HEEGNER POINTS AND BEILINSON–KATO ELEMENTS: A CONJECTURE OF PERRIN-RIOU

by

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Abstract. — A conjecture of Perrin-Riou relating Heegner cycles to Beilinson–Kato elements is proved, by relating both objects to p-adic families of Beilinson–Flach elements in the higher Chow groups of products of two modular curves.

Contents

1. Introduction

Let A be an elliptic curve over the field $\mathbf{Q}$ of rational numbers, having semistable reduction at an odd prime p. Denote by

$$\zeta_A^{\mathrm{Kato}} \in H^1(\mathbf{Q}, V_p(A))$$

the global p-adic Beilinson–Kato element associated in [**Kat04**] to (a fixed modular parametrisation of) A (cf. Section 1.1 below). It lies at the "bottom layer" of Kato's Euler system arising from p-adic families of Beilinson elements in the second K-group of a modular curve, associated to pairs of Eisenstein series. The relevance of this global class to the Birch and Swinnerton-Dyer conjecture stems from the close relationship it enjoys with the Hasse–Weil L-function $L(A/\mathbf{Q}, s)$ of A and its p-adic avatars. More

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precisely, Kato's reciprocity law stated in equation (1) below implies that the image of $\mathrm{res}_p\big(\zeta_A^{\mathrm{Kato}}\big) \in H^1(\mathbf{Q}_p, V_p(A))$ by the Bloch–Kato dual exponential is a non-zero multiple of the central critical value $L(A/\mathbf{Q}, 1)$. Of primary interest for this paper is the scenario where $L(A/\mathbf{Q}, 1) = 0$, in which ζ_A^{Kato} belongs to the p-adic Bloch–Kato Selmer group of A and therefore defines a local point in $A(\mathbf{Q}_p) \otimes \mathbf{Q}_p$. In [PR93] Perrin-Riou predicts that this local point is a prescribed element in the natural image of the group of rational points $A(\mathbf{Q}) \otimes \mathbf{Q}_p$. The main goal of this article is to prove the following theorem, which settles Perrin-Riou's conjecture.

Theorem A. — *Let A be an elliptic curve over the field $\mathbf{Q}$ of rational numbers, having semistable reduction at an odd prime p. If the Hasse–Weil complex L-function $L(A/\mathbf{Q}, s)$ of A vanishes at $s = 1$, then there exists a global point $\boldsymbol{P}$ in $A(\mathbf{Q})$ satisfying the following properties.*

1. *The point $\boldsymbol{P}$ has infinite order if and only if $L(A/\mathbf{Q}, s)$ has a simple zero at $s = 1$.*
2. *The following equality holds in $\mathbf{Q}_p$ up to multiplication by a non-zero rational number:*

$$\log_{\omega_A}\big(\mathrm{res}_p\big(\zeta_A^{\mathrm{Kato}}\big)\big) = \log^2_{\omega_A}(\boldsymbol{P}).$$

Here ω_A is the Néron differential of a global minimal Weierstraß equation for A and $\log_{\omega_A} : A(\mathbf{Q}_p) \longrightarrow \mathbf{Q}_p$ is the corresponding p-adic Lie group logarithm.

The reader is referred to Section 1.3 for a discussion of previous partial results and of related work.

In a more general setting, Theorem B below proves a natural generalisation of Perrin-Riou's conjecture for p-semistable elliptic newforms f of even weight $k_o \geqslant 2$ and trivial Nebentype, which recasts Theorem A when f is the newform of weight two associated with A by the modularity theorem.

1.1. Statement of the main result. — Fix a positive integer N_f, an odd prime p not dividing N_f, algebraic closures $\bar{\mathbf{Q}}$ and $\bar{\mathbf{Q}}_p$ of $\mathbf{Q}$ and $\mathbf{Q}_p$ respectively and field embeddings $i_\infty : \bar{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $i_p : \bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$. Denote by ord_p the p-adic valuation on $\bar{\mathbf{Q}}_p^*$ satisfying $\mathrm{ord}_p(p) = 1$ and by $|\cdot|_p$ the corresponding p-adic absolute value.

Let $f = \sum_{n \geqslant 1} a_n(f) \cdot q^n$ be a newform of even weight $k_o \geqslant 2$ and level $\Gamma_0(N_f p^r)$ for some $r \leqslant 1$. Let L be the finite extension of $\mathbf{Q}_p$ generated by $\mu_{N_f p^r}$ and the (images under i_p) of the Fourier coefficients $a_n(f)$ of f. Let $\alpha = \alpha_f$ and $\beta = \beta_f$ be the roots of the Hecke polynomial $X^2 - a_p(f) \cdot X + \mathbf{1}_{p^r}(p) \cdot p^{k_o - 1}$, ordered in such a way that $\mathrm{ord}_p(\alpha) \leqslant \mathrm{ord}_p(\beta)$. (Here $\mathbf{1}_m$ is the trivial Hecke character modulo m.) We assume that the form f is p-regular, viz. the roots α and β are distinct. Let $f_\alpha = f(q) - \beta_f \cdot f(q^p)$ be the p-stabilisation of f with U_p-eigenvalue α and let

$$L_p(f_\alpha) = L_\alpha(f, \boldsymbol{s}) \in \mathcal{O}(\mathcal{W})$$

be the cyclotomic p-adic L-function associated with f_α and the choice of complex Deligne periods $\Omega_f^{\pm}$, where $\mathcal{O}(\mathcal{W})$ is the ring of analytic functions on the p-adic weight space $\mathcal{W} = \mathrm{Hom}_{\mathrm{cont}}(\mathbf{Z}_p^*, \mathbf{C}_p^*)$ over $\mathbf{Q}_p$. We normalise $L_p(f_\alpha)$ as in Theorem 16.2 of

[**Kat04**], so that $L_p(f_\alpha, s-\mu)$ is an explicit multiple of the algebraic number

$$L(f,\mu,s)/(-2\pi i)^{s-1}\Omega_f^{\pm}$$

for each integer $1 \leqslant s \leqslant k_o - 1$ and each finite order character $\mu : \mathbf{Z}_p^* \longrightarrow \bar{\mathbf{Q}}_p^*$ satisfying $(-1)^{s-1}\mu(-1) = \pm 1$. (We use the additive notation for the product of characters in $\mathcal{W}(\bar{\mathbf{Q}}_p)$, so that $s - \mu$ is a shorthand for the continuous character $\kappa^s \cdot \mu^{-1} : \mathbf{Z}_p^* \longrightarrow \bar{\mathbf{Q}}_p^*$ with κ the inclusion of $\mathbf{Z}_p^*$ in $\bar{\mathbf{Q}}_p^*$.)

According to the work of Kato [**Kat04**] (see in particular Theorem 16.6 and Part 2 of Theorem 12.4) there exists a unique global Iwasawa cohomology class

$$\boldsymbol{\zeta}_f^{\mathrm{Kato}} \in H^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), V(f))$$

satisfying the explicit reciprocity law

$$\big\langle \mathrm{Log}_f\big(\mathrm{res}_p\big(\boldsymbol{\zeta}_f^{\mathrm{Kato}}\big)\big), \eta_f^{\alpha}\big\rangle = L_p(f_\alpha, 1+\boldsymbol{s}), \tag{1}$$

where the notations are as follows. Let $Y = Y_1(N_f p^r)$ be the affine modular curve of level $\Gamma_1(N_f p^r)$ over $\mathbf{Q}$. Assume for simplicity $N_f p^r \geqslant 4$, so that Y represents the functor sending a $\mathbf{Q}$-scheme S to the set of isomorphism classes of elliptic curves over S with a point of exact order $N_f p^r$. Consider the p-adic sheaves

$$\mathscr{L}_{k_o-2} = \mathrm{TSym}^{k_o-2} R^1(E \longrightarrow Y)_* \mathbf{Z}_p(1) \quad \text{and} \quad \mathscr{S}_{k_o-2} = \mathrm{Symm}^{k_o-2} R^1(E \longrightarrow Y)_* \mathbf{Z}_p$$

on Y, where $E \longrightarrow Y$ is the universal elliptic curve, and $\mathrm{TSym}^i\cdot$ and $\mathrm{Symm}^i\cdot$ denote respectively the submodule of symmetric tensors and the symmetric quotient of the i-th tensor power of $\cdot$. Set $Y_{\bar{\mathbf{Q}}} = Y \otimes_{\mathbf{Q}} \bar{\mathbf{Q}}$ and define

$$H^1_{\text{ét}}(Y_{\bar{\mathbf{Q}}}, \mathscr{L}_{k_o-2})(1) \otimes_{\mathbf{Z}_p} L \twoheadrightarrow V(f)$$

to be the maximal L-quotient on which the dual Hecke operator T'_n acts as multiplication by $a_n(f)$ for each $n \geqslant 1$. Dually define

$$V^*(f) \hookrightarrow H^1_{\text{ét,c}}(Y_{\bar{\mathbf{Q}}}, \mathscr{S}_{k_o-2}) \otimes_{\mathbf{Z}_p} L$$

to be the maximal L-submodule on which T_n acts as multiplication by $a_n(f)$ for each positive integer n. (See [**Kat04**, Section 2] or [**BSV21b**, Section 2] for detailed definitions.) The $G_{\mathbf{Q}}$-representation $V^*(f)$ is the Deligne representation of f and Poincaré duality identifies $V(f)$ with the dual of $V^*(f)$. The group $H^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), V(f))$ is the global cyclotomic Iwasawa cohomology of $V(f)$, viz. the $\mathbf{Q}_p$-linear extension of the inverse limit of the groups $H^1(\mathbf{Q}(\mu_{p^n}), \mathtt{V}(f))$, for any $G_{\mathbf{Q}}$-invariant $\mathcal{O}_L$-lattice $\mathtt{V}(f)$ in $V(f)$. The map res_p is restriction from the global Iwasawa cohomology to the similarly defined local Iwasawa cohomology $H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), V(f))$. To define the Perrin-Riou logarithm Log_f and the de Rham class η_f^α, we distinguish two cases.

