BALANCED DIAGONAL CLASSES AND RATIONAL POINTS ON ELLIPTIC CURVES

by

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Abstract. — Let A be an elliptic curve over the rationals with multiplicative reduction at a prime p, and let K be a quadratic field in which p is inert. Under a generalised Heegner assumption, our previous contribution [BSV20] to this volume attaches to (A, p, K) balanced diagonal classes in the Selmer groups of the p-adic Tate module of A over certain ring class fields of K. These classes are obtained as p-adic limits of geometric classes in the cohomology of higher-dimensional Kuga–Sato varieties. The main result of this paper relates these diagonal classes to p-adic logarithms of Heegner or Stark–Heegner points, depending on whether K is complex or real respectively.

To Bernadette Perrin-Riou on her 65th birthday

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1. Description and statement of results

Let $(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ be a triple of p-adic Hida families of common tame level N. Assume that f interpolates the weight 2 cusp form attached to an elliptic curve $A/\mathbf{Q}$ with multiplicative reduction at p, and that $\boldsymbol{g}_\alpha$ and $\boldsymbol{h}_\alpha$ respectively specialise in weight 1 to (p-stabilised) theta-series g_α and h_α associated to the same quadratic extension $K/\mathbf{Q}$, having good reduction at p and inverse characters. Let $\kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ be the diagonal class constructed in our previous contribution [BSV20] to this volume. This article builds on the main results of loc. cit. to relate (a component of) the Bloch–Kato

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logarithm of the specialisation at $(2,1,1)$ of $\kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ to the product of the formal group logarithms of two Heegner points, respectively Stark–Heegner points when K is imaginary, respectively real. See Theorem A below for the precise statement, holding under Assumption 1.1.

Our strategy goes along the following lines. Let $\mathscr{L}_p^f(\boldsymbol{f}, g_\alpha, h_\alpha)$ denote the restriction to the line $(\boldsymbol{k},1,1)$ of the triple product p-adic L-function $\mathscr{L}_p^f(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ defined in loc. cit.. Section 3 shows that $\mathscr{L}_p^f(\boldsymbol{f}, g_\alpha, h_\alpha)^2$ factors as a product of two Hida-Rankin p-adic L-functions attached to A/K. A suitable extension of main result of [BD07], resp. [BD09] for K imaginary quadratic, resp. real quadratic shows that the second derivative at $\boldsymbol{k}=2$ of the above mentioned Hida–Rankin p-adic L-functions is equal to the square of the formal group logarithm of a Heegner point, resp. Stark–Heegner point. Theorem A of [BSV20] describes $\mathscr{L}_p^f(\boldsymbol{f}, g_\alpha, h_\alpha)$ as the image by a branch of the Perrin–Riou logarithm of the restriction of $\kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ to the line $(\boldsymbol{k},1,1)$. Theorem A of this paper then follows from Proposition 2.2, which extends results of [Ven16] to obtain a formula for the second derivative of the Perrin-Riou logarithm of the above class at $\boldsymbol{k}=2$.

More precisely, let $A/\mathbf{Q}$ be an elliptic curve of conductor $N_f p$, having multiplicative reduction at a prime $p>3$ (hence $p \nmid N_f$). Let $K/\mathbf{Q}$ be a quadratic extension of discriminant d_K coprime with $N_f p$ and quadratic character $\varepsilon_K : (\mathbf{Z}/d_K\mathbf{Z})^* \to \mu_2$. Let

$$f = \sum_{n \geqslant 1} a_n(A) \cdot q^n \in S_2(N_f p, \mathbf{Z})^{\mathrm{new}}$$

be the weight-two newform associated with A by the modularity theorem of Wiles, Taylor–Wiles et al., and let

$$\nu_g : G_K \longrightarrow \bar{\mathbf{Q}}^* \quad \text{and} \quad \nu_h : G_K \longrightarrow \bar{\mathbf{Q}}^*$$

be two ray class characters of K. Write $N_f = N_f^+ \cdot N_f^-$, where N_f^- is the product of the prime divisors of N_f which are inert in $K/\mathbf{Q}$. We make the following

Assumption 1.1. —

1. *(Heegner hypothesis) p is* inert *in $K/\mathbf{Q}$, N_f^- is square-free and $\varepsilon_K(-N_f^-) = +1$.*
2. *(Modularity) When $K/\mathbf{Q}$ is real, both ν_g and ν_h have mixed signature.*
3. *(Cuspidality) The characters ν_g and ν_h are not induced by Dirichlet characters.*
4. *(Self-duality) The central characters of ν_g and ν_h are inverse to each other.*
5. *(Local signs) The conductors of ν_g and ν_h are coprime to $p \cdot d_K \cdot N_f$.*
6. *(Residual irreducibility) The $\mathbf{F}_p[G_\mathbf{Q}]$-module $A_p(\bar{\mathbf{Q}})$ of p-torsion points of A is irreducible.*

Let ν_ξ denote either ν_g or ν_h and let $L/\mathbf{Q}_p$ be a finite extension containing the Fourier coefficients of f and the values of ν_ξ. In light of Assumption 1.1, the two-dimensional L-representation $\mathrm{Ind}_\mathbf{Q}^K(\nu_\xi)$ of $G_\mathbf{Q}$ induced by $\nu_\xi : G_K \longrightarrow L^*$ is *odd* and *irreducible*. Thanks to the work of Hecke [Miy06, Section 4.8], it arises from the cuspidal weight-one theta series

$$\xi = \sum_{\mathfrak{a}} \nu_\xi(\mathfrak{a}) \cdot q^{\mathbf{N}\mathfrak{a}} \in S_1(N_\xi, \chi_\xi).$$

Here the sum runs over the ideals $\mathfrak{a}$ of $\mathcal{O}_K$ which are coprime to the conductor $\mathfrak{f}_\xi$ of ν_ξ, $\mathbf{N}\mathfrak{a}$ denotes the norm of $\mathfrak{a}$, $N_\xi = d_K \cdot \mathbf{N}\mathfrak{f}_\xi$ and $\chi_\xi = \varepsilon_K \cdot \nu_\xi^{\mathrm{cen}}$, where $\nu_\xi^{\mathrm{cen}} : G_{\mathbf{Q}} \longrightarrow \bar{\mathbf{Q}}^*$ is the central character of ν_ξ. The form ξ is primitive of conductor N_ξ and the dual of its Deligne–Serre L-representation is isomorphic to $\mathrm{Ind}_{\mathbf{Q}}^K(\nu_\xi)$.

Since p is inert in $K/\mathbf{Q}$, one has $a_p(\xi) = 0$ so that the p-th Hecke polynomial of ξ is equal to

$$X^2 + \chi_\xi(p).$$

Let $\alpha_\xi \in \mathscr{O}^*$ be a fixed square root of $-\chi_\xi(p)$, and write

$$\xi_\alpha = \xi(q) - \beta_\xi \cdot \xi(q^p) \in S_1(N_\xi p, \chi_\xi), \qquad \text{with} \;\; \beta_\xi = \frac{\chi_\xi(p)}{\alpha_\xi} = -\alpha_\xi \tag{1}$$

for the corresponding p-stabilisation. (Here we assume that L contains α_ξ.) Since $\chi_g \cdot \chi_h$ is the trivial character, without loss of generality we may assume that the roots $\alpha_g, \beta_g, \alpha_h, \beta_h$ are ordered in such a way that

$$\alpha_g \cdot \alpha_h = \beta_g \cdot \beta_h = a_p(A) = \pm 1. \tag{2}$$

As explained in Section 5 of our contribution [BSV20], the work of Hida and Wiles implies the existence of a unique triple $(\boldsymbol{f}^\sharp, \boldsymbol{g}_\alpha^\sharp, \boldsymbol{h}_\alpha^\sharp)$ of L-rational primitive Hida families of tame conductors (N_f, N_g, N_h) and tame characters (χ_f, χ_g, χ_h) which specialises to the triple (f, g_α, h_α) at w_o. Note that the triple $(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$ satisfies Assumptions 1.1 and 1.2 stated in Section 1 of [BSV20] (cf. Equation (1) and Assumption 1.1.3), and that $w_o = (2, 1, 1)$ is *exceptional* in the sense of Section 1.2 of loc. cit. (cf. Equation (2)).

With notations as in Section 1.1 of loc. cit., denote by N the least common multiple of N_f, N_g and N_h, by $V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ the big Galois representation attached to any choice of level-N test vector for $(\boldsymbol{f}^\sharp, \boldsymbol{g}_\alpha^\sharp, \boldsymbol{h}_\alpha^\sharp)$ (cf. Remark 1.3(3) of loc. cit.), and by

$$\kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha) \in H^1_{\mathrm{bal}}(\mathbf{Q}, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha))$$

the corresponding diagonal class. In [Hsi20] Hsieh constructs a distinguished level-N test vector $(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ (denoted $(\boldsymbol{f}^\star, \boldsymbol{g}_\alpha^\star, \boldsymbol{h}_\alpha^\star)$ in [BSV20, Section 6.1]) for $(\boldsymbol{f}^\sharp, \boldsymbol{g}_\alpha^\sharp, \boldsymbol{h}_\alpha^\sharp)$, and computes explicitly the local constants which appear in the interpolation formulae satisfied by the p-adic L-function $\mathscr{L}_p^f(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ (cf. Sections 1.1 and 6.1 of loc. cit.).

Let $V_p(A) = \mathrm{Ta}_p(A) \otimes_{\mathbf{Z}} \mathbf{Q}$ be the p-adic Tate module of A with $\mathbf{Q}_p$-coefficients, let $Y_1(N_f p)$ be the open modular curve over $\mathbf{Q}$ of level $\Gamma_1(N_f p)$, and let $V(f)$ be the f-isotypic quotient of $H^1_{\text{ét}}(Y_1(N_f p)_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(1))$ (cf. Sections 2.1 and 2.4 of [BSV20]). Fix a modular parametrisation

$$\wp_\infty : Y_1(N_f p) \longrightarrow A.$$

This induces an isomorphism of $G_{\mathbf{Q}}$-modules

$$\wp_{\infty *} : V(f) \cong V_p(A) \tag{3}$$

which we often consider as an equality in what follows. Set

$$V(f, g, h) = V_p(A) \otimes_{\mathbf{Q}_p} V(g) \otimes_L V(h),$$

where $V(\xi) = V(\xi_\alpha)$ is the canonical model of the dual of the Deligne–Serre representation of $\xi = g, h$ arising from the specialisation of $V(\boldsymbol{\xi}_\alpha)$ at weight one (cf. Section 5 of [**BSV20**]). The fixed test vector $(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ and modular parametrisation $\wp_\infty$ determine a projection $V(\boldsymbol{f}_2, \boldsymbol{g}_{\alpha 1}, \boldsymbol{h}_{\alpha 1}) \twoheadrightarrow V(f,g,h)$ (denoted $\varpi_\star$ in Section 2 below), mapping the specialisation at w_o of $\kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ to a global class

$$\kappa_{\alpha\alpha}(f,g,h) \in H^1(\mathbf{Q}, V(f,g,h)).$$

Let c be the non-trivial element of $\mathrm{Gal}(K/\mathbf{Q})$ and let $\nu_\xi^c : G_K \to L^*$ be the conjugate of ν_ξ by c. By Assumption 1.1(4) the characters

$$\varphi = \nu_g \cdot \nu_h \quad \text{and} \quad \psi = \nu_g \cdot \nu_h^c$$

are *ring class characters* of K (i.e., $\varphi^c = \varphi^{-1}$ and $\psi^c = \psi^{-1}$). Note the factorisation of $G_{\mathbf{Q}}$-representations

$$V(f,g,h) \cong V_p(A) \otimes \mathrm{Ind}_{\mathbf{Q}}^K(\varphi) \oplus V_p(A) \otimes \mathrm{Ind}_{\mathbf{Q}}^K(\psi). \tag{4}$$

In particular the Bloch–Kato Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$ decomposes as

$$\mathrm{Sel}(\mathbf{Q}, V(f,g,h)) \cong \mathrm{Sel}(K_\varphi, V_p(A))^\varphi \oplus \mathrm{Sel}(K_\psi, V_p(A))^\psi, \tag{5}$$

where $K_\cdot/K$ denotes the ring class field having the same conductor as $\cdot$ and $\mathrm{Sel}(K_\cdot, V_p(A))^\cdot$ is the submodule of the Selmer group $\mathrm{Sel}(K_\cdot, V_p(A)) \otimes_{\mathbf{Q}_p} L$ of $V_p(A) \otimes_{\mathbf{Q}_p} L$ over $K_\cdot$ on which $\mathrm{Gal}(K_\cdot/K)$ acts via the inverse of $\cdot$.

It follows from Equation (4) and the Artin formalism that the Garrett triple product L-function $L(f \otimes g \otimes h, s) = L(V(f,g,h), s)$ factors as the product of the Rankin L-functions $L(A/K, \varphi, s)$ and $L(A/K, \psi, s)$, which have both sign -1 in their functional equation by Assumption 1.1.1. In particular $L(f \otimes g \otimes h, s)$ vanishes to order at least two at $s = 1$. Theorem B of [**BSV20**] in the exceptional case then proves that the diagonal class $\kappa_{\alpha\alpha}(f,g,h)$ is crystalline at p, hence belongs to the Bloch–Kato Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$ of the representation $V(f,g,h)$ of $G_{\mathbf{Q}}$:

$$\kappa_{\alpha\alpha}(f,g,h) \in \mathrm{Sel}(\mathbf{Q}, V(f,g,h)).$$

Write ϱ for either φ or ψ. The articles [**BD07**] and [**BD09**] (see also [**GSS16**]) associate to $\boldsymbol{f}$ and ϱ a p-adic L-function

$$L_p(\boldsymbol{f}/K, \varrho) \in \mathscr{O}_{\boldsymbol{f}},$$

interpolating the central values of the L-series $L(f_k/K, \varrho, s)$ of the base change of f_k to K twisted by ϱ. Their definition, which depends only on the primitive family $\boldsymbol{f}^\sharp$, is recalled in Section 3.2 below.

