}$ in $H^0_{\text{ét}}(Y(1,N(p)), \mathscr{S}_{(p),i}(r))$, and similarly elements (denoted by the same symbol) in $H^0_{\text{ét}}(Y(1(p),N), \mathscr{S}_{(p),i}(r))$ and $H^0_{\text{ét}}(Y(1,N), \mathscr{S}_i(r))$. According to Equation (173) and the definition of $\mathrm{Det}^{\boldsymbol{r}}$ in Equation (41) one has

$$\lambda^{\boldsymbol{r}}_{p*}(\mathrm{Det}^{\boldsymbol{r}}) = p^r \cdot \mathrm{Det}^{\boldsymbol{r}}, \tag{175}$$

where $\lambda_{p*}^{\boldsymbol{r}} = \lambda_{p*}^{r_1} \otimes \lambda_{p*}^{r_2} \otimes \lambda_{p*}^{r_3} \otimes \mathrm{id} : S_{\boldsymbol{r}} \otimes \det^{-r} \to S_{\boldsymbol{r}} \otimes \det^{-r}$, hence (since $\check{\nu}_p$ has degree $p+1$)

$$\check{\nu}_{p*} \circ \varphi_{p*} \circ \lambda_{p*}^{\boldsymbol{r}}(\mathtt{Det}^{\boldsymbol{r}}) = (p+1)p^r \cdot \mathtt{Det}^{\boldsymbol{r}} \in H^0_{\text{ét}}(Y(1,N), \mathscr{S}_{\boldsymbol{r}}(r)).$$

Retracing the definitions of Section 2.3 and using Equation (21) this gives

$$\mathrm{pr}_{p*}(\mathtt{Det}^{\boldsymbol{r}}) = (p^2-1)p^r \cdot \mathtt{Det}^{\boldsymbol{r}}.$$

The previous equation and the functoriality of the Hochschild–Serre spectral sequence implies (cf. Section 3)

$$\mathrm{pr}_{ppp*}(\kappa_{Np,\boldsymbol{r}}) = \mathtt{s}_{\boldsymbol{r}*} \circ \mathtt{HS} \circ \mathrm{pr}_{ppp*} \circ d_*(\mathtt{Det}^{\boldsymbol{r}}) = \mathtt{s}_{\boldsymbol{r}*} \circ \mathtt{HS} \circ d_* \circ \mathrm{pr}_{p*}(\mathtt{Det}^{\boldsymbol{r}}) = (p^2-1)p^r \cdot \kappa_{N,\boldsymbol{r}}.$$

This proves the second identity in Equation (174). The first one is proved by a similar (and simpler) argument.

Step 2. The following identities hold in $H^1(\mathbf{Q}, V_{N,\boldsymbol{r}}(r+2))$:

(176)

$$\mathrm{pr}_{p11*}(\kappa_{Np,\boldsymbol{r}}) = (p-1) \cdot T_p \otimes \mathrm{id} \otimes \mathrm{id}(\kappa_{N,\boldsymbol{r}}); \qquad \mathrm{pr}_{1pp*}(\kappa_{Np,\boldsymbol{r}}) = (p-1)p^{r-r_1} \cdot T'_p \otimes \mathrm{id} \otimes \mathrm{id}(\kappa_{N,\boldsymbol{r}});$$

$$\mathrm{pr}_{1p1*}(\kappa_{Np,\boldsymbol{r}}) = (p-1) \cdot \mathrm{id} \otimes T_p \otimes \mathrm{id}(\kappa_{N,\boldsymbol{r}}); \qquad \mathrm{pr}_{p1p*}(\kappa_{Np,\boldsymbol{r}}) = (p-1)p^{r-r_2} \cdot \mathrm{id} \otimes T'_p \otimes \mathrm{id}(\kappa_{N,\boldsymbol{r}});$$

$$\mathrm{pr}_{11p*}(\kappa_{Np,\boldsymbol{r}}) = (p-1) \cdot \mathrm{id} \otimes \mathrm{id} \otimes T_p(\kappa_{N,\boldsymbol{r}}); \qquad \mathrm{pr}_{pp1*}(\kappa_{Np,\boldsymbol{r}}) = (p-1)p^{r-r_3} \cdot \mathrm{id} \otimes \mathrm{id} \otimes T'_p(\kappa_{N,\boldsymbol{r}}).$$

We prove the second identity in the first line. Note that the finite étale cover $\check{\nu}_p$ is not Galois. To remedy this let $\vartheta : \mathcal{Y} \longrightarrow Y(1,N)$ be a finite étale Galois morphism which factors through $\check{\nu}_p \circ \varphi_p : Y(1,N(p)) \longrightarrow Y(1,N)$, say $\vartheta = \check{\nu}_p \circ \varphi_p \circ \alpha$ with $\alpha : \mathcal{Y} \longrightarrow Y(1,N(p))$. Denote by $G = \mathrm{Gal}(\vartheta)$ its Galois group. For each $u \geqslant 1$ denote by $\pi_{1*}^u = \nu_{p*} : H^1(Y(1,N(p)), \mathscr{S}_{(p),u}) \to H^1(Y(1,N), \mathscr{S}_u)$, and similarly set $\pi_1^{u*} = \nu_p^*$. Set

$$\begin{aligned} \pi_{p*}^u &= \check{\nu}_{p*} \circ \varphi_{p*} \circ \lambda_{p*}^u, \\ \pi_p^{u*} &= \lambda_p^{u*} \circ \varphi_p^* \circ \check{\nu}_p^*, \\ \pi_{ijk}^{\boldsymbol{r}*} &= \pi_i^{r_1*} \otimes \pi_j^{r_2*} \otimes \pi_k^{r_3*} \\ \text{and} \quad \pi_{ijk*}^{\boldsymbol{r}} &= \pi_{i*}^{r_1} \otimes \pi_{j*}^{r_2} \otimes \pi_{k*}^{r_3}, \end{aligned}$$

where i,j,k is any triple of elements of $\{1,p\}$. Moreover for each morphism $a : X \to Y$ of curves over $\mathbf{Q}$ write $\boldsymbol{a} = a \times_{\mathbf{Q}} a \times_{\mathbf{Q}} a : X^3 \to Y^3$. With these notations it follows directly from the definitions that

$$\pi_{1pp*}^{\boldsymbol{r}} \circ \pi_{ppp}^{\boldsymbol{r}*} = (p+1)^2 p^{r_2+r_3} \cdot T'_p \otimes \mathrm{id} \otimes \mathrm{id}. \tag{177}$$

On the other hand, after setting

$$\kappa_{Np,\boldsymbol{r}}^\star = \mathtt{s}_{\boldsymbol{r}*} \circ \mathtt{HS} \circ d_* \circ \vartheta^*(\mathtt{Det}^{\boldsymbol{r}}),$$

one has $(p+1)\deg(\alpha) \cdot \kappa_{N,\boldsymbol{r}} = \boldsymbol{\vartheta}_*(\kappa_{Np,\boldsymbol{r}}^\star)$, hence

$$\begin{aligned} (p+1)\deg(\alpha)^4 \cdot \pi_{ppp}^{\boldsymbol{r}*}(\kappa_{N,\boldsymbol{r}}) &= \lambda_p^{\boldsymbol{r}*} \circ \boldsymbol{\alpha}_* \circ \boldsymbol{\vartheta}^* \circ \boldsymbol{\vartheta}_*(\kappa_{Np,\boldsymbol{r}}^\star) \\ &= \sum_{(g_1,g_2,g_3) \in G^3} \lambda_p^{\boldsymbol{r}*} \circ \boldsymbol{\alpha}_* \circ (g_1 \times g_2 \times g_3)_*(\kappa_{Np,\boldsymbol{r}}^\star). \end{aligned}$$

For each $g, h \in G$ one has $\pi_{p*}^{r_i} \circ \lambda_p^{r_i*} \circ \alpha_* \circ g_* = p^{r_i} \cdot \vartheta_* = p^{r_i} \cdot \vartheta_* \circ h_*$, hence the previous equation yields

(178)

$$\begin{aligned}
&(p+1)\deg(\alpha)^4 \cdot \pi_{1pp*}^{\boldsymbol{r}} \circ \pi_{ppp}^{\boldsymbol{r}*}(\kappa_{N,\boldsymbol{r}}) \\
&\quad = p^{r_2+r_3} \sum_{(g_1,g_2,g_3)\in G^3} \big(\nu_{p*} \circ \lambda_p^{r_1*} \otimes \check{\nu}_{p*} \circ \varphi_{p*} \otimes \check{\nu}_{p*} \circ \varphi_{p*}\big) \circ \boldsymbol{\alpha}_* \circ \boldsymbol{g}_{1*}(\kappa_{Np,\boldsymbol{r}}^{\star}) \\
&\quad = (p+1)^3 p^{r_2+r_3} \deg(\alpha)^4 \cdot (\nu_{p*} \otimes \check{\nu}_{p*} \circ \varphi_{p*} \otimes \check{\nu}_{p*} \circ \varphi_{p*}) \circ (\lambda_p^{r_1*} \otimes \mathrm{id} \otimes \mathrm{id})(\kappa_{Np,\boldsymbol{r}}^{\bullet}),
\end{aligned}$$

where $\kappa_{Np,\boldsymbol{r}}^{\bullet} = \mathtt{s}_{\boldsymbol{r}*} \circ \mathtt{HS} \circ d_* \circ (\check{\nu}_p \circ \varphi_p)^*(\mathtt{Det}^{\boldsymbol{r}})$. According to Equations (41) and (173)

$$\lambda_{p*}^{\boldsymbol{r}}(\kappa_{Np,\boldsymbol{r}}^{\bullet}) = \lambda_{p*}^{r_1} \otimes \lambda_{p*}^{r_2} \otimes \lambda_{p*}^{r_3}(\kappa_{Np,\boldsymbol{r}}^{\bullet}) = p^r \cdot \kappa_{Np,\boldsymbol{r}}^{\bullet}$$

$$\text{and} \quad \lambda_p^{r_1*} \circ \lambda_{p*}^{r_1}(P) = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \cdot P = p^{r_1} \cdot P,$$

for P in S_{r_1}, hence (since $2r = r_1 + r_2 + r_3$) one can rewrite Equation (178) as

$$\pi_{1pp*}^{\boldsymbol{r}} \circ \pi_{ppp}^{\boldsymbol{r}*}(\kappa_{N,\boldsymbol{r}}) = (p+1)^2 p^r \cdot \pi_{1pp*}^{\boldsymbol{r}}(\kappa_{Np,\boldsymbol{r}}^{\bullet}). \tag{179}$$

(Note that, regarding the natural isomorphism of Equation (171) and its analogue for $Y(1, N(p))$ as equalities, the pullback by $\check{\nu}_p \circ \varphi_p$ is identified with the identity.) In addition Equation (8) gives

$$\mathrm{pr}_{1pp*}(\kappa_{Np,\boldsymbol{r}}) = \pi_{1pp*}^{\boldsymbol{r}} \circ \boldsymbol{\mu}_{p*}(\kappa_{Np,\boldsymbol{r}}) = (p-1) \cdot \pi_{1pp*}^{\boldsymbol{r}}(\kappa_{Np,\boldsymbol{r}}^{\bullet}). \tag{180}$$

Equations (177), (179) and (180) finally give

$$(p+1)^2 p^r \cdot \mathrm{pr}_{1pp*}(\kappa_{Np,\boldsymbol{r}}) = (p-1)(p+1)^2 p^{r_2+r_3} \cdot T_p' \otimes \mathrm{id} \otimes \mathrm{id}(\kappa_{N,\boldsymbol{r}}).$$

This proves the second identity in the first line of Equation (176). The other equalities in the second column (resp., the equalities in the first column) are proved by a similar (resp., similar and simpler) argument.

Step 3. We can now conclude the proof of the proposition.

Applying the projector $\mathrm{pr}_{f_k g_l h_m}$ (see Equation (48)) to the identities (174) and (176) gives

$$\begin{aligned}
\mathrm{pr}_{111*}(\kappa_{Np,\boldsymbol{r}})_{fgh} &= (p^2-1) \cdot \kappa(f,g,h); \\
\mathrm{pr}_{ppp*}(\kappa_{Np,\boldsymbol{r}})_{fgh} &= p^r(p^2-1) \cdot \kappa(f,g,h); \\
\mathrm{pr}_{p11*}(\kappa_{Np,\boldsymbol{r}})_{fgh} &= (p-1)\bar{\chi}_{\boldsymbol{f}}(p) a_p(f) \cdot \kappa(f,g,h); \\
\mathrm{pr}_{1pp*}(\kappa_{Np,\boldsymbol{r}})_{fgh} &= (p-1)p^{r-r_1} a_p(f) \cdot \kappa(f,g,h); \\
\mathrm{pr}_{1p1*}(\kappa_{Np,\boldsymbol{r}})_{fgh} &= (p-1)\bar{\chi}_{\boldsymbol{g}}(p) a_p(g) \cdot \kappa(f,g,h); \\
\mathrm{pr}_{p1p*}(\kappa_{Np,\boldsymbol{r}})_{fgh} &= (p-1)p^{r-r_2} a_p(g) \cdot \kappa(f,g,h); \\
\mathrm{pr}_{11p*}(\kappa_{Np,\boldsymbol{r}})_{fgh} &= (p-1)\bar{\chi}_{\boldsymbol{h}}(p) a_p(h) \cdot \kappa(f,g,h); \\
\mathrm{pr}_{pp1*}(\kappa_{Np,\boldsymbol{r}})_{fgh} &= (p-1)p^{r-r_3} a_p(h) \cdot \kappa(f,g,h).
\end{aligned} \tag{181}$$

Here $(f,g,h) = (f_k, g_l, h_m)$, $\mathrm{pr}_{ijk*}(\kappa_{Np,\boldsymbol{r}})_{fgh}$ is a shorthand for the image of $\mathrm{pr}_{ijk*}(\kappa_{Np,\boldsymbol{r}})$ under $\mathrm{pr}_{fgh*} = \mathrm{pr}_{f_k g_l h_m *}$, and we used the identity $T_p' = T_p \circ \langle p \rangle'$ as endomorphisms of $H^1_{\text{ét}}(Y_1(N)_{\bar{\mathbf{Q}}}, \mathscr{L}_i(j))_{\mathbf{Q}_p}$. Because the map

$$\mathsf{s}_i : H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{S}_i) \to H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{L}_i)(-i)$$

intertwines the action of the dual Atkin–Lehner operators w'_p on both sides, it follows from the definitions that

$$\Pi^{\alpha}_{fgh*}\big(\kappa^{\dagger}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)\big) = \mathrm{pr}_{fgh*}\Big(\Pi^{\alpha}_{\boldsymbol{r}*}\Big(\big(w'_p \otimes \mathrm{id} \otimes \mathrm{id}\big)_*(\kappa_{Np,\boldsymbol{r}})\Big)\Big). \tag{182}$$

It it easily checked that

$$\mathrm{pr}_{p*} \circ w'_p = p^i \cdot \mathrm{pr}_{1*} \quad \text{ and } \quad \mathrm{pr}_{1*} \circ w'_p = \langle p \rangle' \cdot \mathrm{pr}_{p*}$$

as morphisms from $H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}}, \mathscr{L}_i)$ to $H^1_{\text{ét}}(Y_1(N)_{\bar{\mathbf{Q}}}, \mathscr{L}_i)$. As a consequence, setting $\langle p \rangle'_f = \langle p \rangle' \otimes \mathrm{id} \otimes \mathrm{id}$ and writing $\alpha_f = \alpha_{\boldsymbol{f}_k}, \beta_f = \beta_{\boldsymbol{f}_k}, \alpha_g = \alpha_{\boldsymbol{g}_l}$ et cetera, one has

$$\begin{aligned}
&\Pi^{\alpha}_{\boldsymbol{r}*} \circ \big(w'_p \otimes \mathrm{id} \otimes \mathrm{id}\big) \\
&\quad = \left(\langle p \rangle' \cdot \mathrm{pr}_{p*} - \frac{\beta_f}{p} \cdot \mathrm{pr}_{1*}\right) \otimes \left(\mathrm{pr}_{1*} - \frac{\beta_g}{p^{r_2+1}} \cdot \mathrm{pr}_{p*}\right) \otimes \left(\mathrm{pr}_{1*} - \frac{\beta_h}{p^{r_3+1}} \cdot \mathrm{pr}_{p*}\right) \\
&\quad = \langle p \rangle'_f \cdot \mathrm{pr}_{p11*} - \frac{\beta_f}{p} \cdot \mathrm{pr}_{111*} - \frac{\beta_g \, \langle p \rangle'_f}{p^{r_2+1}} \cdot \mathrm{pr}_{pp1*} - \frac{\beta_h \, \langle p \rangle'_f}{p^{r_3+1}} \cdot \mathrm{pr}_{p1p*} + \frac{\beta_f \beta_g}{p^{r_2+2}} \cdot \mathrm{pr}_{1p1*} \\
&\qquad + \frac{\beta_f \beta_h}{p^{r_3+2}} \cdot \mathrm{pr}_{11p*} + \frac{\beta_g \beta_h \, \langle p \rangle'_f}{p^{r_2+r_3+2}} \cdot \mathrm{pr}_{ppp*} - \frac{\beta_f \beta_g \beta_h}{p^{r_2+r_3+3}} \cdot \mathrm{pr}_{1pp*}.
\end{aligned}$$