Assume first that p does not divide the conductor of f, so that $V^{\cdot}(f)$ (where $\cdot$ denotes either $\emptyset$ of $*$) is crystalline at p. Then

$$\mathrm{Log}_f : H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), V(f)) \longrightarrow \mathcal{O}(\mathcal{W}) \otimes_{\mathbf{Q}_p} V_{\mathrm{cris}}(f)$$

is the Perrin-Riou logarithm associated in [**PR94**] with the restriction (via i_p) of $V(f)$ to the decomposition group $G_{\mathbf{Q}_p}$. Here $V^{\cdot}_{\mathrm{cris}}(f)$ is the crystalline Dieudonné module

$H^0(\mathbf{Q}_p, B_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V^\cdot(f))$ of $V^\cdot(f)$. The pairing

$$\langle \cdot, \cdot \rangle : V_{\mathrm{cris}}(f) \otimes_L V^*_{\mathrm{cris}}(f) \longrightarrow L$$

is the one induced by Poincaré duality and we use again the same symbol for its $\mathcal{O}(\mathcal{W})$-linear extension. The Faltings comparison isomorphism between the étale and the de Rham cohomology of $Y_{\mathbf{Q}_p}$ yields a canonical isomorphism between $\mathrm{Fil}^0 V_{\mathrm{cris}}(f)$ and the f-isotypic component of the space of weight-k_o modular forms of level $\Gamma_1(N_f)$ defined over L. (See for example [**BSV21b**, Section 2.5] for more details.) The form f then corresponds to a canonical generator ω_f of $\mathrm{Fil}^0 V_{\mathrm{cris}}(f)$, and one defines η_f^α to be the unique element of $V^*_{\mathrm{cris}}(f)$ such that $\varphi(\eta_f^\alpha) = \alpha \cdot \eta_f^\alpha$ and $\langle \omega_f, \eta_f^\alpha \rangle = 1$, where φ is the crystalline Frobenius. Here we use the assumptions $\alpha \neq \beta$ and $\mathrm{ord}_p(\alpha) \leqslant \mathrm{ord}_p(\beta)$ to guarantee the existence of η_f^α.

Assume now that p divides the conductor $N_f p$ of f. The representations $V^\cdot(f)$ (with $\cdot = \emptyset, *$) are semi-stable at p and one defines as above the classes ω_f in $\mathrm{Fil}^0 V_{\mathrm{st}}(f)$ and η_f^α in $V^*_{\mathrm{st}}(f)^{\varphi=\alpha}$ satisfying $\langle \omega_f, \eta_f^\alpha \rangle = 1$, where $V^\cdot_{\mathrm{st}}(f)$ is a shorthand for $H^0(\mathbf{Q}_p, V^\cdot(f) \otimes_{\mathbf{Q}_p} B_{\mathrm{st}})$ and the pairing $\langle \cdot, \cdot \rangle$ is induced by Poincaré duality. The maximal quotient $V(f)^-$ of $V(f)$ on which the inertia subgroup $I_{\mathbf{Q}_p}$ of $G_{\mathbf{Q}_p}$ acts trivially is free of rank one over L and a Frobenius acts on it via multiplication by α. Set $V_{\mathrm{cris}}(f)^- = H^0(\mathbf{Q}_p, V(f)^- \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}})$. Then the linear form

$$\langle \cdot, \eta_f^\alpha \rangle : V_{\mathrm{st}}(f) \longrightarrow L$$

factors through $V_{\mathrm{st}}(f) \longrightarrow V_{\mathrm{cris}}(f)^-$, and one defines $\langle \mathrm{Log}_f(\cdot), \eta_f^\alpha \rangle$ by the composition

$$H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), V(f)) \longrightarrow H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), V(f)^-) \longrightarrow V_{\mathrm{cris}}(f)^- \otimes_{\mathbf{Q}_p} \mathcal{O}(\mathcal{W}) \longrightarrow \mathcal{O}(\mathcal{W}),$$

where the first arrow is the natural one, the second is the Perrin-Riou logarithm associated in [**PR94**] with the p-adic representation $V(f)^-$ and the third arises from the linear form $\langle \cdot, \eta_f^\alpha \rangle$ on the semi-stable module $V_{\mathrm{st}}(f)$.

Set $G_\infty = \mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$ and $\Lambda_\infty = \mathbf{Z}_p[\![G_\infty]\!]$. The Shapiro isomorphism identifies $H^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), V(f))$ with $H^1(\mathbf{Q}, V(f) \otimes_{\mathbf{Z}_p} \Lambda_\infty(\varepsilon^{-1}))$, where $\varepsilon : G_{\mathbf{Q}} \longrightarrow \Lambda^*_\infty$ is the tautological character. The morphism of $\mathbf{Z}_p$-algebras $\chi_{\mathrm{cyc}}^{k_0/2-1} : \Lambda_\infty \longrightarrow \mathbf{Z}_p$ arising from the $(k_o/2-1)$-th power of the p-adic cyclotomic character $\chi_{\mathrm{cyc}} : G_{\mathbf{Q}} \longrightarrow \mathbf{Z}_p^*$ then induces a morphism (denoted by the same symbol) from $H^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), V(f))$ to the cohomology $H^1(\mathbf{Q}, \mathcal{V}(f))$ of the central critical twist $\mathcal{V}(f) = V(f)(1-k_o/2)$ of $V(f)$. Define the p-adic *Beilinson–Kato element* of f by

$$\zeta_f^{\mathrm{Kato}} = \chi_{\mathrm{cyc}}^{k_o/2-1}(\boldsymbol{\zeta}_f^{\mathrm{Kato}}) \in H^1(\mathbf{Q}, \mathcal{V}(f)).$$

In the statement of Theorem A, one defines $\zeta_A^{\mathrm{Kato}} = \pi_*(\zeta_{f_A}^{\mathrm{Kato}})$ in $H^1(\mathbf{Q}_p, V_p(A))$ to be the image of $\zeta_{f_A}^{\mathrm{Kato}}$ under the isomorphism $V(f_A) \longrightarrow V_p(A) = H^1_{\mathrm{\acute{e}t}}(A \otimes_{\mathbf{Q}} \bar{\mathbf{Q}}, \mathbf{Q}_p(1))$ induced by a modular parametrisation $\pi : Y \longrightarrow A$. Here f_A is the weight two newform associated with A by the modularity theorem of Wiles, Taylor–Wiles et alii.

Let K be a quadratic imaginary field of odd discriminant d_K, satisfying the Heegner hypothesis relative to pN_f, viz. each prime divisor of pN_f splits in $K/\mathbf{Q}$. As explained in Section 4.4 below, the p-adic Abel–Jacobi image of the Heegner cycle associated with f and K (cf. [Nek92, BDP13]) yields a class

$$z_K(f) \in \mathrm{Sel}(K, \mathcal{V}(f))^{-\varepsilon_f}$$

in the Selmer group of $\mathcal{V}(f)$ over K, on which complex conjugation acts as minus the sign ε_f in the functional equation satisfied by $L(f,s)$. If k_o is equal to 2 then $\mathrm{pr}_f : \mathrm{Ta}_p(J) \otimes_{\mathbf{Z}_p} L \twoheadrightarrow \mathcal{V}(f)$ is naturally isomorphic to the maximal quotient of the p-adic Tate module of the Jacobian J of $X_1(N_f p^r)$ on which $T'_n = a_n(f)$ for each $n \geqslant 1$. In this case $z_K(f) = \mathrm{Trace}_{H/K}\big(\mathrm{pr}_{f*}(z_K)\big)$, where H is the Hilbert class field of K and z_K in $H^1(H, \mathrm{Ta}_p(J))$ is the image under the global p-adic Kummer map of a Heegner divisor with trivial conductor in $J(H)$.

Theorem B. — *Assume that $L(f,s)$ vanishes at $s = k_o/2$. Then ζ_f^{Kato} belongs to the Bloch–Kato Selmer group $\mathrm{Sel}(\mathbf{Q}, \mathcal{V}(f))$ and the equality*

$$L(f, \varepsilon_K, k_o/2)_{\mathrm{alg}} \cdot \log_{\omega_f}\big(\mathrm{res}_p\big(\zeta_f^{\mathrm{Kato}}\big)\big) = \log^2_{\omega_f}\big(\mathrm{res}_{\mathfrak{p}}\big(z_K(f)\big)\big)$$

holds in L up to multiplication by a non-zero scalar in the number field $K((a_n(f_\alpha))_{n\geqslant 1})$.

In the statement we denoted by $L(f, \varepsilon_K, k_o/2)_{\mathrm{alg}}$ the algebraic part of the central critical value of the Hecke L-function $L(f, \varepsilon_K, s)$ of f twisted by the quadratic character ε_K of K. It is defined by

$$L(f, \varepsilon_K, k_o/2)_{\mathrm{alg}} = \frac{(k_o/2-1)! \cdot \sqrt{d_K}}{(-2\pi i)^{k_o/2-1} \cdot \Omega_f^-} \cdot L(f, \varepsilon_K, k_o/2)$$

and belongs to the number field $\mathbf{Q}(a_n(f), n \geqslant 1)$. Moreover we denoted by $\log_{\omega_f}$ the linear form $\langle \log_p(\cdot), \omega_f \rangle$ on the finite subspace of $H^1(\mathbf{Q}_p, \mathcal{V}(f))$, where $\log_p$ is the inverse of the Bloch–Kato exponential and ω_f in $\mathrm{Fil}^1 V^*_{\mathrm{dR}}(f)$ is the class attached to f by the Faltings comparison isomorphism.

Theorem A follows from Theorem B, the Gross–Zagier formula [GZ86] and Waldspurger's theorem on non-vanishing of quadratic twist (cf. Théorème 5 of [Wal84]).

1.2. Outline of the proof. — For simplicity we place ourselves in the setting of Theorem A, in which f is a newform of weight 2 with rational Fourier coefficients. The proof of Theorem A ultimately realises $\boldsymbol{P}$ as a Heegner point $P_K \in A(\mathbf{Q})$ associated to the imaginary quadratic field K introduced in Section 1.1.

The comparison between the Beilinson–Kato element ζ_A^{Kato} and the Heegner point P_K proceeds in two stages, in which the Beilinson-Flach elements defined in Section 2 play the role of a bridge between the two invariants. Roughly speaking, the Beilinson–Flach elements germane to our setting are obtained by replacing one of the families of Eisenstein series underlying the construction of Kato's Euler system with a family of theta-series attached to K. This family specialises in weight one to the Eisenstein series $\mathrm{Eis}_1(\varepsilon_K)$, whose p-adic Galois representation is equal to the sum of the trivial representation and its twist by the Dirichlet character ε_K associated with the extension $K/\mathbf{Q}$ (see Section 4.2 for details). This fact suggests a relation between the Beilinson–Flach elements and the Beilinson–Kato elements attached to the family of Eisenstein series passing through $\mathrm{Eis}_1(\varepsilon_K)$, formalised in Theorem 4.2 below as an equality of global classes in Iwasawa cohomology (and not just of their bottom layers over $\mathbf{Q}$).

The second key comparison relates the Heegner point P_K to the Beilinson–Flach elements. It is achieved in Theorem 4.3 by combining the 3-variable reciprocity law for the Beilinson–Flach elements of Kings–Loeffler–Zerbes [**KLZ17, LZ16**] with the main result of [**BDP13**], which describes the square of the formal group logarithm of P_K as a value of a Hida–Rankin p-adic L-function outside the range of classical interpolation.