Write K_p for the completion of K at the inert prime p. Noting that p splits completely in K_ϱ/K, let Frob_p in $\mathrm{Gal}(K_\varrho/\mathbf{Q})$ be the Frobenius element determined by the fixed embedding of $\bar{\mathbf{Q}}$ into $\bar{\mathbf{Q}}_p$, mapping K_ϱ to K_p. Denote by

$$\log_{\omega_f} : A(K_p)_L = A(K_p) \hat{\otimes} L \longrightarrow K_p \otimes_{\mathbf{Q}_p} L$$

the L-linear extension of the composition

$$A(K_p) \hat{\otimes} \mathbf{Q}_p \cong H^1_{\mathrm{fin}}(K_p, V(f)) \xrightarrow{\log_p} \tan_{K_p}(f) \cong K_p,$$

where H^1_{fin} is the finite subspace of H^1, $\tan_{K_p}(f)$ is the tangent space of the de Rham module $H^0(K_p, V(f) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})$, the first isomorphism arises from the map $\wp_{\infty *}$ and Kummer theory, $\log_p$ is the Bloch–Kato logarithm and the second isomorphism is evaluation at the canonical differential ω_f in the dual of $\tan_{K_p}(f)$ associated with f (see Section 2.5 of [BSV20], in particular Equations (29), (30) and (32)). Under our running assumptions, the p-adic L-function $L_p(\boldsymbol{f}/K,\varrho)$ vanishes at $\boldsymbol{k}=2$ to order at least two. An extension of the main results of [BD07] and [BD09] in the imaginary quadratic and real quadratic setting respectively – see in particular [GSS16, LMH20, LV14, Mok11] – prove the existence of a non-zero algebraic constant $\mathcal{Q}\in\bar{\mathbf{Q}}^*$ such that

$$c_f^2\cdot\frac{d^2}{d\boldsymbol{k}^2}L_p(\boldsymbol{f}/K,\varrho)_{\boldsymbol{k}=2}=\mathcal{Q}\cdot\log^2_{\omega_f}(P_\varrho^\varepsilon), \tag{6}$$

where $c_f=c_f(\wp_\infty)\in K_p^*$ is an explicit non-zero p-adic constant (depending on $\wp_\infty$) introduced in Section 2.2 below (see also Remark 1.2), and the point P_ϱ^ε in $A(K_p)_L$ are defined as follows.

If K is imaginary quadratic, choose a primitive Heegner point P in $A(K_\varrho)$ and let

$$P_\varrho=\sum_{\sigma\in\mathrm{Gal}(K_\varrho/K)}\varrho(\sigma)^{-1}\cdot P^\sigma\quad\text{and}\quad P_\varrho^\varepsilon=P_\varrho+\varepsilon\cdot P_\varrho^{\mathrm{Frob}_p}\quad\text{for }\varepsilon=a_p(A).$$

Note that the *global* point P_ϱ^ε is viewed in Equation (6) as a local point via our fixed embedding of $\bar{\mathbf{Q}}$ into $\bar{\mathbf{Q}}_p$. When ϱ is quadratic one checks that Frob_p acts on P_ϱ via a sign ε_ϱ (see for example the discussion in Section 4 of [BD07]).

If K is real quadratic, the *local point* P_ϱ in $A(K_p)$ is defined as in the above formula, by exploiting the action of $\mathrm{Pic}(\mathcal{O}_\varrho)$ on a *Stark–Heegner point* $P\in A(K_p)$ attached to K_ϱ, where $\mathrm{Pic}(\mathcal{O}_\varrho)\cong\mathrm{Gal}(K_\varrho/K)$ is the Picard group of the order $\mathcal{O}_\varrho$ of K corresponding to K_ϱ via class field theory.

Remark 1.2. — The main results of [BD07, BD09] are stated in terms of the logarithm

$$\log_A=\log_{q_A}\circ\varphi_{\mathrm{Tate}}^{-1}:A(K_p)\longrightarrow K_p,$$

where q_A is the Tate period of $A_{\mathbf{Q}_p}$, $\varphi_{\mathrm{Tate}}:K_p^*/q_A^{\mathbf{Z}}\cong A(K_p)$ is the Tate parametrisation and $\log_{q_A}:K_p^*\longrightarrow K_p$ is the branch of the p-adic logarithm which vanishes at q_A (see Section 2.2 below for more details). The p-adic constant $c_f\in K_p^*$ (defined in Equation (14) below) accounts for the discrepancy between $\log_A$ and the logarithm $\log_{\omega_f}$ introduced above (cf. Lemma 2.1 below). The nontrivial element of $\mathrm{Gal}(K_p/\mathbf{Q}_p)$ acts on c_f as multiplication by $\varepsilon=a_p(A)$, hence c_f^2 belongs to $\mathbf{Q}_p^*$. Similarly $\log^2_{\omega_f}(P_\varrho^\varepsilon)$ belongs to L, so that the identity (6) takes place in L.

Denote by

$$\mathscr{L}_p^f(\boldsymbol{f},g_\alpha,h_\alpha)\in\mathscr{O}_{\boldsymbol{f}}$$

the restriction of $\mathscr{L}_p^f(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)$ to the line $(\boldsymbol{k},1,1)$. Theorem 3.1 below shows the factorisation formula

$$\mathscr{L}_p^f(\boldsymbol{f},g_\alpha,h_\alpha)^2=\mathscr{A}\cdot L_p(\boldsymbol{f}/K,\varphi)\cdot L_p(\boldsymbol{f}/K,\psi), \tag{7}$$

where $\mathscr{A}$ is a bounded analytic function on $U_{\boldsymbol{f}}$ such that $\mathscr{A}(2)$ is an element of $\bar{\mathbf{Q}}^*$.

Under the assumptions of this section, Proposition 2.2 gives a formula for the second derivative of the Perrin-Riou big logarithm of a balanced class along the line

$(\boldsymbol{k}, 1, 1)$ at the point $\boldsymbol{k} = 2$. Combined with [BSV20, Theorem A], this gives the equality

$$c_f^2 \cdot \frac{d^2}{d\boldsymbol{k}^2} \mathscr{L}_p^f(\boldsymbol{f}, g_\alpha, h_\alpha)_{\boldsymbol{k}=2} = \mathcal{Q} \cdot \log_{\beta\beta}\big(\mathrm{res}_p(\kappa_{\alpha\alpha}(f, g, h))\big), \tag{8}$$

where $\mathcal{Q}$ is an explicit constant in $\mathbf{Q}^*$ and $\log_{\beta\beta}\big(\mathrm{res}_p(\kappa_{\alpha\alpha}(f, g, h))\big)$ is the evaluation of the p-adic Bloch–Kato logarithm of $\mathrm{res}_p(\kappa_{\alpha\alpha}(f, g, h))$ at a canonical differential $\omega_f \otimes \omega_{g_\alpha} \otimes \omega_{h_\alpha}$ (see Section 2 for details).

Combining Equations (6), (7) and (8) yields

Theorem A. — *For $\mathcal{Q}$ in $\bar{\mathbf{Q}}^*$ one has the equality*

$$\log_{\beta\beta}\big(\mathrm{res}_p\big(\kappa_{\alpha\alpha}(f, g, h)\big)\big) = \mathcal{Q} \cdot \log_{\omega_f}(P_\varphi^\varepsilon) \cdot \log_{\omega_f}(P_\psi^\varepsilon).$$

Recall that the complex L-function $L(f \otimes g \otimes h, s)$ attached to $V(f, g, h)$ vanishes to order at least 2 at $s = 1$ by Assumption 1.1.

Corollary B. — *Let K be imaginary quadratic. If $\varrho = \varphi$ or ψ is quadratic, assume that $\varepsilon = \varepsilon_\varrho$. Then*

$$\frac{d^2}{ds^2} L(f \otimes g \otimes h, s)_{s=1} \neq 0 \qquad \Longleftrightarrow \qquad \log_{\beta\beta}\big(\mathrm{res}_p\big(\kappa_{\alpha\alpha}(f, g, h)\big)\big) \neq 0.$$

Proof. — Under the current assumptions P_ϱ^ε is non-zero whenever P_ϱ is non-zero. Corollary B then follows from Theorem A combined with S.-W. Zhang's proof of the Gross–Zagier formula for Shimura curves [Zha01]. ☐

Remark C. — Theorem A and a suitable converse to the Gross–Zagier–Kolyvagin theorem show that the equivalent statements of Corollary B are also equivalent to the equality

$$\mathrm{Sel}(\mathbf{Q}, V(f, g, h)) = L \cdot \kappa_{\alpha\alpha}(f, g, h) \oplus L \cdot \kappa_{\beta\beta}(f, g, h), \tag{9}$$

that is the Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f, g, h))$ is generated by the global class $\kappa_{\alpha\alpha}(f, g, h)$ and its counterpart $\kappa_{\beta\beta}(f, g, h)$ defined by replacing the pair $(\boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ with $(\boldsymbol{g}_\beta, \boldsymbol{h}_\beta)$ (cf. Equation (2)).

To show that the equality (9) follows from the non-vanishing of the second derivative of $L(f \otimes g \otimes h, s)$, one notes that this condition implies that $\mathrm{Sel}(\mathbf{Q}, V(f, g, h))$ is two-dimensional by the Gross–Zagier–Kolyvagin theorem. The classes $\kappa_{\alpha\alpha}(f, g, h)$ and $\kappa_{\beta\beta}(f, g, h)$ are both non-trivial by Corollary B, hence one is reduced to prove that they are linearly independent. This follows again from Corollary B, noting that

$$\log_{\beta\beta}\big(\mathrm{res}_p(\kappa_{\beta\beta}(f, g, h))\big) = 0$$

since the Selmer class $\kappa_{\beta\beta}(f, g, h)$ arises from the *balanced* class $\kappa(\boldsymbol{f}, \boldsymbol{g}_\beta, \boldsymbol{h}_\beta)$.

Conversely, assume that the classes $\kappa_{\alpha\alpha}(f, g, h)$ and $\kappa_{\beta\beta}(f, g, h)$ generate the Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f, g, h))$, so that

$$\dim_L \mathrm{Sel}(\mathbf{Q}, V(f, g, h)) \leqslant 2. \tag{10}$$

Granting a *converse of the Gross–Zagier–Kolyvagin theorem* of the form

$$\dim_L \mathrm{Sel}(K_\varrho, V_p(A))^\varrho \leqslant 1 \quad \Longrightarrow \quad \mathrm{ord}_{s=1} L(f/K, \varrho, s) = \dim_L \mathrm{Sel}(K_\varrho, V_p(A))^\varrho \tag{11}$$

for ϱ equal to φ and ψ as above, one concludes readily as follows. Since the sign of the functional equation of $L(f/K, \varrho, s)$ is -1, Equations (10) and (11) imply that $L(f/K, \varrho, s)$ has a simple zero at $s = 1$ for $\varrho = \varphi$ and ψ, hence $L(f \otimes g \otimes h, s)$ has a double zero at $s = 1$. The above converse theorem may be approached by an extension of the methods of the forthcoming work [BLV17], which prove Birch and Swinnerton-Dyer formulae for general families of anticyclotomic characters of p-power conductor and are suited to extend such formulae to arbitrary ring class characters.

In the real quadratic setting, the next result relates the (local) Stark–Heegner points to the (global) Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f, g, h))$.

Corollary D. — *Assume that K is real quadratic. If the Stark–Heegner points P_φ^ε and P_ψ^ε are both non-trivial, then* $\dim_L \mathrm{Sel}(\mathbf{Q}, V(f, g, h)) \geqslant 2$.

Proof. — Theorem A implies that $\kappa_{\alpha\alpha}(f, g, h)$ and $\kappa_{\beta\beta}(f, g, h)$ are non-zero. The same argument as in Remark C shows that these classes are linearly independent. $\square$

Remark E. — Under the assumptions of Corollary D, the definition of $\kappa_{\alpha\alpha}(f, g, h)$ and $\kappa_{\beta\beta}(f, g, h)$ combined with Theorem A imply that the Stark–Heegner point P_ϱ^ε ($\varrho = \varphi, \psi$) arises as the restriction at p of a Selmer class in $\mathrm{Sel}(K_\varrho, V_p(A))^\varrho$. We refer the reader to the contribution [DR20] by Darmon–Rotger to this volume for an extensive discussion of this application (see in particular Theorem A of loc. cit.).

2. Derivatives of big logarithms II

This section should be regarded as a continuation of [BSV20, Section 6], where a study of multivariable Perrin-Riou logarithms is undertaken. After the preliminary Sections 2.1 and 2.2, Proposition 2.2 in Section 2.3 establishes a formula for the second derivative of the Perrin-Riou big logarithm of a balanced class along the line $(\boldsymbol{k}, 1, 1)$ at the point $\boldsymbol{k} = 2$, which constitutes a crucial ingredient in the proof of Theorem A.

Let (f, g, h) and $(\boldsymbol{f}^\sharp, \boldsymbol{g}_\alpha^\sharp, \boldsymbol{h}_\alpha^\sharp)$ be as in Section 1. Denote by $(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$, or more simply $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$, any level-$N$ test vector for $(\boldsymbol{f}^\sharp, \boldsymbol{g}_\alpha^\sharp, \boldsymbol{h}_\alpha^\sharp)$ (where N is as in Section 1). Throughout this section Assumption 1.1 is in force. In particular Assumption 6.3 of loc. cit. is satisfied (as $A_p(\bar{\mathbf{Q}})$ is p-distinguished by Tate's theory, since $p \geqslant 5$, cf. Section 2.2 below), hence one can consider the distinguished level-N test vector $(\boldsymbol{f}^\star, \boldsymbol{g}_\alpha^\star, \boldsymbol{h}_\alpha^\star)$ introduced in Section 6.1 of loc. cit.. (To ease notations, the latter was simply denoted $(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ in Section 1).

2.1. The projection $\varpi_{\boldsymbol{fgh}}$ and the class $\kappa_{\alpha\alpha}(f, g, h)$.

— Associated with the choice of a test vector $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) = (\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ we define a $G_{\mathbf{Q}}$-equivariant projection

$$\varpi_{\boldsymbol{fgh}} : V(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1) \longrightarrow V(f, g_\alpha, h_\alpha) \tag{12}$$

by the following recipe. Let $\boldsymbol{\xi}$ denote one of $\boldsymbol{f}, \boldsymbol{g}_\alpha$ or $\boldsymbol{h}_\alpha$. For each positive integer d dividing N/N_ξ denote by

$$v_d : Y_1(N, p) \longrightarrow Y_1(N_\xi, p)$$

the degeneracy map corresponding to multiplication by d on $\mathbf{H}$ under the analytic isomorphism defined in Equation (6) of loc. cit.. The $\mathbf{Q}$-rational map v_d induces pull-backs $v_d^* : V^*(\boldsymbol{\xi}^\sharp)^\cdot \longrightarrow V^*(\boldsymbol{\xi})^\cdot$ (for $\cdot = \emptyset, \pm$), which in turn induce morphisms $v_d^* : D^*(\boldsymbol{\xi}^\sharp)^\pm \longrightarrow D^*(\boldsymbol{\xi})^\pm$ and $v_d^* : H^1(\mathbf{Q}_p, V^*(\boldsymbol{\xi}^\sharp)^\cdot) \longrightarrow H^1(\mathbf{Q}_p, V^*(\boldsymbol{\xi})^\cdot)$ between the associated period rings and Galois cohomology groups. As d runs over the positive divisors of N/N_ξ, the images of $D^*(\boldsymbol{\xi}^\sharp)^\pm$ under the operators v_d^* generate $D^*(\boldsymbol{\xi})^\pm$ over $\mathscr{O}_{\boldsymbol{\xi}}$. As a consequence, if $\omega_{\boldsymbol{\xi}}^\cdot$ and $\eta_{\boldsymbol{\xi}}^\cdot$ (for $\cdot = \emptyset, \sharp$) denote the $\mathscr{O}_{\boldsymbol{\xi}}$-adic differentials associated to $\boldsymbol{\xi}^\cdot$ in Equations (118) and (122) of loc. cit. respectively, one has

$$\eta_{\boldsymbol{f}} = v_{\boldsymbol{f}}^*(\eta_{\boldsymbol{f}}^\sharp), \quad \omega_{\boldsymbol{g}} = v_{\boldsymbol{g}}^*(\omega_{\boldsymbol{g}}^\sharp) \quad \text{and} \quad \omega_{\boldsymbol{h}} = v_{\boldsymbol{h}}^*(\omega_{\boldsymbol{h}}^\sharp)$$

with $\mathscr{O}_{\boldsymbol{\xi}}$-linear combinations $v_{\boldsymbol{\xi}}^*$ of the operators v_d^*. (See Section 5 of [BSV20], especially Equation (95), Equations (117)–(123) and the discussion following them, for more details.) Denote by $v_{\boldsymbol{\xi}*} : V(\boldsymbol{\xi}) \longrightarrow V(\boldsymbol{\xi}^\sharp)$ the dual of $v_{\boldsymbol{\xi}}^*$ under the perfect pairing (103) of loc. cit. and set

$$\varpi_{\boldsymbol{fgh}} = v_{\boldsymbol{f}*} \otimes v_{\boldsymbol{g}*} \otimes v_{\boldsymbol{h}*} : V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \longrightarrow V(\boldsymbol{f}^\sharp, \boldsymbol{g}_\alpha^\sharp, \boldsymbol{h}_\alpha^\sharp).$$

With a slight abuse of notation, the map (12) is defined as the base change of $\varpi_{\boldsymbol{fgh}}$ under evaluation at $w_o = (2, 1, 1)$ on $\mathscr{O}_{\boldsymbol{fgh}}$ (cf. Equations (106) and (107) of [BSV20]).