Together with Equations (181) and (182) this yields

$$\Pi^{\alpha}_{fgh*}\big(\kappa^{\dagger}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)\big) = (p-1) \cdot \mathscr{E}_f(f,g,h) \cdot \kappa(f,g,h),$$

where (recalling that $a_p(\xi) = \alpha_\xi + \beta_\xi$ and $\alpha_\xi \beta_\xi = \chi_{\boldsymbol{\xi}}(p) p^{s-1}$ for $\xi \in S_s(N, \chi_{\boldsymbol{\xi}})$, that $2r = r_1 + r_2 + r_3$ and that $\chi_{\boldsymbol{f}} \chi_{\boldsymbol{g}} \chi_{\boldsymbol{h}}(p) = 1$ by Assumption 1.2)

$$\begin{aligned}
\mathscr{E}_f(f,g,h) &= \alpha_f + \beta_f - \beta_f - \frac{\beta_f}{p} - \frac{\chi_{\boldsymbol{f}}(p)\beta_g\alpha_h}{p^{r_2+r_3-r+1}} - \frac{\chi_{\boldsymbol{f}}(p)\beta_g\beta_h}{p^{r_2+r_3-r+1}} - \frac{\chi_{\boldsymbol{f}}(p)\alpha_g\beta_h}{p^{r_3+r_2-r+1}} \\
&\quad - \frac{\chi_{\boldsymbol{f}}(p)\beta_g\beta_h}{p^{r_3+r_2-r+1}} + \frac{\bar{\chi}_{\boldsymbol{g}}(p)\beta_f\alpha_g\beta_g}{p^{r_2+2}} + \frac{\bar{\chi}_{\boldsymbol{g}}(p)\beta_f\beta_g^2}{p^{r_2+2}} + \frac{\bar{\chi}_{\boldsymbol{h}}(p)\beta_f\beta_h\alpha_h}{p^{r_3+2}} + \frac{\bar{\chi}_{\boldsymbol{h}}(p)\beta_f\beta_h^2}{p^{r_3+2}} \\
&\quad + \frac{\chi_{\boldsymbol{f}}(p)\beta_g\beta_h}{p^{r_2+r_3-r+1}} + \frac{\chi_{\boldsymbol{f}}(p)\beta_g\beta_h}{p^{r_2+r_3-r+2}} - \frac{\alpha_f\beta_f\beta_g\beta_h}{p^{r_1+r_2+r_3-r+3}} - \frac{\beta_f^2\beta_g\beta_h}{p^{r_1+r_2+r_3-r+3}} \\
&= \alpha_f \cdot \left(1 - \frac{\beta_f\beta_g\alpha_h}{p^{r+2}} - \frac{\beta_f\alpha_g\beta_h}{p^{r+2}} - \frac{\beta_f\beta_g\beta_h}{p^{r+2}} + \frac{\chi_{\boldsymbol{h}}(p)\beta_f^2\beta_g^2}{p^{r_1+r_2+3}}\right. \\
&\qquad\qquad \left. + \frac{\bar{\chi}_{\boldsymbol{f}}(p)\beta_f^2}{p^{r_1+2}} + \frac{\chi_{\boldsymbol{g}}(p)\beta_f^2\beta_h^2}{p^{r_1+r_3+3}} - \frac{\bar{\chi}_{\boldsymbol{f}}(p)\beta_f^3\beta_g\beta_h}{p^{r+r_1+4}}\right) \\
&= \alpha_f \cdot \left(1 - \frac{\beta_f\alpha_g\beta_h}{p^{r+2}}\right)\left(1 - \frac{\beta_f\beta_g\alpha_h}{p^{r+2}}\right)\left(1 - \frac{\beta_f\beta_g\beta_h}{p^{r+2}}\right).
\end{aligned} \tag{183}$$

This concludes the proof of the proposition. $\square$

8.4. p**-stabilisation of de Rham classes.** — Let $w=(k,l,m)$ be a classical triple in Σ, such that p does not divide the conductors of $\boldsymbol{f}_k, \boldsymbol{g}_l$ and $\boldsymbol{h}_m$. As in the previous section denote by f_k, g_l and h_m the modular forms of level $\Gamma_1(N)$ with ordinary p-stabilisations $\boldsymbol{f}_k$, $\boldsymbol{g}_l$ and $\boldsymbol{h}_m$ respectively. For each integer $M \geqslant 3$ denote by $V^*_{\mathrm{dR},\boldsymbol{r}}(M)$ the $(k+l+m-2)/2$-th Tate twist of the tensor product of the de Rham cohomology groups $H^1_{\mathrm{dR}}(Y_1(M)_{\mathbf{Q}_p}, \mathscr{S}_{\mathrm{dR},r_j})_L$, for $j=1,2,3$. Then the restriction of the morphism

$$V^*_{\mathrm{dR},\boldsymbol{r}}(N) \longrightarrow V^*_{\mathrm{dR},\boldsymbol{r}}(Np)$$

defined by

$$\left(\mathrm{pr}_1^* - \frac{\beta_{\boldsymbol{f}_k}}{p^{k-1}} \cdot \mathrm{pr}_p^*\right) \otimes \left(\mathrm{pr}_1^* - \frac{\beta_{\boldsymbol{g}_l}}{p^{l-1}} \cdot \mathrm{pr}_p^*\right) \otimes \left(\mathrm{pr}_1^* - \frac{\beta_{\boldsymbol{h}_m}}{p^{m-1}} \cdot \mathrm{pr}_p^*\right)$$

to the (f,g,h)-isotypic component of $V^*_{\mathrm{dR},\boldsymbol{r}}(N)$ gives a *p-stabilisation* isomorphism

$$\Pi^{\alpha *}_{f_k g_l h_m} : V^*_{\mathrm{dR}}(f_k, g_l, h_m) \cong V^*_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m).$$

Lemma 8.4. — *Assume that p does not divide the conductors of $\boldsymbol{f}_k, \boldsymbol{g}_l$ and $\boldsymbol{h}_m$. Then*

$$\Pi^{\alpha *}_{f_k g_l h_m}\big(\eta^{\alpha}_{f_k} \otimes \omega_{g_l} \otimes \omega_{h_m}\big) = (p-1)\alpha_{\boldsymbol{f}_k}\left(1-\frac{\beta_{\boldsymbol{f}_k}}{\alpha_{\boldsymbol{f}_k}}\right)\left(1-\frac{\beta_{\boldsymbol{f}_k}}{p\alpha_{\boldsymbol{f}_k}}\right) \cdot \eta^{\alpha}_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_l} \otimes \omega_{\boldsymbol{h}_m}.$$

Proof. — Set $\Pi^{\alpha*}_k = \mathrm{pr}_1^* - \frac{\beta_{\boldsymbol{f}_k}}{p^{k-1}} \cdot \mathrm{pr}_p^*$, set $\Pi^{\alpha}_{k*} = \mathrm{pr}_{1*} - \frac{\beta_{\boldsymbol{f}_k}}{p^{k-1}} \cdot \mathrm{pr}_{p*}$ and define similarly $\Pi^{\alpha*}_l$ and $\Pi^{\alpha*}_m$. By the definition of p-stabilisation (cf. Equation (54)), one has $\Pi^{\alpha*}_k(\omega_\xi) = \omega_{\xi_\alpha}$ for any $\xi \in S_k(N,L)_{f_k}$, and similarly for $\Pi^{\alpha*}_l$ and $\Pi^{\alpha*}_m$. In particular

$$\Pi^{\alpha*}_l(\omega_{g_l}) = \omega_{\boldsymbol{g}_l} \quad \text{and} \quad \Pi^{\alpha*}_m(\omega_{h_m}) = \omega_{\boldsymbol{h}_m}. \tag{184}$$

According to Equation (3.4.5) on Page 76 of [Shi71], one has

$$(a^w, b^w)_M = M^{n-2} \cdot (a,b)_M$$

for any cuspidal forms a and b of weight n and level $\Gamma_1(M)$, where we recall that $\cdot^w = w_M(\cdot)$ is a shorthand for the image of $\cdot$ under the Atkin–Lehner operator w_M defined in Equation (33), and $(\cdot,\cdot)_M$ is the Petersson product on $S_n(M,\mathbf{C})$ defined after Equation (35). It follows that (cf. Equation (34) and the discussion following it)

$$\big\langle \eta_{\boldsymbol{f}_k}, w_{Np} \circ \Pi^{\alpha*}_k(\omega_\xi)\big\rangle_{\boldsymbol{f}_k} = \frac{(\boldsymbol{f}_k^w, \xi_\alpha^w)_{Np}}{(\boldsymbol{f}_k^w, \boldsymbol{f}_k^w)_{Np}} = \frac{(\boldsymbol{f}_k, \xi_\alpha)_{Np}}{(\boldsymbol{f}_k, \boldsymbol{f}_k)_{Np}} = \frac{(f_k, \xi)_N}{(f_k, f_k)_N}. \tag{185}$$

for each ξ in $S_k(N,L)_{f_k}$, where $\xi^w_\alpha = w_{Np}(\xi_\alpha)$.

The (easily verified) relations $w_{Np} \circ \mathrm{pr}_1^* = \mathrm{pr}_p^* \circ w_N$ and $w_{Np} \circ \mathrm{pr}_p^* = p^{k-2} \cdot \mathrm{pr}_1^* \circ w_N$ yield

$$\begin{aligned}\Pi^{\alpha}_{k*} \circ w_{Np} \circ \Pi^{\alpha*}_k &= \left(\mathrm{pr}_{1*} - \frac{\beta_{\boldsymbol{f}_k}}{p^{k-1}} \cdot \mathrm{pr}_{p*}\right) \circ \left(\mathrm{pr}_p^* - \frac{\beta_{\boldsymbol{f}_k}}{p} \cdot \mathrm{pr}_1^*\right) \circ w_N \\ &= (p-1)\left(T_p' - \frac{2(p+1)\beta_{\boldsymbol{f}_k}}{p} + \frac{\beta^2_{\boldsymbol{f}_k}}{p^k} \cdot T_p\right) \circ w_N.\end{aligned}$$

As $a_p(f_k) = \alpha_{\boldsymbol{f}_k} + \beta_{\boldsymbol{f}_k}$ and $T'_p \circ w_N$ and $T_p \circ w_N$ act respectively as $a_p(f_k) \cdot w_N$ and $\bar{\chi}_{\boldsymbol{f}}(p) a_p(f_k) \cdot w_N$ on $V^*_{\mathrm{dR}}(f_k)$, a direct computation then gives (cf. Equation (183))

$$\Pi^{\alpha}_{k*} \circ w_{Np} \circ \Pi^{\alpha*}_{k} = (p-1)\alpha_{\boldsymbol{f}_k} \left(1 - \frac{\beta_{\boldsymbol{f}_k}}{\alpha_{\boldsymbol{f}_k}}\right)\left(1 - \frac{\beta_{\boldsymbol{f}_k}}{p\alpha_{\boldsymbol{f}_k}}\right) \cdot w_N$$

as morphisms from $V^*_{\mathrm{dR}}(f_k)$ to $V^*_{\mathrm{dR}}(f^*_k)$. Because $\Pi^{\alpha*}_k$ and Π^{α}_{k*} are adjoint to each other under the pairings $\langle \cdot, \cdot \rangle_{f_k}$ and $\langle \cdot, \cdot \rangle_{\boldsymbol{f}_k}$, this implies

$$\frac{\langle \Pi^{\alpha*}_k(\eta_{f_k}), w_{Np} \circ \Pi^{\alpha*}_k(\omega_\xi)\rangle_{\boldsymbol{f}_k}}{(p-1)\alpha_{\boldsymbol{f}_k}\left(1 - \frac{\beta_{\boldsymbol{f}_k}}{\alpha_{\boldsymbol{f}_k}}\right)\left(1 - \frac{\beta_{\boldsymbol{f}_k}}{p\alpha_{\boldsymbol{f}_k}}\right)} = \langle \eta_{f_k}, w_N(\omega_\xi)\rangle_{f_k} = \frac{(f^w_k, \xi^w)_N}{(f^w_k, f^w_k)_N} = \frac{(f_k, \xi)_N}{(f_k, f_k)_N} \tag{186}$$

for each ξ in $S_k(N, L)_{f_k} = \mathrm{Fil}^1 V^*_{\mathrm{dR}}(f_k)$. As the composition $w_{Np} \circ \Pi^{\alpha*}_k$ gives an isomorphism between $S_k(N, L)_{f_k}$ and $S_k(Np, L)_{\boldsymbol{f}^*_k}$, and the isomorphism

$$\Pi^{\alpha*}_k : V^*_{\mathrm{dR}}(f_k) \cong V^*_{\mathrm{dR}}(\boldsymbol{f}_k)$$

commutes with the action of the Frobenius endomorphism on both sides, comparing Equation (185) with Equation (186) yields the identity

$$\Pi^{\alpha*}_k(\eta^{\alpha}_{f_k}) = (p-1)\alpha_{\boldsymbol{f}_k}\left(1 - \frac{\beta_{\boldsymbol{f}_k}}{\alpha_{\boldsymbol{f}_k}}\right)\left(1 - \frac{\beta_{\boldsymbol{f}_k}}{p\alpha_{\boldsymbol{f}_k}}\right) \cdot \eta^{\alpha}_{\boldsymbol{f}_k}$$

(cf. Equation (37) for the definition of the differential $\eta^{\alpha}_{\boldsymbol{f}_k}$). The lemma follows from the previous equation and Equation (184). □

8.5. Conclusion of the proof. — This section concludes the proof of Theorem A.

According to Corollary 8.2 the class $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ belongs to $H^1_{\mathrm{bal}}(\mathbf{Q}, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$. Let Σ^o_{bal} be the set of balanced triples (k, l, m) such that $k, l, m \geqslant 3$ and p does not divide the conductors of $\boldsymbol{f}_k, \boldsymbol{g}_l$ and $\boldsymbol{h}_m$. Let $\boldsymbol{\xi}$ denote one of $\boldsymbol{f}, \boldsymbol{g}$ and $\boldsymbol{h}$. Because Σ^o_{bal} is dense in $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}}$, in order to prove Theorem A it is sufficient to show that

$$\mathscr{L}_{\boldsymbol{\xi}}\big(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})\big)(w) = \mathscr{L}^{\xi}_p(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m) \tag{187}$$

for every $w = (k, l, m)$ in Σ^o_{bal}, where to ease the notation one writes

$$\mathscr{L}_{\boldsymbol{\xi}}(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) = \mathscr{L}_{\boldsymbol{\xi}}(\mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))).$$

Fix such a triple w and to ease notation set $\alpha_f = \alpha_{\boldsymbol{f}_k}, \beta_f = \beta_{\boldsymbol{f}_k}, \alpha_g = \alpha_{\boldsymbol{g}_l}$ et cetera.

Consider first the case $\boldsymbol{\xi} = \boldsymbol{f}$. Write as usual $\boldsymbol{r} = (r_1, r_2, r_3) = (k-2, l-2, m-2)$. Since p does not divide the conductor of $\boldsymbol{f}_k, \boldsymbol{g}_l$ and $\boldsymbol{h}_m$, the Ramanujan–Petersson conjecture gives

$$\left(1 - \frac{\beta_f}{\alpha_f}\right)\left(1 - \frac{\beta_f}{p\alpha_f}\right)\left(1 - \frac{\alpha_f \beta_g \beta_h}{p^{r+2}}\right) \neq 0.$$

Moreover $\boldsymbol{f}_k = f_\alpha$ (resp., $\boldsymbol{g}_l = g_\alpha$, $\boldsymbol{h}_m = h_\alpha$) is the ordinary p-stabilisation of a cusp form $f = f_k$ (resp., $g = g_l$, $h = h_m$) of level $\Gamma_1(N)$. Proposition 7.3, the definition of $\log_p(\cdot)_f$ and Lemma 8.4 then prove that

$$(-1)^{r-r_1}(r - r_1)! \cdot \mathscr{L}_{\boldsymbol{f}}\big(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})\big)(w)$$

is equal to

$$\frac{\left(1-\frac{\beta_f\alpha_g\alpha_h}{p^{r+2}}\right)}{\left(1-\frac{\beta_f}{\alpha_f}\right)\left(1-\frac{\beta_f}{p\alpha_f}\right)\left(1-\frac{\alpha_f\beta_g\beta_h}{p^{r+2}}\right)}\cdot\log_p\big(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_w\big)\big(\Pi^{\alpha*}_{fgh}\big(\eta^\alpha_f\otimes\omega_g\otimes\omega_h\big)\big),$$

where $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_w\in H^1_{\mathrm{bal}}(\mathbf{Q}_p,V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m))$ is the image of $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ under the specialisation map ρ_w (and as usual $\log_p(\cdot)$ is a shorthand for $\log_p(\mathrm{res}_p(\cdot))$ for all global classes $\cdot$ in $H^1_{\mathrm{bal}}(\mathbf{Q},V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m))$). As $\Pi^{\alpha*}_{fgh}$ is the transpose of Π^α_{fgh*}, the functoriality under correspondences of the Faltings comparison isomorphism for $E_1^\cdot(N)$ and of the Leray spectral sequence (from which Equation (26) is deduced) imply that

$$\log_p\big(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_w\big)\circ\Pi^{\alpha*}_{fgh}=\log_p\Big(\Pi^\alpha_{fgh*}\big(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_w\big)\Big)\tag{188}$$

as functionals on $\mathrm{Fil}^0V^*_{\mathrm{dR}}(f,g,h)$. According to Theorem 8.1 and Proposition 8.3

$$\Pi^\alpha_{fgh*}\big(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_w\big)$$

equals

$$\left(1-\frac{\alpha_f\beta_g\beta_h}{p^{r+2}}\right)\left(1-\frac{\beta_f\alpha_g\beta_h}{p^{r+2}}\right)\left(1-\frac{\beta_f\beta_g\alpha_h}{p^{r+2}}\right)\left(1-\frac{\beta_f\beta_g\beta_h}{p^{r+2}}\right)\cdot\kappa(f,g,h).\tag{189}$$

The previous three equations show that $\mathscr{L}_{\boldsymbol{f}}\big(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})\big)(w)$ is equal to the product of

$$\frac{(-1)^{r-r_1}}{(r-r_1)!}\frac{\left(1-\frac{\beta_f\alpha_g\alpha_h}{p^{r+2}}\right)\left(1-\frac{\beta_f\alpha_g\beta_h}{p^{r+2}}\right)\left(1-\frac{\beta_f\beta_g\alpha_h}{p^{r+2}}\right)\left(1-\frac{\beta_f\beta_g\beta_h}{p^{r+2}}\right)}{\left(1-\frac{\beta_f}{\alpha_f}\right)\left(1-\frac{\beta_f}{p\alpha_f}\right)}$$

and

$$\log_p\big(\kappa(f,g,h)\big)(\eta^\alpha_f\otimes\omega_g\otimes\omega_h),$$

which in turn is equal to $\mathscr{L}^f_p(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)$ by the explicit reciprocity law Proposition 3.6. This proves Equation (187), and with it Theorem A, for $\boldsymbol{\xi}=\boldsymbol{f}$.

The proofs of Equation (187) for $\boldsymbol{\xi}=\boldsymbol{g},\boldsymbol{h}$ are similar. We give the details for $\boldsymbol{\xi}=\boldsymbol{g}$. Exchanging the roles of $\boldsymbol{f}$ and $\boldsymbol{g}$ in the constructions of Sections 7.1, 7.3, and 8.4, (the resulting) Propositions 7.3 and 8.4 proves that

$$(-1)^{r-r_2}(r-r_2)!\cdot\mathscr{L}_{\boldsymbol{g}}\big(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})\big)(w)$$

is equal to

$$\frac{\left(1-\frac{\alpha_f\beta_g\alpha_h}{p^{r+2}}\right)}{\left(1-\frac{\beta_g}{\alpha_g}\right)\left(1-\frac{\beta_g}{p\alpha_g}\right)\left(1-\frac{\beta_f\alpha_g\beta_h}{p^{r+2}}\right)}\cdot\log_p\big(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_w\big)\big(\Pi^{\alpha*}_{fgh}\big(\omega_f\otimes\eta^\alpha_g\otimes\omega_h\big)\big).$$

Equations (188)–(189) (which are symmetric in $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$) then prove that the special value $\mathscr{L}_{\boldsymbol{g}}(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))(w)$ is the product of

$$\frac{(-1)^{r-r_2}}{(r-r_2)!}\cdot\frac{\left(1-\frac{\alpha_f\beta_g\alpha_h}{p^{r+2}}\right)\left(1-\frac{\alpha_f\beta_g\beta_h}{p^{r+2}}\right)\left(1-\frac{\beta_f\beta_g\alpha_h}{p^{r+2}}\right)\left(1-\frac{\beta_f\beta_g\beta_h}{p^{r+2}}\right)}{\left(1-\frac{\beta_g}{\alpha_g}\right)\left(1-\frac{\beta_g}{p\alpha_g}\right)}.$$

and

$$\log_p\big(\kappa(f,g,h)\big)(\omega_f\otimes\eta_g^\alpha\otimes\omega_h)$$

This is precisely the formula for $\mathscr{L}_p^g(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)$ obtained by replacing the triple $(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)$ with $(\boldsymbol{g}_l,\boldsymbol{f}_k,\boldsymbol{h}_m)$ in the statement of the explicit reciprocity law Proposition 3.6, thus concluding the proof of Theorem A.