The comparison between the Beilinson–Kato element ζ_A^{Kato} and the Heegner point P_K is carried out in Section 4 in the case where p is not a prime of split multiplicative reduction for A, while a discussion of the split multiplicative case is postponed to Section 5. The equality arising from our two-stage comparison of global classes involves the appearance of a ratio of p-adic periods, which is a priori a purely p-adic quantity. In order to show that this quantity is in fact a non-zero rational number, we reduce to the validity of Perrin-Riou's conjecture for elliptic curves A with complex multiplication by K. This special setting is treated separately in Section 3, by exploiting the relation between Kato's Euler system and the Euler system of elliptic units.

1.3. Remarks and relations with previous work on Theorem A. —

- When A has complex multiplication and p is a prime of good ordinary reduction, Theorem A follows from the work of Perrin-Riou, Rubin and Bertrand [**PR93, PR87, Rub92, Ber77**]. Here Perrin-Riou's p-adic Gross–Zagier formula and Bertrand's proof of the non-triviality of the canonical p-adic height for CM elliptic curves play a fundamental role.

  Section 3 below (cf. Theorem 3.1) presents a different proof of Theorem A in this setting, which generalises to the CM abelian varieties of GL_2-type associated with p-ordinary canonical Hecke characters (for which the non-triviality of the p-adic height is not known). This proof is based on two main ingredients: the comparison between the Euler system of Beilinson–Kato elements and that of elliptic units, studied by Kato in [**Kat04**, Section 12.5], and the p-adic Gross–Zagier formula proved by the first two authors and Prasanna in [**BDP13, BDP12**], which links the Euler system of elliptic units and that of Heegner points. The proof of Theorem 3.1 is a simpler variant in the CM setting of that of Theorem B (cf. Section 1.2).
- When A has good supersingular reduction at p, Theorem A is equivalent to the main result of [**Kob13**]. More precisely, in this setting (cf. the CM case) the canonical cyclotomic p-adic heights on $A(\mathbf{Q})$ are non-trivial, hence the results of [**PR93**] show that the p-adic Gross–Zagier formula proved by Kobayashi in [**Kob13**] implies Theorem A and that, vice versa, the main result of [**Kob13**] is a consequence of Theorem A when ζ_A^{Kato} is non-zero. On the other hand, the recent work of Skinner, Urban, X. Wan, W. Zhang et alii on the cyclotomic Main Conjecture and on the p-converses to the theorem of Gross–Zagier–Kolyvagin prove that the vanishing at $s = 1$ of the first derivative of $L(A/\mathbf{Q}, s)$ forces that of the first derivatives of the cyclotomic p-adic L-functions associated with A.

In particular, in the special case $k_o = 2$, our main result Theorem B gives a different proof of the main result of [Kob13].

- Theorem A in the exceptional case (viz. when A has split multiplicative reduction at p) is proved in [Ven16] using the main result of [BD07] as a crucial ingredient. Once again, the non-triviality of a suitable (central critical) p-adic height pairing is used in [Ven16] to deduce Theorem A from the p-adic Gross–Zagier formula of [BD07]. When $k_o = 2$, our argument gives a different proof of the main results of [Ven16] which does not use (and indeed easily recovers) the p-adic Gross–Zagier formula of [BD07].
- Our proof treats the supersingular and exceptional cases on the same footing as the good ordinary case. A central role is played by the p-adic Gross–Zagier formula proved in [BDP13]. This formula relates the special value of an anticyclotomic Rankin–Selberg p-adic L-function *outside* the range of classical interpolation to the p-adic *logarithm* of a Heegner point, which in the ordinary case is a much simpler invariant than its cyclotomic p-adic height (cf. [PR87]). Not surprisingly, the exceptional case is particularly intriguing and our argument requires a more delicate analysis in this setting.
- With the notations of Section 1.1, assume that f is p-old, let γ denote either α or β, and let f_γ be the p-stabilisation of f with U_p-eigenvalue γ. When f_γ has non-critical slope (i.e., $\mathrm{ord}_p(\gamma) < k_o - 1$), S. Kobayashi [Kob21] announced a proof of the p-adic Gross–Zagier formula for f_γ, relating the derivative of $L_p(f_\gamma)$ at k_o to $h_{p,\gamma}(z_K(f))$, where $h_{p,\gamma}$ is the the cyclotomic p-adic height on $\mathrm{Sel}(\mathbf{Q}, \mathcal{V}(f))$ attached to the γ-splitting $V_{\mathrm{cris}}(f) = \mathrm{Fil}^0 V_{\mathrm{cris}}(f) \oplus V_{\mathrm{cris}}(f)^{\varphi=\gamma\cdot p^{-k_o/2}}$ of the Hodge filtration on $V_{\mathrm{cris}}(f)$ (cf. [Nek93]). When $z_K(f)$ is non-zero, such a formula is a direct consequence of Theorem B and the p-adic height formalism developed by Nekovář and Benois (cf. the Rubin-style formula proved in Section 11.5.10 of [Nek06], which readily generalises to the non-ordinary setting considered in [Ben21]). Theorem B (and loc. cit.) applies more generally when f_γ is not θ-critical. The non-triviality of $z_K(f)$ is needed to guarantee that the p-adic logarithm of ζ_f^{Kato} (which appears in the aforementioned Rubin's formula) is non-zero. Thanks to the results of Cornut and Vatsal [CV07], this assumption can be removed by a slight extension of the results of Section 4 below (viz. by "enlarging" the Hida family $\boldsymbol{g}$ in order to include weight-one theta series associated with non-trivial ring class characters of K among its classical specialisations).

Grounding on Kobayashi's announcement, the article [BPS21] by Büyükboduk, Pollack and Sasaki also proves the p-adic Gross–Zagier (p-GZ) formula for f_γ. More precisely, it extends Kobayashi's announced result to non-θ-critical newforms via a p-adic variation argument, using the fact that the quantities in the p-GZ formula (for small slope newforms) are known to vary in Coleman families. When f is the weight-two newform associated with a rational elliptic curve with good ordinary reduction at p and the relevant Heegner point is assumed to be non-trivial, it then deduces Perrin-Riou's conjecture from the p-GZ formulas for f_α and f_β, combined with previous computations of Perrin-Riou (cf. [**PR93**]).

Organisation of the paper. — Section 2 develops the needed facts on Rankin–Selberg convolutions and the Euler system of Beilinson–Flach elements. The reader may skip this section at a first reading and come back to it only when needed. Section 3 proves Theorem B in the special case of a weight-two theta series arising from a p-ordinary canonical Hecke character of a quadratic imaginary field. Section 4 proves Theorem B in the generic case, using Section 3 to handle a rationality question. Section 5 sketches the proof of Theorem B in the exceptional case.

2. Rankin–Selberg convolutions and Beilinson–Flach elements

2.1. Coleman families. — Let $\boldsymbol{f}$ and $\boldsymbol{g}$ be two Coleman families of tame levels $N_{\boldsymbol{f}}$ and $N_{\boldsymbol{g}}$ and tame characters $\chi_{\boldsymbol{f}}$ and $\chi_{\boldsymbol{g}}$, parametrised by connected affinoid discs $U_{\boldsymbol{f}}$ and $U_{\boldsymbol{g}}$ centred at integers $k_o \geqslant 1$ and $l_o \geqslant 1$ in the weight space $\mathcal{W}_L = \mathcal{W} \times_{\mathbf{Q}_p} L$ over a finite extension L of $\mathbf{Q}_p$. Let $\boldsymbol{\xi}$ denote either $\boldsymbol{f}$ or $\boldsymbol{g}$. By definition $\boldsymbol{\xi} = \sum_{n\geqslant 1} a_n(\boldsymbol{\xi})\cdot q^n$ is a formal q-expansion with coefficients in the ring $\mathscr{O}_{\boldsymbol{\xi}} = \mathcal{O}(U_{\boldsymbol{\xi}})$ of analytic functions on $U_{\boldsymbol{\xi}}$, such that the weight-u specialisation $\boldsymbol{\xi}_u = \sum_{n\geqslant 1} a_n(\boldsymbol{\xi})(u)\cdot q^n$ in $L[\![q]\!]$ is the q-expansion of a p-stabilised newform of weight u, level $\Gamma_1(N_{\boldsymbol{\xi}})\cap\Gamma_0(p)$ and character $\chi_{\boldsymbol{\xi}} : (\mathbf{Z}/N_{\boldsymbol{\xi}}\mathbf{Z})^* \longrightarrow L^*$ for all integers u in a cofinite subset $U_{\boldsymbol{\xi}}^{\mathrm{cl}}$ of $U_{\boldsymbol{\xi}} \cap \mathbf{Z}_{\geqslant u_o}$ (with $u_o = k_o, l_o$). If $\boldsymbol{\xi}_u$ is old at p, it is a p-stabilisation of a newform ξ_u of level $\Gamma_1(N_{\boldsymbol{\xi}})$. If $\boldsymbol{\xi}_u$ is new at p, set $\xi_u = \boldsymbol{\xi}_u$.

2.2. Deligne representations. — Let $u \geqslant 2$ be a classical point in $U_{\boldsymbol{\xi}}^{\mathrm{cl}}$. Define the representations $V(\boldsymbol{\xi}_u)$, $V^*(\boldsymbol{\xi}_u)$, $V(\xi_u)$ and $V^*(\xi_u)$ similarly as $V(f)$ and $V^*(f)$ in Section 1.1. For example, the Deligne representation $V^*(\boldsymbol{\xi}_u)$ of $\boldsymbol{\xi}_u$ is the maximal L-submodule of $H^1_{\text{ét,c}}(Y_1(N_{\boldsymbol{\xi}},p)\otimes_{\mathbf{Q}}\bar{\mathbf{Q}},\mathscr{S}_{u-2})\otimes_{\mathbf{Z}_p} L$ on which the Hecke operator T_n acts as multiplication by $a_n(\boldsymbol{\xi}_u) = a_n(\boldsymbol{\xi})(u)$ for each $n \geqslant 1$. Here $Y_1(N_{\boldsymbol{\xi}},p)$ is the affine modular curve of level $\Gamma_1(N_{\boldsymbol{\xi}})\cap\Gamma_0(p)$ over $\mathbf{Q}$ and $\mathscr{S}_{u-2}$ is the $(u-2)$-th symmetric power of the relative first p-adic cohomology $R^1(E \longrightarrow Y(N_{\boldsymbol{\xi}},p))_*\mathbf{Z}_p$ of the universal elliptic curve $E \longrightarrow Y_1(N_{\boldsymbol{\xi}},p)$. Here we assume for simplicity that $N_{\boldsymbol{\xi}}+p$ is at most 5, so that $Y_1(N_{\boldsymbol{\xi}},p)$ represents the appropriate moduli functor (cf. Section 2.1 of [**Kat04**]). Similarly, when working with $Y_1(N_{\boldsymbol{\xi}})$, we implicitly assume $N_{\boldsymbol{\xi}} \geqslant 4$. The interested reader should have no difficulty in extending the constructions and the arguments below to the case of eigenforms of small level.