Recall the modular parametrisation

$$\wp_\infty : Y_1(N_f p) \longrightarrow A$$

fixed in Section 1 (cf. Equation (3)) and set

$$\varpi_\star = \wp_{\infty*} \otimes \mathrm{id} \circ \varpi_{\boldsymbol{f}^\star \boldsymbol{g}_\alpha^\star \boldsymbol{h}_\alpha^\star} : V(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1) \longrightarrow V(f, g_\alpha, h_\alpha) \cong V(f, g, h),$$

(where id denotes the identity on $V(g_\alpha) \otimes_L V(h_\alpha) = V(g) \otimes_L V(h)$.) Then with the notation of Section 1 (cf. Remark 1.3(3) and Theorem B of [BSV20])

$$\kappa_{\alpha\alpha}(f, g, h) = \varpi_\star(\kappa(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1)) \in \mathrm{Sel}(\mathbf{Q}, V(f, g, h)).$$

For each local crystalline class $\mathfrak{z}$ in $H^1_{\mathrm{fin}}(\mathbf{Q}_p, V(f, g_\alpha, h_\alpha))$ define the *$\beta\beta$-component* of its p-adic logarithm by

$$\log_{\beta\beta}(\mathfrak{z}) = \left\langle \log_p(\mathfrak{z}), \omega_f \otimes \omega_{g_\alpha} \otimes \omega_{h_\alpha} \right\rangle_{f g_\alpha h_\alpha},$$

where ω_f is the differential associated with f in Equation (30) of [BSV20], the weight-one differentials ω_{g_α} and ω_{h_α} are the specialisations of $\omega_{\boldsymbol{g}_\alpha}^\sharp$ and $\omega_{\boldsymbol{h}_\alpha}^\sharp$ at weight one (cf. Equation (129) of [BSV20]), and the pairing $\langle \cdot, \cdot \rangle_{f g_\alpha h_\alpha}$ arises from the product of perfect dualities $\langle \cdot, \cdot \rangle_\xi$ introduced in Equations (31) and (128) of [BSV20], for $\xi = f, g_\alpha, h_\alpha$. Finally for any global Selmer class κ in $\mathrm{Sel}(\mathbf{Q}, V(f, g, h))$ define (cf. Equation (8))

$$\log_{\beta\beta}(\mathrm{res}_p(\kappa)) = \log_{\beta\beta}(\kappa_p),$$

where $\kappa_p \in H^1_{\mathrm{fin}}(\mathbf{Q}_p, V(f, g_\alpha, h_\alpha))$ is defined by $\wp_{\infty*} \otimes \mathrm{id}(\kappa_p) = \mathrm{res}_p(\kappa)$.

2.2. Tate's theory and the constant c_f. — The Tate parametrisation (cf. Chapter V of [Sil94]) yields a rigid analytic isomorphism

$$\varphi_{\mathrm{Tate}} : E_{q_A} \longrightarrow A_{K_p}$$

between the Tate curve

$$E_{q_A} = \mathbf{G}_{m,K_p}^{\mathrm{rig}}/q_A^{\mathbf{Z}}$$

over K_p and the base change A_{K_p} of A to K_p. Here $\mathbf{G}_{m,K_p}^{\mathrm{rig}}$ is the rigid multiplicative group over K_p and $q_A \in p\mathbf{Z}_p$ is the Tate period of $A_{\mathbf{Q}_p}$ (cf. loc. cit.).

Denote again by

$$\varphi_{\mathrm{Tate}} : V_p(E_{q_A}) \cong V_p(A)$$

the isomorphism of G_{K_p}-modules induced by the Tate parametrisation on the p-adic Tate modules with $\mathbf{Q}_p$-coefficients, and define

$$\wp_{\mathrm{Tate}} = \varphi_{\mathrm{Tate}}^{-1} \circ \wp_{\infty *} : V(f) \cong V_p(E_{q_A})$$

as the composition of its inverse with $\wp_{\infty *} : V(f) \cong V_p(A)$ (cf. Equation (3)). It induces a morphism of filtered modules (denoted by the same symbol)

$$\wp_{\mathrm{Tate}} : D_{\mathrm{dR},K_p}(V(f)) \cong D_{\mathrm{dR},K_p}(V_p(E_{q_A})),$$

where $D_{\mathrm{dR},K_p}(\cdot) = H^0(K_p, \cdot \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})$ is Fontaine's de Rham functor.

The projection $\mathbf{G}_{m,K_p}^{\mathrm{rig}} \longrightarrow E_{q_A}$ gives rise to an exact sequence of G_{K_p}-modules

$$0 \longrightarrow \mathbf{Q}_p(1) \longrightarrow V_p(E_{q_A}) \longrightarrow \mathbf{Q}_p \longrightarrow 0. \tag{13}$$

Applying Fontaine's de Rham functor $D_{\mathrm{dR},K_p}(\cdot) = H^0(K_p, \cdot \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})$ to the previous exact sequence yields a morphism $D_{\mathrm{dR},K_p}(V_p(E_{q_A})) \longrightarrow D_{\mathrm{dR},K_p}(\mathbf{Q}_p) = K_p$, which restricts to an isomorphism $\mathrm{Fil}^0 D_{\mathrm{dR},K_p}(V_p(E_{q_A})) \cong K_p$. Define

$$\mathbf{1}_A \in \mathrm{Fil}^0 D_{\mathrm{dR},K_p}(V_p(E_{q_A}))$$

for the generator corresponding to the identity of K_p under this isomorphism. On the other hand, the newform f corresponds (under Faltings' comparison isomorphism) to a canonical generator ω_f of $\mathrm{Fil}^0 D_{\mathrm{dR},K_p}(V(f)) = \mathrm{Fil}^1 V_{\mathrm{dR}}^*(f) \otimes_{\mathbf{Q}_p} K_p$ (cf. Equations (29) and (30) of [BSV20], noting that $V(f)(-1) = V^*(f)$). The non-zero p-adic constant

$$c_f \in K_p^*$$

which appears in Equation (6) of Section 1 is defined by the identity

$$\wp_{\mathrm{Tate}}(\omega_f) = c_f \cdot \mathbf{1}_A. \tag{14}$$

With the notations of Section 1, the following lemma shows that Equation (6) is a restatement of the main results of [BD07, BD09] (cf. Remark 1.2).

Lemma 2.1. — *Up to sign, one has the identity*

$$\log_{\omega_f} = \frac{c_f}{\deg(\wp_\infty)} \cdot \log_A .$$

Proof. — Let $u \in \mathcal{O}_{K_p}^*$ be a p-adic unit and let $P = \varphi_{\mathrm{Tate}}(u)$ be its image in $A(K_p)$ under the Tate parametrisation, so that

$$\log_A(P) = \log_p(u), \tag{15}$$

where $\log_p : K_p^* \longrightarrow K_p$ is the p-adic logarithm.

For V equal to one of $\mathbf{Q}_p(1), V_p(A), V_p(E_{q_A})$ and $V(f)$, denote by $\mathrm{tang}_{K_p}(V)$ the tangent space of $D_{\mathrm{dR},K_p}(V)$ and by

$$\log_V : H^1_{\mathrm{fin}}(K_p, V) \longrightarrow \mathrm{tang}_{K_p}(V)$$

the Bloch–Kato logarithm (viz. the inverse of the Bloch–Kato exponential map for V, which is an isomorphism). After identifying $\mathcal{O}_{K_p}^* \hat{\otimes} \mathbf{Q}_p$, resp. $A(K_p)\hat{\otimes}\mathbf{Q}_p$ with the finite subspace of $H^1(K_p, \mathbf{Q}_p(1))$, resp. $H^1(K_p, V_p(A))$ via Kummer theory, one has

$$\log_p(u) = \big\langle \log_{\mathbf{Q}_p(1)}(u), 1 \big\rangle_m = \big\langle \log_{V_p(E_{q_A})}(u), \mathbf{1}_A \big\rangle_W = \big\langle \log_{V_p(A)}(P), \varphi_{\mathrm{Tate}}(\mathbf{1}_A) \big\rangle_W, \tag{16}$$

where

$$\langle \cdot, \cdot \rangle_m : D_{\mathrm{dR},K_p}(\mathbf{Q}_p(1)) \otimes_{K_p} D_{\mathrm{dR},K_p}(\mathbf{Q}_p) \longrightarrow D_{\mathrm{dR},K_p}(\mathbf{Q}_p(1)) = K_p$$

is the pairing associated with the multiplication $m : \mathbf{Q}_p(1) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p \longrightarrow \mathbf{Q}_p(1)$, and for $\mathcal{A}$ equal to either A_{K_p} or E_{q_A}, the morphism

$$\langle \cdot, \cdot \rangle_W : \mathrm{tang}_{K_p}(V_p(\mathcal{A})) \otimes_{K_p} \mathrm{Fil}^0 D_{\mathrm{dR},K_p}(V_p(\mathcal{A})) \longrightarrow D_{\mathrm{dR},K_p}(\mathbf{Q}_p(1)) = K_p$$

is the one induced by the Weil pairing $W : V_p(\mathcal{A}) \otimes_{\mathbf{Q}_p} V_p(\mathcal{A}) \longrightarrow \mathbf{Q}_p(1)$. (The first identity in Equation (16) is well known, while the others follow from the functoriality of the Bloch–Kato logarithm and of the Weil pairing, after noting that the Weil pairing on E_{q_A} and the multiplication map m are compatible via the exact sequence (13).)

Under the natural isomorphism between $V_p(A)$ and $H^1_{\text{ét}}(A_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(1))$, the Weil pairing agrees (up to sign) with the cup-product pairing

$$H^1_{\text{ét}}(A_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(1)) \otimes_{\mathbf{Q}_p} H^1_{\text{ét}}(A_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(1)) \longrightarrow H^2_{\text{ét}}(A_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(2)) \cong \mathbf{Q}_p(1)$$

associated with the multiplication map $\mathbf{Q}_p(1) \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(1) \longrightarrow \mathbf{Q}_p(2)$, hence

$$\big\langle \log_{V_p(A)}(P), \varphi_{\mathrm{Tate}}(\mathbf{1}_A) \big\rangle_W = \deg(\wp_\infty) \cdot \big\langle \log_{V(f)}(\wp_{\infty *}^{-1}(P)), \wp_{\infty *}^{-1} \circ \varphi_{\mathrm{Tate}}(\mathbf{1}_A) \big\rangle_f.$$

By the definitions of $\log_{\omega_f}$ and c_f, the right hand side of the previous equation equals

$$\frac{\deg(\wp_\infty)}{c_f} \cdot \log_{\omega_f}(P).$$

Together with Equations (15)–(16), this prove that $\log_{\omega_f}(P)$ and $\frac{c_f}{\deg(\wp_\infty)} \cdot \log_A(P)$ are equal for each point $P \in A(K_p)$ in the image of $\mathcal{O}_{K_p}^*$ under the Tate parametrisation. Since $\mathcal{O}_{K_p}^*$ has finite index in $E_{q_A}(K_p)$, this concludes the proof. □

2.3. An exceptional zero formula and Equation (8). — As above, denote by $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = (\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)$ a level-N test vector for $(\boldsymbol{f}^\sharp,\boldsymbol{g}^\sharp_\alpha,\boldsymbol{h}^\sharp_\alpha)$. Let

$$\mathfrak{Z} \in H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))$$

be a local balanced class such that

$$\mathfrak{z} \stackrel{\mathrm{def}}{=} \rho_{w_o}(\mathfrak{Z}) \in H^1_{\mathrm{fin}}(\mathbf{Q}_p, V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)).$$

In other words we assume that the specialisation $\mathfrak{z}$ of $\mathfrak{Z}$ at $w_o = (2,1,1)$ belongs to the Bloch–Kato Selmer finite subspace of $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1))$. The aim of this section is to prove the following *exceptional zero formula* for the analytic function

$$\mathscr{L}_{\boldsymbol{f}}(\mathfrak{Z};\boldsymbol{k},1,1) = \mathscr{L}og(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})(\mathfrak{Z})|_{(\boldsymbol{k},\boldsymbol{l},\boldsymbol{m})=(\boldsymbol{k},1,1)} \in \mathscr{O}_{\boldsymbol{f}},$$

viz. the restriction to the line $(\boldsymbol{k},1,1)$ of the image of $\mathfrak{Z}$ under the Perrin-Riou logarithm $\mathscr{L}_{\boldsymbol{f}} = \mathscr{L}og(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ (cf. [**Ven16**]). In light of Theorems A and B of our article [**BSV20**], taking $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = (\boldsymbol{f}^\star,\boldsymbol{g}^\star_\alpha,\boldsymbol{h}^\star_\alpha)$ and $\mathfrak{Z} = \mathrm{res}_p(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))$ in its statement yields the key Equation (8) used in Section 1 to derive Theorem A.