9. Proof of Theorem B

This section proves Theorem B stated in the Introduction. The notations and assumptions are as in Section 1.2. Then $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ is a level-N test vector for $(\boldsymbol{f}^\sharp,\boldsymbol{g}^\sharp,\boldsymbol{h}^\sharp)$ and $w_o=(k,l,m)$ is an *unbalanced* triple in $\Sigma_{\boldsymbol{f}}$.

For the convenience of the reader, we briefly describe the contents of the different subsections. Section 9.1 proves Theorem B assuming that w_o is not exceptional in the sense of Section 1.2. Section 9.2 proves an exceptional zero formula for the big logarithm $\mathscr{L}_{\boldsymbol{f}}$ when w_o is *exceptional of type* (5), viz. in the exceptional case arising from the vanishing at w_o of the analytic f-Euler factor $\mathcal{E}_f^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ introduced in Equation (4). Section 9.3 constructs the improved diagonal classes $\kappa_g^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ and $\kappa_h^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ introduced in Section 1.2. Their construction is nontrivial only when the g-Euler factor $\mathcal{E}_g(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ defined in Equation (1) vanishes at w_o, that is when w_o is exceptional of type (3) (cf. Section 1.2). Section 9.4 finally proves Theorem B when w_o is exceptional.

9.1. Proof in the non-exceptional case. — This section proves Theorem B when w_o is not exceptional.

Lemma 9.1. — *The Bloch–Kato finite, exponential and geometric subspaces of the local cohomology group $H^1(\mathbf{Q}_p,V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m))$ are all equal.*

Proof. — We use the notations introduced in the proof of Lemma 3.5. As in loco citato, it is sufficient to prove that $D_{\mathrm{st}}^{\varphi=1,\mathrm{N}=0}$ vanishes.

Since $k\geqslant l+m$, one has $\mathrm{ord}_p(\alpha_w^f)\leqslant -1$ and $\mathrm{ord}_p(\beta_w^\cdot)\leqslant -1$ for $\cdot=\emptyset,g,h$, hence $D_{\mathrm{st}}^{\varphi=1}$ is contained in the L-module generated by $\mathbf{a}_w,\mathbf{a}_w^g,\mathbf{a}_w^h$ and $\mathbf{b}_w^f$. Moreover

$$|\alpha_w|_\infty=p^{(\varepsilon_w-1)/2},\quad |\alpha_w^\xi|_\infty=p^{(\varepsilon_w-2\cdot\varepsilon_\xi-1)/2}\quad\text{and}\quad |\beta_w^f|_\infty=p^{(2\cdot\varepsilon_f-\varepsilon_w-1)/2}$$

for $\xi=g,h$ (cf. loco citato for the notation). It follows that $D_{\mathrm{st}}^{\varphi=1}$ is equal to zero if $\varepsilon_w=0$ or $\varepsilon_w=2$. If $\varepsilon_w=3$, then $D_{\mathrm{st}}^{\varphi=1}$ is contained in $L\cdot\mathbf{a}_w^g\oplus L\cdot\mathbf{a}_w^h$ and

$$\mathrm{N}(r\cdot\mathbf{a}_w^g+s\cdot\mathbf{a}_w^h)=(r+s)\cdot\mathbf{b}_w^f+r\cdot\mathbf{b}_w^h+s\cdot\mathbf{b}_w^g,$$

for each r,s in L, hence $D_{\mathrm{st}}^{\varphi=1,\mathrm{N}=0}=0$. If $\varepsilon_w=\varepsilon_\xi=1$ for $\xi=g,h$ and $\{\xi,\zeta\}=\{g,h\}$, then $D_{\mathrm{st}}^{\varphi=1}$ is contained in the L-module generated by $\mathbf{a}_w$ and $\mathbf{a}_w^\zeta$, and

$$\mathrm{N}(r\cdot\mathbf{a}_w+s\cdot\mathbf{a}_w^\zeta)=r\cdot\mathbf{a}_w^\xi+s\cdot\mathbf{b}_w^f,$$

hence $D_{\mathrm{st}}^{\varphi=1,\mathrm{N}=0}=0$. Finally, if $\varepsilon_w=\varepsilon_f=1$, one has

$$\mathrm{N}(r\cdot\mathbf{a}_w+s\cdot\mathbf{a}_w^g+t\cdot\mathbf{a}_w^h+u\cdot\mathbf{b}_w^f)=r\cdot\mathbf{a}_w^f+s\cdot\mathbf{b}_w^h+t\cdot\mathbf{b}_w^g+u\cdot\mathbf{b}_w,$$

hence $D^{\varphi=1,\mathrm{N}=0}$ vanishes also in this case, concluding the proof of the lemma. □

In light of Lemma 9.1, in order to prove Theorem B it is sufficient to show that

$$\exp_p^*(\kappa(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m))=0 \quad \text{if and only if} \quad L(f_k^\sharp\otimes g_l^\sharp\otimes h_m^\sharp,(k+l+m-2)/2)=0, \tag{190}$$

where $\exp_p^*$ is the Bloch–Kato dual exponential and $\exp_p^*(\cdot)=\exp_p^*(\mathrm{res}_p(\cdot))$ for any $\cdot$ in the global cohomology group $H^1(\mathbf{Q},V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m))$.

Set

$$V^{\cdot}(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)^{\pm}=V^{\cdot}(\boldsymbol{f}_k)^{\pm}\otimes_L V^{\cdot}(\boldsymbol{g}_l)\otimes_L V^{\cdot}(\boldsymbol{h}_m)(c.), \tag{191}$$

where $c.=(4-k-l-m)/2$ and $c.=(k+l+m-2)/2$ if $\cdot=\emptyset$ and $\cdot=*$ respectively. Because $k\geqslant l+m$ the inclusion $V^*(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)^+\hookrightarrow V^*(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)$ and the projection $V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)\twoheadrightarrow V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)^-$ induce isomorphisms

$$\begin{aligned} & D_{\mathrm{st}}(V^*(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)^+)\cong V^*_{\mathrm{dR}}(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)/\mathrm{Fil}^0 \\ \text{and}\qquad & \mathrm{Fil}^0V_{\mathrm{dR}}(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)\cong D_{\mathrm{st}}(V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)^-)\end{aligned} \tag{192}$$

respectively. (If $\boldsymbol{g}_l$ or $\boldsymbol{h}_m$ is a weight-one modular form, the modules $V_{\mathrm{dR}}(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)$ and $V^*_{\mathrm{dR}}(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)$ are defined using the conventions introduced in the last item of Sections 5, cf. Equations (127) and (129) and Section 7.1.1.1.) Let

$$\langle\cdot,\cdot\rangle_{\boldsymbol{f}_k\boldsymbol{g}_l\boldsymbol{h}_m}:\mathrm{Fil}^0V_{\mathrm{dR}}(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)\otimes_L V^*_{\mathrm{dR}}(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)/\mathrm{Fil}^0\longrightarrow L$$

be the perfect pairing induced on the de Rham modules by the specialisation at w_o (cf. Equations (106)–(109)) of the tensor product of the pairings $\langle\cdot,\cdot\rangle_{\boldsymbol{\xi}}$ defined in Equation (103), for $\boldsymbol{\xi}=\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}$. (According to Equation (109), if k,l and m are all geometric this is also induced by the tensor product of the pairings $\langle\cdot,\cdot\rangle_{\xi}$ introduced in Equation (31), for $\xi=\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m$.) By construction $V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f$ is a $G_{\mathbf{Q}_p}$-submodule of $V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)^-$, and the image of

$$D_{\mathrm{cris}}(V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f)\hookrightarrow D_{\mathrm{st}}(V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)^-)\cong\mathrm{Fil}^0V_{\mathrm{dR}}(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)$$

(cf. Equation (192)) is orthogonal under $\langle\cdot,\cdot\rangle_{\boldsymbol{f}_k\boldsymbol{g}_l\boldsymbol{h}_m}$ to the kernel of the projection

$$V^*_{\mathrm{dR}}(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)/\mathrm{Fil}^0\cong D_{\mathrm{st}}(V^*(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)^+)\twoheadrightarrow D_{\mathrm{cris}}(V^*(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f),$$

where $V^*(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f$ is the c_*-th Tate twist of $V^*(\boldsymbol{f}_k)^+\otimes_L V^*(\boldsymbol{g}_l)^-\otimes_L V^*(\boldsymbol{h}_m)^-$. Moreover, after setting $x_o=(w_o,(k-l-m)/2)$ (and identifying $D_{\mathrm{cris}}(\mathbf{Q}_p(i))$ with $\mathbf{Q}_p\cdot t^i$), one has by definition (cf. Section 7)

$$\begin{aligned} & D_{\mathrm{cris}}(V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f)=\bar{D}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f\otimes_{x_o}L \\ \text{and}\quad & D_{\mathrm{cris}}(V^*(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f)=\bar{D}^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f\otimes_{x_o}L.\end{aligned}$$

By Corollary 8.2 the class $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ is balanced, viz. its restriction at p is the image of a (unique) class $\breve{\kappa}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ in $H^1(\mathbf{Q}_p,\mathscr{F}^2V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))$. Let $\breve{\kappa}(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)$ be the specialisation of $\breve{\kappa}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ at w_o, and let $\kappa(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f$ be its image in $H^1(\mathbf{Q}_p,V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f)$ under the morphism p_{f*} (cf. Section 7.2). As the diagram

$$\begin{tikzcd} H^1(\mathbf{Q}_p,\mathscr{F}^2V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)) \arrow[r] \arrow[d, "p_{f*}"'] & H^1(\mathbf{Q}_p,V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)) \arrow[d] \\ H^1(\mathbf{Q}_p,V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f) \arrow[r] & H^1(\mathbf{Q}_p,V(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)^-) \end{tikzcd} \tag{193}$$

commutes, the previous paragraph reduces the proof of Equation (190) to the following claim.

($\boldsymbol{\alpha}$) The Garrett L-function $L(f_k^\sharp \otimes g_l^\sharp \otimes h_m^\sharp, s)$ vanishes at $s = (k+l+m-2)/2$ if and only if

$$\big\langle \exp_p^*(\kappa(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)_f), \mu\big\rangle_{\boldsymbol{f}_k\boldsymbol{g}_l\boldsymbol{h}_m} = 0$$

for all differentials μ in $\bar{D}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f \otimes_{x_o} L$. Here $\exp_p^*$ is the Bloch–Kato dual exponential on $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)_f)$ and $\langle \cdot, \cdot \rangle_{\boldsymbol{f}_k\boldsymbol{g}_l\boldsymbol{h}_m}$ is the specialisation at x_o of the bilinear form $\langle \cdot, \cdot \rangle_{\boldsymbol{fgh}}$ defined in Equation (139).

As $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ varies through the level-N test vectors for $(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$, the specialisations at x_o of the associated $\bar{\mathscr{O}}_{\boldsymbol{fgh}}$-adic differentials $\eta_{\boldsymbol{f}}\omega_{\boldsymbol{g}}\omega_{\boldsymbol{h}}$ (cf. Equation (142)) generate $\bar{D}^*(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)_f \otimes_{x_o} L$. This follows from the results of Sections 2.5, 5 and 7.1.1. As a consequence the claim ($\boldsymbol{\alpha}$) is equivalent to

($\boldsymbol{\beta}$) The Garrett L-function $L(f_k^\sharp \otimes g_l^\sharp \otimes h_m^\sharp, s)$ vanishes at $s = (k+l+m-2)/2$ if and only if

$$\big\langle \exp_p^*(\kappa(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)_f), \eta_{\boldsymbol{f}_k}\omega_{\boldsymbol{g}_l}\omega_{\boldsymbol{h}_m}\big\rangle_{\boldsymbol{f}_k\boldsymbol{g}_l\boldsymbol{h}_m} = 0$$

for all level-N test vectors $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ for $(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$, where $\eta_{\boldsymbol{f}_k}\omega_{\boldsymbol{g}_l}\omega_{\boldsymbol{h}_m}$ in $D_{\mathrm{cris}}(V^*(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)_f)$ is the specialisation of $\eta_{\boldsymbol{f}}\omega_{\boldsymbol{g}}\omega_{\boldsymbol{h}}$ at x_o (cf. Section 7.1.1).

Remark 9.2. — As explained in Remark 1.3(3), the class $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$, hence $\breve{\kappa}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ and a fortiori $\kappa(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)_f$, is independent of the choice of the level-N test vector $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ for $(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$.

Assume in the rest of this section that w_o is *not* exceptional. This implies that

$$\beta_{\boldsymbol{f}_k}\alpha_{\boldsymbol{g}_l}\alpha_{\boldsymbol{h}_m} \neq p^{(k+l+m-2)/2}$$

for each test vector $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$. (As usual $\beta_{\boldsymbol{f}_k} = \chi_{\boldsymbol{f}}(p)p^{k-1}/a_p(k)$, hence the previous equation is a consequence of Equation (5) and the Ramanujan–Petersson conjecture.) According to Theorem A, (the proof of) Proposition 7.3 and the previous equation, for each level-N test vector $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ one has

$$\mathscr{L}_p^f(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m) = \mathscr{E}_{w_o} \cdot \big\langle \exp_p^*(\kappa(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)_f), \eta_{\boldsymbol{f}_k}\omega_{\boldsymbol{g}_l}\omega_{\boldsymbol{h}_m}\big\rangle_{\boldsymbol{f}_k\boldsymbol{g}_l\boldsymbol{h}_m}$$

for a *non-zero* algebraic number $\mathscr{E}_{w_o}$. The statement ($\boldsymbol{\beta}$) can then be rephrased as

($\boldsymbol{\gamma}$) $L(f_k^\sharp \otimes g_l^\sharp \otimes h_m^\sharp, (k+l+m-2)/2) = 0$ if and only if $\mathscr{L}_p^f(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m) = 0$ for all level-N test vectors $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ for $(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$.

Under the current Assumption 1.7 on the local signs $\varepsilon_\ell(f_k^\sharp, g_l^\sharp, h_m^\sharp)$, the claim ($\boldsymbol{\gamma}$) is a consequence of Jacquet's conjecture proved by Harris–Kudla in [**HK91**]. Indeed, as w_o is not exceptional, there exist test vectors $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ such that $\mathscr{L}_p^f(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ is a non-zero multiple of the complex central value $L(f_k^\sharp \otimes g_l^\sharp \otimes h_m^\sharp, (k+l+m-2)/2)$ (cf. Section 6 and [**DR14**, Theorems 4.2 and 4.7]).

9.2. Derivatives of big logarithms I. — Assume in this section that the unbalanced classical triple w_o in $\Sigma_{\boldsymbol{f}}$ satisfies the conditions displayed in Equation (5) of Section 1.2. In particular $w_o = (2, 1, 1)$.

Denote by $\mathscr{I} = \mathscr{I}_{w_o}$ the ideal of functions in $\mathscr{O}_{\boldsymbol{fgh}}$ which vanish at w_o. The exceptional zero condition (5) and Proposition 7.3 imply that the big logarithm $\mathscr{L}_{\boldsymbol{f}}$ takes values in $\mathscr{I}$. According to loc. cit. $\mathscr{L}_{\boldsymbol{f}}$ factors through the morphism induced by the projection $p_f : \mathscr{F}^2 V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) \longrightarrow V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f$ and we write again

$$\mathscr{L}_{\boldsymbol{f}} : H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f) \longrightarrow \mathscr{I}$$

for the resulting map. The aim of this section is to prove Proposition 9.3 below, which gives a formula for the derivative of $\mathscr{L}_{\boldsymbol{f}}$ at w_o, namely for the the composition of $\mathscr{L}_{\boldsymbol{f}}$ with the projection $\mathscr{I} \to \mathscr{I}/\mathscr{I}^2$. In order to state it we need to introduce further notations.