For $h = \boldsymbol{\xi}_u, \xi_u$, the morphism $\mathscr{L}_{u-2}\otimes\mathscr{S}_{u-2} \longrightarrow \mathbf{Z}_p$ arising from the relative Weil pairing and Poincaré duality yield a perfect duality

$$\langle\cdot,\cdot\rangle_h : V(h)\otimes_L V^*(h) \longrightarrow L.$$

Write pr_1 and pr_p for the degeneracy maps $Y_1(N_{\boldsymbol{\xi}},p) \longrightarrow Y_1(N_{\boldsymbol{\xi}})$ sending an elliptic curve (E,P,C) with $\Gamma_1(N_{\boldsymbol{\xi}})\cap\Gamma_0(p)$-level structure to (E,P) and $(E/C,P+C)$ respectively. If $\boldsymbol{\xi}_u$ is p-old, the map

$$\Pi_{\boldsymbol{\xi}_u *} = \mathrm{pr}_{1*} - \chi_{\boldsymbol{\xi}}(p)\cdot a_p(\boldsymbol{\xi}_u)^{-1}\cdot \mathrm{pr}_{p*} : H^1_{\text{ét}}(Y_1(N_{\boldsymbol{\xi}},p),\mathscr{L}_{u-2}) \longrightarrow H^1_{\text{ét}}(Y_1(N_{\boldsymbol{\xi}}),\mathscr{L}_{u-2})$$

induces an isomorphism between $V(\boldsymbol{\xi}_u)$ and $V(\xi_u)$. Its adjoint

$$\Pi^*_{\boldsymbol{\xi}_u} = \mathrm{pr}_1^* - \chi_{\boldsymbol{\xi}}(p) \cdot a_p(\boldsymbol{\xi}_u)^{-1} \cdot \mathrm{pr}_p^*$$

with respect to the Poincaré dualities $\langle \cdot, \cdot \rangle_{\boldsymbol{\xi}_u}$ and $\langle \cdot, \cdot \rangle_{\xi_u}$ yields an isomorphism between $V^*(\xi_u)$ and $V^*(\boldsymbol{\xi}_u)$. When p divides the conductor of $\boldsymbol{\xi}_u$, so that by definition $\boldsymbol{\xi}_u = \xi_u$, we define $\Pi_{\boldsymbol{\xi}_u *}$ to be the identity on $V(\xi_u)$.

For $\bullet = \mathrm{cris}, \mathrm{st}, \mathrm{dR}$, $\cdot = \emptyset, *$ and $h = \boldsymbol{\xi}_u, \xi_u$ set

$$V^{\cdot}_{\bullet}(h) = H^0(\mathbf{Q}_p, V^{\cdot}(h) \otimes_{\mathbf{Q}_p} B_{\bullet}).$$

Since $V^{\cdot}(h)$ is semistable at p, we often identify $V^{\cdot}_{\mathrm{st}}(h)$ and $V^{\cdot}_{\mathrm{dR}}(h)$, which equips the latter with the action a semistable Frobenius φ. We denote again by

$$\langle \cdot, \cdot \rangle_h : V_{\bullet}(h) \otimes_L V^*_{\bullet}(h) \longrightarrow L$$

the perfect pairing induced by the Poincaré duality in étale cohomology. Assuming that L contains a primitive $N_{\boldsymbol{\xi}}$-th root of unit, the Faltings–Tsuji comparison isomorphism identifies canonically $\mathrm{Fil}^0 V_{\mathrm{dR}}(h)$ (resp., $\mathrm{Fil}^1 V^*_{\mathrm{dR}}(h)$) with the h^w-isotypic (resp., h-isotypic) component of $S_u(\Gamma_1(N_{\boldsymbol{\xi}} p^r), L)$. Here $r = 1$ if $h = \boldsymbol{\xi}_u$, $r = 0$ if $\boldsymbol{\xi}_u$ is p-old and $h = \xi_u$, and $h^w = w_{N_{\boldsymbol{\xi}} p^r}(h)$ is the image of h under the Atkin–Lehner operator $w_{N_{\boldsymbol{\xi}} p^r}$. (We refer to Section 2.5 of [**BSV21b**] and the references therein for more details.) Write ω_{h^w} (resp., ω_h) for the canonical basis of $\mathrm{Fil}^0 V_{\mathrm{dR}}(h)$ (resp., $\mathrm{Fil}^1 V^*_{\mathrm{dR}}(h)$) corresponding to h^w (resp., h) and define η_h in $V^*_{\mathrm{dR}}(h)/\mathrm{Fil}^1$ by the identity

$$\langle \omega_{h^w}, \eta_h \rangle_h = 1.$$

One says that a classical point $u \geqslant 2$ in $U^{\mathrm{cl}}_{\boldsymbol{\xi}}$ is *good* if p does not divide the conductor of $\boldsymbol{\xi}_u$, the p-th Hecke polynomial $X^2 - a_p(\xi_u) \cdot X + \chi_{\boldsymbol{\xi}}(p) p^{u-1}$ of ξ_u has distinct roots and $\boldsymbol{\xi}_u$ is not θ-critical (viz. is not the image of an overconvergent modular form of weight $2-u$ and tame level $N_{\boldsymbol{\xi}}$ under the $(u-1)$-th power of Serre's theta operator $\theta = q\frac{d}{dq}$, cf. [**Bel12**]). The p-adic valuation of $a_p(\boldsymbol{\xi})$ is constant on $U_{\boldsymbol{\xi}}$, equal to the *slope* $\lambda_{\boldsymbol{\xi}}$ in $\mathbf{Q}_{\geqslant 0}$ of $\boldsymbol{\xi}$, and each classical point u in $U^{\mathrm{cl}}_{\boldsymbol{\xi}}$ satisfying $2\lambda_{\boldsymbol{\xi}} < u - 1$ is good. For each good point u and $h = \boldsymbol{\xi}_u, \xi_u$, the de Rham module $V^*_{\mathrm{dR}}(h) = V^*_{\mathrm{cris}}(h)$ is the direct sum of $\mathrm{Fil}^1 V^*_{\mathrm{dR}}(h)$ and the φ-eigenspace $V_{\mathrm{dR}}(h)^{\varphi = \alpha_h}$ with eigenvalue $\alpha_h = a_p(\boldsymbol{\xi}_u)$. In this case one defines

$$\eta^{\alpha}_h \in V^*_{\mathrm{dR}}(h)^{\varphi = \alpha_h}$$

to be the unique element which lifts η_h.

Being semistable, the restriction to $G_{\mathbf{Q}_p}$ of the representations $V^{\cdot}(h)$ are trianguline, for $h = \boldsymbol{\xi}_u, \xi_u$. Precisely, set $\mathscr{R}_L = \mathscr{R} \otimes_{\mathbf{Q}_p} L$, where $\mathscr{R} = \mathbf{B}^{\dagger}_{\mathrm{rig}, \mathbf{Q}_p}$ is the Robba ring over $\mathbf{Q}_p$, equipped with its natural Frobenius endomorphism φ and its natural continuous action of the group $\Gamma = \mathrm{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p)$. According to results of Fontaine, Cherbonnier–Colmez, Kedlaya et alii there is a fully faithful exact functor $\mathbf{D}^{\dagger}_{\mathrm{rig}, L}$ from the category of L-adic representations of $G_{\mathbf{Q}_p}$ to that of (φ, Γ)-modules over $\mathscr{R}_L$, whose essential image is the category of étale (φ, Γ)-modules. (We refer to [**Pot13**, Section 2] and the references quoted there for detailed definitions.) If

$$D(h) = \mathbf{D}^{\dagger}_{\mathrm{rig}, L}(V(h)),$$

then there exists a short exact sequence

$$0 \longrightarrow D(h)^+_\alpha \longrightarrow D(h) \longrightarrow D(h)^-_\alpha \longrightarrow 0 \tag{2}$$

of (φ, Γ)-modules over $\mathscr{R}_L$, with $D(h)^\pm_\alpha$ isomorphic to the (φ, Γ)-modules $\mathscr{R}_L(\delta^\pm_{h,\alpha})$ associated with the characters $\delta^\pm_{h,\alpha} : \mathbf{Q}_p^* \longrightarrow L^*$ defined by the formulae

$$\delta^+_{h,\alpha}(p^r t) = \chi_{\boldsymbol{\xi}}(p)^{-r} \cdot \alpha_h^r \cdot t^{u-1} \quad \text{and} \quad \delta^-_{h,\alpha}(p^r t) = \alpha_h^{-r}$$

for each r in $\mathbf{Z}$ and t in $\mathbf{Z}_p^*$. The (φ, Γ)-module $D(h)^+_\alpha$ is étale precisely if $\lambda_{\boldsymbol{\xi}} = 0$, i.e. if $\alpha_h = a_p(\boldsymbol{\xi}_u)$ is a p-adic unit. Similarly

$$D^*(h) = \mathbf{D}^\dagger_{\mathrm{rig},L}(V^*(h))$$

admits a triangulation

$$0 \longrightarrow D^*(h)^+_\alpha \longrightarrow D^*(h) \longrightarrow D^*(h)^-_\alpha \longrightarrow 0,$$

with $D^*(h)^\pm_\alpha$ isomorphic to the (φ, Γ)-modules $\mathscr{R}_L(\gamma^\pm_{h,\alpha})$ associated with the characters $\gamma^\pm_{h,\alpha} : \mathbf{Q}_p^* \longrightarrow L^*$ defined for each r in $\mathbf{Z}$ and t in $\mathbf{Z}_p^*$ by the formulae

$$\gamma^+_{h,\alpha}(p^r t) = \alpha_h^r \quad \text{and} \quad \gamma^-_{h,\alpha}(p^r t) = \chi_{\boldsymbol{\xi}}(p)^r \cdot \alpha_h^{-r} \cdot t^{1-u}.$$

The perfect Poincaré duality $\langle \cdot, \cdot \rangle_h$ induces a perfect duality

$$\langle \cdot, \cdot \rangle_h : D(h) \otimes_{\mathscr{R}_L} D^*(h) \longrightarrow \mathscr{R}_L,$$

which entails perfect dualities between $D(h)^\pm_\alpha$ and $D^*(h)^\mp_\alpha$.