Proposition 2.2. — *One has* $\mathrm{ord}_{\boldsymbol{k}=2}\mathscr{L}_{\boldsymbol{f}}(\mathfrak{Z};k,1,1) \geqslant 2$ *and (up to sign)*

$$c_{\boldsymbol{f}}^2 \cdot \frac{d^2}{d\boldsymbol{k}^2}\mathscr{L}_{\boldsymbol{f}}(\mathfrak{Z};\boldsymbol{k},1,1)_{\boldsymbol{k}=2} = \frac{\deg(\wp_\infty)}{2\mathrm{ord}_p(q_A)}\left(1-\frac{1}{p}\right)^{-1} \cdot \log_{\beta\beta}(\varpi_{\boldsymbol{fgh}}(\mathfrak{z})).$$

We first prove a simple lemma. As in Section 1.1 of [**BSV20**], denote by $\Lambda_{\boldsymbol{f}}$ the ring of analytic functions on $U_{\boldsymbol{f}}$ bounded by one, so that $\mathscr{O}_{\boldsymbol{f}} = \Lambda_{\boldsymbol{f}}[1/p]$. Let

$$\Phi : G_{\mathbf{Q}_p} \longrightarrow \Lambda_{\boldsymbol{f}}^*$$

be a continuous character such that $\Phi(\cdot)_{\boldsymbol{k}=2}$ is the trivial character, and let $\boldsymbol{V}$ be a free $\mathscr{O}_{\boldsymbol{f}}$-module of finite rank on which $G_{\mathbf{Q}_p}$ acts via $\Phi\cdot\chi_{\mathrm{cyc}}$. Let $V = \boldsymbol{V}\otimes_2 L$ be the base change of $\boldsymbol{V}$ under evaluation at $\boldsymbol{k}=2$ on $\mathscr{O}_{\boldsymbol{f}}$. Multiplication by $\boldsymbol{k}-2$ on $\boldsymbol{V}$ gives rise to an exact sequence

$$(17)\quad \cdots \longrightarrow H^i(\mathbf{Q}_p,\boldsymbol{V}) \xrightarrow{\boldsymbol{k}-2} H^i(\mathbf{Q}_p,\boldsymbol{V}) \longrightarrow H^i(\mathbf{Q}_p,V) \xrightarrow{\delta} H^{i+1}(\mathbf{Q}_p,\boldsymbol{V}) \longrightarrow \cdots.$$

As $\Phi(\cdot)_{\boldsymbol{k}=2}$ is the trivial character of $G_{\mathbf{Q}_p}$ the representation V is the direct sum of a finite number of copies of $L(1)$, hence there are natural isomorphisms

$$H^1(\mathbf{Q}_p,V) \cong \mathbf{Q}_p^*\hat{\otimes}V(-1) \quad\text{and}\quad H^2(\mathbf{Q}_p,V)\cong V(-1)$$

arising from Kummer's theory and the invariant map $\mathrm{inv}_p : H^2(\mathbf{Q}_p,\mathbf{Q}_p(1)) \cong \mathbf{Q}_p$ respectively. One considers the previous isomorphisms as identities in the rest of this section. Define

$$\beta_{\boldsymbol{V}} : \mathbf{Q}_p^*\hat{\otimes}V(-1) \xrightarrow{\delta} H^2(\mathbf{Q}_p,\boldsymbol{V}) \longrightarrow H^2(\mathbf{Q}_p,\boldsymbol{V})\otimes_2 L \cong V(-1),$$

where the second map is the natural projection (and the isomorphism comes from the exact sequence (17), since $H^3(\mathbf{Q}_p,\boldsymbol{V})$ vanishes). Because $\Phi(\cdot)_{\boldsymbol{k}=2}$ is the trivial character its derivative defines a morphism

$$\frac{d}{d\boldsymbol{k}}\Phi(\cdot)_{\boldsymbol{k}=2} \in H^1(\mathbf{Q}_p,L) \cong \mathrm{Hom}_{\mathrm{cont}}(\mathbf{Q}_p^*,L),$$

where the isomorphism is induced by the reciprocity map

$$\mathrm{rec}_p : \mathbf{Q}_p^*\hat{\otimes}\mathbf{Q}_p \cong G_{\mathbf{Q}_p}^{\mathrm{ab}}\hat{\otimes}\mathbf{Q}_p$$

(normalised as in [**BSV20**, Section 9.2]). Taking the tensor product over L with $V(-1)$ this induces a morphism (denoted by the same symbol)

$$\frac{d}{d\boldsymbol{k}}\Phi(\cdot)_{\boldsymbol{k}=2} : \mathbf{Q}_p^*\hat{\otimes}V(-1) \longrightarrow V(-1).$$

Lemma 2.3. — $\beta_{\boldsymbol{V}} = \frac{d}{d\boldsymbol{k}}\Phi(\cdot)_{\boldsymbol{k}=2}$.

Proof. — Without loss of generality one can assume that $\boldsymbol{V}$ is equal to $\mathscr{O}_{\boldsymbol{f}}(\Phi\cdot\chi_{\mathrm{cyc}})$, hence $V = L(1)$. Let $x = q\hat{\otimes}v$ be an element of $\mathbf{Q}_p^*\hat{\otimes}L$ and let $c_x : G_{\mathbf{Q}_p} \to L(1)$ be a 1-cocycle representing it. Let $\tilde{c}_x : G_{\mathbf{Q}_p} \to \mathscr{O}_{\boldsymbol{f}}(\Phi\cdot\chi_{\mathrm{cyc}})$ be the 1-cochain defined by viewing c_x as a function with values in $\mathscr{O}_{\boldsymbol{f}}$. Clearly $\tilde{c}_x(\cdot)_{\boldsymbol{k}=2} = c_x$. If d denotes the differential in the complex $\mathrm{C}^\bullet_{\mathrm{cont}}(\mathbf{Q}_p, \mathscr{O}_{\boldsymbol{f}}(\Phi\cdot\chi_{\mathrm{cyc}}))$ of inhomogeneous continuous cochains of $G_{\mathbf{Q}_p}$ with values in $\mathscr{O}_{\boldsymbol{f}}(\Phi\cdot\chi_{\mathrm{cyc}})$, then

$$d\tilde{c}_x(\sigma,\tau) = (\Phi(\sigma)-1)\cdot\chi_{\mathrm{cyc}}(\sigma)\cdot c_x(\tau) = \frac{d}{d\boldsymbol{k}}\Phi(\sigma)_{\boldsymbol{k}=2}\cdot\big(\chi_{\mathrm{cyc}}(\sigma)\cdot c_x(\tau)\big)\cdot(\boldsymbol{k}-2)+\cdots,$$

where the dots denote higher terms in the Taylor expansion at $\boldsymbol{k}=2$. This and local class field theory yield

$$\beta_{\boldsymbol{V}}(x) = \mathrm{inv}_p\Big(\frac{d}{d\boldsymbol{k}}\Phi(\cdot)_{\boldsymbol{k}=2}\cup cl(c_x)\Big) = \frac{d}{d\boldsymbol{k}}\Phi(q)_{\boldsymbol{k}=2}\cdot v,$$

where $\cup$ is the cup-product associated with the multiplication map $L\otimes_L L(1) \longrightarrow L(1)$. The lemma follows. □

Proof of Proposition 2.2. — By assumption $\mathfrak{Z} = \imath_*(\mathfrak{Y})$ is the image of a (unique) cohomology class $\mathfrak{Y}$ in $H^1(\mathbf{Q}_p, \mathscr{F}^2V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))$ under the map induced by the inclusion $\imath : \mathscr{F}^2V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \to V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$. Set

$$\mathfrak{y} = \rho_{w_o*}(\mathfrak{Y}) \in H^1(\mathbf{Q}_p, \mathscr{F}^2V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)),$$

so that $\mathfrak{z} = \rho_{w_o*}(\mathfrak{Z})$ is the image of $\mathfrak{y}$ under the natural map. By construction (cf. [**BSV20**, Proposition 7.3])

$$\mathscr{L}_{\boldsymbol{f}}(\mathfrak{Z}) = \mathscr{L}_{\boldsymbol{f}}(p_{f*}(\mathfrak{Y})). \tag{18}$$

If $\bullet$ and $\circ$ denote either α or β, define as in Section 9.2 of loc. cit. (cf. the proof of Proposition 9.3 of loc. cit.)

$$V(\boldsymbol{f}_2)^{\cdot}_{\bullet\circ} = V(\boldsymbol{f}_2)^{\cdot}\otimes_L V(\boldsymbol{g}_1)_\bullet\otimes_L V(\boldsymbol{h}_1)_\circ,$$

where $\cdot = \emptyset, \pm$ and $V(\boldsymbol{\xi}_1)_\beta = V(\boldsymbol{\xi}_1)^+$ and $V(\boldsymbol{\xi}_1)_\alpha = V(\boldsymbol{\xi}_1)^-$ for $\boldsymbol{\xi} = \boldsymbol{g},\boldsymbol{h}$. In the present setting the form $\boldsymbol{\xi}_1$ is *regular*, viz. $\alpha_{\boldsymbol{\xi}_1}$ and $\beta_{\boldsymbol{\xi}_1} = -\alpha_{\boldsymbol{\xi}_1}$ are distinct, hence $V(\boldsymbol{\xi}_1)_\bullet$ is equal to the subspace $V(\boldsymbol{\xi}_1)^{\mathrm{Frob}_p=\bullet}$ of $V(\boldsymbol{\xi}_1)$ on which an arithmetic Frobenius Frob_p acts as multiplication by $\bullet_{\boldsymbol{\xi}_1}$ (cf. Section 9.2 of loc. cit.). It follows that for $\cdot = \emptyset$ and $\cdot = \pm$ there are *canonical* direct sum decompositions

$$V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)^{\cdot} = V(\boldsymbol{f}_2)^{\cdot}_{\alpha\alpha}\oplus V(\boldsymbol{f}_2)^{\cdot}_{\alpha\beta}\oplus V(\boldsymbol{f}_2)^{\cdot}_{\beta\alpha}\oplus V(\boldsymbol{f}_2)^{\cdot}_{\beta\beta} \tag{19}$$

of $L[G_{\mathbf{Q}_p}]$-modules. In particular $V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f = V(\boldsymbol{f}_2)^-_{\beta\beta}$ is a direct summand of $V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)^-$ (cf. Equation (191) of loc. cit.), hence

$$p_{f*}(\mathfrak{y}) = 0$$

since by assumption $\mathfrak{z}$ is crystalline (cf. Section 9.1 of loc. cit., in particular Equation (193)). As a consequence

$$p_{f*}(\mathfrak{Y}) = (\boldsymbol{k}-2)\cdot\mathfrak{Y}_{\boldsymbol{k}} + (\boldsymbol{l}-1)\cdot\mathfrak{Y}_{\boldsymbol{l}} + (\boldsymbol{m}-1)\cdot\mathfrak{Y}_{\boldsymbol{m}} \tag{20}$$

for classes $\mathfrak{Y}_\cdot$ in $H^1(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})_f)$ (cf. the proof of Proposition 7.3 of loc. cit. or [**Ven16**, Lemma 5.6]). Set

$$\mathfrak{y}_{\boldsymbol{k}} = \rho_{w_o*}(\mathfrak{Y}_{\boldsymbol{k}}) \in H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)_f).$$

Because $\mathscr{L}_{\boldsymbol{f}}$ is $\mathscr{O}_{\boldsymbol{fgh}}$-linear, Equation (18), Proposition 9.3(1) of loco citato and Theorem 3.14 of [**GS93**] give

$$\left(1-\frac{1}{p}\right)\cdot\frac{d^2}{d\boldsymbol{k}^2}\mathscr{L}_{\boldsymbol{f}}(\mathfrak{Z},\boldsymbol{k},1,1)_{\boldsymbol{k}=2} = \mathfrak{y}_{\boldsymbol{k}}(p^{-1})_f - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}}\cdot\mathfrak{y}_{\boldsymbol{k}}(e(1))_f = \frac{-1}{\mathrm{ord}_p(q_A)}\cdot\mathfrak{y}_{\boldsymbol{k}}(q_A)_f, \tag{21}$$

where

$$-\frac{1}{2}\cdot\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} = d\mathrm{log}a_p(\boldsymbol{k})_{\boldsymbol{k}=2}$$

is the logarithmic derivative at $\boldsymbol{k}=2$ of the p-th Fourier coefficient $a_p(\boldsymbol{k})$ of $\boldsymbol{f}^\sharp$ (cf. Section 9.2 of [**BSV20**]). In particular this implies that the quantity $\mathfrak{y}_{\boldsymbol{k}}(q_A)_f$ is independent of the choice of $\mathfrak{Y}_{\boldsymbol{k}}$ satisfying Equation (20).

As shown in the proof of Proposition 9.3 of loc. cit. the class of the extension

$$0 \longrightarrow V(\boldsymbol{f}_2)^+_{\beta\beta} \longrightarrow V(\boldsymbol{f}_2)_{\beta\beta} \longrightarrow V(\boldsymbol{f}_2)^-_{\beta\beta} \longrightarrow 0 \tag{22}$$

in

$$\mathrm{Ext}^1_{L[G_{\mathbf{Q}_p}]}(V(\boldsymbol{f}_2)^-_{\beta\beta}, V(\boldsymbol{f}_2)^+_{\beta\beta}) \cong \mathbf{Q}_p^*\hat{\otimes}_{\mathbf{Q}_p}\mathrm{Hom}_L(V(\boldsymbol{f}_2)^-_{\beta\beta}, V(\boldsymbol{f}_2)^+_{\beta\beta}(-1))$$

is equal to

$$q_{\boldsymbol{f}_2} = q_A\hat{\otimes}\delta_{\boldsymbol{f}_2}$$

for an isomorphism $\delta_{\boldsymbol{f}_2} : V(\boldsymbol{f}_2)^-_{\beta\beta} \to V(\boldsymbol{f}_2)^+_{\beta\beta}(-1)$, and the connecting morphisms $\partial^i_{\boldsymbol{f}_2}$ associated to (22) satisfy

$$\partial^0_{\boldsymbol{f}_2}(v) = q_A\hat{\otimes}\delta_{\boldsymbol{f}_2}(v) = q_{\boldsymbol{f}_2}\cup v \quad\text{and}\quad \partial^1_{\boldsymbol{f}_2}(\varphi\otimes v) = -\varphi(q_A)\cdot\delta_{\boldsymbol{f}_2}(v) = -q_{\boldsymbol{f}_2}\cup(\varphi\otimes v) \tag{23}$$

for all φ in $\mathrm{Hom}_{\mathrm{cont}}(\mathbf{Q}_p^*, L)$ and v in $V(\boldsymbol{f}_2)^-_{\beta\beta}$, where $\cup$ is the cup-product induced by the evaluation map. Define

$$V(\boldsymbol{f})^\cdot_{\beta\beta} = \big(V(\boldsymbol{f})^\cdot \otimes_{\mathscr{O}_{\boldsymbol{f}}} \kappa_{\mathrm{cyc}}^{1-\boldsymbol{k}/2}\big)\otimes_L V(\boldsymbol{g}_1)^+ \otimes_L V(\boldsymbol{h}_1)^+.$$