Since $\bar{\chi}_{\boldsymbol{f}}(p) = \chi_{\boldsymbol{g}}\chi_{\boldsymbol{h}}(p)$ and $\bar{\chi}_{\boldsymbol{f}}(p) \cdot a_p(2) = b_p(1) \cdot c_p(1)$ under the current assumptions, the $G_{\mathbf{Q}_p}$-representation

$$V(\boldsymbol{f}_2)^-_{\beta\beta} \stackrel{\text{def}}{=} V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f \otimes_{w_o} L = V(\boldsymbol{f}_2)^- \otimes_L V(\boldsymbol{g}_1)^+ \otimes_L V(\boldsymbol{h}_1)^+$$

is isomorphic to the direct sum of a finite number of copies of the trivial p-adic representation of $G_p = G_{\mathbf{Q}_p}$ (cf. Section 7.2). Let G_p^{ab} be the Galois group of the maximal abelian extension of $\mathbf{Q}_p$, and let

$$\text{rec}_p : \mathbf{Q}_p^* \hat{\otimes} \mathbf{Q}_p \cong G_p^{\text{ab}} \hat{\otimes} \mathbf{Q}_p$$

be the reciprocity map of local class field theory, normalised by requiring that $\text{rec}_p(p^{-1})$ is an arithmetic Frobenius. Identify $H^1(\mathbf{Q}_p, \mathbf{Q}_p) = \text{Hom}_{\text{cont}}(G_p^{\text{ab}}, \mathbf{Q}_p)$ with $\text{Hom}_{\text{cont}}(\mathbf{Q}_p^*, \mathbf{Q}_p)$ under rec_p, so that

$$H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2)^-_{\beta\beta}) = \text{Hom}_{\text{cont}}(\mathbf{Q}_p^*, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} V(\boldsymbol{f}_2)^-_{\beta\beta} \tag{194}$$
$$\text{and} \qquad D_{\text{cris}}(V(\boldsymbol{f}_2)^-_{\beta\beta}) = V(\boldsymbol{f}_2)^-_{\beta\beta}.$$

Under these identifications the Bloch–Kato dual exponential $\exp_p^*$ on $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2)^-_{\beta\beta})$ satisfies

$$\exp_p^*(\psi \otimes v) = \psi(e(1)) \cdot v \in V(\boldsymbol{f}_2)^-_{\beta\beta} \tag{195}$$

for all $\psi \otimes v$ in $\text{Hom}_{\text{cont}}(\mathbf{Q}_p^*, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} V(\boldsymbol{f}_2)^-_{\beta\beta}$, where

$$e(1) = (1+p) \hat{\otimes} \log_p(1+p)^{-1} \in \mathbf{Z}_p^* \hat{\otimes} \mathbf{Q}_p.$$

Similarly the $G_{\mathbf{Q}_p}$-module

$$V^*(\boldsymbol{f}_2)^+_{\beta\beta} \stackrel{\text{def}}{=} V^*(\boldsymbol{f}_2)^+ \otimes_L V^*(\boldsymbol{g}_1)^- \otimes_L V^*(\boldsymbol{h}_1)^-$$

is isomorphic to the direct sum of several copies of the trivial representation of $G_{\mathbf{Q}_p}$, hence $D_{\text{cris}}(V^*(\boldsymbol{f}_2)^+_{\beta\beta}) = V^*(\boldsymbol{f}_2)^+_{\beta\beta}$ and Paragraph 7.1.1.1 give a perfect pairing

$$\langle \cdot, \cdot \rangle_{\boldsymbol{f}_2 \boldsymbol{g}_1 \boldsymbol{h}_1} : V(\boldsymbol{f}_2)^-_{\beta\beta} \otimes_L V^*(\boldsymbol{f}_2)^+_{\beta\beta} \longrightarrow L.$$

For each $\mathfrak{z} = \psi \otimes v$ in $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2)^-_{\beta\beta})$, with $\psi \in \mathrm{Hom}_{\mathrm{cont}}(\mathbf{Q}_p^*, \mathbf{Q}_p)$ and $v \in V(\boldsymbol{f}_2)^-_{\beta\beta}$, and each q in $\mathbf{Q}^*$, define (cf. Equation (129) and the discussion preceding it)

$$\mathfrak{z}(q) = \psi(q) \cdot v \in V(\boldsymbol{f}_2)^-_{\beta\beta}$$

and

$$\mathfrak{z}(q)_f = (p-1)a_p(2) \cdot \big\langle \mathfrak{z}(q), \eta_{\boldsymbol{f}_2} \otimes \omega_{\boldsymbol{g}_1} \otimes \omega_{\boldsymbol{h}_1} \big\rangle_{\boldsymbol{f}_2\boldsymbol{g}_1\boldsymbol{h}_1} \in L.$$

Let $\mathfrak{z}$ in $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1))$ be the specialisation at w_o of a balanced class $\mathfrak{Z}$ in $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$, that is $\mathfrak{z} = \rho_{w_o*}(\mathfrak{Z})$. Then $\mathfrak{Z}$ is the natural image of a unique class $\mathfrak{Y}$ in $H^1(\mathbf{Q}_p, \mathscr{F}^2 V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$. Define

$$\mathfrak{y}_f = p_{f*}\big(\rho_{w_o*}(\mathfrak{Y})\big) \in H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2)^-_{\beta\beta}) \tag{196}$$

$$\text{and} \quad \exp_p^*(\mathfrak{z})_f = (p-1)a_p(2) \cdot \big\langle \exp_p^*(\mathfrak{y}_f), \eta_{\boldsymbol{f}_2} \otimes \omega_{\boldsymbol{g}_1} \otimes \omega_{\boldsymbol{h}_1} \big\rangle_{\boldsymbol{f}_2\boldsymbol{g}_1\boldsymbol{h}_1}.$$

The following key proposition studies the derivatives of the logarithm $\mathscr{L}_{\boldsymbol{f}}$, extending some of the results of [**Ven16**]. Its proof exploits the existence of an *improved* big logarithm for the restriction of $\mathscr{L}_{\boldsymbol{f}}$ to the *improving plane* $\mathcal{H}_{\boldsymbol{f}}$ defined by the equation $\boldsymbol{k} = \boldsymbol{l} + \boldsymbol{m}$. Part 1 of the proposition is a crucial ingredient in the proof of the main result of our contribution [**BSV20a**], and Part 3 is essential for the ongoing proof of Theorem B in the exceptional case (cf. Section 9.4). Part 2 is not used elsewhere in the paper and is stated for completeness (and with future applications of this work in mind). Before stating the proposition, we introduce some notation.

For the proof of Theorem B, we are especially interested in the *improving line* $\mathcal{H}_{\boldsymbol{fg}}$ in $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}}$ defined by the equations $\boldsymbol{k} = \boldsymbol{l} + 1$ and $\boldsymbol{m} = 1$; it is the intersection of the improving planes $\mathcal{H}_{\boldsymbol{g}}$ (introduced in Section 1.2) and $\mathcal{H}_{\boldsymbol{f}}$. Let $\mathrm{res}_{\boldsymbol{fg}} : \mathscr{O}_{\boldsymbol{fgh}} \longrightarrow \mathscr{O}_{\boldsymbol{g}}$ be the morphism sending the analytic function $F(\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m})$ to its restriction $F(\boldsymbol{l}+1, \boldsymbol{l}, 1)$ to the improving line $\mathcal{H}_{\boldsymbol{fg}}$. For each $\mathscr{O}_{\boldsymbol{fgh}}$-module M, denote by $M|_{\mathcal{H}_{\boldsymbol{fg}}} = M \otimes_{\mathrm{res}_{\boldsymbol{fg}}} \mathscr{O}_{\boldsymbol{g}}$ the base chance of M along $\mathrm{res}_{\boldsymbol{fg}}$, and for each m in M denote by $m|_{\mathcal{H}_{\boldsymbol{fg}}}$ the image of m under the projection $M \longrightarrow M|_{\mathcal{H}_{\boldsymbol{fg}}}$. Set

$$V(\boldsymbol{fg}, \boldsymbol{h}_1) = V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{fg}}} \quad \text{and} \quad V(\boldsymbol{fg}, \boldsymbol{h}_1)_f = V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f|_{\mathcal{H}_{\boldsymbol{fg}}}.$$

Shrinking $U_{\boldsymbol{g}}$ and $U_{\boldsymbol{h}}$ if necessary, assume that $l + m$ belongs to $U_{\boldsymbol{f}}$ for each (l, m) in $U_{\boldsymbol{g}} \times U_{\boldsymbol{h}}$, and recall the analytic f-Euler factor

$$\mathcal{E}_f^*(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) = 1 - \frac{b_p(\boldsymbol{l}) \cdot c_p(\boldsymbol{m})}{\bar{\chi}_{\boldsymbol{f}}(p) \cdot a_p(\boldsymbol{l} + \boldsymbol{m})} \in \mathscr{O}_{\boldsymbol{g}} \hat{\otimes}_L \mathscr{O}_{\boldsymbol{h}} \tag{197}$$

introduced in Equation (4). (We also recall that $a_p(\boldsymbol{k})$, $b_p(\boldsymbol{l})$ and $c_p(\boldsymbol{m})$ are the p-th Fourier coefficients of the primitive Hida families $\boldsymbol{f}^\sharp$, $\boldsymbol{g}^\sharp$ and $\boldsymbol{h}^\sharp$ associated respectively with $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$.) In the present exceptional zero scenario (cf. Equation (5)) it vanishes at $(\boldsymbol{l}, \boldsymbol{m}) = (1, 1)$. Denote by

$$\mathcal{E}_f^*(\boldsymbol{fg}, \boldsymbol{h}_1) = \mathcal{E}_f^*(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{fg}}} \in \mathscr{O}_{\boldsymbol{g}}$$

the restriction of $\mathcal{E}_f^*(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ to $\mathcal{H}_{\boldsymbol{fg}}$. Finally define the *analytic $\mathscr{L}$-invariants*

$$\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}} = -2 \cdot d \log a_p(\boldsymbol{k})|_{\boldsymbol{k}=2}, \quad \mathfrak{L}_{\boldsymbol{g}}^{\mathrm{an}} = -2 \cdot d \log b_p(\boldsymbol{l})|_{\boldsymbol{l}=1} \quad \text{and} \quad \mathfrak{L}_{\boldsymbol{h}}^{\mathrm{an}} = -2 \cdot d \log c_p(\boldsymbol{m})|_{\boldsymbol{m}=1}.$$

We can now state the main result of this section.

Proposition 9.3. —

1. *Let* $\mathfrak{Z} \in H^1(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f)$ *and let* $\mathfrak{z} = \rho_{w_o}(\mathfrak{Z}) \in H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2)^-_{\beta\beta})$. *Then*
$$\begin{aligned}2(1-1/p)\cdot \mathscr{L}_{\boldsymbol{f}}(\mathfrak{Z}) \equiv & \Big(\mathfrak{z}(p^{-1})_f - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}}\cdot \mathfrak{z}(e(1))_f\Big)\cdot(\boldsymbol{k}-2)\\ &+\Big(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{g}}\cdot \mathfrak{z}(e(1))_f - \mathfrak{z}(p^{-1})_f\Big)\cdot(\boldsymbol{l}-1)\\ &+\Big(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{h}}\cdot \mathfrak{z}(e(1))_f - \mathfrak{z}(p^{-1})_f\Big)\cdot(\boldsymbol{m}-1) \qquad \big(\mathrm{mod}\ \mathscr{I}^2\big).\end{aligned}$$
2. *Let* $\mathfrak{Z}$ *be a local balanced class in* $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))$ *and let* $\mathfrak{z} = \rho_{w_o}(\mathfrak{Z})$ *be its* w_o*-specialisation in* $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1))$. *Then*
$$2(1-1/p)\cdot\mathscr{L}_{\boldsymbol{f}}(\mathfrak{Z})$$
is congruent modulo $\mathscr{I}^2$ *to*
$$\Big(\big(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{g}} - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}}\big)\cdot(\boldsymbol{l}-1) + \big(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{h}} - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}}\big)\cdot(\boldsymbol{m}-1)\Big)\cdot \exp^*_p(\mathfrak{z})_f.$$
3. *There exists a morphism*
$$\mathscr{L}^*_{V(\boldsymbol{fg},\boldsymbol{h}_1)_f} : H^1(\mathbf{Q}_p, V(\boldsymbol{fg},\boldsymbol{h}_1)_f) \longrightarrow \mathscr{O}_{\boldsymbol{g}}$$
such that, for each local class $\mathfrak{Z}$ *in* $H^1(\mathbf{Q}_p, V(\boldsymbol{fg},\boldsymbol{h}_1)_f)$ *and each positive integer* $l \geqslant 1$ *in* $U_{\boldsymbol{g}}$ *congruent to* 1 *modulo* $p-1$, *one has*
$$\mathscr{E}(l)\cdot\mathscr{L}^*_{V(\boldsymbol{fg},\boldsymbol{h}_1)_f}(\mathfrak{Z})(l) = (p-1)a_p(l+1)\cdot\langle \exp^*_p(\mathfrak{z}), \eta_{\boldsymbol{f}_{l+1}}\omega_{\boldsymbol{g}_l}\omega_{\boldsymbol{h}_1}\rangle_{\boldsymbol{f}_{l+1}\boldsymbol{g}_l\boldsymbol{h}_1},$$
where $\mathscr{E}(l) = 1 - \frac{\bar{\chi}_{\boldsymbol{f}}(p)\cdot a_p(l+1)}{p\cdot b_p(l)\cdot c_p(1)}$ *and* $\mathfrak{z} = \rho_l(\mathfrak{Z})$ *in* $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_{l+1},\boldsymbol{g}_l,\boldsymbol{h}_1)_f)$ *is the weight-*l *specialisation of* $\mathfrak{Z}$. *Moreover, the following diagram commutes.*
$$\begin{tikzcd} H^1(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f) \arrow[r, "\mathscr{L}_{\boldsymbol{f}}"] \arrow[d, "\mathrm{res}_{\boldsymbol{fg}*}"'] & \mathscr{O}_{\boldsymbol{fgh}} \arrow[d, "\mathrm{res}_{\boldsymbol{fg}}"] \\ H^1(\mathbf{Q}_p, V(\boldsymbol{fg},\boldsymbol{h}_1)_f) \arrow[r, "\mathcal{E}^*_f(\boldsymbol{fg},\boldsymbol{h}_1)\cdot\mathscr{L}^*_{V(\boldsymbol{fg},\boldsymbol{h}_1)_f}"] & \mathscr{O}_{\boldsymbol{g}} \end{tikzcd}$$

Proof. — Let $\varepsilon : \bar{\mathscr{O}}_{\boldsymbol{fgh}} \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}$ be the map which sends the analytic function $F(\boldsymbol{k},\boldsymbol{l},\boldsymbol{m},\boldsymbol{j})$ in $\bar{\mathscr{O}}_{\boldsymbol{fgh}}$ to its restriction $F(\boldsymbol{k},\boldsymbol{l},\boldsymbol{m},0) \in \mathscr{O}_{\boldsymbol{fgh}}$ to the hyperplane $\boldsymbol{j}=0$ (see Section 7.1 and note that $j_o = 0$). Because $M(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f$ is equal (by definition) to the base change $\bar{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f \otimes_\varepsilon \mathscr{O}_{\boldsymbol{fgh}}$, this induces in cohomology
$$\varepsilon_* : H^1(\mathbf{Q}_p, \bar{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f) \longrightarrow H^1(\mathbf{Q}_p, M(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f).$$
A slight generalisation of [**Ven16**, Proposition 3.8] stated in Lemma 9.4 below gives an *improved big dual exponential*
$$\mathcal{L}^*_{\boldsymbol{f}} : H^1(\mathbf{Q}_p, M(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f) \longrightarrow D(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f$$
such that, for all classes $\mathfrak{Z}$ in $H^1(\mathbf{Q}_p, M(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f)$ and all $w=(k,l,m)\in\Sigma$, one has
$$\big(1-p^{-1}\cdot\Psi_w(\mathrm{Frob}_p)\big)\cdot\mathcal{L}^*_{\boldsymbol{f}}(\mathfrak{Z})(w) = \exp^*(\mathfrak{Z}_w), \tag{198}$$
where Ψ_w is the composition of the unramified character $\Psi : G_{\mathbf{Q}_p} \longrightarrow \mathscr{O}^*_{\boldsymbol{fgh}}$ introduced in Equation (136) with evaluation at w, $\exp^*$ is the Bloch–Kato dual exponential on $H^1(\mathbf{Q}_p, M(\boldsymbol{f}_k,\boldsymbol{g}_l,\boldsymbol{h}_m)_f)$, and $\mathfrak{Z}_w$ is a shorthand for $\rho_{w*}(\mathfrak{Z})$. (Precisely, after setting

$\mathscr{R} = \mathscr{O}_{\boldsymbol{fgh}}$, $\mathscr{M} = M(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f$ and $\Phi = \Psi$, then one has $\mathcal{L}^*_{\boldsymbol{f}} = \mathscr{E}xp^*_\Psi$ with the notations of Lemma 9.4.) Recall the big logarithm $\bar{\mathscr{L}}_{\boldsymbol{f}}$ introduced in Equation (144), and let

$$\mathscr{L}^*_{\boldsymbol{f}} : H^1(\mathbf{Q}_p, M(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f) \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}$$

be the composition of $\mathcal{L}^*_{\boldsymbol{f}}$ with the base change

$$\langle\cdot, \eta_{\boldsymbol{f}}\omega_{\boldsymbol{g}}\omega_{\boldsymbol{h}}\rangle_{\boldsymbol{fgh}} \otimes_\varepsilon \mathscr{O}_{\boldsymbol{fgh}} : D(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f \to \mathscr{O}_{\boldsymbol{fgh}}$$

of the linear form $\langle\cdot, \eta_{\boldsymbol{f}}\omega_{\boldsymbol{g}}\omega_{\boldsymbol{h}}\rangle_{\boldsymbol{fgh}}$ along ε. Equation (198) and Proposition 7.1 yield

$$\varepsilon \circ \bar{\mathscr{L}}_{\boldsymbol{f}} = \big(1 - \Psi(\mathrm{Frob}_p)^{-1}\big) \cdot \mathscr{L}^*_{\boldsymbol{f}} \circ \varepsilon_*. \tag{199}$$

Define $\varrho = \rho_{w_o} : \bar{\mathscr{O}}_{\boldsymbol{fgh}} \longrightarrow \mathscr{O}_{\mathrm{cyc}}$ by $\varrho(F(\boldsymbol{k},\boldsymbol{l},\boldsymbol{m},\boldsymbol{j})) = F(w_o,\boldsymbol{j})$ and denote by $\bar{M}(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f$ the base change $\bar{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f \otimes_\varrho \mathscr{O}_{\mathrm{cyc}}$. Note that in the present setting $G_{\mathbf{Q}_p}$ acts on $\bar{M}(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f$ via the character $\kappa_{\mathrm{cyc}}^{-\boldsymbol{j}}$, and for all integers j divisible by $p-1$, evaluation at j on $\mathscr{O}_{\mathrm{cyc}}$ induces a natural isomorphism (cf. Sect. 7.1)

$$V(\boldsymbol{f}_2)^-_{\beta\beta}(-j) = \bar{M}(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f \otimes_j L. \tag{200}$$

The results of Coleman and Perrin-Riou (see Section 4 of [**PR94**]) then give a morphism of $\mathscr{O}_{\mathrm{cyc}}$-modules

$$\mathscr{L}_{\mathrm{cyc}} : H^1(\mathbf{Q}_p, \bar{M}(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f) \longrightarrow \mathscr{O}_{\mathrm{cyc}}$$

such that, for all classes $\mathfrak{Z}$ in $H^1(\mathbf{Q}_p, \bar{M}(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f)$ and all integers $j \geqslant 0$ satisfying $j \equiv 0 \pmod{p-1}$, one has

$$\mathscr{L}_{\mathrm{cyc}}(\mathfrak{Z})(j) = j!\frac{(1-p^j)}{(1-p^{-j-1})}\exp^*(\mathfrak{Z}_j)_f. \tag{201}$$

Here $\mathfrak{Z}_j$ is the image of $\mathfrak{Z}$ in $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2)^-_{\beta\beta}(-j))$ under the morphism induced by (200) and one writes again

$$\exp^*(\cdot)_f = (p-1)a_p(2) \cdot \big\langle \exp^*(\cdot), \eta_{\boldsymbol{f}_2}\omega_{\boldsymbol{g}_1}\omega_{\boldsymbol{h}_1}\big\rangle_{\boldsymbol{f}_2\boldsymbol{g}_1\boldsymbol{h}_1}$$

for the composition of the linear form $(p-1)a_p(2) \cdot \big\langle \cdot, \eta_{\boldsymbol{f}_2}\omega_{\boldsymbol{g}_1}\omega_{\boldsymbol{h}_1}\big\rangle_{\boldsymbol{f}_2\boldsymbol{g}_1\boldsymbol{h}_1}$ on $V(\boldsymbol{f}_2)^-_{\beta\beta}$ with the Bloch–Kato dual exponential map

$$\exp^* : H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2)^-_{\beta\beta}(-j)) \longrightarrow V(\boldsymbol{f}_2)^-_{\beta\beta} \otimes_{\mathbf{Q}_p} \mathbf{Q}_p \cdot t^{-j} \cong V(\boldsymbol{f}_2)^-_{\beta\beta}$$