2.3. Big Galois representations. — Let $\mathcal{U}_{\boldsymbol{\xi}} \hookrightarrow \mathcal{W}_L$ be a connected open disc centred at u_o. Assume that $\mathcal{U}_{\boldsymbol{\xi}}$ is contained in an affinoid disc in $\mathcal{W}_L$, and that $U_{\boldsymbol{\xi}}$ is contained in $\mathcal{U}_{\boldsymbol{\xi}}$. Denote by $\Lambda_{\boldsymbol{\xi}}$ the ring of bounded analytic functions on $\mathcal{U}_{\boldsymbol{\xi}}$. Set $\Gamma_{\boldsymbol{\xi}} = \Gamma_1(N_{\boldsymbol{\xi}}) \cap \Gamma_0(p)$ and let

$$\boldsymbol{\mathcal{L}}_{\boldsymbol{\xi}} = \mathcal{D}'_{\mathcal{U}_{\boldsymbol{\xi}},m}[1/p]$$

be the $\Lambda_{\boldsymbol{\xi}}[\Gamma_{\boldsymbol{\xi}}]$-module of locally m-analytic distributions on $\mathsf{T}' = p\mathbf{Z}_p \times \mathbf{Z}_p^*$ associated in [**BSV21b**, Section 4.1] with $\mathcal{U}_{\boldsymbol{\xi}}$ and a fixed sufficiently large integer $m = m(\mathcal{U}_{\boldsymbol{\xi}})$. (See also [**GS93**] and [**AIS15**], where slight variants of these distributions spaces were introduced.) The cohomology group $H^1(\Gamma_{\boldsymbol{\xi}}, \boldsymbol{\mathcal{L}}_{\boldsymbol{\xi}})$ and its compactly supported counterpart $H^1_{\mathrm{c}}(\Gamma, \boldsymbol{\mathcal{L}}_{\boldsymbol{\xi}})$ (viz. the space of $\Gamma_{\boldsymbol{\xi}}$-invariant $\boldsymbol{\mathcal{L}}_{\boldsymbol{\xi}}$-valued modular symbols) carry natural commuting actions of the Galois group $G_{\mathbf{Q}}$ and of a Hecke algebra generated by the dual Hecke operators T'_n for $n \geqslant 1$ (cf. loco citato). Denote by $H^1_{\mathrm{par}}(\Gamma_{\boldsymbol{\xi}}, \boldsymbol{\mathcal{L}}_{\boldsymbol{\xi}})$ the image of $H^1_{\mathrm{c}}(\Gamma_{\boldsymbol{\xi}}, \boldsymbol{\mathcal{L}}_{\boldsymbol{\xi}})$ in $H^1(\Gamma_{\boldsymbol{\xi}}, \boldsymbol{\mathcal{L}}_{\boldsymbol{\xi}})$, and define

$$H^1_{\mathrm{par}}(\Gamma_{\boldsymbol{\xi}}, \boldsymbol{\mathcal{L}}_{\boldsymbol{\xi}})(1) \otimes_{\Lambda_{\boldsymbol{\xi}}} \mathscr{O}_{\boldsymbol{\xi}} \twoheadrightarrow V(\boldsymbol{\xi})$$

to be the maximal $\mathscr{O}_{\boldsymbol{\xi}}$-quotient on which the dual Hecke operator T'_n acts as multiplication by $a_n(\boldsymbol{\xi})$ for each positive integer n. Dually define

$$V^*(\boldsymbol{\xi}) \hookrightarrow H^1_{\mathrm{par}}(\Gamma_{\boldsymbol{\xi}}, \boldsymbol{\mathcal{S}}_{\boldsymbol{\xi}})(-\boldsymbol{\kappa}_{\boldsymbol{\xi}}) \otimes_{\Lambda_{\boldsymbol{\xi}}} \mathscr{O}_{\boldsymbol{\xi}}$$

to be the maximal $\mathscr{O}_{\boldsymbol{\xi}}$-submodule on which the Hecke operator T_n acts as multiplication by $a_n(\boldsymbol{\xi})$ for each $n \geqslant 1$, where $\boldsymbol{\mathcal{S}}_{\boldsymbol{\xi}} = \mathcal{D}_{\mathcal{U}_{\boldsymbol{\xi}},m}[1/p]$ is the $\Lambda_{\boldsymbol{\xi}}[\Gamma_{\boldsymbol{\xi}}]$-module of locally m-analytic distributions on $\mathsf{T} = \mathbf{Z}_p^* \times \mathbf{Z}_p$ introduced in [**BSV21b**, Section 4.1], and where $\boldsymbol{\kappa}_{\boldsymbol{\xi}} : G_{\mathbf{Q}} \longrightarrow \Lambda_{\boldsymbol{\xi}}^*$ is the composition of the p-adic cyclotomic character and the universal character $\mathbf{Z}_p^* \longrightarrow \Lambda_{\boldsymbol{\xi}}^*$. In the rest of this section we make the following crucial assumption. One says that a normalised eigenform $\xi = \sum_{n \geqslant 0} a_n(\xi) q^n$ of weight u, level $\Gamma_1(N_{\boldsymbol{\xi}})$ and character χ_ξ is *p-regular* if its p-th Hecke polynomial $T^2 - a_p(\xi)T + p^{u-1}\chi_\xi(p)$ has distinct roots. One says that ξ has *p-split real multiplication* if it is the weight-one theta series attached to a ray class character of a real quadratic field in which p splits.

Assumption 2.1. — *Let $\boldsymbol{\xi}$ denote either $\boldsymbol{f}$ or $\boldsymbol{g}$, and let $u_o \geqslant 1$ be the centre of the affinoid disc $U_{\boldsymbol{\xi}}$. Then one of the following statements $\mathbf{E}_1$–$\mathbf{E}_3$ is satisfied.*

- $\mathbf{E}_1$. *$u_o \geqslant 2$ and $\boldsymbol{\xi}_{u_o}$ is a non-critical p-regular eigenform.*
- $\mathbf{E}_2$. *$u_o = 1$ and $\boldsymbol{\xi}_1$ is a p-stabilisation of a classical, p-regular cuspidal weight one newform of level $N_{\boldsymbol{\xi}}$ without p-split real multiplication.*
- $\mathbf{E}_3$. *$u_o = 1$ and $\boldsymbol{\xi}_1$ is the p-stabilisation of a p-irregular weight one Eisenstein series of conductor $N_{\boldsymbol{\xi}}$.*

Assumption 2.1 guarantees that the eigenform $\boldsymbol{\xi}_{u_o}$ is an étale point of the cuspidal part $\kappa^{\mathrm{cusp}} : \mathscr{C}^{\mathrm{cusp}}(N_{\boldsymbol{\xi}}) \longrightarrow \mathcal{W}_L$ of the Coleman–Mazur–Buzzard p-adic eigencurve $\kappa : \mathscr{C}(N_{\boldsymbol{\xi}}) \longrightarrow \mathcal{W}_L$ of tame level $N_{\boldsymbol{\xi}}$. More precisely, in case $\mathbf{E}_1$ the work of Hida and Coleman imply that κ is étale at $\boldsymbol{\xi}_{u_o}$ (cf. Proposition 2.11 of [**Bel12**]). In case $\mathbf{E}_2$ the main result of [**BD16**] proves that κ is étale at $\boldsymbol{\xi}_1$. Finally in case $\mathbf{E}_3$ Theorem A of [**BDP21**] proves that the map κ^{cusp} is étale at the cuspidal-overconvergent p-stabilised Eisenstein series $\boldsymbol{\xi}_1$.

Let $V^{\cdot}(\boldsymbol{\xi})$ denote either $V(\boldsymbol{\xi})$ or $V^*(\boldsymbol{\xi})$. The étaleness of κ^{cusp} at $\boldsymbol{\xi}_{u_o}$ implies that $V^{\cdot}(\boldsymbol{\xi})$ is a free $\mathscr{O}_{\boldsymbol{\xi}}$-module of rank two (cf. Sections 4.3 and 5 of [**BSV21b**]). For each good point u in $U_{\boldsymbol{\xi}}^{\mathrm{cl}}$ there are canonical specialisation isomorphisms

$$\rho_u : V^{\cdot}(\boldsymbol{\xi}) \otimes_u L \cong V^{\cdot}(\boldsymbol{\xi}_u),$$

where $\cdot \otimes_u L$ denotes base change along evaluation at u on $\mathscr{O}_{\boldsymbol{\xi}}$. We refer to Section 5 of [**BSV21b**] for the definition of ρ_u and to [**BSV21b**, Proposition 4.2] and [**PS13**, Theorems 1.1 and 1.2] for the proof that they are isomorphisms at good points. There exists a perfect $G_{\mathbf{Q}}$-equivariant pairing (cf. [**BSV21b**, Section 5])

$$\langle \cdot, \cdot \rangle_{\boldsymbol{\xi}} : V(\boldsymbol{\xi}) \otimes_{\mathscr{O}_{\boldsymbol{\xi}}} V^*(\boldsymbol{\xi}) \longrightarrow \mathscr{O}_{\boldsymbol{\xi}},$$

compatible with the dualities $\langle \cdot, \cdot \rangle_{\boldsymbol{\xi}_u}$ under the specialisation maps ρ_u at good points.

2.3.1. *Weight-one specialisations.* — Assume in this subsection $u_o = 1$, so that either condition $\mathbf{E}_2$ or condition $\mathbf{E}_3$ in Assumption 2.1 is satisfied. Set

$$V^*(\boldsymbol{\xi}_1) = V^*(\boldsymbol{\xi}) \otimes_1 L \quad \text{and} \quad V(\boldsymbol{\xi}_1) = V(\boldsymbol{\xi}) \otimes_1 L,$$

where $\cdot \otimes_1 L$ denotes the base change along evaluation at 1 on $\mathscr{O}_{\boldsymbol{\xi}}$, and denote by $\rho_1 : V^{\cdot}(\boldsymbol{\xi}) \longrightarrow V^{\cdot}(\boldsymbol{\xi}_1)$ the projection (also called *specialisation*) map. The weight-one specialisation of the pairing $\langle \cdot, \cdot \rangle_{\boldsymbol{\xi}}$ yields a canonical perfect duality

$$\langle \cdot, \cdot \rangle_{\boldsymbol{\xi}_1} : V(\boldsymbol{\xi}_1) \otimes_L V^*(\boldsymbol{\xi}_1) \longrightarrow L. \tag{3}$$

The following proposition will be crucial for the proof of the main result of this paper.