These are $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}_p}]$-modules, sitting in a short exact sequence

$$0 \longrightarrow V(\boldsymbol{f})^+_{\beta\beta} \longrightarrow V(\boldsymbol{f})_{\beta\beta} \longrightarrow V(\boldsymbol{f})^-_{\beta\beta} \longrightarrow 0$$

which specialises to (22) under evaluation at $\boldsymbol{k}=2$ on $\mathscr{O}_{\boldsymbol{f}}$. Identify the $\mathscr{O}_{\boldsymbol{f}}$-module $V(\boldsymbol{f})_{\beta\beta}$ with the direct sum of $V(\boldsymbol{f})^+_{\beta\beta}$ and $V(\boldsymbol{f})^-_{\beta\beta}$ under a fixed $\mathscr{O}_{\boldsymbol{f}}$-splitting of the previous exact sequence. There is then a continuous map

$$q_{\boldsymbol{f}} : G_{\mathbf{Q}_p} \longrightarrow \mathrm{Hom}_{\mathscr{O}_{\boldsymbol{f}}}(V(\boldsymbol{f})^-_{\beta\beta}, V(\boldsymbol{f})^+_{\beta\beta})$$

satisfying the following properties. For all $\boldsymbol{v}^{\pm} \in V(\boldsymbol{f})^{\pm}_{\beta\beta}$ and $\sigma \in G_{\mathbf{Q}_p}$ (cf. Equation (101) of loc. cit.)

$$\sigma(\boldsymbol{v}^{+}) = \frac{\omega_{\mathrm{cyc}}(\sigma) \cdot \kappa_{\mathrm{cyc}}^{\boldsymbol{k}/2}(\sigma)}{\psi_{\boldsymbol{f}}\psi_{\boldsymbol{g}_1}\psi_{\boldsymbol{h}_1}(\sigma)} \cdot \boldsymbol{v}^{+} \quad \text{and} \quad \sigma(\boldsymbol{v}^{-}) = \frac{\psi_{\boldsymbol{f}}(\sigma)\kappa_{\mathrm{cyc}}^{1-\boldsymbol{k}/2}(\sigma)}{\psi_{\boldsymbol{g}_1}\psi_{\boldsymbol{h}_1}(\sigma)} \cdot \boldsymbol{v}^{-} + q_{\boldsymbol{f}}(\sigma, \boldsymbol{v}^{-}), \tag{24}$$

where $\psi_{\boldsymbol{f}} : G^{\mathrm{nr}}_{\mathbf{Q}_p} \longrightarrow \Lambda^{*}_{\boldsymbol{f}}$ is the unramified character of $G_{\mathbf{Q}_p}$ which sends an arithmetic Frobenius Frob_p to $a_p(\boldsymbol{k})$, and similarly $\psi_{\boldsymbol{g}_1}, \psi_{\boldsymbol{h}_1} : G^{\mathrm{nr}}_{\mathbf{Q}_p} \longrightarrow \mathscr{O}^{*}$ are defined by $\psi_{\boldsymbol{g}_1}(\mathrm{Frob}_p) = b_p(1)$ and $\psi_{\boldsymbol{h}_1}(\mathrm{Frob}_p) = c_p(1)$ respectively. (Here one uses that both $\chi_{\boldsymbol{f}}$ and $\chi_{\boldsymbol{g}} \cdot \chi_{\boldsymbol{h}}$ are equal to the trivial character.) Moreover the specialisation

$$q_{\boldsymbol{f}}(\cdot)_{\boldsymbol{k}=2} : G_{\mathbf{Q}_p} \longrightarrow \mathbf{Q}_p(1) \otimes_{\mathbf{Q}_p} \mathrm{Hom}_{\mathscr{O}_f}(V(\boldsymbol{f}_2)^{-}_{\beta\beta}, V(\boldsymbol{f}_2)^{+}_{\beta\beta}(-1))$$

of $q_{\boldsymbol{f}}$ at $\boldsymbol{k} = 2$ (via $\mathrm{Hom}_{\mathscr{O}_{\boldsymbol{f}}}(V(\boldsymbol{f})^{-}_{\beta\beta}, V(\boldsymbol{f})^{+}_{\beta\beta}) \otimes_2 L \cong \mathrm{Hom}_L(V(\boldsymbol{f}_2)^{-}_{\beta\beta}, V(\boldsymbol{f}_2)^{+}_{\beta\beta})$) is a 1-cocycle satisfying

$$cl\big(q_{\boldsymbol{f}}(\cdot)_{\boldsymbol{k}=2}\big) = q_{\boldsymbol{f}_2}. \tag{25}$$

For future reference denote by $\Phi_{\boldsymbol{f}} : G_{\mathbf{Q}_p} \longrightarrow \Lambda^{*}_{\boldsymbol{f}}$ the character

$$\Phi_{\boldsymbol{f}} = \kappa_{\mathrm{cyc}}^{\boldsymbol{k}/2-1} \cdot \psi_{\boldsymbol{f}}^{-1} \cdot \psi_{\boldsymbol{g}_1}^{-1} \cdot \psi_{\boldsymbol{h}_1}^{-1}, \tag{26}$$

so that $\Phi_{\boldsymbol{f}}(\cdot)_{\boldsymbol{k}=2}$ is the trivial character and $G_{\mathbf{Q}_p}$ acts on $V(\boldsymbol{f})^{+}_{\beta\beta}$ via $\chi_{\mathrm{cyc}} \cdot \Phi_{\boldsymbol{f}}$.

Denote by

$$\mathfrak{Y}_{\beta\beta} \in H^1(\mathbf{Q}_p, V(\boldsymbol{f})_{\beta\beta}) \quad \text{and} \quad \mathfrak{Y}_{\boldsymbol{k},\beta\beta} \in H^1(\mathbf{Q}_p, V(\boldsymbol{f})^{-}_{\beta\beta})$$

the images of $\mathfrak{Y}$ and $\mathfrak{Y}_{\boldsymbol{k}}$ under the maps induced by the projections

$$\mathscr{F}^2 V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \longrightarrow\!\!\!\!\!\rightarrow \mathscr{F}^2 V(\boldsymbol{f}, \boldsymbol{g}_1, \boldsymbol{h}_1) \longrightarrow\!\!\!\!\!\rightarrow V(\boldsymbol{f})_{\beta\beta}$$

and

$$V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})_f \longrightarrow\!\!\!\!\!\rightarrow V(\boldsymbol{f}, \boldsymbol{g}_1, \boldsymbol{h}_1)_f = V(\boldsymbol{f})^{-}_{\beta\beta}$$

respectively. (Here $V(\boldsymbol{f}, \boldsymbol{g}_1, \boldsymbol{h}_1) = V(\boldsymbol{f})^{\cdot} \otimes_L V(\boldsymbol{g}_1) \otimes_L V(\boldsymbol{h}_1)(\kappa_{\mathrm{cyc}}^{1-\boldsymbol{k}/2})$. Note that the discussion leading to Equation (19) yields a similar canonical decomposition of the $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}}]$-module $V(\boldsymbol{f}, \boldsymbol{g}_1, \boldsymbol{h}_1)$.) According to Equation (20) the cohomology class $\mathfrak{Y}_{\beta\beta}$ is represented by a 1-cocycle of the form

$$Y_{\beta\beta} = Y^{+}_{\beta\beta} \oplus (\boldsymbol{k} - 2) \cdot Y^{-}_{\beta\beta} : G_{\mathbf{Q}_p} \longrightarrow V(\boldsymbol{f})_{\beta\beta},$$

for 1-cochains $Y^{\cdot}_{\beta\beta} : G_{\mathbf{Q}_p} \to V(\boldsymbol{f})^{\cdot}_{\beta\beta}$. Using Equation (24) the cocycle relation for $Y_{\beta\beta}$ gives

$$dY^{+}_{\beta\beta}(\sigma, \tau) = -(\boldsymbol{k} - 2) \cdot q_{\boldsymbol{f}}(\sigma, Y^{-}_{\beta\beta}(\tau)) \quad \text{and} \quad dY^{-}_{\beta\beta} = 0. \tag{27}$$

In particular the specialisations $y^{\cdot}_{\beta\beta} : G_{\mathbf{Q}_p} \to V(\boldsymbol{f}_2)^{\cdot}_{\beta\beta}$ of $Y^{\cdot}_{\beta\beta}$ at $\boldsymbol{k} = 2$ are both 1-cocycles and by construction

$$i^{+}_{*}(\mathfrak{y}^{+}_{\beta\beta}) = \mathfrak{y}_{\beta\beta} \quad \text{and} \quad (\boldsymbol{k} - 2) \cdot cl(Y^{-}_{\beta\beta}) = (\boldsymbol{k} - 2) \cdot \mathfrak{Y}_{\boldsymbol{k},\beta\beta}, \tag{28}$$

where $\mathfrak{y}^{\pm}_{\beta\beta} = cl(y^{\pm}_{\beta\beta}) \in H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2)^{\pm}_{\beta\beta})$ are the classes represented by $y^{\pm}_{\beta\beta}$, the map i^{+}_{*} is the one induced by the inclusion $i^{+} : V(\boldsymbol{f}_2)^{+}_{\beta\beta} \hookrightarrow V(\boldsymbol{f}_2)_{\beta\beta}$ and $\mathfrak{y}_{\beta\beta}$ in $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_2)_{\beta\beta})$ is the image of $\mathfrak{y}$ under the map induced by the projection onto the direct summand $V(\boldsymbol{f}_2)_{\beta\beta}$ of $\mathscr{F}^2V(\boldsymbol{f}_2,\boldsymbol{g}_1,\boldsymbol{h}_1)$. The second identity in Equation (28) implies

$$\mathfrak{y}_{\boldsymbol{k}}(q_A)_f=\mathfrak{y}^-_{\beta\beta}(q_A)_f$$

(cf. the remark after Equation (21)), hence Equation (21) can be rephrased as

$$\left(1-\frac{1}{p}\right)\cdot\frac{d^2}{d\boldsymbol{k}^2}\mathscr{L}_{\boldsymbol{f}}(\mathfrak{z},\boldsymbol{k},1,1)_{\boldsymbol{k}=2}=\frac{-1}{\mathrm{ord}_p(q_A)}\cdot\mathfrak{y}^-_{\beta\beta}(q_A)_f. \tag{29}$$

In light of Equations (24)–(26) and Lemma 2.3, the first equalities in Equations (27) and (28) yield

$$\begin{aligned}-\partial^1_{\boldsymbol{f}_2}(\mathfrak{y}^-_{\beta\beta})&=\mathrm{inv}_p\big(cl\big(q_{\boldsymbol{f}_2}(\sigma,y^-_{\beta\beta}(\tau))\big)\big)\\&=-\beta_{V(\boldsymbol{f})^+_{\beta\beta}}(\mathfrak{y}^+_{\beta\beta})=-\frac{d}{d\boldsymbol{k}}\Phi_{\boldsymbol{f}}(\mathfrak{y}^+_{\beta\beta})_{\boldsymbol{k}=2}=-\frac{1}{2}\cdot\log_{q_A}(\mathfrak{y}^+_{\beta\beta}).\end{aligned} \tag{30}$$

More precisely, the first equality follows from Equation (23), the second from Equations (25) and (27) and the definition of $\beta_{V(\boldsymbol{f})^+_{\beta\beta}}$, and the third from Lemma 2.3. Finally, for each unit u in $\mathbf{Z}_p^*$, one has (cf. Equation (26))

$$\frac{d}{d\boldsymbol{k}}\Phi_{\boldsymbol{f}}(u)_{\boldsymbol{k}=2}=\frac{d}{d\boldsymbol{k}}\kappa_{\mathrm{cyc}}^{\boldsymbol{k}/2-1}(\mathrm{rec}_p(u))_{\boldsymbol{k}=2}=\frac{d}{d\boldsymbol{k}}\big(u^{\boldsymbol{k}/2-1}\big)_{\boldsymbol{k}=2}=\frac{1}{2}\cdot\log_p(u)$$

and

$$\frac{d}{d\boldsymbol{k}}\Phi_{\boldsymbol{f}}(p)_{\boldsymbol{k}=2}=\alpha_g\cdot\alpha_h\cdot\frac{d}{d\boldsymbol{k}}a_p(\boldsymbol{k})_{\boldsymbol{k}=2}=-\frac{1}{2}\cdot\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}},$$

which in light of the identity $\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}}=\frac{\log_p(q_A)}{\mathrm{ord}_p(q_A)}$ proved in [GS93, Theorem 3.14] yields the last equality in Equation (30). (Here one denotes again by

$$\log_{q_A}:\mathbf{Q}_p^*\hat{\otimes}V(\boldsymbol{f}_2)^+_{\beta\beta}(-1)\longrightarrow V(\boldsymbol{f}_2)^+_{\beta\beta}(-1)\cong D_{\mathrm{cris}}(V(\boldsymbol{f}_2)^+_{\beta\beta})$$

the morphism induced by $\log_{q_A}=\log_p-\frac{\log_p(q_A)}{\mathrm{ord}_p(q_A)}\cdot\mathrm{ord}_p:\mathbf{Q}_p^*\to\mathbf{Q}_p$).