(cf. Section 7.1 and Equation (194)). According to Proposition 3.6 of [**Ven16**] (see also [**Ben14**, Proposition 2.2.2]), for all classes $\mathfrak{Z}$ in $H^1(\mathbf{Q}_p, \bar{M}(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f)$ one has

$$\frac{d}{d\boldsymbol{j}}\mathscr{L}_{\mathrm{cyc}}(\mathfrak{Z})_{\boldsymbol{j}=0} = (1-1/p)^{-1} \cdot \mathfrak{z}(p^{-1})_f, \tag{202}$$

where $\mathfrak{z}$ is a shorthand for $\mathfrak{Z}_0$. Moreover Proposition 7.1 and Equation (201) yield the identity

$$\varrho \circ \bar{\mathscr{L}}_{\boldsymbol{f}} = \mathscr{L}_{\mathrm{cyc}} \circ \varrho_*. \tag{203}$$

Let $\mathfrak{Z}$ be a class in $H^1(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f)$ and let $\mathfrak{z} = \rho_{w_o}(\mathfrak{Z}) \in H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2)^-_{\beta\beta})$ be its specialisation at w_o. As explained in the proof of Proposition 7.3 (see in particular Equations (151) and (152)), the class $\mathfrak{Z}$ can by lifted to an element $\mathscr{Z}$ in $H^1(\mathbf{Q}_p, \bar{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f)$ via the map induced in cohomology by the isomorphism

$$\bar{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f/(2\boldsymbol{j}-\boldsymbol{k}+\boldsymbol{l}+\boldsymbol{m})\cdot\bar{M}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \cong V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f,$$

and one has

$$\mathscr{L}_{\boldsymbol{f}}(\mathfrak{Z})(\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}) = \bar{\mathscr{L}}_{\boldsymbol{f}}(\mathscr{Z})(\boldsymbol{k},\boldsymbol{l},\boldsymbol{m},(\boldsymbol{k}-\boldsymbol{l}-\boldsymbol{m})/2), \tag{204}$$

for any such lift $\mathscr{Z}$. As (cf. Equation (136))

$$2\cdot\left(1-\Psi(\mathrm{Frob}_p)^{-1}\right) = \mathfrak{L}_{\boldsymbol{g}}^{\mathrm{an}}\cdot(\boldsymbol{l}-1)+\mathfrak{L}_{\boldsymbol{h}}^{\mathrm{an}}\cdot(\boldsymbol{m}-1)-\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}}\cdot(\boldsymbol{k}-2)+\cdots,$$

where the dots denote the terms of higher degree in the Taylor expansion at w_o, Equations (199) and (203) yield that $2(1-1/p)\cdot\bar{\mathscr{L}}_{\boldsymbol{f}}(\mathscr{Z})$ is equal to

$$2\left(1-\Psi(\mathrm{Frob}_p)^{-1}\right)(1-1/p)\cdot\mathscr{L}_{\boldsymbol{f}}^{*}(\varepsilon_*(\mathscr{Z}))+2(1-1/p)\cdot\mathscr{L}_{\mathrm{cyc}}(\varrho_*(\mathscr{Z}))+\cdots,$$

which in turn agrees with

$$\mathfrak{z}(e(1))_f\cdot\left(\mathfrak{L}_{\boldsymbol{g}}^{\mathrm{an}}\cdot(\boldsymbol{l}-1)+\mathfrak{L}_{\boldsymbol{h}}^{\mathrm{an}}\cdot(\boldsymbol{m}-1)-\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}}\cdot(\boldsymbol{k}-2)\right)+2\cdot\mathfrak{z}(p^{-1})_f\cdot\boldsymbol{j}+\cdots$$

by Equations (195), (198) and (202). This proves Part 1 in the statement.

To prove Part 2 let $\mathfrak{Z},\mathfrak{Y},\mathfrak{z}$ and $\mathfrak{y}_f$ be as in Equation (196), so that

$$\exp_p^*(\mathfrak{z})_f = \mathfrak{y}_f(e(1))_f \tag{205}$$

(cf. Equation (195)). Note that the $L[G_p]$-module $\mathscr{F}^2V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)$ splits as the direct sum of its submodules $V(\boldsymbol{f}_2)^+_{\alpha\beta}=V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_g$, $V(\boldsymbol{f}_2)^+_{\beta\alpha}=V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_h$ and

$$V(\boldsymbol{f}_2)_{\beta\beta}=V(\boldsymbol{f}_2)\otimes_L V(\boldsymbol{g}_1)^+\otimes_L V(\boldsymbol{h}_1)^+$$

(cf. Section 7.2). Moreover, if $V(\boldsymbol{f}_2)^+_{\beta\beta}$ denotes the tensor product of $V(\boldsymbol{f}_2)^+, V(\boldsymbol{g}_1)^+$ and $V(\boldsymbol{h}_1)^+$ (that is $\mathscr{F}^3V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)$ with the notations of Section 7.2), the projection $V(\boldsymbol{f}_2)_{\beta\beta}\twoheadrightarrow V(\boldsymbol{f}_2)^-_{\beta\beta}$ gives rise to a short exact sequence of $G_{\mathbf{Q}_p}$-modules

$$0\longrightarrow V(\boldsymbol{f}_2)^+_{\beta\beta}\xrightarrow{i^+}V(\boldsymbol{f}_2)_{\beta\beta}\xrightarrow{\pi^-}V(\boldsymbol{f}_2)^-_{\beta\beta}\longrightarrow 0. \tag{206}$$

It follows that the image of $H^1(\mathbf{Q}_p,\mathscr{F}^2V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1))$ under p_{f*} equals that of $H^1(\mathbf{Q}_p,V(\boldsymbol{f}_2)_{\beta\beta})$ under π^-, hence

$$\mathfrak{y}_f\in\pi^-_*\left(H^1(\mathbf{Q}_p,V(\boldsymbol{f}_2)_{\beta\beta})\right). \tag{207}$$

The short exact sequence (206) defines an extension class q_f in

$$\mathrm{Ext}^1_{L[G_p]}\left(V(\boldsymbol{f}_2)^-_{\beta\beta},V(\boldsymbol{f}_2)^+_{\beta\beta}\right)\cong H^1(\mathbf{Q}_p,L(1))\otimes_L\mathrm{Hom}_L\left(V(\boldsymbol{f}_2)^-_{\beta\beta},V(\boldsymbol{f}_2)^+_{\beta\beta}(-1)\right).$$

After identifying $H^1(\mathbf{Q}_p,L(1))$ with $\mathbf{Q}_p^*\hat{\otimes}L$ under the Kummer isomorphism, this defines a morphism

$$\begin{aligned}\mathrm{L}_{q_f}: H^1(\mathbf{Q}_p,V(\boldsymbol{f}_2)^-_{\beta\beta}) &\cong \mathrm{Hom}_{\mathrm{cont}}(\mathbf{Q}_p^*,L)\otimes_L V(\boldsymbol{f}_2)^-_{\beta\beta}\\ &\longrightarrow V(\boldsymbol{f}_2)^+_{\beta\beta}(-1)\cong H^2(\mathbf{Q}_p,V(\boldsymbol{f}_2)^+_{\beta\beta}),\end{aligned}$$

where the last isomorphism arises from the invariant map $H^2(\mathbf{Q}_p,L(1))\cong L$ of local class field theory. A direct computation, carried out in Lemma 9.5 below, shows that $\mathtt{L}_{q_f}$ is equal to the connecting morphism $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2)^-_{\beta\beta}) \longrightarrow H^2(\mathbf{Q}_p, V(\boldsymbol{f}_2)^+_{\beta\beta})$ associated with the exact sequence (206). It then follows from Equation (207) that

$$\mathtt{L}_{q_f}(\mathfrak{y}_f) = 0. \tag{208}$$

According to Theorem 3.18 of [GS93] q_f is of the form $\mathfrak{q}_f \otimes \delta_f$ for some linear form $\delta_f : V(\boldsymbol{f}_2)^-_{\beta\beta} \longrightarrow V(\boldsymbol{f}_2)^+_{\beta\beta}$ and $\mathfrak{q}_f$ in $\mathbf{Q}_p^* \hat{\otimes} L$ such that $\mathrm{ord}_p(\mathfrak{q}_f) \neq 0$ and

$$\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} = \log_p(\mathfrak{q}_f)/\mathrm{ord}_p(\mathfrak{q}_f).$$

Then

$$\log_{\mathfrak{q}_f} = \log_p - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} \cdot \mathrm{ord}_p \in \mathrm{Hom}_{\mathrm{cont}}(\mathbf{Q}_p^*, L)$$

is the branch of the p-adic logarithm which vanishes at $\mathfrak{q}_f$ and $L \cdot \log_{\mathfrak{q}_f} \otimes_L V(\boldsymbol{f}_2)^-_{\beta\beta}$ is contained in the kernel of $\mathtt{L}_{q_f}$. Taking the long exact sequence associated with (206) one easily checks that the kernel of $\mathtt{L}_{q_f}$ has the same dimension as $V(\boldsymbol{f}_2)^-_{\beta\beta}$, hence $L \cdot \log_{\mathfrak{q}_f} \otimes_L V(\boldsymbol{f}_2)^-_{\beta\beta}$ is equal to the kernel of $\mathtt{L}_{q_f}$. Equation (208) then yields $\mathfrak{y}_f = \log_{\mathfrak{q}_f} \otimes v_f$ for some v_f in $V(\boldsymbol{f}_2)^-_{\beta\beta}$, hence

$$\mathfrak{y}_f(p^{-1}) = \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} \cdot v_f = \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} \cdot \mathfrak{y}_f(e(1)). \tag{209}$$

Part 1 of the proposition and Equations (205) and (209) give

$$\begin{aligned}
&2(1-1/p)\cdot \mathscr{L}_{\boldsymbol{f}}(\mathfrak{Z}) = 2(1-1/p)\cdot \mathscr{L}_{\boldsymbol{f}} \circ p_{f*}(\mathfrak{Y}) \\
&\overset{\text{Part 1}}{\equiv} \Big(\mathfrak{y}_f(p^{-1})_f - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} \cdot \mathfrak{y}_f(e(1))_f\Big)\cdot(\boldsymbol{k}-2) \\
&+\Big(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{g}} \cdot \mathfrak{y}_f(e(1))_f - \mathfrak{y}_f(p^{-1})_f\Big)\cdot(\boldsymbol{l}-1) + \Big(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{h}} \cdot \mathfrak{y}_f(e(1))_f - \mathfrak{y}_f(p^{-1})_f\Big)\cdot(\boldsymbol{m}-1) \\
&\overset{\text{Eq. (209)}}{\equiv} \mathfrak{y}_f(e(1))_f \cdot \Big(\big(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{g}} - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}}\big)\cdot(\boldsymbol{l}-1) + \big(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{h}} - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}}\big)\cdot(\boldsymbol{m}-1)\Big) \\
&\overset{\text{Eq. (205)}}{\equiv} \exp^*(\mathfrak{z})_f \cdot \Big(\big(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{g}} - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}}\big)\cdot(\boldsymbol{l}-1) + \big(\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{h}} - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}}\big)\cdot(\boldsymbol{m}-1)\Big) \pmod{\mathscr{I}^2},
\end{aligned}$$

as was to be shown.

We finally prove Part 3. Taking $\mathscr{R} = \mathscr{O}_{\boldsymbol{g}}$, $\mathscr{M} = V(\boldsymbol{fg}, \boldsymbol{h}_1)_f$ and $\Phi = \mathrm{res}_{\boldsymbol{fg}} \circ \Psi$ in Lemma 9.4 gives an improved big dual exponential

$$\mathscr{E}xp^*_{V(\boldsymbol{fg},\boldsymbol{h}_1)_f} : H^1(\mathbf{Q}_p, V(\boldsymbol{fg}, \boldsymbol{h}_1)_f) \longrightarrow D(\boldsymbol{fg}, \boldsymbol{h}_1),$$

where $D(\boldsymbol{fg}, \boldsymbol{h}_1)_f = (\mathtt{V}(\boldsymbol{fg}, \boldsymbol{h}_1)_f \hat{\otimes}_{\mathbf{Z}_p} \hat{\mathbf{Z}}_p^{\mathrm{nr}})^{G_{\mathbf{Q}_p}}[1/p]$ and $\mathtt{V}(\boldsymbol{fg}, \boldsymbol{h}_1)_f$ is a $G_{\mathbf{Q}_p}$-invariant $\Lambda_{\boldsymbol{g}}$-lattice in $V(\boldsymbol{fg}, \boldsymbol{h}_1)_f$. Note that $D(\boldsymbol{fg}, \boldsymbol{h}_1)_f$ is naturally isomorphic to the base change of $D(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f$ along $\mathrm{res}_{\boldsymbol{fg}} : \mathscr{O}_{\boldsymbol{fgh}} \longrightarrow \mathscr{O}_{\boldsymbol{g}}$, and define

$$\mathscr{L}^*_{V(\boldsymbol{fg},\boldsymbol{h}_1)_f} : H^1(\mathbf{Q}_p, V(\boldsymbol{fg}, \boldsymbol{h}_1)_f) \longrightarrow \mathscr{O}_{\boldsymbol{g}}$$

to be the composition of $\mathscr{E}xp^*_{V(\boldsymbol{fg},\boldsymbol{h}_1)_f}$ with the base change

$$\langle \cdot, \eta_{\boldsymbol{f}} \omega_{\boldsymbol{g}} \omega_{\boldsymbol{h}} \rangle \otimes_{\mathrm{res}_{\boldsymbol{fg}}} \mathscr{O}_{\boldsymbol{g}} : D(\boldsymbol{fg}, \boldsymbol{h}_1)_f \longrightarrow \mathscr{O}_{\boldsymbol{g}}$$

along $\mathrm{res}_{\boldsymbol{fg}}$ of the linear form $\langle \cdot, \eta_{\boldsymbol{f}} \omega_{\boldsymbol{g}} \omega_{\boldsymbol{h}} \rangle_{\boldsymbol{fgh}}$ on $D(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f$. After noting that

$$1 - \Psi(\mathrm{Frob}_p)^{-1}(\boldsymbol{l}+\boldsymbol{m}, \boldsymbol{l}, \boldsymbol{m}) = \mathcal{E}^*_f(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \quad \text{and} \quad 1 - p^{-1}\cdot \Psi_w(\mathrm{Frob}_p) = \mathscr{E}(l)$$

for each positive integer $l \geqslant 1$ in $U_{\boldsymbol{g}}$ congruent to 1 modulo $p-1$, where $w = (l+1, l, 1)$ in $\mathcal{H}_{\boldsymbol{fg}}$, the interpolation property satisfied by $\mathscr{L}_{V(\boldsymbol{fg},\boldsymbol{h}_1)_f}$ and the commutativity of the diagram in the statement follow directly from Equation (143) (cf. Section 7.1.1.1 for the case $l = 1$), Proposition 7.3 (and its proof) and Lemma 9.4. □

The following two lemmas have been invoked in the proof of Proposition 9.3.

Lemma 9.4. — *Let R be a complete local Noetherian ring with finite residue field of characteristic p, and let $\mathscr{R} = R[1/p]$. Let M be a free R-module of finite rank, equipped with the action of $G_{\mathbf{Q}_p}$ given by a continuous* unramified *character $\Phi : G_{\mathbf{Q}_p} \longrightarrow R^*$. Set $\mathscr{M} = M[1/p]$. Then there exists a morphism of $\mathscr{R}$-modules*

$$\mathscr{E}xp^*_\Phi : H^1(\mathbf{Q}_p, \mathscr{M}) \longrightarrow (M \hat{\otimes}_{\mathbf{Z}_p} \hat{\mathbf{Z}}_p^{\mathrm{nr}})^{G_{\mathbf{Q}_p}}[1/p]$$

such that, for each continuous morphism of $\mathbf{Z}_p$-algebras $\nu : R \to \bar{\mathbf{Q}}_p$ and each class $\mathfrak{Z} \in H^1(\mathbf{Q}_p, \mathscr{M})$, one has

$$\nu\big(\mathscr{E}xp^*_\Phi(\mathfrak{Z})\big) = \big(1 - p^{-1} \cdot \Phi_\nu(\mathrm{Frob}_p)\big)^{-1} \cdot \exp^*_p(\mathfrak{Z}_\nu),$$

where the notations are as follows. Set $\mathscr{O}_\nu = \nu(R)$ and $L_\nu = \mathrm{Frac}(\mathscr{O}_\nu)$. The unramified character $\Phi_\nu : G_{\mathbf{Q}_p} \longrightarrow \mathscr{O}_\nu^*$ is the composition of Φ with ν, the class $\mathfrak{Z}_\nu$ in $H^1(\mathbf{Q}_p, L_\nu(\Phi_\nu))$ is the image of $\mathfrak{Z}$ under the map induced in cohomology by ν, and

$$\exp^*_p : H^1(\mathbf{Q}_p, L_\nu(\Phi_\nu)) \longrightarrow D_{\mathrm{cris}}(L_\nu(\Phi)) = (\mathscr{O}_\nu(\Phi_\nu) \hat{\otimes}_{\mathbf{Z}_p} \hat{\mathbf{Z}}_p^{\mathrm{nr}})^{G_{\mathbf{Q}_p}}[1/p]$$

is the Bloch–Kato dual exponential.

Proof. — When $\mathscr{R} = \mathscr{O}_{\boldsymbol{f}}$ and $\mathscr{M} = \mathscr{O}_{\boldsymbol{f}}(\check{a}_p(\boldsymbol{k}))$, this is [Ven16, Proposition 3.8]. Mutatis mutandis, the proof of loco citato works in this more general setting. □

Lemma 9.5. — *Let M and N be two finite dimensional L-vector spaces, equipped with the trivial action of the absolute Galois group G_p of $\mathbf{Q}_p$, let*

$$0 \longrightarrow M(1) \xrightarrow{\alpha} V \xrightarrow{\beta} N \longrightarrow 0 \tag{210}$$

be a short exact sequence of (continuous) $L[G_p]$-modules, and let

$$q_V \in \mathrm{Ext}^1_{L[G_p]}(N, M(1)) \cong \hat{\mathbf{Q}}_p^* \otimes_{\mathbf{Z}_p} \mathrm{Hom}_L(N, M)$$

be the corresponding extension class (where one identifies $H^1(\mathbf{Q}_p, \mathbf{Z}_p(1))$ with the p-adic completion $\hat{\mathbf{Q}}_p^*$ of $\mathbf{Q}_p$ via the Kummer map). Then the connecting morphism

$$\delta_V : H^1(\mathbf{Q}_p, N) \longrightarrow H^1(\mathbf{Q}_p, M(1))$$

associated with the short exact sequence is equal to the composition

$$\mathtt{L}_V : H^1(\mathbf{Q}_p, N) \cong \mathrm{Hom}_{\mathrm{cont}}(\hat{\mathbf{Q}}_p^*, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} N \xrightarrow{e_V} M \cong H^2(\mathbf{Q}_p, M(1)),$$

where the first isomorphism arises from the local Artin map $\mathrm{rec}_p : \mathbf{Q}_p^* \longrightarrow G_p^{\mathrm{ab}}$ (sending p^{-1} to an arithmetic Frobenius), the second isomorphism arises from the invariant map $\mathrm{inv}_p : H^2(\mathbf{Q}_p, \mathbf{Z}_p(1)) \cong \mathbf{Z}_p$, and e_V is evaluation at q_V (under the product of the natural dualities $\hat{\mathbf{Q}}_p^* \otimes_{\mathbf{Z}_p} \mathrm{Hom}_{\mathrm{cont}}(\hat{\mathbf{Q}}_p^*, \mathbf{Z}_p) \longrightarrow \mathbf{Z}_p$ and $\mathrm{Hom}_L(N, M) \otimes_L N \longrightarrow M$).