Proposition 2.2. — *$V^*(\boldsymbol{\xi}_1)$ and $V(\boldsymbol{\xi}_1)$ afford the Deligne–Serre Artin representation of $G_{\mathbf{Q}}$ associated with $\boldsymbol{\xi}_1$ and its dual respectively.*

Proof. — It is sufficient to prove the statement for $V(\boldsymbol{\xi}_1)$ (cf. Equation (3)). According to the results recalled above, for each prime ℓ not dividing $pN_{\boldsymbol{\xi}}$, a Frobenius at ℓ in $G_{\mathbf{Q}}$ acts on $V(\boldsymbol{\xi}_1)$ with trace $a_\ell(\boldsymbol{\xi}_1)$. It follows that the semi-simplification $V(\boldsymbol{\xi}_1)^{ss}$ of $V(\boldsymbol{\xi}_1)$ is isomorphic to the dual of the Deligne–Serre representation of $\boldsymbol{\xi}_1$. We have to show that $V(\boldsymbol{\xi}_1) = V(\boldsymbol{\xi}_1)^{ss}$ is semi-simple.

If condition $\mathbf{E}_2$ is satisfied, then $\boldsymbol{\xi}_1$ is a cuspidal eigenform, hence $V(\boldsymbol{\xi}_1)^{ss}$ is irreducible. The equality $V(\boldsymbol{\xi}_1) = V(\boldsymbol{\xi}_1)^{ss}$ follows in this case.

Assume that condition $\mathbf{E}_3$ is satisfied, so that $V(\boldsymbol{\xi}_1)^{ss} = L \oplus L(\chi)$ is the direct sum of the trivial representation L of $G_{\mathbf{Q}}$ and its twist $L(\chi)$ by an odd Dirichlet character of conductor coprime to $pN_{\boldsymbol{\xi}}$ such that $\chi(p) = 1$. In this case $V(\boldsymbol{\xi}_1)$ represents an element of $H^1(\mathbf{Q}, L(\psi))$ with $\psi = \chi$ or $\psi = \chi^{-1}$, and we have to show that this element is trivial. Since ($H^1(\mathbf{Q}, L(\psi))$ is 1-dimensional and) the restriction at p map $H^1(\mathbf{Q}, L(\psi)) \longrightarrow H^1(\mathbf{Q}_p, L(\psi))$ is injective (cf. Sections 3.1 and 3.2 of [BD16]), it is sufficient to prove that $G_{\mathbf{Q}_p}$ acts trivially on $V(\boldsymbol{\xi}_1)$, namely

$$V(\boldsymbol{\xi}_1) \simeq L^2 \quad \text{as } G_{\mathbf{Q}_p}\text{-modules.} \tag{4}$$

We prove this statement using the results of [Oht00] and [BDP21].

Set $\boldsymbol{V} = H^1(\Gamma_{\boldsymbol{\xi}}, \boldsymbol{\mathcal{L}}_{\boldsymbol{\xi}})^{\leqslant 0}(1) \otimes_{\Lambda_{\boldsymbol{\xi}}} \mathscr{O}_{\boldsymbol{\xi}}$ and $\boldsymbol{V}_{\mathrm{par}} = H^1_{\mathrm{par}}(\Gamma_{\boldsymbol{\xi}}, \boldsymbol{\mathcal{L}}_{\boldsymbol{\xi}})^{\leqslant 0}(1) \otimes_{\Lambda_{\boldsymbol{\xi}}} \mathscr{O}_{\boldsymbol{\xi}}$, where $\leqslant 0$ refers to the slope zero part for the action of the dual Hecke operator U'_p (cf. Section 4.1.4 of [BSV21b]). Denote by $\boldsymbol{V}^+$ the maximal submodule of $\boldsymbol{V}$ on which the inertia subgroup of $G_{\mathbf{Q}_p}$ acts via the character $\chi_{\mathrm{cyc}}^{\boldsymbol{u}-1} : G_{\mathbf{Q}_p} \longrightarrow \mathscr{O}_{\boldsymbol{\xi}}^*$ whose composition with evaluation at u in $U_{\boldsymbol{\xi}} \cap \mathbf{Z}$ is the u-th power of the p-adic cyclotomic character. Define similarly $\boldsymbol{V}^+_{\mathrm{par}}$ and set $\boldsymbol{V}^- = \boldsymbol{V}/\boldsymbol{V}^+$ and $\boldsymbol{V}^-_{\mathrm{par}} = \boldsymbol{V}_{\mathrm{par}}/\boldsymbol{V}^+_{\mathrm{par}}$. The article [Oht00] (together with Section 4.3 of [BSV21b]) proves the following facts.

- $\boldsymbol{O}_1$. The modules $\boldsymbol{V}^\pm$ and $\boldsymbol{V}^\pm_{\mathrm{par}}$ are free of finite rank over $\mathscr{O}_{\boldsymbol{\xi}}$, and $\boldsymbol{V}^+ = \boldsymbol{V}^+_{\mathrm{par}}$.
- $\boldsymbol{O}_2$. The Galois group $G_{\mathbf{Q}_p}$ acts on $\boldsymbol{V}^-$ via the unramified character sending an arithmetic Frobenius to the dual Hecke operator U'_p.
- $\boldsymbol{O}_3$. Let $M = M^{\mathrm{ord}}_{U_{\boldsymbol{\xi}}}(N_{\boldsymbol{\xi}})$ be the module of $\mathscr{O}_{\boldsymbol{\xi}}$-adic Hida families of tame level $N_{\boldsymbol{\xi}}$ and let $S = S^{\mathrm{ord}}_{U_{\boldsymbol{\xi}}}(N_{\boldsymbol{\xi}})$ be its cuspidal subspace (cf. Section 5 of [BSV21b]). There are canonical isomorphisms of $\mathscr{O}_{\boldsymbol{\xi}}$-modules
$$\big(\boldsymbol{V}^-_{\mathrm{par}} \hat{\otimes}_{\mathbf{Q}_p} \hat{\mathbf{Q}}_p^{\mathrm{nr}}\big)^{G_{\mathbf{Q}_p}} \simeq S \quad \text{and} \quad \big(\boldsymbol{V}^- \hat{\otimes}_{\mathbf{Q}_p} \hat{\mathbf{Q}}_p^{\mathrm{nr}}\big)^{G_{\mathbf{Q}_p}} \simeq M$$
(compatible with the inclusions $S \hookrightarrow M$ and $\boldsymbol{V}_{\mathrm{par}} \hookrightarrow \boldsymbol{V}$ and) intertwining the actions of the n-th Hecke operator T_n on the left hand sides with those of the dual Hecke operator T'_n on the right hand sides, for each integer $n \geqslant 1$.

Define $V(\boldsymbol{\xi})^{\cdot}$ (resp., $\tilde{V}(\boldsymbol{\xi})^{\cdot}$) to be the maximal quotient of $\boldsymbol{V}^{\cdot}_{\mathrm{par}}$ (resp., $\boldsymbol{V}^{\cdot}$) on which the dual Hecke operator U'_n acts as multiplication by $a_n(\boldsymbol{\xi})$, for each positive integer n. The étaleness of κ^{cusp} at $\boldsymbol{\xi}_1$ (cf. the discussion following Assumption 2.1), Property $\boldsymbol{O}_2$ and the identity $\chi(p) = 1$ yield isomorphisms of $\mathscr{O}_{\boldsymbol{\xi}}[G_{\mathbf{Q}_p}]$-modules

$$V(\boldsymbol{\xi})^+ \simeq \mathscr{O}_{\boldsymbol{\xi}}(\chi_{\mathrm{cyc}}^{\boldsymbol{u}-1} \cdot \breve{a}_p(\boldsymbol{\xi})^{-1}) \quad \text{and} \quad V(\boldsymbol{\xi})^- \simeq \mathscr{O}_{\boldsymbol{\xi}}(\breve{a}_p(\boldsymbol{\xi})), \tag{5}$$

where $\check{a}_p(\boldsymbol{\xi}) : G_{\mathbf{Q}_p} \longrightarrow \mathscr{O}_{\boldsymbol{\xi}}^*$ is the unramified character sending an arithmetic Frobenius to $a_p(\boldsymbol{\xi})$, and $\chi_{\mathrm{cyc}}^{\boldsymbol{u}-1} : G_{\mathbf{Q}_p} \longrightarrow \mathscr{O}_{\boldsymbol{\xi}}^*$ satisfies $\chi_{\mathrm{cyc}}^{\boldsymbol{u}-1}(\sigma)(u) = \chi_{\mathrm{cyc}}(\sigma)^{u-1}$ for each σ in $G_{\mathbf{Q}_p}$ and each integer u in $U_{\boldsymbol{\xi}}$. One has the following exact and commutative diagram of $\mathscr{O}_{\boldsymbol{\xi}}[G_{\mathbf{Q}_p}]$-modules, where $i_{\mathrm{par}}^{\cdot} : V(\boldsymbol{\xi})^{\cdot} \longrightarrow \tilde{V}(\boldsymbol{\xi})^{\cdot}$ (for $\cdot$ in $\{\emptyset, +, -\}$) are the maps induced on the $\boldsymbol{\xi}$-isotypic quotients by the inclusion of $\boldsymbol{V}_{\mathrm{par}}$ into $\boldsymbol{V}$.

(6)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & V(\boldsymbol{\xi})^+ & \longrightarrow & V(\boldsymbol{\xi}) & \longrightarrow & V(\boldsymbol{\xi})^- & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle i_{\mathrm{par}}} & & \downarrow{\scriptstyle i_{\mathrm{par}}^-} & & \\
0 & \longrightarrow & \tilde{V}(\boldsymbol{\xi})^+ & \longrightarrow & \tilde{V}(\boldsymbol{\xi}) & \longrightarrow & \tilde{V}(\boldsymbol{\xi})^- & \longrightarrow & 0
\end{array}$$

Indeed, the exactness of the first row follows from the freeness of $V(\boldsymbol{\xi})^-$, and Property $\boldsymbol{O}_1$ gives the equality $V(\boldsymbol{\xi})^+ = \tilde{V}(\boldsymbol{\xi})^+$. Since $\boldsymbol{\xi}$ is cuspidal, for each u in $U \cap \mathbf{Z}_{\geqslant 3}$ the base change of i_{par} along evaluation at u is an isomorphism, hence $\mathrm{rank}_{\mathcal{O}_{\boldsymbol{\xi}}} \tilde{V}(\boldsymbol{\xi}) = 2$ and $\mathrm{rank}_{\mathcal{O}_{\boldsymbol{\xi}}} \tilde{V}(\boldsymbol{\xi})^{\pm} = 1$. Because $\tilde{V}(\boldsymbol{\xi})^+$ (resp., $V(\boldsymbol{\xi})$) is free over $\mathcal{O}_{\boldsymbol{\xi}}$, one deduces that the second row is exact (resp., i_{par} and i_{par}^- are injective). In particular the projection $\tilde{V}(\boldsymbol{\xi}) \longrightarrow \tilde{V}(\boldsymbol{\xi})^-$ induces an isomorphism of $\mathcal{O}_{\boldsymbol{\xi}}[G_{\mathbf{Q}_p}]$-modules

$$\tilde{V}(\boldsymbol{\xi})/V(\boldsymbol{\xi}) \simeq \tilde{V}(\boldsymbol{\xi})^-/V(\boldsymbol{\xi})^-,$$

where we identify $V(\boldsymbol{\xi})^{\cdot}$ with a submodule of $\tilde{V}(\boldsymbol{\xi})^{\cdot}$ under the injective map $i_{\mathrm{par}}^{\cdot}$.