As the connecting morphisms $\partial^0_{\boldsymbol{f}_2}$ and $-\partial^1_{\boldsymbol{f}_2}$ are adjoint to each other under the cup-product induced by $\langle\cdot,\cdot\rangle_{\boldsymbol{f}_2\boldsymbol{g}_1\boldsymbol{h}_1}$, Equations (23), (29) and (30) combine to give

$$\left(1-\frac{1}{p}\right)\cdot\frac{d^2}{d\boldsymbol{k}^2}\mathscr{L}_{\boldsymbol{f}}(\mathfrak{z},\boldsymbol{k},1,1)_{\boldsymbol{k}=2}=\frac{1}{2\mathrm{ord}_p(q_A)}\cdot\big\langle\log_{q_A}(\mathfrak{y}^+_{\beta\beta}),\delta^{-1}_{\boldsymbol{f}_2}\big(\eta_{\boldsymbol{f}_2}\otimes\omega_{\boldsymbol{g}_1}\otimes\omega_{\boldsymbol{h}_1}\big)\big\rangle_{\boldsymbol{f}_2\boldsymbol{g}_1\boldsymbol{h}_1}. \tag{31}$$

Since f has trivial character, one has $V^*(f)^{\cdot}=V(f)^{\cdot}(-1)$ for $\cdot=\emptyset,\pm$ (cf. Sections 2.5 and 5 of [BSV20]). There are then natural $\mathrm{Gal}(K_p/\mathbf{Q}_p)$-equivariant isomorphisms

$$\mathrm{Fil}^1D_{\mathrm{dR},K_p}(V^*(f))\cong\mathrm{Fil}^0D_{\mathrm{dR},K_p}(V(f))\cong D_{\mathrm{cris},K_p}(V(f)^-)=V(f)^-\otimes_{\mathbf{Q}_p}K_p,$$

under which we identify the differential (cf. Section 2.5 of loco citato)

$$\omega_f\in\mathrm{Fil}^1V^*_{\mathrm{dR}}(f)=\mathrm{Fil}^1D_{\mathrm{dR},K_p}(V^*(f))^{\mathrm{Gal}(K_p/\mathbf{Q}_p)}$$

with an element of $V(f)^-$. Lemma 2.4 below proves that

$$\delta_f(\omega_f\otimes\omega_{g_\alpha}\otimes\omega_{h_\alpha})=\pm\frac{c_f^2}{\deg(\wp_\infty)}\cdot\eta_f\otimes\omega_{g_\alpha}\otimes\omega_{h_\alpha}$$

in $V^*(f)^+_{\beta\beta} = V(f)^+(-1) \otimes_{\mathbf{Q}_p} V^*(g)^- \otimes_L V(h)^-$, hence by construction

$$\delta^{-1}_{\boldsymbol{f}_2}\big(\eta_{\boldsymbol{f}_2} \otimes \omega_{\boldsymbol{g}_1} \otimes \omega_{\boldsymbol{h}_1}\big) = \pm \frac{\deg(\wp_\infty)}{c_f^2} \cdot \varpi^*_{\boldsymbol{fgh}}\big(\omega_f \otimes \omega_{g_\alpha} \otimes \omega_{h_\alpha}\big), \tag{32}$$

where $\varpi^*_{\boldsymbol{fgh}} = v^*_{\boldsymbol{f}} \otimes v^*_{\boldsymbol{g}} \otimes v^*_{\boldsymbol{h}}$ is the adjoint of $\varpi_{\boldsymbol{fgh}}$ under the Poicaré dualities $\langle \cdot, \cdot \rangle_{fg_\alpha h_\alpha}$ and $\langle \cdot, \cdot \rangle_{\boldsymbol{f}_2 \boldsymbol{g}_1 \boldsymbol{h}_1}$. Finally, the first identity in Equation (28) gives

$$\log_{q_A}(\mathfrak{y}^+_{\beta\beta}) = \pi_{\beta\beta}(\log_p(\mathfrak{z})), \tag{33}$$

where $\pi_{\beta\beta}$ is the composition

$$D_{\mathrm{dR}}(V(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1))/\mathrm{Fil}^0 \cong D_{\mathrm{st}}(V(\boldsymbol{f}_2, \boldsymbol{g}_1, \boldsymbol{h}_1)^+) \twoheadrightarrow D_{\mathrm{cris}}(V(\boldsymbol{f}_2)^+_{\beta\beta})$$

arising from Equations (191) and (192) of [BSV20] and Equation (19). Since by construction the $\beta\beta$-logarithm $\log_{\beta\beta}$ factors through the projection $\pi_{\beta\beta}$, the proposition is a direct consequence of Equations (31)–(33). □

Lemma 2.4. — *Let*

$$\partial_f : V(f)^- \longrightarrow K_p^* \hat{\otimes} V(f)^+(-1)$$

be the connecting morphism associated with the exact sequence of G_{K_p}-modules

$$0 \longrightarrow V(f)^+ \longrightarrow V(f) \longrightarrow V(f)^- \longrightarrow 0.$$

Then $\partial_f = q_A \hat{\otimes} \delta_f$ for an isomorphism

$$\delta_f : V(f)^- \longrightarrow V(f)^+(-1)$$

satisfying, up to sign, the following identity in $V(f)^+(-1)$:

$$\delta_f(\omega_f) = \frac{c_f^2}{\deg(\wp_\infty)} \cdot \eta_f.$$

Proof. — Consider the following diagram of $\mathbf{Q}_p[G_{K_p}]$-modules with exact rows, in which all the vertical maps are isomorphisms.

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{Q}_p(1) \arrow[r] \arrow[d, "\varphi^+_{\mathrm{Tate}}"'] & V_p(E_{q_A}) \arrow[r] \arrow[d, "\varphi_{\mathrm{Tate}}"'] & \mathbf{Q}_p \arrow[r] \arrow[d, "\varphi^-_{\mathrm{Tate}}"'] & 0 \\ 0 \arrow[r] & V_p(A)^+ \arrow[r] & V_p(A) \arrow[r] & V_p(A)^- \arrow[r] & 0 \\ 0 \arrow[r] & V(f)^+ \arrow[r] \arrow[u, "\wp^+_{\infty*}"] & V(f) \arrow[r] \arrow[u, "\wp_{\infty*}"] & V(f)^- \arrow[r] \arrow[u, "\wp^-_{\infty*}"] & 0 \end{tikzcd} \tag{34}$$

Here φ_{Tate} is the map induced on the p-adic Tate modules by the Tate uniformisation $E_{q_A} \cong A_{K_p}$, and the first row is the short exact sequence induced by the natural projection $\mathbf{G}^{\mathrm{rig}}_{m,K_p} \longrightarrow E_{q_A}$ (cf. Introduction).

The class in

$$\mathrm{Ext}^1_{\mathbf{Q}_p[G_{K_p}]}(\mathbf{Q}_p, \mathbf{Q}_p(1)) = H^1(K_p, \mathbf{Q}_p(1)) \cong K_p^* \hat{\otimes} \mathbf{Q}_p$$

represented by the first row equals $q_A \hat{\otimes} 1$, hence the associated connecting morphism

$$\partial_{\mathrm{Tate}} : \mathbf{Q}_p \longrightarrow K_p^* \hat{\otimes} \mathbf{Q}_p$$

satisfies

$$\partial_{\mathrm{Tate}}(1) = q_A \hat{\otimes} 1. \tag{35}$$

After setting

$$\gamma_{q_A} = \frac{-1}{\mathrm{ord}_p(q_A)} \cdot \mathrm{ord}_p \in \mathrm{Hom}_{\mathrm{cont}}(K_p^*, \mathbf{Q}_p) \cong H^1(K_p, \mathbf{Q}_p),$$

this implies

$$\langle \gamma_{q_A}, \partial_{\mathrm{Tate}}(1) \rangle_m = 1, \tag{36}$$

where

$$\langle \cdot, \cdot \rangle_m : H^1(K_p, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} H^1(K_p, \mathbf{Q}_p(1)) \longrightarrow K_p$$

is the local Tate pairing attached to the multiplication $m : \mathbf{Q}_p \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(1) \longrightarrow \mathbf{Q}_p(1)$. Moreover, the Diagram (34) and Equation (35) imply that the connecting morphisms

$$\partial_A : V_p(A)^- \longrightarrow K_p^* \hat{\otimes} V_p(A)^+(-1) \quad \text{and} \quad \partial_f : V(f)^- \longrightarrow K_p^* \hat{\otimes} V(f)^+(-1)$$

associated respectively to the second and third rows of Diagram (34) are of the form

$$\partial_A = q_A \otimes \delta_A \quad \text{and} \quad \partial_f = q_A \otimes \delta_f \tag{37}$$

for isomorphisms $\delta_A : V_p(A)^- \longrightarrow V_p(A)^+(-1)$ and $\delta_f : V(f)^- \longrightarrow V(f)^+(-1)$.

Up to sign, one has the identities

$$\begin{aligned}
\langle \omega_f, \delta_f(\omega_f) \rangle_f &= \langle \gamma_{q_A} \otimes \omega_f, \partial_f(\omega_f) \rangle_f \\
&= \frac{1}{\deg(\wp_\infty)} \cdot \left\langle \gamma_{q_A} \otimes \wp_{\infty *}^-(\omega_f), \partial_A(\wp_{\infty *}^-(\omega_f)) \right\rangle_{\mathrm{Weil}} \\
&= \frac{c_f^2}{\deg(\wp_\infty)} \cdot \left\langle \gamma_{q_A} \otimes \varphi_{\mathrm{Tate}}^-(1), \partial_A(\varphi_{\mathrm{Tate}}^-(1)) \right\rangle_{\mathrm{Weil}} \\
&= \frac{c_f^2}{\deg(\wp_\infty)} \cdot \left\langle \gamma_{q_A} \otimes \varphi_{\mathrm{Tate}}^-(1), \varphi_{\mathrm{Tate}}^+(\partial_{\mathrm{Tate}}(1)) \right\rangle_{\mathrm{Weil}} \\
&= \frac{c_f^2}{\deg(\wp_\infty)} \cdot \langle \gamma_{q_A}, \partial_{\mathrm{Tate}}(1) \rangle_m ,
\end{aligned} \tag{38}$$

where $\langle \cdot, \cdot \rangle_{\mathrm{Weil}} : H^1(K_p, V_p(A)^+) \otimes_{\mathbf{Q}_p} H^1(K_p, V_p(A)^-) \longrightarrow K_p$ is the local Tate pairing associated with the Weil paring on $V_p(A)$. Indeed, the first equality follows from Equation (37). The second equality follows (up to sign) from the functoriality of Poincaré duality under finite morphisms of curves and its compatibility with the Weil pairing on elliptic curves. The third equality follows from the definition of c_f (cf. Equation (14)). The fourth equality follows from Diagram (34). The fifth and last equality follows from the functoriality of the Weil paring under isogenies, after noting that the Kummer duality between $\mathbf{Q}_p(1)$ and $\mathbf{Q}_p$ induced by the Weil pairing on $V_p(E_{q_A})$ is equal (up to sign) to the multiplication map m.

Since $V(f)^+(-1) = D_{\mathrm{cris}}(V(f)^+)$ is a one-dimensional $\mathbf{Q}_p$-vector space generated by η_f and $\langle \omega_f, \eta_f \rangle_f = 1$, the lemma follows from Equations (36) and (38). □

3. Factorisations of p-adic L-functions

This section is devoted to the proof of Theorem 3.1 below, viz. the crucial factorisation formula (7) of Section 1, under the assumptions listed therein. In light of the discussion of Section 1 (see Equations (7) and (8)) and of Section 2, this is the last step in our proof of Theorem A.

The reader is cautioned that the notations for p-adic L-functions in force here are consistent with those of [**BSV20**, Section 6] and differ slightly from those of Section 1. Thus $L_p(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$ denotes the square of the triple product square-root p-adic L-function $\mathscr{L}_p^f(\boldsymbol{f}^\star, \boldsymbol{g}^\star, \boldsymbol{h}^\star)$ attached to our fixed choice of test vector $(\boldsymbol{f}^\star, \boldsymbol{g}^\star, \boldsymbol{h}^\star)$, and the restriction of $L_p(\boldsymbol{f}^\sharp, \boldsymbol{g}^\sharp, \boldsymbol{h}^\sharp)$ to the line $(\boldsymbol{k}, 1, 1)$ is denoted

$$L_p(\boldsymbol{f}^\sharp, \boldsymbol{g}_1^\sharp, \boldsymbol{h}_1^\sharp) = L_p(\boldsymbol{f}^\sharp, g_\alpha, h_\alpha)$$

(recall that $\boldsymbol{g}^\sharp$ and $\boldsymbol{h}^\sharp$ interpolate the chosen p-stabilisations g_α and h_α respectively). Accordingly, the Hida–Rankin p-adic L-functions associated to the ring class characters φ and ψ are denoted by $L_p(\boldsymbol{f}^\sharp, \varphi)$ and $L_p(\boldsymbol{f}^\sharp, \psi)$ (as observed in Section 1, they depend only on the primitive family $\boldsymbol{f}^\sharp$).

Theorem 3.1. — *Up to shrinking $U_{\boldsymbol{f}}$ if necessary, there is a factorisation*

$$L_p(\boldsymbol{f}^\sharp, \boldsymbol{g}_1^\sharp, \boldsymbol{h}_1^\sharp) = \mathscr{A} \cdot L_p(\boldsymbol{f}^\sharp/K, \varphi) \cdot L_p(\boldsymbol{f}^\sharp/K, \psi),$$

where $\mathscr{A} \in \mathscr{O}_{\boldsymbol{f}}^*$ is a bounded analytic function on $U_{\boldsymbol{f}}$ such that

$$\mathscr{A}(2) \in \mathbf{Q}(\boldsymbol{g}_1^\sharp, \boldsymbol{h}_1^\sharp)^*,$$

$\mathbf{Q}(\boldsymbol{g}_1^\sharp, \boldsymbol{h}_1^\sharp)$ being the field generated by the Fourier coefficients of $\boldsymbol{g}_1^\sharp$ and $\boldsymbol{h}_1^\sharp$.

3.1. The Mazur–Kitagawa p-adic L-function. — Let χ be a Dirichlet character of conductor coprime to $N_f p$. For every classical point $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$ let $L(f_k^\sharp, \chi, s)$ be the Hecke L-series of $f_k^\sharp \otimes \chi$, defined as the analytic continuation of the Dirichlet series $\sum_{n \geqslant 1} \chi(n) a_n(f_k^\sharp) \cdot n^{-s}$ converging absolutely for $\Re(s) > (k+1)/2$. A result of Shimura gives *complex periods* $\Omega_\infty(f_k^\sharp)^+$ and $\Omega_\infty(f_k^\sharp)^-$ in $\mathbf{C}^*$ satisfying the following properties. One has

$$\Omega_\infty(f_k^\sharp)^+ \cdot \Omega_\infty(f_k^\sharp)^- = (f_k^\sharp, f_k^\sharp)_{N_f p^{r(k)}},$$

where $r(k)$ is equal to one if $k = 2$ and to zero otherwise. Upon setting

$$\Omega_\infty(f_k^\sharp, \chi) = \Omega_\infty(f_k^\sharp)^{\mathrm{sign}(\chi)}$$

($\mathrm{sign}(\chi)$ being the sign of $\chi(-1)$) the quantity

$$L(f_k^\sharp, \chi, k/2)_{\mathrm{alg}} = \frac{(k/2-1)! \cdot \mathfrak{g}(\bar{\chi}) \cdot L(f_k^\sharp, \chi, k/2)}{(-2\pi i)^{k/2-1} \cdot \Omega_\infty(f_k^\sharp, \chi)} \in \mathbf{Q}(f_k^\sharp, \chi) \tag{39}$$

belongs to the number field $\mathbf{Q}(f_k^\sharp, \chi)$ generated over $\mathbf{Q}$ by the Fourier coefficients of $f_k^\sharp$ and the values of χ. Here $\mathfrak{g}(\bar{\chi}) = \sum_{a \in (\mathbf{Z}/c_\chi \mathbf{Z})^*} \bar{\chi}(a) \cdot \zeta_{c_\chi}^a$ is the Gauß sum of $\bar{\chi} = \chi^{-1}$, where c_χ is the conductor of χ and $\zeta_{c_\chi} = e^{2\pi i/c_\chi}$. One calls $L(f_k^\sharp, \chi, k/2)_{\mathrm{alg}}$ the *algebraic part* of the central critical value $L(f_k^\sharp, \chi, k/2)$.