Proof. — Identify $M(1)$ with a subspace of V via the injective morphism α, and fix an L-linear section $\sigma : N \longrightarrow V$ of β. Under the natural isomorphisms

$$\mathrm{Ext}^1_{L[G_p]}(N, M(1)) = \mathrm{Ext}^1_{L[G_p]}(L, \mathrm{Hom}_L(N, M)(1)) = H^1(\mathbf{Q}_p, \mathrm{Hom}_L(N, M)(1)),$$

the extension class of (210) is represented by the 1-cocylce

$$\xi_V = \xi_{V,\sigma} : G_p \longrightarrow \mathrm{Hom}_L(M, N)(1)$$

defined by the formulae

$$g(\sigma(n)) - \sigma(n) = \xi_V(g)(n)$$

for each g in G_p and each n in N.

For each 1-cocycle (id est continuous morphism of groups) $\varphi : G_p \longrightarrow N$, the image of φ under the connecting map δ_V is represented by the 2-cocycle $\delta_V^o(\varphi)$ defined by

$$\delta_V^o(\varphi)(g, h) = g\big(\sigma(\varphi(h))\big) - \sigma\big(\varphi(gh)\big) + \sigma(\varphi(g)) = \xi_V(g)(\varphi(h)) = \xi_V \cup_{\mathrm{ev}} \varphi\,(g, h),$$

where $\cup_{\mathrm{ev}} : C^\bullet_{\mathrm{cont}}(G_p, \mathrm{Hom}_L(N, M)(1)) \otimes_L C^\bullet_{\mathrm{cont}}(G_p, N) \longrightarrow C^\bullet_{\mathrm{cont}}(G_p, M(1))$ denotes the cup-product induced on continuous cochains by the evaluation pairing

$$\mathrm{ev} : \mathrm{Hom}_L(N, M) \otimes_L N \longrightarrow M$$

(cf. Sections 3.4.1.2 and 3.4.5.1 of [**Nek06**]). If $\langle \cdot, \cdot \rangle_{\mathrm{ev}}$ denotes the composition of the cup-product pairing induced in $(1, 1)$-cohomology by $\cup_{\mathrm{ev}}$ with the M-linear extension

$$\mathrm{inv}_M : H^2(\mathbf{Q}_p, M(1)) = H^2(\mathbf{Q}_p, \mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} M \cong M$$

of the local invariant map inv_p, it follows that

$$\mathrm{inv}_M(\delta_V(\varphi)) = \langle cl(\xi_V), \varphi \rangle_{\mathrm{ev}}\,, \tag{211}$$

where $cl(\cdot)$ denotes the class represented by $\cdot$. Under the natural isomorphisms

$$H^1(\mathbf{Q}_p, \mathrm{Hom}_L(N, M)(1)) = H^1(\mathbf{Q}_p, \mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} \mathrm{Hom}_L(N, M)$$

and $H^1(\mathbf{Q}_p, N) = H^1(\mathbf{Q}_p, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} N$, the pairing $\langle \cdot, \cdot \rangle_{\mathrm{ev}}$ corresponds to the product of ev and the local Tate duality

$$\langle \cdot, \cdot \rangle : H^1(\mathbf{Q}_p, \mathbf{Z}_p(1)) \otimes_{\mathbf{Z}_p} H^1(\mathbf{Q}_p, \mathbf{Z}_p) \xrightarrow{\cup} H^2(\mathbf{Q}_p, \mathbf{Z}_p(1)) \xrightarrow{\mathrm{inv}_p} \mathbf{Z}_p$$

associated with the multiplication pairing $\mathbf{Z}_p(1) \otimes_{\mathbf{Z}_p} \mathbf{Z}_p \longrightarrow \mathbf{Z}_p$. Finally one has

$$\langle \kappa(q), \chi \rangle = \chi(\mathrm{rec}_p(q))$$

for each χ in $H^1(\mathbf{Q}_p, \mathbf{Z}_p)$ and each q in $\mathbf{Q}_p^*$, where $\kappa : \mathbf{Q}_p^* \longrightarrow H^1(\mathbf{Q}_p, \mathbf{Z}_p(1))$ denotes the Kummer map (cf. Proposition 1 in Section 2.3 of [**Ser67**]), hence

$$\langle cl(\xi_V), \varphi \rangle_{\mathrm{ev}} = e_V(\varphi),$$

which combined with Equation (211) concludes the proof. □

9.3. Improved diagonal classes. — This section proves the existence of the big g-improved diagonal class introduced in Equation (2) of Section 1.2.

Section 8.1 associates to the *ordered* triple of Hida families $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ the big diagonal class $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ (which is symmetric in the forms $\boldsymbol{g}$ and $\boldsymbol{h}$). After identifying the big $G_{\mathbf{Q}}$-representations $V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}), V(\boldsymbol{g}, \boldsymbol{f}, \boldsymbol{h})$ and $V(\boldsymbol{h}, \boldsymbol{f}, \boldsymbol{g})$ under the natural isomorphisms, a priori the three classes

$$\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}), \quad \kappa(\boldsymbol{g}, \boldsymbol{f}, \boldsymbol{h}) \quad \text{and} \quad \kappa(\boldsymbol{h}, \boldsymbol{f}, \boldsymbol{g})$$

in $H^1(\mathbf{Q}, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$ may be different. This is indeed *not* the case.

Lemma 9.6. — *The classes $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$, $\kappa(\boldsymbol{g}, \boldsymbol{f}, \boldsymbol{h})$ and $\kappa(\boldsymbol{h}, \boldsymbol{f}, \boldsymbol{g})$ are equal.*

Proof. — Let Σ^o_{bal} be the set of balanced triples $w = (k, l, m)$ such that p does not divide the conductors of $\boldsymbol{f}_k$, $\boldsymbol{g}_l$ and $\boldsymbol{h}_m$. Since $H^1(\mathbf{Q}, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$ is a torsion-free $\mathscr{O}_{\boldsymbol{fgh}}$-module and Σ^o_{bal} is dense in $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}}$, one has

$$\bigcap_{w \in \Sigma^o_{\mathrm{bal}}} (\boldsymbol{k} - k, \boldsymbol{l} - l, \boldsymbol{m} - m) \cdot H^1(\mathbf{Q}, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) = 0.$$

It is then sufficient to prove that the three classes in the statement have the same specialisation at each balanced classical triple w in Σ^o_{bal}. Because the map $\Pi^{\alpha}_{f_k g_l h_m *}$ (defined after Equation (169)) is an isomorphism at each point (k, l, m) of Σ^o_{bal}, this is a consequence of Theorem 8.1 and Proposition 8.3. □

We now construct the g-improved balanced diagonal class

$$\kappa^*_g(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \in H^1_{\mathrm{bal}}(\mathbf{Q}, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}}) \tag{212}$$

satisfying Equation (2) of Section 1.2.

Set $\Lambda_{\boldsymbol{gh}} = \Lambda_{\boldsymbol{g}} \hat{\otimes}_{\mathscr{O}} \Lambda_{\boldsymbol{h}}$, so that $\mathscr{O}_{\boldsymbol{gh}} = \Lambda_{\boldsymbol{gh}}[1/p]$. For every $\Lambda_{\boldsymbol{gfh}}$-module M, define

$$M|_{\mathcal{H}_{\boldsymbol{g}}} = M \otimes_{\nu_{\boldsymbol{g}}} \Lambda_{\boldsymbol{gh}}$$

to be the base change of the $\Lambda_{\boldsymbol{gfh}}$-module M under the morphism $\nu_{\boldsymbol{g}} : \Lambda_{\boldsymbol{gfh}} \longrightarrow \Lambda_{\boldsymbol{gh}}$ sending the analytic function $F(\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m})$ to its restriction $F(\boldsymbol{l} - \boldsymbol{m} + 2, \boldsymbol{l}, \boldsymbol{m})$ to the g-improving plane $\mathcal{H}_{\boldsymbol{g}}$ (cf. Section 1.2). A similar notation applies to $\mathscr{O}_{\boldsymbol{gfh}}$-modules and sheaves of $\Lambda_{\boldsymbol{gfh}}$ or $\mathscr{O}_{\boldsymbol{gfh}}$-modules.

Remark 9.7. — The space $\mathcal{A}'_{\boldsymbol{g}} \hat{\otimes} \mathcal{A}_{\boldsymbol{f}} \hat{\otimes} \mathcal{A}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}$ is identified with a subspace of the $\Lambda_{\boldsymbol{gh}}$-valued functions f on $\mathsf{T}' \times \mathsf{T} \times \mathsf{T}$ that are locally analytic and such that

$$f\left(t_x \cdot x, t_y \cdot y, t_z \cdot z\right) = \nu_{\boldsymbol{g}}(t_x^{\kappa_{\boldsymbol{f}}} t_y^{\kappa_{\boldsymbol{g}}} t_z^{\kappa_{\boldsymbol{h}}}) \cdot f\left(x, y, z\right).$$

(This can be seen by applying [GS16, Lemma 7.3] with $X = \mathsf{T}' \times \mathsf{T} \times \mathsf{T}$ to reduce the statement to the fact that the formation of locally analytic function - without the homogeneity property imposed - is compatible with base change.) Conversely, such a function f can be assumed to be in the image of $\mathcal{A}'_{\boldsymbol{f}} \hat{\otimes} \mathcal{A}_{\boldsymbol{g}} \hat{\otimes} \mathcal{A}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}$, by increasing the radius of convergence in the definition of $\mathcal{A}^{\cdot}_{\boldsymbol{f}} = \mathcal{A}^{\cdot}_{U_{\boldsymbol{f}}, \imath}$, $\mathcal{A}^{\cdot}_{\boldsymbol{g}} = \mathcal{A}^{\cdot}_{U_{\boldsymbol{g}}, \imath}$ and $\mathcal{A}^{\cdot}_{\boldsymbol{h}} = \mathcal{A}^{\cdot}_{U_{\boldsymbol{h}}, \imath}$.

Consider the analytic function $\mathtt{D}^*_{\boldsymbol{g}} : \mathsf{T}' \times \mathsf{T} \times \mathsf{T} \longrightarrow \Lambda_{\boldsymbol{gfh}}$ defined by the formula

$$\mathtt{D}^*_{\boldsymbol{g}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = \det(\boldsymbol{x}, \boldsymbol{y})^{\kappa^*_{\boldsymbol{h}}} \cdot \det(\boldsymbol{x}, \boldsymbol{z})^{\kappa^*_{\boldsymbol{f}}} \cdot \det(\boldsymbol{y}, \boldsymbol{z})^{(k+m-l-2)/2}$$

for each $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ in $\mathsf{T}' \times \mathsf{T} \times \mathsf{T}$ with $\boldsymbol{a} = (a_1, a_2)$ for $\boldsymbol{a} = \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$. (Because we apply an integer power to the last determinant, there is no need to restrict to the domain $\mathsf{T}' \times (\mathsf{T} \times \mathsf{T})_0$ as we did in the definition of **Det** in Section 8.1.) Then $\mathbf{Det}^*_g := \nu_{\boldsymbol{g}} \circ \mathtt{D}^*_{\boldsymbol{g}} : \mathsf{T}' \times \mathsf{T} \times \mathsf{T} \longrightarrow \Lambda_{\boldsymbol{gh}}$ is a locally analytic function satisfying the homogeneity property of Remark 9.7. It also satisfies the invariance property

$$\mathbf{Det}^*_g(\boldsymbol{x} \cdot \gamma, \boldsymbol{y} \cdot \gamma, \boldsymbol{z} \cdot \gamma) = \det(\gamma)^{\nu_{\boldsymbol{g}} \circ \kappa^*_{\boldsymbol{gfh}}} \cdot \mathbf{Det}^*_g(\boldsymbol{x} \cdot \gamma, \boldsymbol{y} \cdot \gamma, \boldsymbol{z} \cdot \gamma).$$

Applying Remark 9.7 and recalling that $\kappa_{\boldsymbol{g}} = \nu_{\boldsymbol{g}} \circ \kappa^*_{\boldsymbol{gfh}}$, we have thus defined

$$\mathbf{Det}^*_{\boldsymbol{g}} \in H^0(\Gamma_0(p\mathbf{Z}_p), \mathcal{A}'_{\boldsymbol{g}} \hat{\otimes} \mathcal{A}_{\boldsymbol{f}} \hat{\otimes} \mathcal{A}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}(-\kappa_{\boldsymbol{g}})). \tag{213}$$

With the notations of Sections 4.2 and 8.1, let

$$\boldsymbol{\mathcal{A}}'_{\boldsymbol{g}} \boxtimes \boldsymbol{\mathcal{A}}_{\boldsymbol{f}} \boxtimes \boldsymbol{\mathcal{A}}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}} = \mathcal{A}'_{\boldsymbol{g}} \hat{\otimes} \mathcal{A}_{\boldsymbol{f}} \hat{\otimes} \mathcal{A}_{\boldsymbol{h}}|^{\text{ét}}_{\mathcal{H}_{\boldsymbol{g}}} \quad \text{and} \quad \boldsymbol{\mathcal{A}}'_{\boldsymbol{g}} \otimes \boldsymbol{\mathcal{A}}_{\boldsymbol{f}} \otimes \boldsymbol{\mathcal{A}}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}} = d^*\big(\boldsymbol{\mathcal{A}}'_{\boldsymbol{g}} \boxtimes \boldsymbol{\mathcal{A}}_{\boldsymbol{f}} \boxtimes \boldsymbol{\mathcal{A}}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}\big)$$

be the étale sheaf on Y^3 associated with the representation $\mathcal{A}_{\boldsymbol{f}} \hat{\otimes} \mathcal{A}'_{\boldsymbol{g}} \hat{\otimes} \mathcal{A}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}$ in $\mathbf{M}(\Gamma_0(p\mathbf{Z}_p)^3)$ and its pull back under the diagonal embedding $d : Y \longrightarrow Y^3$ respectively, so that one has a natural inclusion

$$H^0(\Gamma_0(p\mathbf{Z}_p), \mathcal{A}'_{\boldsymbol{g}} \hat{\otimes} \mathcal{A}_{\boldsymbol{f}} \hat{\otimes} \mathcal{A}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}(-\kappa_{\boldsymbol{g}})) \hookrightarrow H^0_{\text{ét}}(Y, \boldsymbol{\mathcal{A}}'_{\boldsymbol{g}} \otimes \boldsymbol{\mathcal{A}}_{\boldsymbol{f}} \otimes \boldsymbol{\mathcal{A}}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}(-\kappa_{\boldsymbol{g}})). \tag{214}$$

On the other hand, consider the following composition.

$$\begin{aligned} & H^0_{\text{ét}}(Y, \boldsymbol{\mathcal{A}}'_{\boldsymbol{g}} \otimes \boldsymbol{\mathcal{A}}_{\boldsymbol{f}} \otimes \boldsymbol{\mathcal{A}}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}(-\kappa_{\boldsymbol{g}})) \\ & \xrightarrow{d_*} H^4_{\text{ét}}(Y^3, \boldsymbol{\mathcal{A}}'_{\boldsymbol{g}} \boxtimes \boldsymbol{\mathcal{A}}_{\boldsymbol{f}} \boxtimes \boldsymbol{\mathcal{A}}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}(-\kappa_{\boldsymbol{g}}) \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(2)) \\ & \xrightarrow{\mathtt{HS}} H^1\big(\mathbf{Q}, H^3_{\text{ét}}\big(Y^3_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}'_{\boldsymbol{g}} \boxtimes \boldsymbol{\mathcal{A}}_{\boldsymbol{f}} \boxtimes \boldsymbol{\mathcal{A}}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}\big)(2 + \boldsymbol{\kappa}_{\boldsymbol{g}})\big) \end{aligned} \tag{215}$$

Because $H^4_{\text{ét}}(Y^3_{\bar{\mathbf{Q}}}, \mathscr{F})$ vanishes for every pro-sheaf $\mathscr{F} \in \mathbf{S}(Y^3_{\text{ét}})$ (cf. the discussion following Equation (156)), one has a natural isomorphism

$$H^3_{\text{ét}}\big(Y^3_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}'_{\boldsymbol{g}} \boxtimes \boldsymbol{\mathcal{A}}_{\boldsymbol{f}} \boxtimes \boldsymbol{\mathcal{A}}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}\big) = H^3_{\text{ét}}\big(Y^3_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}'_{\boldsymbol{f}} \boxtimes \boldsymbol{\mathcal{A}}_{\boldsymbol{g}} \boxtimes \boldsymbol{\mathcal{A}}_{\boldsymbol{h}}\big)|_{\mathcal{H}_{\boldsymbol{g}}}.$$

Moreover, as in Equation (156), the base change along $\nu_{\boldsymbol{g}}$ of the projection arising from the Künneth decomposition et cetera induce a map

$$H^1\big(\mathbf{Q}, H^3_{\text{ét}}\big(Y^3_{\bar{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}'_{\boldsymbol{f}} \boxtimes \boldsymbol{\mathcal{A}}_{\boldsymbol{g}} \boxtimes \boldsymbol{\mathcal{A}}_{\boldsymbol{h}}\big)|_{\mathcal{H}_{\boldsymbol{g}}}(2 + \boldsymbol{\kappa}_{\boldsymbol{g}})\big) \longrightarrow H^1(\mathbf{Q}, V(\boldsymbol{g}, \boldsymbol{f}, \boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}}), \tag{216}$$

and we denote by

$$\mathrm{AJ}^{\boldsymbol{gfh}}_{\text{ét}} : H^0_{\text{ét}}(Y, \boldsymbol{\mathcal{A}}'_{\boldsymbol{g}} \otimes \boldsymbol{\mathcal{A}}_{\boldsymbol{f}} \otimes \boldsymbol{\mathcal{A}}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}(-\kappa_{\boldsymbol{g}})) \longrightarrow H^1(\mathbf{Q}, V(\boldsymbol{g}, \boldsymbol{f}, \boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}}) \tag{217}$$

the composition of the maps (215) and (216).

Identifying $V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}}$ and $V(\boldsymbol{g}, \boldsymbol{f}, \boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}}$, one defines the sought for g-improved diagonal class (212) to be the image of $\mathbf{Det}^*_{\boldsymbol{g}}$ under the big Abel–Jacobi map defined in Equation (217), multiplied by the normalising factor $\frac{1}{b_p(\boldsymbol{l})}$ (cf. Equation (155)):

$$\kappa^*_g(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) = \frac{1}{b_p(\boldsymbol{l})} \cdot \mathrm{AJ}^{\boldsymbol{gfh}}_{\text{ét}}\big(\mathbf{Det}^*_{\boldsymbol{g}}\big).$$

(Here one views $\mathbf{Det}^*_{\boldsymbol{g}}$ as a global section of the étale sheaf $\boldsymbol{\mathcal{A}}'_{\boldsymbol{g}} \otimes \boldsymbol{\mathcal{A}}_{\boldsymbol{f}} \otimes \boldsymbol{\mathcal{A}}_{\boldsymbol{h}}|_{\mathcal{H}_{\boldsymbol{g}}}(-\kappa_{\boldsymbol{g}})$ via the inclusion (214).) The balancedness of $\kappa^*_{\boldsymbol{g}}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ follows from a similar argument as the one in the proof of Corollary 8.2.