Set $V(\boldsymbol{\xi}_1)^{\cdot} = V(\boldsymbol{\xi})^{\cdot} \otimes_1 L$ and $\tilde{V}(\boldsymbol{\xi}_1)^{\cdot} = \tilde{V}(\boldsymbol{\xi})^{\cdot} \otimes_1 L$. Applying $\cdot \otimes_1 L$ to Diagram (6) yields the following exact and commutative diagram of $L[G_{\mathbf{Q}_p}]$-module, where $\mathfrak{m}_1$ is the ideal of functions in $\mathscr{O}_{\boldsymbol{\xi}}$ which vanish at $\boldsymbol{u} = 1$.

(7)
$$\begin{array}{ccccccccc}
 & & & & \tilde{V}(\boldsymbol{\xi})^-/V(\boldsymbol{\xi})^-[\mathfrak{m}_1] & & & & \\
 & & & & \downarrow{\scriptstyle \delta} & & & & \\
0 & \longrightarrow & V(\boldsymbol{\xi}_1)^+ & \longrightarrow & V(\boldsymbol{\xi}_1) & \longrightarrow & V(\boldsymbol{\xi}_1)^- & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle i_{\mathrm{par}} \otimes_1 L} & & \downarrow{\scriptstyle i_{\mathrm{par}}^- \otimes_1 L} & & \\
 & & \tilde{V}(\boldsymbol{\xi}_1)^+ & \longrightarrow & \tilde{V}(\boldsymbol{\xi}_1) & \longrightarrow & \tilde{V}(\boldsymbol{\xi}_1)^- & \longrightarrow & 0
\end{array}$$

We claim that the map i_{par}^- takes values in $\mathfrak{m}_1 \cdot \tilde{V}(\boldsymbol{\xi})^-$, i.e.

(8)
$$i_{\mathrm{par}}^- \otimes_1 L = 0.$$

Assuming the claim, we conclude the proof as follows. As $a_p(\boldsymbol{\xi}) - 1 = a_p(\boldsymbol{\xi}) - a_p(\boldsymbol{\xi}_1)$ belongs to $\mathfrak{m}_1$, Property $\boldsymbol{O}_2$ and Equation (5) imply that $G_{\mathbf{Q}_p}$ acts trivially on $V(\boldsymbol{\xi}_1)^+$, $V(\boldsymbol{\xi}_1)^-$ and $\tilde{V}(\boldsymbol{\xi})^-/V(\boldsymbol{\xi})^-[\mathfrak{m}_1]$. Fix an L-basis $\{v^+, v^-\}$ of $V(\boldsymbol{\xi}_1)$ with v^+ in the image of $V(\boldsymbol{\xi}_1)^+ \hookrightarrow V(\boldsymbol{\xi}_1)$. By Equation (8) and Diagram (7) $v^- - q \cdot v^+$ belongs to the image of δ for some q in L, hence $G_{\mathbf{Q}_p}$ acts trivially on v^-, thus proving (4).

We now prove the claim (8). Define $S(\boldsymbol{\xi})$ and $M(\boldsymbol{\xi})$ to be the maximal quotients of S and M respectively on which the n-th Hecke operator acts as multiplication by $a_n(\boldsymbol{\xi})$, for each integer $n \geqslant 1$. According to Property $\boldsymbol{O}_3$, it is sufficient to prove that the image of the map $S(\boldsymbol{\xi}) \longrightarrow M(\boldsymbol{\xi})$ (induced by the inclusion $S \hookrightarrow M$) takes values in $\mathfrak{m}_1 \cdot M(\boldsymbol{\xi})$. Shrinking $U_{\boldsymbol{\xi}}$ if necessary, Theorem A.(i) of [BDP21] shows that $S(\boldsymbol{\xi}) = \mathcal{O}_{\boldsymbol{\xi}} \cdot \boldsymbol{\xi}$ is the free rank-one $\mathcal{O}_{\boldsymbol{\xi}}$-module generated by $\boldsymbol{\xi}$. We are then reduced to prove that the image of $\boldsymbol{\xi}$ under the projection $[\cdot] : M \longrightarrow M(\boldsymbol{\xi})$ belongs to $\mathfrak{m}_1 \cdot M(\boldsymbol{\xi})$:

$$[\boldsymbol{\xi}] \text{ belongs to } \mathfrak{m}_1 \cdot M(\boldsymbol{\xi}). \tag{9}$$

Let $\boldsymbol{E}$ be the normalised Eisenstein eigenfamily in M specialising to $\boldsymbol{\xi}_1$ in weight one and having T_ℓ-eigenvalues $1 + \chi(\ell) \cdot \ell^{\boldsymbol{u}-1}$ for each prime ℓ different from p. Define

$$\boldsymbol{e} = \frac{\boldsymbol{\xi} - \boldsymbol{E}}{\pi},$$

where π is a fixed generator of $\mathfrak{m}_1$. One has $(U_p - a_p(\boldsymbol{\xi})) \cdot \boldsymbol{e} = a'_p(\boldsymbol{\xi}) \cdot \boldsymbol{E}$ with $\pi \cdot a'_p(\boldsymbol{\xi}) = a_p(\boldsymbol{\xi}) - 1$. Propositions 2.6 and 5.7 of [BDP21] prove that $a'_p(\boldsymbol{g})$ does not vanish at $\boldsymbol{u} = 1$. Shrinking the disc $U_{\boldsymbol{\xi}}$ further if necessary, we can then assume that $a'_p(\boldsymbol{\xi})$ is a unit in $\mathcal{O}_{\boldsymbol{\xi}}$, hence $[\boldsymbol{E}] = 0$ and $[\boldsymbol{\xi}] = \pi \cdot [\boldsymbol{e}]$ in $M(\boldsymbol{\xi})$. This proves the claim (9) and concludes the proof of the proposition. □

2.3.2. *Triangulations.* — Set $\mathscr{R}_{\boldsymbol{\xi}} = \mathscr{R} \hat{\otimes}_{\mathbf{Q}_p} \mathscr{O}_{\boldsymbol{\xi}}$. A construction of Berger and Colmez [BC08] associates with the restriction of $V^{\cdot}(\boldsymbol{\xi})$ to $G_{\mathbf{Q}_p}$ a (φ, Γ)-module

$$D^{\cdot}(\boldsymbol{\xi}) = \mathbf{D}^{\dagger}_{\mathrm{rig}, \mathscr{O}_{\boldsymbol{\xi}}}(V^{\cdot}(\boldsymbol{\xi}))$$

over $\mathscr{R}_{\boldsymbol{\xi}}$, together with specialisation isomorphisms

$$\rho_u : D^{\cdot}(\boldsymbol{\xi}) \otimes_u L \cong D^{\cdot}(\boldsymbol{\xi}_u) \tag{10}$$

for each good point u in $U^{\mathrm{cl}}_{\boldsymbol{\xi}}$. (See [Pot13, Theorem 2.2] and the references therein for the definition of the functor $\mathbf{D}^{\dagger}_{\mathrm{rig}, \cdot}$ with $\cdot$ an affinoid L-algebra.)

There are exact sequences

$$0 \longrightarrow D^{\cdot}(\boldsymbol{\xi})^{+} \longrightarrow D^{\cdot}(\boldsymbol{\xi}) \longrightarrow D^{\cdot}(\boldsymbol{\xi})^{-} \longrightarrow 0 \tag{11}$$

of (φ, Γ)-modules over $\mathscr{R}_{\boldsymbol{\xi}}$, which recast the triangulations on $D^{\cdot}(\boldsymbol{\xi}_u)$ described in Section 2.2 after base change along evaluation at a good point u in $U^{\mathrm{cl}}_{\boldsymbol{\xi}}$. If condition $\mathbf{E}_1$ (cf. Assumption 2.1) is satisfied, this follows from the results of Kisin and Liu [Kis03, Liu15]. If either condition $\mathbf{E}_2$ or condition $\mathbf{E}_3$ is satisfied, then $\boldsymbol{\xi}$ is ordinary and the restriction of $V(\boldsymbol{\xi})$ to $G_{\mathbf{Q}_p}$ is nearly-ordinary: there exists a short exact sequence of $\mathscr{O}_{\boldsymbol{\xi}}[G_{\mathbf{Q}_p}]$-modules $\Delta_{\boldsymbol{\xi}} : V(\boldsymbol{\xi})^{+} \hookrightarrow V(\boldsymbol{\xi}) \twoheadrightarrow V(\boldsymbol{\xi})^{-}$, where $V(\boldsymbol{\xi})^{+}$ is the submodule on which $G_{\mathbf{Q}_p}$ acts via the character $\chi_{\boldsymbol{\xi}} \cdot \chi_{\mathrm{cyc}}^{\boldsymbol{u}-1} \cdot \breve{a}_p(\boldsymbol{\xi})^{-1} : G_{\mathbf{Q}_p} \longrightarrow \mathscr{O}^{*}_{\boldsymbol{\xi}}$ (see the proof of Proposition 2.2 for the notation), and $V(\boldsymbol{\xi})^{-} = V(\boldsymbol{\xi})/V(\boldsymbol{\xi})^{+}$ is unramified. The étaleness of the cuspidal eigencurve $\mathscr{C}^{\mathrm{cusp}}(N_{\boldsymbol{\xi}}) \longrightarrow \mathscr{W}_L$ at $\boldsymbol{\xi}_1$ (cf. the discussion following Assumption 2.1) guarantees that the $G_{\mathbf{Q}_p}$-modules $V(\boldsymbol{\xi})^{\pm}$ are free of rank one over $\mathscr{O}_{\boldsymbol{\xi}}$. The sought for triangulation (11) is obtained by applying the Berger–Colmez functor $\mathbf{D}^{\dagger}_{\mathrm{rig}, \mathscr{O}_{\boldsymbol{\xi}}}$ to the short exact sequence $\Delta_{\boldsymbol{\xi}}$.