According to a result of Mazur and Kitagawa (cf. [Kit94, GS93, BD07]) the algebraic central values $L(f_k^\sharp, \chi, k/2)_{\mathrm{alg}}$, defined for $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$, can be interpolated by an analytic function

$$L_p(\boldsymbol{f}^\sharp, \chi) \in \mathscr{O}_{\boldsymbol{f}},$$

which we call the *Mazur–Kitagawa* p-adic L-function of $(\boldsymbol{f}^\sharp, \chi)$. More precisely, up to shrinking $U_{\boldsymbol{f}}$ if necessary, there exist for every $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$ non-zero p-adic periods

$$\lambda_k^+, \lambda_k^- \in \bar{\mathbf{Q}}_p^*, \quad \text{with} \quad \lambda_2^\pm = 1,$$

such that

(40)

$$L_p(\boldsymbol{f}^\sharp, \chi)(k) = \lambda_k^{\mathrm{sign}(\chi)} \cdot \left(1 - \frac{p^{k/2-1}\chi(p)}{a_p(k)}\right) \cdot \left(1 - \varepsilon_k(p) \cdot \frac{p^{k/2-1}\bar{\chi}(p)}{a_p(k)}\right) \cdot L(f_k^\sharp, \chi, k/2)_{\mathrm{alg}},$$

where $\varepsilon_k(p) = 0$ if $k = 2$ (i.e. if $f_k^\sharp$ is p-new) and $\varepsilon_k(p) = 1$ otherwise (i.e. if $f_k^\sharp$ is p-old).

Remark 3.2. — 1. The p-adic L-function $L_p(\boldsymbol{f}^\sharp, \chi)$ is the restriction to the *central critical line* $s = k/2$ of a two-variable p-adic L-function

$$L_p^{\mathrm{MK}}(\boldsymbol{f}^\sharp, \chi) = L_p^{\mathrm{MK}}(\boldsymbol{f}^\sharp, \chi)(\boldsymbol{k}, \boldsymbol{j}) \in \mathscr{O}_{\boldsymbol{f}} \hat{\otimes} \mathscr{O}_{\mathrm{cyc}}$$

of the weight variable $\boldsymbol{k} \in U_{\boldsymbol{f}}$ and *cyclotomic* variable $\boldsymbol{j}$ (cf. [BSV20, Section 7.1]). For every classical point $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$ one has

$$L_p^{\mathrm{MK}}(\boldsymbol{f}^\sharp, \chi)(k, \boldsymbol{j}) = \lambda_k^{\mathrm{sign}(\chi)} \cdot L_p(f_k^\sharp, \chi)(\boldsymbol{j}),$$

where $L_p(f_k^\sharp, \chi) = L_p(f_k^\sharp, \chi)(\boldsymbol{j}) \in \mathscr{O}_{\mathrm{cyc}}$ is the cyclotomic p-adic L-function of $f_k^\sharp \otimes \chi$ (cf. [MTT86]) defined as the Mellin transform of a measure on $\mathbf{Z}_p^* \times (\mathbf{Z}/c_\chi \mathbf{Z})^*$ associated to the sign(χ)-modular symbol attached to $f_k^\sharp$. In order to construct $L_p^{\mathrm{MK}}(\boldsymbol{f}^\sharp, \chi)$ one interpolates these modular symbols, and the p-adic periods $\lambda_k^\pm$ are the *error terms* arising from the p-adic interpolation.

2. If $k = 2$ and

$$\chi(p) = a_p(2)$$

(with $a_p(2) = a_p(A) = \pm 1$), the Euler factor $1 - \frac{p^{k/2-1}\chi(p)}{a_p(k)}$ which appears in Equation (40) vanishes. In this *exceptional zero* situation (cf. [MTT86]) $L_p(\boldsymbol{f}^\sharp, \chi)$ vanishes at $k = 2$ independently of whether the complex L-series $L(f, \chi, s)$ vanishes at $s = 1$ or not.

3.2. Hida–Rankin p-adic L-functions attached to quadratic fields. — Let $K/\mathbf{Q}$ be a quadratic field of discriminant coprime to $N_f p$, satisfying the Heegner hypothesis given in Assumption 1.1(1). To lighten notations, assume in the real quadratic case that $N_f^- = 1$ (so that one works with forms on GL_2).

The Hida–Rankin p-adic L-function attached to the pair $(\boldsymbol{f}^\sharp, \varrho)$ ($\varrho = \varphi$ or ψ) introduced in [BD07] and [BD09] is an analytic function

$$L_p(\boldsymbol{f}^\sharp/K, \varrho) \in \mathscr{O}_{\boldsymbol{f}}$$

satisfying the following interpolation property. For every classical point $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$

$$L_p(\boldsymbol{f}^\sharp/K,\varrho)(k) = \Omega_p(f_k^\sharp,\varrho)^2\left(1-\frac{p^{k-2}}{a_p(k)^2}\right)^2 L(f_k^\sharp/K,\varrho,k/2)_{\mathrm{alg}}, \tag{41}$$

where the *algebraic part* of $L(f_k^\sharp/K,\varrho,k/2)$ is defined by

$$L(f_k^\sharp/K,\varrho,k/2)_{\mathrm{alg}} = \frac{(k/2-1)!^2\cdot d_K^{(k-1)/2}}{(2\pi i)^{k-2}\cdot\Omega_\infty(f_k^\sharp,\varrho)}\cdot L(f_k^\sharp/K,\varrho,k/2)\in L. \tag{42}$$

Here $L(f_k^\sharp/K,\varrho,s) = L(f_k^\sharp\otimes\vartheta_\varrho,s)$ is the Rankin–Selberg convolution of $f_k^\sharp$ and the weight-one theta series ϑ_ϱ associated to ϱ, and the complex and p-adic *periods* $\Omega_\infty(f_k^\sharp,\varrho)$ and $\Omega_p(f_k^\sharp,\varrho)$ are defined as follows.

When K is real quadratic, then

$$\Omega_\infty(f_k^\sharp,\varrho) = \left(\Omega_\infty(f_k^\sharp)^{\mathrm{sign}(\varrho)}\right)^2, \qquad \Omega_p(f_k^\sharp,\varrho) = \left(\lambda_k^{\mathrm{sign}(\varrho)}\right)^2.$$

When $K/\mathbf{Q}$ is imaginary quadratic, one sets

$$\Omega_\infty(f_k^\sharp,\varrho) = (f_k^\sharp,f_k^\sharp)_{N_f p^{r(k)}},$$

where $r(k)=1$ if $k=2$ and $r(k)=0$ otherwise.

We finally recall the definition of the p-adic periods $\Omega_p(f_k^\sharp,\varrho)$ in the imaginary case. With the notations of Assumption 1.1 let $B_{/\mathbf{Q}}$ be the definite quaternion algebra with discriminant $N_f^-\infty$. As explained in Section 2 of [BD07] the form $f_k^\sharp$ gives rise, via the Jacquet–Langlands correspondence, to a weight-k eigenform ϕ_k on $\hat{B}^*$ of level $\Sigma_0(pN^+,N^-)\subset\hat{B}^*$, having the same system of Hecke eigenvalues as $f_k^\sharp$. This form is unique up to multiplication by a non-zero scalar. As in loc. cit., for every $k>2$ (resp., $k=2$) normalise ϕ_k by requiring that its Petersson norm is equal to 1 (resp., that it takes values in $\mathbf{Z}$). This characterises ϕ_k up to sign for $k>2$. According to Theorem 2.5 of loc. cit. (up to shrinking $U_{\boldsymbol{f}}$ if necessary) there exists an $\mathscr{O}_{\boldsymbol{f}}$-adic family ϕ_∞ of eigenforms on $\hat{B}^*$ whose specialisation at a classical point $k\in U^{\mathrm{cl}}$ is equal to $\lambda_B(k)\cdot\phi_k$, for some

$$\lambda_B(k)\in L^* \quad\text{with}\quad \lambda_B(2)=1$$

(see Section 2 of loc. cit. for the details). The definition of $L_p(\boldsymbol{f}^\sharp/K)$ given in Section 3 of loc. cit. depends on ϕ_∞, and one sets $\Omega_p(f_k^\sharp,\varrho)=\lambda_B(k)$. In particular $\Omega_p(f,\varrho)=1$.

3.3. Proof of Theorem 3.1. — The decomposition of Galois representations

$$V(g)\otimes_L V(h) = \mathrm{Ind}_{\mathbf{Q}}^K(\nu_g)\otimes_L \mathrm{Ind}_{\mathbf{Q}}^K(\nu_h) = \mathrm{Ind}_{\mathbf{Q}}^K(\varphi)\oplus\mathrm{Ind}_{\mathbf{Q}}^K(\psi)$$

yields for every $k\in U_{\boldsymbol{f}}^{\mathrm{cl}}$ a factorisation of complex L-functions

$$L(f_k^\sharp\otimes g\otimes h,s) = L(f_k^\sharp/K,\varphi,s)\cdot L(f_k^\sharp/K,\psi,s). \tag{43}$$

The imaginary case. Assume that $K/\mathbf{Q}$ is imaginary quadratic and let k be a classical point in $U_{\boldsymbol{f}}^{\mathrm{cl}}\cap\mathbf{Z}_{>2}$. Then the complex period $\Omega_\infty(f_k^\sharp,\varrho)$ is equal to the

Petersson norm $\langle f_k^\sharp, f_k^\sharp\rangle_{N_f p^{r(k)}}$, hence Equations (42), (43) and [BSV20, (133)], give

$$\frac{\Gamma(k,1,1)}{2^{\alpha(k,1,1)}}\cdot\frac{L(f_k^\sharp\otimes g\otimes h,k/2)}{\pi^{2(k-2)}\cdot(f_k^\sharp,f_k^\sharp)_{N_f}^2}=\frac{2^{2k-4-\alpha(k,1,1)}}{d_K^{k-1}}\cdot L(f_k^\sharp/K,\varphi,k/2)_{\mathrm{alg}}\cdot L(f_k^\sharp/K,\psi,k/2)_{\mathrm{alg}}. \tag{44}$$

With notations as in [BSV20, Section 6], one finds from Equations (1) and (2)

$$\mathcal{E}(\boldsymbol{f}_k^\sharp,\boldsymbol{g}_1^\sharp,\boldsymbol{h}_1^\sharp)=\left(1-\frac{p^{k/2-1}}{a_p(k)}\right)^2\left(1+\frac{p^{k/2-1}}{a_p(k)}\right)^2=\left(1-\frac{p^{k-2}}{a_p(k)^2}\right)^2. \tag{45}$$

Since $\Omega_p(f_k^\sharp,\varrho)$ is equal to the quaternionic period $\lambda_B(k)$ for both $\varrho=\varphi$ and $\varrho=\psi$ (cf. the discussion following Equation (41)), Equations (42), (41), (44), (45) and [BSV20, (132), (135)] yield

$$L_p(\boldsymbol{f}^\sharp,\boldsymbol{g}_1^\sharp,\boldsymbol{h}_1^\sharp)(k)=\mathscr{A}_{B,k}^2\cdot\mathscr{A}_k^o\cdot L_p(\boldsymbol{f}^\sharp/K,\varphi)(k)\cdot L_p(\boldsymbol{f}^\sharp/K,\psi)(k) \tag{46}$$

for every $k\in U_{\boldsymbol{f}}^{\mathrm{cl}}\cap\mathbf{Z}_{>2}$, where one writes

$$\mathscr{A}_{B,k}=\frac{1}{\lambda_B(k)^2\cdot\mathcal{E}_0(\boldsymbol{f}_k^\sharp)\cdot\mathcal{E}_1(\boldsymbol{f}_k^\sharp)}\qquad\text{and}\qquad\mathscr{A}_k^o=\frac{2^{2k-4-\alpha(k,1,1)}}{d_K^{k-1}}\prod_{v|N}\mathrm{Loc}_v.$$

Since Loc_v is a non-zero constant in $\mathbf{Q}^*$ for every $v|N$, and p does not divide d_K, the values $\mathscr{A}_k^o\in\mathbf{Q}^*$ for $k\in U_{\boldsymbol{f}}^{\mathrm{cl}}$ are interpolated by a unit in $\mathscr{O}_{\boldsymbol{f}}^*$. Equation (46) then reduces the proof of Theorem 3.1 to the following statement.

Lemma 3.3. — *There exists a bounded analytic function $\mathscr{A}_B\in\mathscr{O}_{\boldsymbol{f}}$ satisfying the following properties.*

1. *$\mathscr{A}_B(k)=\mathscr{A}_{B,k}$ for infinitely many classical points $k\in U_{\boldsymbol{f}}^{\mathrm{cl}}$.*
2. *$\mathscr{A}_B(2)$ is a non-zero element in $\mathbf{Q}^*$.*

We defer the proof of Lemma 3.3 to Section 3.4 below.

The real case. Assume that K is real quadratic and let $k\in U_{\boldsymbol{f}}^{\mathrm{cl}}\cap\mathbf{Z}_{>2}$. Define the quantity

$$\mathscr{A}_{\mathrm{GL}_2,k}=\frac{1}{\lambda_k^+\cdot\lambda_k^-\cdot\mathcal{E}_0(\boldsymbol{f}_k)\cdot\mathcal{E}_1(\boldsymbol{f}_k)}. \tag{47}$$

By a similar argument as in the imaginary case, one reduces the proof of Theorem 3.1 to the following statement.

Lemma 3.4. — *There exists a bounded analytic function $\mathscr{A}_{\mathrm{GL}_2}\in\mathscr{O}_{\boldsymbol{f}}$ satisfying the following properties.*

1. *$\mathscr{A}_{\mathrm{GL}_2}(k)=\mathscr{A}_{\mathrm{GL}_2,k}$ for infinitely many classical points $k\in U_{\boldsymbol{f}}^{\mathrm{cl}}$.*
2. *$\mathscr{A}_{\mathrm{GL}_2}(2)$ is a non-zero element in $\mathbf{Q}^*$.*

3.4. Proofs of Lemma 3.3 and Lemma 3.4. — According to Proposition 5.2 of [BD07] there exists an analytic function $\mathscr{A}_{\mathrm{GL}_2}^B\in\mathscr{O}_{\boldsymbol{f}}$ (denoted η in loc. cit.) such that, for every $k\in U_{\boldsymbol{f}}^{\mathrm{cl}}\cap\mathbf{Z}_{>2}$

$$\mathscr{A}_{\mathrm{GL}_2}^B(k)=\frac{\lambda_B(k)^2}{\lambda_k^+\cdot\lambda_k^-}=\frac{\mathscr{A}_{\mathrm{GL}_2,k}}{\mathscr{A}_{B,k}}\qquad\text{and}\qquad\mathscr{A}_{\mathrm{GL}_2}^B(2)\in\mathbf{Q}^*.$$

In particular, after shrinking $U_{\boldsymbol{f}}$ if necessary, the analytic function $\mathscr{A}^{B}_{\mathrm{GL}_2}$ is a unit in $\mathscr{O}_{\boldsymbol{f}}$. This implies that Lemma 3.3 follows from Lemma 3.4, hence to conclude the proof of Theorem 3.1 it is sufficient to prove the latter.

To prove Lemma 3.4 we consider triple product p-adic L-functions associated to $\boldsymbol{f}^{\sharp}$ and two weight one Eisenstein series attached to the characters which appear in the following lemma.