We now verify that $\kappa^*_g(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ satisfies the identity displayed in Equation (2):

$$\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}} = \mathcal{E}_g(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \cdot \kappa^*_g(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}). \tag{218}$$

Let $\mathcal{H}^{\mathrm{cl}}_{\boldsymbol{g}}$ be the intersection of $\mathcal{H}_{\boldsymbol{g}}$ with $U^{\mathrm{cl}}_{\boldsymbol{f}} \times U^{\mathrm{cl}}_{\boldsymbol{g}} \times U^{\mathrm{cl}}_{\boldsymbol{h}}$. As $H^1(\mathbf{Q}, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}})$ is a torsion-free $\mathscr{O}_{\boldsymbol{gh}}$-module, in order to prove the previous equation it is sufficient to show that

$$\rho_{w*}(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) = \mathcal{E}_g(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m) \cdot \rho_{w*}(\kappa^*_g(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) \tag{219}$$

for each classical triple $w = (k, l, m)$ in the subset

$$\mathcal{H}^{\mathrm{bal}}_{\boldsymbol{g}} = \{(k, l, m) \in \mathcal{H}^{\mathrm{cl}}_{\boldsymbol{g}} \mid m \geqslant 3\}$$

of $\mathcal{H}^{\mathrm{cl}}_{\boldsymbol{g}}$, where $\rho_w : V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \longrightarrow V(\boldsymbol{f}_l, \boldsymbol{g}_l, \boldsymbol{h}_m)$ is the specialisation map (cf. Equation (145)) and $\mathcal{E}_g(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$ is the value of $\mathcal{E}_g(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ at (l, m). The set $\mathcal{H}^{\mathrm{bal}}_{\boldsymbol{g}}$ is the intersection of $\mathcal{H}_{\boldsymbol{g}}$ with the balanced region Σ_{bal}. Moreover Lemma 9.6 and Theorem 8.1 yield

$$(p-1)b_p(l) \cdot \varrho_{w*}(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) = \mathcal{E}_g(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m) \cdot \kappa^\dagger(\boldsymbol{g}_l, \boldsymbol{f}_k, \boldsymbol{h}_m)$$

for each $w = (k, l, m)$ in $\mathcal{H}^{\mathrm{bal}}_{\boldsymbol{g}}$. (Recall from Equation (157) that the definition of the twisted diagonal class $\kappa^\dagger(\boldsymbol{g}_l, \boldsymbol{f}_k, \boldsymbol{h}_m)$ is *not* symmetric in the forms $\boldsymbol{f}_k, \boldsymbol{g}_l$ and $\boldsymbol{h}_m$. Indeed, after identifying $V(\boldsymbol{g}_l, \boldsymbol{f}_k, \boldsymbol{h}_m)$ with $V(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$, it follows from Theorem 8.1 and Lemma 9.6 that the class $\kappa^\dagger(\boldsymbol{g}_l, \boldsymbol{f}_k, \boldsymbol{h}_m)$ is in general *not* equal to $\kappa^\dagger(\boldsymbol{f}_k, \boldsymbol{g}_l, \boldsymbol{h}_m)$.) To prove Equation (219), and with it Equation (218), it then remains to prove that

$$(p-1)b_p(l) \cdot \rho_{w*}\big(\kappa^*_g(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})\big) = \kappa^\dagger(\boldsymbol{g}_l, \boldsymbol{f}_k, \boldsymbol{h}_m)$$

for each $w = (k, l, m)$ in $\mathcal{H}^{\mathrm{bal}}_{\boldsymbol{g}}$. After unwinding the definition, this is in turn a direct consequence of the identity

$$\rho_w(\mathbf{Det}^*_g) = \mathrm{Det}^{\boldsymbol{r}(w)}_{Np},$$

where $\boldsymbol{r}(w) = (l-2, k-2, m-2)$, which holds true in $S_{\boldsymbol{r}(w)} \longrightarrow \mathcal{A}'_{l-2} \hat{\otimes} \mathcal{A}_{k-2} \hat{\otimes} \mathcal{A}_{m-2}$ for each balanced triple $w = (k, l, m)$ in $\mathcal{H}^{\mathrm{bal}}_{\boldsymbol{g}}$ by the very definitions of the invariants $\mathbf{Det}^*_g$ and $\mathrm{Det}^{\boldsymbol{r}}_{Np}$ (cf. Equations (213) and (41)).

9.4. Conclusion of the proof. — Assume that $w_o = (2, 1, 1)$ is exceptional. As in Section 9.2, denote by $\mathcal{H}_{\boldsymbol{fg}}$ the intersection of the improving planes $\mathcal{H}_{\boldsymbol{g}}$ and $\mathcal{H}_{\boldsymbol{f}}$, that is the set of triples in $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}}$ of the form $(\boldsymbol{l}+1, \boldsymbol{l}, 1)$. Denote by

$$\mathscr{L}^f_p(\boldsymbol{fg}, \boldsymbol{h}_1) = \mathscr{L}^f_p(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{fg}}} \in \mathscr{O}_{\boldsymbol{g}}$$

the analytic function on $U_{\boldsymbol{g}}$ which on $\boldsymbol{l}$ takes the value $\mathscr{L}^f_p(\boldsymbol{f}_{l+1}, \boldsymbol{g}_l, \boldsymbol{h}_1)$ (cf. Equation (55)). Define similarly

$$\mathcal{E}^*_f(\boldsymbol{fg}, \boldsymbol{h}_1) = \mathcal{E}^*_f(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{fg}}} \in \mathscr{O}_{\boldsymbol{g}} \quad \text{and} \quad \mathcal{E}_g(\boldsymbol{fg}, \boldsymbol{h}_1) = \mathcal{E}_g(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{fg}}} \in \mathscr{O}_{\boldsymbol{g}}.$$

Lemma 9.8. — *Let h_1 be the modular form of weight one and level $\Gamma_1(N)$ with p-stabilisation $\boldsymbol{h}_1$. One has*

$$\mathscr{L}_p^f(\boldsymbol{f}\boldsymbol{g},\boldsymbol{h}_1)=\mathcal{E}_f^*(\boldsymbol{f}\boldsymbol{g},\boldsymbol{h}_1)\cdot\mathcal{E}_g(\boldsymbol{f}\boldsymbol{g},\boldsymbol{h}_1)\cdot\mathscr{L}_p^{f*}(\boldsymbol{f}\boldsymbol{g},h_1),$$

where $\mathscr{L}_p^{f*}(\boldsymbol{f}\boldsymbol{g},h_1)$ is the analytic function in $\mathscr{O}_{\boldsymbol{g}}$ which on the classical point $l\geqslant 1$ in $U_{\boldsymbol{g}}^{\mathrm{cl}}$ takes the value

$$\mathscr{L}_p^{f*}(\boldsymbol{f}_{l+1},\boldsymbol{g}_l,h_1)=\frac{(w_N(\boldsymbol{f})_{l+1},h_1\cdot\boldsymbol{g}_l)_{Np}}{(w_N(\boldsymbol{f})_{l+1},w_N(\boldsymbol{f})_{l+1})_{Np}}.$$

Moreover, the following two conditions are equivalent.

1. *$\mathscr{L}_p^{f*}(\boldsymbol{f}_2,\boldsymbol{g}_1,h_1)$ is zero for all level-N test vectors $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ for $(\boldsymbol{f}^\sharp,\boldsymbol{g}^\sharp,\boldsymbol{h}^\sharp)$.*
2. *The complex central value $L(f_2^\sharp\otimes g_1^\sharp\otimes h_1^\sharp,1)$ vanishes.*

Proof. — Set $U=U_{\boldsymbol{g}}$, denote by $(\cdot,\cdot)_U : S_U^{\mathrm{ord}}(N,\bar{\chi}_{\boldsymbol{f}})\otimes_{\mathcal{O}(U)}S_U^{\mathrm{ord}}(N,\bar{\chi}_{\boldsymbol{f}})\longrightarrow\mathcal{O}(U)$ the $\mathcal{O}(U)$-adic Petersson product (cf. Section 7 of [**Hid93**]) and define

$$\mathscr{L}_p^{f*}(\boldsymbol{f}\boldsymbol{g},h_1)=\frac{(w_N(\boldsymbol{f})_{+1},e_{\mathrm{ord}}(h_1\cdot\boldsymbol{g}))_U}{(w_N(\boldsymbol{f})_{+1},w_N(f)_{+1})_U}.$$

Here $w_N(\boldsymbol{f})$ is the Hida family introduced in Lemma 6.1, $w_N(\boldsymbol{f})_{+1}$ is the family in $S_U^{\mathrm{ord}}(N,\bar{\chi}_{\boldsymbol{f}})$ whose specialisation at the classical point $m\geqslant 2$ equals $w_N(\boldsymbol{f}_{m+1})$ and e_{ord} is Hida's ordinary projector from the space of $\mathcal{O}(U)$-adic cusp forms of tame level N and character $\bar{\chi}_{\boldsymbol{f}}$ onto $S_U^{\mathrm{ord}}(N,\bar{\chi}_{\boldsymbol{f}})$, cf. [**Hid93**]. (Concretely $e_{\mathrm{ord}}(h_1\cdot\boldsymbol{g})_l$ equals $e_{\mathrm{ord}}(h_1\cdot\boldsymbol{g}_l)$ for each classical point l in U^{cl}, where the idempotent e_{ord} occurring in the right hand side is equal to $\lim_{n\to\infty}U_p^{n!}$.) By construction the value of $\mathscr{L}_p^{f*}(\boldsymbol{f}\boldsymbol{g},h_1)$ at a classical point $m\geqslant 1$ equals $\mathscr{L}_p^{f*}(\boldsymbol{f}_{l+1},\boldsymbol{g}_l,h_1)$.

Recall the operator $V=V_p$ on $L[\![q]\!]$ defined by $V\left(\sum c_nq^n\right)=\sum c_nq^{np}$. Then

$$\boldsymbol{h}_1=(1-\beta_{\boldsymbol{h}_1}\cdot V)h_1\quad\text{and}\quad\boldsymbol{h}_1^{[p]}=(1-\alpha_{\boldsymbol{h}_1}\cdot V)\boldsymbol{h}_1$$

with $\alpha_{\boldsymbol{h}_1}\cdot\beta_{\boldsymbol{h}_1}=\chi_{\boldsymbol{h}}(p)$, and similarly $\boldsymbol{g}_l^{[p]}=(1-\alpha_{\boldsymbol{g}_l}\cdot V)\boldsymbol{g}_l$. Since $\boldsymbol{g}_l^{[p]}\cdot V(\boldsymbol{h}_1)$ is p-depleted (viz. its n-th Fourier coefficient is zero if $p|n$), it is killed by e_{ord}, hence

$$\begin{aligned}\big(w_N(\boldsymbol{f}_{l+1}),\boldsymbol{g}_l\cdot V(\boldsymbol{h}_1)\big)_{Np}&=\alpha_{\boldsymbol{g}_l}\cdot\big(w_N(\boldsymbol{f}_{l+1}),V(\boldsymbol{g}_l\cdot\boldsymbol{h}_1)\big)_{Np}\\&=\frac{\alpha_{\boldsymbol{g}_l}}{\bar{\chi}_{\boldsymbol{f}}(p)\alpha_{\boldsymbol{f}_{l+1}}}\cdot\big(w_N(\boldsymbol{f}_{l+1}),\boldsymbol{g}_l\cdot\boldsymbol{h}_1\big)_{Np}.\end{aligned}$$

(To justify the last equality, note that $e_{\mathrm{ord}}\circ V=U_p^{-1}\cdot e_{\mathrm{ord}}$ and U_p acts on $w_N(\boldsymbol{f}_{l+1})$ as $\bar{\chi}_{\boldsymbol{f}}(p)\cdot\alpha_{\boldsymbol{f}_{l+1}}$.) Then

$$\big(w_N(\boldsymbol{f}_{l+1}),e_{\mathrm{ord}}\big(\boldsymbol{g}_l\cdot\boldsymbol{h}_1^{[p]}\big)\big)_{Np}=\left(1-\frac{\alpha_{\boldsymbol{g}_l}\alpha_{\boldsymbol{h}_1}}{\bar{\chi}_{\boldsymbol{f}}(p)\alpha_{\boldsymbol{f}_{l+1}}}\right)\cdot\big(w_N(\boldsymbol{f}_{l+1}),\boldsymbol{g}_l\cdot\boldsymbol{h}_1\big)_{Np}.$$

Similarly the vanishing of $e_{\mathrm{ord}}(\boldsymbol{g}_l^{[p]}\cdot V(h_1))$ yields

$$\big(w_N(\boldsymbol{f}_{l+1}),\boldsymbol{g}_l\cdot\boldsymbol{h}_1\big)_{Np}=\left(1-\frac{\bar{\chi}_{\boldsymbol{g}}(p)\alpha_{\boldsymbol{g}_l}}{\alpha_{\boldsymbol{h}_1}\alpha_{\boldsymbol{f}_{l+1}}}\right)\cdot\big(w_N(\boldsymbol{f}_{l+1}),\boldsymbol{g}_l\cdot h_1\big)_{Np}.$$

Using once again the identity $e_{\mathrm{ord}}(\boldsymbol{g}_l^{[p]} \cdot V(\boldsymbol{h}_1)) = 0$ one deduces that $\boldsymbol{g}^{[p]} \cdot \boldsymbol{h}_1 - \boldsymbol{g}_l \cdot \boldsymbol{h}_1^{[p]}$ is killed by e_{ord}, hence the previous two equations give (cf. Equations (55) and (131))

$$\begin{aligned}\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})(w) &= \frac{(w_N(\boldsymbol{f}_{l+1}), e_{\mathrm{ord}}(\boldsymbol{g}_l^{[p]} \cdot \boldsymbol{h}_1))_{Np}}{(w_N(\boldsymbol{f}_{l+1}), w_N(\boldsymbol{f}_{l+1}))} \\ &= \left(1 - \frac{\alpha_{\boldsymbol{g}_l}\alpha_{\boldsymbol{h}_1}}{\bar{\chi}_{\boldsymbol{f}}(p)\alpha_{\boldsymbol{f}_{l+1}}}\right)\left(1 - \frac{\bar{\chi}_{\boldsymbol{g}}(p)\alpha_{\boldsymbol{g}_l}}{\alpha_{\boldsymbol{h}_1}\alpha_{\boldsymbol{f}_{l+1}}}\right) \cdot \frac{(w_N(\boldsymbol{f}_{l+1}), \boldsymbol{g}_l \cdot h_1)_{Np}}{(w_N(\boldsymbol{f}_{l+1}), w_N(\boldsymbol{f}_{l+1}))_{Np}} \\ &= \mathcal{E}_f^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})(w) \cdot \mathcal{E}_g(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})(w) \cdot \mathscr{L}_p^{f*}(\boldsymbol{f}_{l+1}, \boldsymbol{g}_l, h_1)\end{aligned}$$

for each $l \geqslant 1$, where $w = (l+1, l, 1)$. (See Equations (1) and (197) for the definitions of $\mathcal{E}_g(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ and $\mathcal{E}_f^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ respectively.) This proves the first statement.

The second statement follows from the main result of [HK91] and Theorem 3 of [DN10]. (Note that $(w_N(\boldsymbol{f}_2), g_1 \cdot h_1)_{Np} = 0$ for each level-N test vectors $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ for $(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$, cf. the discussion preceding the statement of [DN10, Theorem 3].) $\square$

As in Section 9.2, for each $\mathscr{O}_{\boldsymbol{fgh}}$-module M denote by $M|_{\mathcal{H}_{\boldsymbol{fg}}} = M \otimes_{\mathrm{res}_{\boldsymbol{fg}}} \mathscr{O}_{\boldsymbol{g}}$ the base change of M along the morphism $\mathrm{res}_{\boldsymbol{fg}} : \mathscr{O}_{\boldsymbol{fgh}} \longrightarrow \mathscr{O}_{\boldsymbol{g}}$ sending $F(\boldsymbol{k},\boldsymbol{l},\boldsymbol{m})$ to $F(\boldsymbol{l}+1, \boldsymbol{l}, 1)$, and for each m in M denote by $m|_{\mathcal{H}_{\boldsymbol{fg}}}$ the natural image of m in the quotient $M|_{\mathcal{H}_{\boldsymbol{fg}}}$ of M. Finally, if ξ is equal to one of f, g and h, define

$$\mathscr{F}^\bullet V(\boldsymbol{fg}, \boldsymbol{h}_1) = \mathscr{F}^\bullet V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{fg}}} \quad \text{and} \quad V(\boldsymbol{fg}, \boldsymbol{h}_1)_\xi = V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_\xi|_{\mathcal{H}_{\boldsymbol{fg}}}.$$

Lemma 9.9. — *The map*

$$H^1(\mathbf{Q}_p, \mathscr{F}^2 V(\boldsymbol{fg}, \boldsymbol{h}_1)) \longrightarrow H^1(\mathbf{Q}_p, V(\boldsymbol{fg}, \boldsymbol{h}_1))$$

induced by the inclusion $\mathscr{F}^2 V(\boldsymbol{fg}, \boldsymbol{h}_1) \hookrightarrow V(\boldsymbol{fg}, \boldsymbol{h}_1)$ is injective.

Proof. — Set $M = V(\boldsymbol{fg}, \boldsymbol{h}_1)$ and $M_\xi = V(\boldsymbol{fg}, \boldsymbol{h}_1)_\xi$. The statement follows from the vanishing of $H^0(\mathbf{Q}_p, V(\boldsymbol{fg}, \boldsymbol{h}_1)/\mathscr{F}^2)$, which in turn follows from the claim:

$$H^0(\mathbf{Q}_p, \mathrm{gr}^0 M) = H^0(\mathbf{Q}_p, \mathrm{gr}^1 M) = 0. \tag{220}$$

To prove the claim, recall from Section 7.2 that the inertia subgroup of $G_{\mathbf{Q}(\mu_p)}$ acts on $\mathrm{gr}^0 M = M/\mathscr{F}^1 M$ via the character $\kappa_{\mathrm{cyc}}^{1-\boldsymbol{l}}$, hence $H^0(\mathbf{Q}_p, \mathrm{gr}^0 M) = 0$. Moreover, denote by Φ_f, Φ_g and Φ_h the $\mathscr{O}_{\boldsymbol{g}}$-valued unramified characters of $G_{\mathbf{Q}_p}$ sending an arithmetic Frobenius to $\frac{\bar{\chi}_{\boldsymbol{f}}(p) \cdot a_p(\boldsymbol{l}+1)}{b_p(\boldsymbol{l}) \cdot c_p(1)}$, $\frac{\bar{\chi}_{\boldsymbol{g}}(p) \cdot b_p(\boldsymbol{l})}{a_p(\boldsymbol{l}+1) \cdot c_p(1)}$ and $\frac{\bar{\chi}_{\boldsymbol{h}}(p) \cdot c_p(1)}{a_p(\boldsymbol{l}+1) \cdot b_p(\boldsymbol{l})}$ respectively. Then $G_{\mathbf{Q}_p(\mu_p)}$ acts on M_f, M_g and M_h via the characters Φ_f, $\Phi_g \cdot \kappa_{\mathrm{cyc}}^{\boldsymbol{l}}$ and $\Phi_h \cdot \kappa_{\mathrm{cyc}}$ respectively (cf. Section 7.2). According to the Ramanujan–Petersson conjecture the complex numbers $a_p(l+1)$ and $b_p(l)$ have absolute values $p^{l/2}$ and $p^{(l-1)/2}$ respectively for each classical point $l \geqslant 3$ in $U_{\boldsymbol{g}}$, hence $H^0(\mathbf{Q}_p, M_\xi(j)) = 0$ for $\xi = f, g, h$ and each integer j. Since $\mathrm{gr}^2 M$ is isomorphic to the direct sum of M_f, M_g and M_h, and since $\mathrm{gr}^1 M$ is isomorphic to the Kummer $\mathscr{O}_{\boldsymbol{g}}$-dual of $\mathrm{gr}^2 M$ (cf. Section 7.2), the claim follows. $\square$

We can now conclude the proof of Theorem B in the exceptional case.

Recall the g-improved balanced class $\kappa_g^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ in $H^1_{\mathrm{bal}}(\mathbf{Q}, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{g}}})$ constructed in Section 9.3. By the definition of the balanced condition (cf. Section 7.2), the restrictions at p of the classes $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ and $\kappa_g^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ are the images of classes

$$\breve{\kappa}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \in H^1(\mathbf{Q}_p, \mathscr{F}^2 V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) \quad \text{and} \quad \breve{\kappa}_g^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \in H^1(\mathbf{Q}_p, \mathscr{F}^2 V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_g})$$

respectively. Denote by

$$\breve{\kappa}(\boldsymbol{fg},\boldsymbol{h}_1) = \breve{\kappa}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{fg}}} \quad \text{and} \quad \breve{\kappa}_g^*(\boldsymbol{fg},\boldsymbol{h}_1) = \breve{\kappa}_g^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})|_{\mathcal{H}_{\boldsymbol{fg}}}$$

their restrictions to the improving line $\mathcal{H}_{\boldsymbol{fg}}$, and set

$$\kappa(\boldsymbol{fg},\boldsymbol{h}_1)_f = p_{f*}(\breve{\kappa}(\boldsymbol{fg},\boldsymbol{h}_1)) \quad \text{and} \quad \kappa_g^*(\boldsymbol{fg},\boldsymbol{h}_1) = p_{f*}(\breve{\kappa}_g^*(\boldsymbol{fg},\boldsymbol{h}_1)),$$

where $p_f : \mathscr{F}^2 V(\boldsymbol{fg},\boldsymbol{h}_1) \longrightarrow V(\boldsymbol{fg},\boldsymbol{h}_1)_f$ is the natural projection (cf. Section 7.2). According to Equation (218) and Lemma 9.9 one has

$$\kappa(\boldsymbol{fg},\boldsymbol{h}_1)_f = \mathcal{E}_g(\boldsymbol{fg},\boldsymbol{h}_1) \cdot \kappa_g^*(\boldsymbol{fg},\boldsymbol{h}_1)_f.$$

It then follows from Theorem A, Part 3 of Proposition 9.3 and Lemma 9.8 that

$$\mathscr{L}_p^{f*}(\boldsymbol{fg},h_1) = \mathscr{L}^*_{V(\boldsymbol{fg},\boldsymbol{h}_1)_f}(\kappa_g^*(\boldsymbol{fg},\boldsymbol{h}_1)).$$

Evaluating both sides of the previous equation at $\boldsymbol{l}=1$ and using once again Part 3 of Proposition 9.3 one gets the identity

$$\mathscr{L}_p^{f*}(\boldsymbol{f}_2,\boldsymbol{g}_1,h_1) = p \cdot a_p(2) \cdot \left\langle \exp_p^*(\kappa_g^*(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f), \eta_{\boldsymbol{f}_2}\omega_{\boldsymbol{g}_1}\omega_{\boldsymbol{h}_1} \right\rangle_{\boldsymbol{f}_2\boldsymbol{g}_1\boldsymbol{h}_1} \tag{221}$$

where $\kappa_g^*(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f$ is the weight-1 specialisation of $\kappa_g^*(\boldsymbol{fg},\boldsymbol{h}_1)_f$:

$$\kappa_g^*(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f = \rho_{1*}(\kappa_g^*(\boldsymbol{fg},\boldsymbol{h}_1)_f) \in H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2)^-_{\beta\beta}).$$

Similarly as in Section 9.1, we claim that the following statements are equivalent.
(a) The complex central value $L(f_2^\sharp \otimes g_1^\sharp \otimes h_1^\sharp, 1)$ vanishes.
(b) $\mathscr{L}_p^{f*}(\boldsymbol{f}_2,\boldsymbol{g}_1,h_1) = 0$ for all level-N test vectors $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ for $(\boldsymbol{f}^\sharp,\boldsymbol{g}^\sharp,\boldsymbol{h}^\sharp)$.
(c) $\exp_p^*(\kappa_g^*(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f) = 0$.
(d) $\exp_p^*(\mathrm{res}_p(\kappa_g^*(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1))) = 0$.
(e) $\kappa_g^*(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)$ is crystalline at p.

(As usual, here $\kappa_g^*(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)$ in $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1))$ denotes the specialisation of $\kappa_g^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ at w_o.) The equivalence between (a) and (b) is proved in Lemma 9.8.

As $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ varies through the level-N test vectors for $(\boldsymbol{f}^\sharp,\boldsymbol{g}^\sharp,\boldsymbol{h}^\sharp)$, the differentials $\eta_{\boldsymbol{f}_2}\omega_{\boldsymbol{g}_1}\omega_{\boldsymbol{h}_1}$ generate the L-module $V^*(\boldsymbol{f}_2)^+_{\beta\beta} = D_{\mathrm{dR}}(V^*(\boldsymbol{f}_2)^+_{\beta\beta})$ (cf. Section 9.2). Equation (221) then proves that (b) and (c) are equivalent to each other. (Recall that $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$, hence $\kappa_g^*(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$, is independent of the choice of the level-N test vectors $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ for $(\boldsymbol{f}^\sharp,\boldsymbol{g}^\sharp,\boldsymbol{h}^\sharp)$, cf. Remark 1.3(3).)

The equivalence between (c) and (d) follows, as in Section 9.1, from the balancedness of the improved diagonal class. More precisely, the projection

$$p^- : V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1) \longrightarrow V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)^-$$

induces an isomorphism between $\mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)$ and $D_{\mathrm{dR}}(V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)^-)$, hence ($d$) is equivalent to the vanishing of the dual exponential of $p^-_*(\mathrm{res}_p(\kappa(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)))$. In addition, since $V(\boldsymbol{f}_2)^-_{\beta\beta} = V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f$ is a $G_{\mathbf{Q}_p}$-direct summand of $V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)^-$ (cf. Section 9.2), and since $\kappa_g^*(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)$ is balanced at p, the diagram (193) yields

$$p^-_*\big(\mathrm{res}_p\big(\kappa_g^*(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)\big)\big) = \kappa_g^*(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f,$$

thus proving the equivalence between (c) and (d).

Finally, the equivalence between (d) and (e) follows from Lemma 9.1. This concludes the proof of Theorem B in the exceptional case.

References

[AI21a] Fabrizio Andreatta and Adrian Iovita. Katz type p-adic L-functions for primes p non-split in the cm field. *Preprint*, (available at http://www.mat.unimi.it/users/andreat/research.html), 2021. 6

[AI21b] Fabrizio Andreatta and Adrian Iovita. Triple product p-adic L-functions associated to finite slope p-adic families of modular forms. *Duke Math. J.*, to appear (available at http://www.mat.unimi.it/users/andreat/research.html), 2021. 5

[AIS15] Fabrizio Andreatta, Adrian Iovita, and Glenn Stevens. Overconvergent Eichler-Shimura isomorphisms. *J. Inst. Math. Jussieu*, 14(2):221–274, 2015. 27, 30, 31, 33, 34, 36, 37, 39, 40, 46

[AL78] Arthur Oliver Lonsdale Atkin and Wen Ch'ing Winnie Li. Twists of newforms and pseudo-eigenvalues of W-operators. *Invent. Math.*, 48(3):221–243, 1978. 49

[AS86a] Avner Ash and Glenn Stevens. Cohomology of arithmetic groups and congruences between systems of Hecke eigenvalues. *J. Reine Angew. Math.*, 365:192–220, 1986. 29

[AS86b] Avner Ash and Glenn Stevens. Modular forms in characteristic l and special values of their L-functions. *Duke Math. J.*, 53(3):849–868, 1986. 34

[AS08] Avner Ash and Glenn Stevens. p-adic deformations of arithmetic cohomology. *Preprint*, 2008. 27, 31

[BD07] Massimo Bertolini and Henri Darmon. Hida families and rational points on elliptic curves. *Invent. Math.*, 168(2):371–431, 2007. 6

[BD16] Joël Bellaïche and Mladen Dimitrov. On the eigencurve at classical weight 1 points. *Duke Math. J.*, 165(2):245–266, 2016. 41

[BDP13] Massimo Bertolini, Henri Darmon, and Kartik Prasanna. Generalized Heegner cycles and p-adic Rankin L-series. *Duke Math. J.*, 162(6):1033–1148, 2013. With an appendix by Brian Conrad. 6, 15, 23

[BDR15] Massimo Bertolini, Henri Darmon, and Victor Rotger. Beilinson-Flach elements and Euler systems I: Syntomic regulators and p-adic Rankin L-series. *J. Algebraic Geom.*, 24(2):355–378, 2015. 23

[Ben14] Denis Benois. Trivial zeros of p-adic L-functions at near-central points. *J. Inst. Math. Jussieu*, 13(3):561–598, 2014. 80

[Bes00] Amnon Besser. A generalization of Coleman's p-adic integration theory. *Invent. Math.*, 142(2):397–434, 2000. 23

[BK90] Spencer Bloch and Kazuya Kato. L-functions and Tamagawa numbers of motives. In *The Grothendieck Festschrift, Vol. I*, volume 86 of *Progr. Math.*, pages 333–400. Birkhäuser Boston, Boston, MA, 1990. 19, 20, 21, 23, 55

[Bro94] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original. 36

[BSV20a] Massimo Bertolini, Marco Adamo Seveso, and Rodolfo Venerucci. Balanced diagonal classes and rational points on elliptic curves. *This Volume*, 2020. 2, 5, 40, 78

[BSV20b] Massimo Bertolini, Marco Adamo Seveso, and Rodolfo Venerucci. Diagonal classes and the Bloch-Kato conjecture. *Münster J. Math.*, 13(2):317–352, 2020. 20, 23

[Col79] Robert F. Coleman. Division values in local fields. *Invent. Math.*, 53(2):91–116, 1979. 52

[Col97] Robert F. Coleman. p-adic Banach spaces and families of modular forms. *Invent. Math.*, 127(3):417–479, 1997. 31

[Del71] Pierre Deligne. Formes modulaires et représentations l-adiques. In *Séminaire Bourbaki. Vol. 1968/69: Exposés 347–363*, volume 175 of *Lecture Notes in Math.*, pages Exp. No. 355, 139–172. Springer, Berlin, 1971. 24, 25

[dJ95] Aise Johan de Jong. Crystalline Dieudonné module theory via formal and rigid geometry. *Inst. Hautes Études Sci. Publ. Math.*, (82):5–96 (1996), 1995. 39

[DN10] Mladen Dimitrov and Louise Nyssen. Test vectors for trilinear forms when at least one representation is not supercuspidal. *Manuscripta Math.*, 133(3-4):479–504, 2010. 89

[DR14] Henri Darmon and Victor Rotger. Diagonal cycles and Euler systems I: A p-adic Gross-Zagier formula. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(4):779–832, 2014. 23, 24, 47, 49, 76

[DR16] Henri Darmon and Victor Rotger. Diagonal cycles and Euler systems II: the Birch and Swinnerton-Dyer conjecture for Hasse-Weil-Artin l-series. *J. Amer. Math. Soc.*, in press, 2016. 5

[DR20] Henri Darmon and Victor Rotger. p-adic families of diagonal cycles. *This Volume*, 2020. 5

[Fal88] Gerd Faltings. p-adic Hodge theory. *J. Amer. Math. Soc.*, 1(1):255–299, 1988. 15

[FK88] Eberhard Freitag and Reinhardt Kiehl. *Étale cohomology and the Weil conjecture*, volume 13 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1988. Translated from the German by Betty S. Waterhouse and William C. Waterhouse, With an historical introduction by J. A. Dieudonné. 10, 17, 62

[Gar87] Paul B. Garrett. Decomposition of Eisenstein series: Rankin triple products. *Ann. of Math. (2)*, 125(2):209–235, 1987. 50

[GS93] Ralph Greenberg and Glenn Stevens. p-adic L-functions and p-adic periods of modular forms. *Invent. Math.*, 111(2):407–447, 1993. 38, 40, 82

[GS16] Matthew Greenberg and Marco Adamo Seveso. p-adic families of modular forms and p-adic Abel-Jacobi maps. *Ann. Math. Qué.*, 40(2):397–434, 2016. 85

[GS20] Matthew Greenberg and Marco Adamo Seveso. Triple product p-adic L-functions for balanced weights. *Math. Ann.*, 376(1-2):103–176, 2020. 6, 62, 63

[Hid85] Haruzo Hida. A p-adic measure attached to the zeta functions associated with two elliptic modular forms. I. *Invent. Math.*, 79(1):159–195, 1985. 16, 23, 49

[Hid86] Haruzo Hida. Galois representations into $\mathrm{GL}_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms. *Invent. Math.*, 85(3):545–613, 1986. 37, 38, 41, 42

[Hid93] Haruzo Hida. *Elementary theory of L-functions and Eisenstein series*, volume 26 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1993. 88

[HK91] Michael Harris and Stephen S. Kudla. The central critical value of a triple product L-function. *Ann. of Math. (2)*, 133(3):605–672, 1991. 7, 50, 76, 89

[Hsi20] Ming-Lun Hsieh. Hida families and p-adic triple product L-functions. *Amer. J. Math.*, to appear (available at http://www.math.sinica.edu.tw/mlhsieh/research.htm), 2020. 6, 49, 50

[HT01] Michael Harris and Jacques Tilouine. p-adic measures and square roots of special values of triple product L-functions. *Math. Ann.*, 320(1):127–147, 2001. 49

[Ich08] Atsushi Ichino. Trilinear forms and the central values of triple product L-functions. *Duke Math. J.*, 145(2):281–307, 2008. 50

[Jan88] Uwe Jannsen. Continuous étale cohomology. *Math. Ann.*, 280(2):207–245, 1988. 33

[JLZ20] Dimitar Jetchev, David Loeffler, and Sarah Livia Zerbes. Heegner points in Coleman families. *Proceedings of the London Mathematical Society,* to appear, 2020. 6

[Kat04] Kazuya Kato. p-adic Hodge theory and values of zeta functions of modular forms. *Astérisque*, (295):ix, 117–290, 2004. Cohomologies p-adiques et applications arithmétiques. III. 8

[KLZ17] Guido Kings, David Loeffler, and Sarah Livia Zerbes. Rankin-Eisenstein classes and explicit reciprocity laws. *Camb. J. Math.*, 5(1):1–122, 2017. 46, 49, 52

[KLZ20] Guido Kings, David Loeffler, and Sarah Livia Zerbes. Rankin–Eisenstein classes and explicit reciprocity laws. *Amer. J. Math.*, 142(1):79–138, 2020. 23

[LZ14] David Loeffler and Sarah Livia Zerbes. Iwasawa theory and p-adic L-functions over $\mathbb{Z}_p^2$-extensions. *Int. J. Number Theory*, 10(8):2045–2095, 2014. 52

[Mil80] James S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series.* Princeton University Press, Princeton, N.J., 1980. 11, 14, 59

[Mil06] James S. Milne. *Arithmetic duality theorems.* BookSurge, LLC, Charleston, SC, second edition, 2006. 61, 62, 63

[Miy06] Toshitsune Miyake. *Modular forms.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, english edition, 2006. Translated from the 1976 Japanese original by Yoshitaka Maeda. 21

[Nek04] Jan Nekovář. Syntomic cohomology and p-adic regulators. *Preprint*, available at https://webusers.imj-prg.fr/ jan.nekovar/pu/syn.pdf, 2004. 23

[Nek06] Jan Nekovář. Selmer complexes. *Astérisque*, (310):viii+559, 2006. 84

[Niz97] Wiesława Nizioł. On the image of p-adic regulators. *Invent. Math.*, 127(2):375–400, 1997. 23

[Niz01] Wiesława Nizioł. Cohomology of crystalline smooth sheaves. *Compositio Math.*, 129(2):123–147, 2001. 23

[NN16] Jan Nekovář and Wiesława Nizioł. Syntomic cohomology and p-adic regulators for varieties over p-adic fields. *Algebra Number Theory*, 10(8):1695–1790, 2016. 19

[Och03] Tadashi Ochiai. A generalization of the Coleman map for Hida deformations. *Amer. J. Math.*, 125(4):849–892, 2003. 52

[Oht95] Masami Ohta. On the p-adic Eichler-Shimura isomorphism for Λ-adic cusp forms. *J. Reine Angew. Math.*, 463:49–98, 1995. 37, 38

[Oht00] Masami Ohta. Ordinary p-adic étale cohomology groups attached to towers of elliptic modular curves. II. *Math. Ann.*, 318(3):557–583, 2000. 37, 42, 43, 46

[Pil13] Vincent Pilloni. Overconvergent modular forms. *Ann. Inst. Fourier (Grenoble)*, 63(1):219–239, 2013. 39, 41

[PR94] Bernadette Perrin-Riou. Théorie d'Iwasawa des représentations p-adiques sur un corps local. *Invent. Math.*, 115(1):81–161, 1994. With an appendix by Jean-Marc Fontaine. 52, 80

[Sai97] Takeshi Saito. Modular forms and p-adic Hodge theory. *Invent. Math.*, 129(3):607–620, 1997. 17

[Ser67] Jean-Pierre Serre. Local class field theory. In *Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965)*, pages 128–161. Thompson, Washington, D.C., 1967. 84

[Shi71] Goro Shimura. *Introduction to the arithmetic theory of automorphic functions*. Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo; Princeton University Press, Princeton, N.J., 1971. Kanô Memorial Lectures, No. 1. 12, 41, 71

[Tsu99] Takeshi Tsuji. p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case. *Invent. Math.*, 137(2):233–411, 1999. 15

[Ven16] Rodolfo Venerucci. Exceptional zero formulae and a conjecture of Perrin-Riou. *Invent. Math.*, 203(3):923–972, 2016. 78, 79, 80, 83

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