Lemma 3.5. — *There exists two Dirichlet characters χ and ψ satisfying the following properties.*

1. *The conductors c_χ and c_ψ of χ and ψ are coprime to each other and coprime to $N_f p$.*
2. *χ is* even *and $\chi(p)$ is different from ± 1.*
3. *ψ is* odd *and $\psi(p) = -a_p(f)$.*
4. *Both $L(f,\chi,s)$ and $L(f,\psi,s)$ do not vanish at $s=1$.*

Proof. — Let ℓ be a prime which does not divide $N_f p$. According to the main result of [Roh84] there exists $n_o \in \mathbf{N}$ such that $L(f,\chi,1) \neq 0$ for every primitive Dirichlet character χ of $\mathrm{Gal}(\mathbf{Q}(\mu_{\ell^n})^+/\mathbf{Q}) = (\mathbf{Z}/\ell^n\mathbf{Z})^*/\{\pm 1\}$ with $n \geqslant n_o$, where $\mathbf{Q}(\mu_{\ell^n})^+$ is the maximal totally real subfield of the ℓ^n-th cyclotomic extension of $\mathbf{Q}$. If $n \geqslant n_o$ is such that $\ell^n \nmid p^4 - 1$, this shows that there exists a character χ such that

(a) the conductor $c_\chi = \ell^n$ of χ is coprime to $N_f p$.
(b) $\chi(-1) = +1$ and $\chi(p) \neq \pm 1$.
(c) $L(f,\chi,s)$ does not vanish at $s=1$.

Let q be a fixed prime which divides N_f exactly, whose existence is guaranteed by Assumption 1.1. For every quadratic character σ denote by $\mathrm{sign}(f \otimes \sigma)$ the sign at $s=1$ in the functional equation satisfied by the Hecke L-function $L(f,\sigma,s)$. Choose any quadratic Dirichlet character ψ_1 satisfying the following properties.

(d) The conductor $c(\psi_1)$ of ψ_1 is coprime with $\ell \cdot N_f p$.
(e) $\psi_1(-1) = +1$ and $\psi_1(t) = +1$ for every prime t which divides N_f/q.
(f) $\psi_1(p) = -a_p(f)$ and $\psi_1(q) = a_p(f) \cdot \mathrm{sign}(f)$.

One has (cf. Theorem 3.66 of [Shi71])

$$\mathrm{sign}(f \otimes \psi_1) = \mathrm{sign}(f) \cdot \psi_1(-N_f p) = -1,$$

hence the main result of [BFH90] shows that there exists a quadratic Dirichlet character ψ_2 such that

(g) the conductor of ψ_2 is coprime to $\ell \cdot c(\psi_2) \cdot N_f p$.
(h) $\psi_2(-1) = -1$ and $\psi_2(t) = +1$ for every prime divisor t of $N_f p$.
(i) $L(f, \psi_1 \cdot \psi_2, s)$ does not vanish at $s=1$.

According to (a)–(i) the characters χ and $\psi = \psi_1 \cdot \psi_2$ satisfy the required properties. $\square$

Fix two characters χ and ψ satisfying the conclusions of the previous lemma, and set $N = N_f c_\chi c_\psi$ and

$$\xi = \chi^{-1} \cdot \psi^{-1}.$$

Since χ, ψ and ξ are non-trivial and ξ is odd, one can consider the weight one Eisenstein series

$$E(\chi,\psi) = \sum_{n=1}^{\infty} \sigma(\chi,\psi)(n) \cdot q^n \in M_1(N, \xi^{-1})$$

and

$$E(\xi) = E(\mathbf{1},\xi) = \frac{L(\xi,0)}{2} + \sum_{n\geqslant 1} \sigma(\mathbf{1},\xi) \cdot q^n \in M_1(N,\xi),$$

where $\sigma(\alpha,\beta)(n) = \sum_{d|n} \alpha(n/d)\cdot\beta(d)$ for every Dirichlet characters α and β, and $\mathbf{1}$ is the trivial character. Following Section 3 of [BD14], for every classical point $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$ define

$$L_p(\boldsymbol{f}_k^\sharp, E(\chi,\psi)) = \frac{\left(\boldsymbol{f}_k^\sharp, e_{\mathrm{ord}}\left(d^{k/2-1}\check{E}(\xi)\times \check{E}(\chi,\psi)\right)\right)_{Np}}{(\boldsymbol{f}_k^\sharp, \boldsymbol{f}_k^\sharp)_{Np}}, \tag{48}$$

where $\check{E}(\xi) = E(\xi)^{[p]} \in \mathbf{M}_1(N,\xi)$ and $\check{E}(\chi,\psi) = E(\chi,\psi)^{[p]} \in \mathbf{M}_1(N,\xi^{-1})$ are the p-depletions of $E(\xi)$ and $E(\chi,\psi)$ (cf. [BSV20, Section 3.1]). The article [BD14] shows that the function which to $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$ associates $L_p(\boldsymbol{f}_k^\sharp, E(\chi,\psi))$ extends to an analytic function

$$L_p(\boldsymbol{f}^\sharp, E(\chi,\psi)) \in \mathscr{O}_{\boldsymbol{f}}.$$

(The notation is justified by the following lemma, cf. Remark 3.7.) For all $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$ define

$$C_{\chi,\psi}(k) = \frac{-iN_f}{2^{k-2}\cdot\chi(c_\psi)\cdot\psi(c_\chi)\cdot[\Gamma_1(N_f):\Gamma_1(N)]}.$$

For $\cdot = \chi,\psi$ Section 3.1 associates to $(\boldsymbol{f}^\sharp,\cdot)$ the Mazur–Kitagawa p-adic L-function $L_p(\boldsymbol{f}^\sharp,\cdot) \in \mathscr{O}_{\boldsymbol{f}}$.

Lemma 3.6. — 1. *Let $\mathbf{Q}(\chi,\psi)$ be the field generated over $\mathbf{Q}$ by the values of χ and ψ. Then*

$$L_p(\boldsymbol{f}^\sharp, E(\chi,\psi))(2) = (p+1)\cdot C_{\chi,\psi}(2)\cdot L_p(\boldsymbol{f}^\sharp,\chi)(2)\cdot L_p(\boldsymbol{f}^\sharp,\psi)(2) \in \mathbf{Q}(\chi,\psi)^*.$$

In particular the p-adic L-function $L_p(\boldsymbol{f}^\sharp, E(\chi,\psi))$ does not vanish at $k=2$.

2. *(cf. [BD14]) For every classical point $k \in U_{\boldsymbol{f}}^{\mathrm{cl}}$ (strictly) greater than 2 one has*

$$L_p(\boldsymbol{f}^\sharp, E(\chi,\psi))(k) = \mathscr{A}_{\mathrm{GL}_2,k}\cdot C_{\chi,\psi}(k)\cdot L_p(\boldsymbol{f}^\sharp,\chi)(k)\cdot L_p(\boldsymbol{f}^\sharp,\psi)(k), \tag{49}$$

Proof. — 1. Write for simplicity $g = E(\xi)$ and $h = E(\chi,\psi)$, and consider the p-stabilisations

$$g_\alpha(q) = g(q) - \xi(p)\cdot g(q^p), \quad g_\beta(q) = g(q) - g(q^p) \quad\text{and}\quad h_\alpha(q) = h(q) - \psi(p)\cdot h(q^p).$$

Then f (resp., g_α, g_β, h_α) is an eigenvector for the U_p-operator with eigenvalue $\alpha_f = a_p(2) = \pm 1$ (resp., 1, $\xi(p)$, $\chi(p)$), hence Lemma 3.5 and the same computations as in the proof of [DR14, Lemma 4.10] show that

$$2\cdot (f, g_\beta\cdot h_\alpha)_{Np} = (1-\chi(p)/a_p(2))\cdot (f, g_\alpha\cdot h_\alpha)_{Np}.$$

As $\xi(p) \neq 1$ by Lemma 3.5, one can write $g = (g_\alpha - \xi(p) \cdot g_\beta)/(1 - \xi(p))$, which together with the previous equation and a direct computation gives the identity

$$L_p(\boldsymbol{f}^\sharp, E(\chi,\psi))(2) = 2\left(1 - \frac{\chi(p)}{a_p(2)}\right) \cdot \frac{(f, g \cdot h_\alpha)_{Np}}{(f,f)_{Np}}. \tag{50}$$

The L-series of the forms f and h_α admit Euler product expansions, hence the Rankin method (see the argument leading to Equation (18) of [BD14], or [Shi76, Theorem 2 and Lemma 1]) gives

$$(f, g \cdot h_\alpha)_{Np} = -i\mathfrak{g}(\xi) N_f p \cdot L(f \otimes h_\alpha, 1), \tag{51}$$

where $\mathfrak{g}(\cdot)$ is the Gauß sum of the character $\cdot$. (Note that $(\cdot,\cdot)_{Np}$ equals $8\pi^2$ times the Petersson product defined in Equation 9 of [BD14].) Since the characters χ and ψ have opposite parity, one has

$$\Omega_\infty(f,\chi) \cdot \Omega_\infty(f,\psi) = (f,f)_{N_f p} = [\Gamma_1(N_f) : \Gamma_1(N)]^{-1} \cdot (f,f)_{Np}. \tag{52}$$

Moreover a direct comparison of Euler factors (cf. [Shi76, Lemma 1]) and Lemma 3.5 give

$$L(f \otimes h_\alpha, 1) = \left(1 - \frac{a_p(2)\psi(p)}{p}\right) L(f \otimes h, 1) = \left(1 + \frac{1}{p}\right) L(f,\chi,1) \cdot L(f,\psi,1). \tag{53}$$

As $\mathfrak{g}(\xi) = \mathfrak{g}(\chi^{-1}) \cdot \mathfrak{g}(\psi^{-1}) \cdot \chi^{-1}(c_\psi)\psi^{-1}(c_\chi)$ (since $(c_\chi, c_\psi) = 1$), the statement is a direct consequence of Equations (39)–(40), Equations (50)–(53) and Lemma 3.5.

2. This is proved in Proposition 3.3 of [BD14]. Since the setting of loc. cit. is slightly different from ours, for the convenience of the reader we briefly review the argument. Equations (35) and (41) and Proposition 3.2 of [BD14], together with Proposition 4.6 of [DR14], show that for every classical point $k > 2$ one has

$$L_p(\boldsymbol{f}^\sharp, E(\chi,\psi))(k) = \frac{\mathcal{E}(\boldsymbol{f}_k^\sharp, \chi, \psi)}{\mathcal{E}_0(\boldsymbol{f}_k^\sharp) \cdot \mathcal{E}_1(\boldsymbol{f}_k^\sharp)} \cdot \frac{\left(f_k^\sharp, \delta^{k/2-1} E(\xi) \cdot E(\chi,\psi)\right)_N}{(f_k^\sharp, f_k^\sharp)_N},$$

where

$$\delta^{k/2-1} : M_1(N,\xi) \longrightarrow M_{k-1}^{\mathrm{an}}(N,\xi)$$

is the $(k/2-1)$-th iterate of the Shimura–Maaß derivative operator. Here $\mathcal{E}_0(\boldsymbol{f}_k^\sharp)$ and $\mathcal{E}_1(\boldsymbol{f}_k^\sharp)$ are as in Equation [BSV20, (135)], and

$$\mathcal{E}(\boldsymbol{f}_k^\sharp, \chi, \psi) = \left(1 - \frac{p^{k/2-1}\chi(p)}{a_p(k)}\right)\left(1 - \frac{p^{k/2-1}\bar{\chi}(p)}{a_p(k)}\right)\left(1 - \frac{p^{k/2-1}\psi(p)}{a_p(k)}\right)^2.$$

(Recall that $\psi = \psi^{-1}$ is a quadratic character, cf. Lemma 3.5, and that $\mathcal{E}_i(\boldsymbol{f}_k^\sharp)$ is non-zero for $k > 2$.) The Rankin method (see Equations (18) and (19) of [BD14]) yields

$$\left(f_k^\sharp, \delta^{k/2-1} E(\xi) \cdot E(\chi,\psi)\right)_N = \frac{-iN_f \mathfrak{g}(\xi) \cdot (k/2-1)!^2}{2^{k-2} \cdot (-2\pi i)^{k-2}} \cdot L(f_k^\sharp, \chi, k/2) \cdot L(f_k^\sharp, \psi, k/2).$$

As in the proof of Part 1 the statement follows easily from the definitions and the previous three equations. □

Since the analytic functions $L_p(\boldsymbol{f}^\sharp, E(\chi,\psi))$, $L_p(\boldsymbol{f}^\sharp,\chi)$ and $L_p(\boldsymbol{f}^\sharp,\psi)$ do not vanish at $k=2$ by Lemma 3.6(1), and since $C_{\chi,\psi}(k)$ is clearly an invertible element of $\mathscr{O}_{\boldsymbol{f}}$, Lemma 3.6(2) implies that the values $\mathscr{A}_{\mathrm{GL}_2,k}$, defined for $k \in U^{\mathrm{cl}} \cap \mathbf{Z}_{>2}$, are interpolated by an analytic function $\mathscr{A}_{\mathrm{GL}_2}(k)$ which does not vanish at $k=2$. In addition, the explicit formula for the value of $L_p(\boldsymbol{f}^\sharp, E(\chi,\psi))$ at $k=2$ displayed in Lemma 3.6(1) gives

$$\mathscr{A}_{\mathrm{GL}_2}(2) = p+1.$$

This concludes the proof of Lemma 3.4, and with it the proofs of Lemma 3.3 and Theorem 3.1.

Remark 3.7. — 1. The previous lemma (or better its proof) shows that $L_p(\boldsymbol{f}^\sharp, E(\chi,\psi))$ can be though of as a p-adic Rankin–Selberg convolution, which interpolates the critical values $L(f_k^\sharp \otimes E(\chi,\psi), k/2)$ of the convolution of $f_k^\sharp$ with $E(\chi,\psi)$. One can also think of $L_p(\boldsymbol{f}^\sharp, E(\chi,\psi)) = \mathscr{L}_p(\boldsymbol{f}^\sharp, E(\xi), E(\chi,\psi))$ as a square-root triple-product p-adic L-function (cf. Equations (48) and [**BSV20**, (55)]), whose square interpolates the complex central values $L(f_k^\sharp \otimes E(\xi) \otimes E(\chi,\psi), k/2)$.

2. Note that the Euler factor $\mathcal{E}_1(\boldsymbol{f}_k^\sharp) = 1 - \frac{p^{k-2}}{a_p(k)^2}$ vanishes at $k=2$, as a manifestation of the presence of an exceptional zero for $L_p(\boldsymbol{f}^\sharp, s)$ and $L_p(\boldsymbol{f}^\sharp, \boldsymbol{g}_1^\sharp, \boldsymbol{h}_1^\sharp)$ in the sense of [**MTT86**] (cf. Remark 3.2(2)).

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