The duality $\langle \cdot, \cdot \rangle_{\boldsymbol{\xi}}$ between $V(\boldsymbol{\xi})$ and $V^{*}(\boldsymbol{\xi})$ induces a perfect duality

$$\langle \cdot, \cdot \rangle_{\boldsymbol{\xi}} : D(\boldsymbol{\xi}) \otimes_{\mathscr{R}_{\boldsymbol{\xi}}} D^{*}(\boldsymbol{\xi}) \longrightarrow \mathscr{R}_{\boldsymbol{\xi}}$$

on the associated (φ,Γ)-modules, which in turn induces perfect dualities (denoted again by $\langle\cdot,\cdot\rangle_{\boldsymbol{\xi}}$) between $D(\boldsymbol{\xi})^{\pm}$ and $D^*(\boldsymbol{\xi})^{\mp}$. The base change of $\langle\cdot,\cdot\rangle_{\boldsymbol{\xi}}$ along evaluation at a good point u corresponds to the pairing $\langle\cdot,\cdot\rangle_{\boldsymbol{\xi}_u}$ defined in Section 2.2 via the specialisation isomorphism ρ_u.

2.3.3. *Overconvergent Eichler–Shimura isomorphisms.* — Let $\mu_{\boldsymbol{\xi}} : \mathbf{Z}_p^* \longrightarrow \mathscr{O}_{\boldsymbol{\xi}}^*$ be the character sending t in $\mathbf{Z}_p^*$ to the analytic function $\mu_{\boldsymbol{\xi}}(t)$ which on x in $U_{\boldsymbol{\xi}}$ takes the value $x(t)\cdot t^{-1}$. Then the rank-one (φ,Γ)-modules $D^*(\boldsymbol{\xi})^+$ and $D^*(\boldsymbol{\xi})^-(\mu_{\boldsymbol{\xi}})$ are unramified, and the $\mathscr{O}_{\boldsymbol{\xi}}$-modules

$$\mathrm{Fil}^1 V_{\mathrm{dR}}^*(\boldsymbol{\xi}) = \big(D^*(\boldsymbol{\xi})^-(\mu_{\boldsymbol{\xi}})\big)^{\Gamma=1} \quad \text{and} \quad \mathrm{gr}_{\mathrm{dR}}^*(\boldsymbol{\xi}) = \big(D^*(\boldsymbol{\xi})^+\big)^{\Gamma=1}$$

are free of rank one. For each good point u in $U_{\boldsymbol{\xi}}^{\mathrm{cl}}$, the specialisation map ρ_u induces natural isomorphisms of L-vector spaces

$$\mathrm{Fil}^1 V_{\mathrm{dR}}^*(\boldsymbol{\xi})\otimes_u L \cong \mathrm{Fil}^1 V_{\mathrm{dR}}^*(\boldsymbol{\xi}_u) \quad \text{and} \quad \mathrm{gr}_{\mathrm{dR}}^*(\boldsymbol{\xi})\otimes_u L \cong V_{\mathrm{dR}}^*(\boldsymbol{\xi}_u)/\mathrm{Fil}^1,$$

thus justifying the notation. The overconvergent Eichler–Shimura isomorphisms mentioned in the title of this subsection yield canonical generators

$$\omega_{\boldsymbol{\xi}} \in \mathrm{Fil}^1 V_{\mathrm{dR}}^*(\boldsymbol{\xi}) \quad \text{and} \quad \eta_{\boldsymbol{\xi}} \in \mathrm{gr}_{\mathrm{dR}}^*(\boldsymbol{\xi}),$$

which specialise to $\omega_{\boldsymbol{\xi}_u}$ and $\eta_{\boldsymbol{\xi}_u}$ respectively at each good classical point u in $U_{\boldsymbol{\xi}}^{\mathrm{cl}}$. When condition $\mathbf{E}_1$ in Assumption 2.1 is satisfied, this follows from the main result of [**AIS15**] (cf. [**LZ16**, Section 6.4]). When either condition $\mathbf{E}_2$ or condition $\mathbf{E}_3$ is satisfied, this follows from Ohta's Eichler–Shimura isomorphism [**Oht00**] (cf. Property $\boldsymbol{O}_3$ in the proof of Proposition 2.2) and its compatibility with the Faltings–Tsuji comparison isomorphism proved in Theorem 9.5.2 of [**KLZ17**]. We refer the reader to Section 5 of [**BSV21b**] for more details in the ordinary setting.

Similarly one defines

$$\mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{\xi}) = \big(D(\boldsymbol{\xi})^-\big)^{\Gamma=1} \quad \text{and} \quad \mathrm{tg}_{\mathrm{dR}}(\boldsymbol{\xi}) = \big(D(\boldsymbol{\xi})^+(\mu_{\boldsymbol{\xi}}^{-1})\big)^{\Gamma=1},$$

which are in perfect duality with $\mathrm{gr}_{\mathrm{dR}}^*(\boldsymbol{\xi})$ and $\mathrm{Fil}^1 V_{\mathrm{dR}}^*(\boldsymbol{\xi})$ respectively under $\langle\cdot,\cdot\rangle_{\boldsymbol{\xi}}$.

2.3.3.1. *Weight-one differentials.* — If $u_o = 1$, i.e. if either $\mathbf{E}_1$ or $\mathbf{E}_2$ in Assumption 2.1 is satisfied, we define $\omega_{\boldsymbol{\xi}_1}$ and $\eta_{\boldsymbol{\xi}_1}$ in $V_{\mathrm{dR}}^*(\boldsymbol{\xi}_1) = D_{\mathrm{dR}}(V^*(\boldsymbol{\xi}_1))$ to be the weight-one specialisations of $\omega_{\boldsymbol{\xi}}$ and $\eta_{\boldsymbol{\xi}}$ respectively. In this case we set $\eta_{\boldsymbol{\xi}_1}^{\alpha} = \eta_{\boldsymbol{\xi}_1}$.

2.4. Perrin-Riou logarithms. — For $\cdot = \emptyset, *$ set

$$V^{\cdot}(\boldsymbol{f},\boldsymbol{g}) = V^{\cdot}(\boldsymbol{f})\hat{\otimes}_L V^{\cdot}(\boldsymbol{g}) \quad \text{and} \quad \mathscr{O}_{\boldsymbol{fg}} = \mathscr{O}_{\boldsymbol{f}}\hat{\otimes}_L\mathscr{O}_{\boldsymbol{g}}.$$

Denote by

$$D^{\cdot}(\boldsymbol{f},\boldsymbol{g}) = \mathbf{D}_{\mathrm{rig},\mathscr{O}_{\boldsymbol{fg}}}^{\dagger}(V^{\cdot}(\boldsymbol{f},\boldsymbol{g}))$$

the (φ,Γ)-module over $\mathscr{R}_{\boldsymbol{fg}} = \mathscr{R}\hat{\otimes}_{\mathbf{Q}_p}\mathscr{O}_{\boldsymbol{fg}}$ associated by Berger–Colmez with the restriction of $V^{\cdot}(\boldsymbol{f},\boldsymbol{g})$ to $G_{\mathbf{Q}_p}$. This is naturally isomorphic to $D^{\cdot}(\boldsymbol{f})\hat{\otimes}_{\mathscr{R}_L}D^{\cdot}(\boldsymbol{g})$ and for each symbol a and b in $\{\emptyset,+,-\}$ one writes $\mathscr{F}^{ab}D^{\cdot}(\boldsymbol{f},\boldsymbol{g})$ for the completed tensor product over $\mathscr{R}_L$ of $D^{\cdot}(\boldsymbol{f})^a$ and $D^{\cdot}(\boldsymbol{g})^b$, where $D^{\cdot}(\boldsymbol{\xi})^{\emptyset} = D^{\cdot}(\boldsymbol{\xi})$. Define

$$H^1_{\mathrm{Iw,bal}}(\mathbf{Q}_p(\mu_{p^\infty}),V(\boldsymbol{f},\boldsymbol{g})) \longrightarrow H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),V(\boldsymbol{f},\boldsymbol{g}))\otimes_{\Lambda_\infty}\mathcal{O}(\mathcal{W})$$

to be the submodule of classes which map to zero under the morphism

$$\begin{array}{c} H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), V(\boldsymbol{f},\boldsymbol{g}))\otimes_{\Lambda_\infty}\mathcal{O}(\mathcal{W}) = H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), D(\boldsymbol{f},\boldsymbol{g})) \\ \big\downarrow \\ H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_p^\infty), \mathscr{F}^{--}D(\boldsymbol{f},\boldsymbol{g})) \end{array}$$

induced by the projection $D(\boldsymbol{f},\boldsymbol{g}) \longrightarrow \mathscr{F}^{--}D(\boldsymbol{f},\boldsymbol{g})$. Here $H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), V(\boldsymbol{f},\boldsymbol{g}))$ is defined as in Section 1.1. One equips $\mathcal{O}(\mathcal{W})$ with the structure of Λ_∞-algebra via the continuous character $[\cdot] : G_\infty \longrightarrow \mathcal{O}(\mathcal{W})^*$ defined by $[g](x) = x(\chi_{\mathrm{cyc}}(g))$ for g in G_∞ and x in $\mathcal{W}$. For each affinoid $\mathbf{Q}_p$-algebra B and each (φ,Γ)-module D over $\mathscr{R}_B = \mathscr{R}\hat{\otimes}_{\mathbf{Q}_p}B$, one writes $H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu), D) = D^{\psi=1}$ for the analytic Iwasawa cohomology of D, which is canonically isomorphic to $H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), V)\otimes_{\Lambda_\infty}\mathcal{O}(\mathcal{W})$ if $D = \mathbf{D}^\dagger_{\mathrm{rig},B}(V)$ arises from a B-adic representation V of $G_{\mathbf{Q}_p}$ via the Berger–Colmez functor. (We refer to [KPX14] for more details on the analytic Iwasawa cohomology.)

Since the map induced by the inclusion $\mathscr{F}^{-+}D(\boldsymbol{f},\boldsymbol{g}) \longrightarrow \mathscr{F}^{-\emptyset}D(\boldsymbol{f},\boldsymbol{g})$ in Iwasawa cohomology is injective, the projection

$$p_f^- : D(\boldsymbol{f},\boldsymbol{g}) \longrightarrow \mathscr{F}^{-\emptyset}D(\boldsymbol{f},\boldsymbol{g})$$

induces a morphism of $\mathscr{O}_{\boldsymbol{fg}}\hat{\otimes}_{\mathbf{Q}_p}\mathcal{O}(\mathcal{W})$-modules (denoted by the same symbol)

$$p_f^- : H^1_{\mathrm{Iw,bal}}(\mathbf{Q}_p(\mu_{p^\infty}), V(\boldsymbol{f},\boldsymbol{g})) \longrightarrow H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), \mathscr{F}^{-+}D(\boldsymbol{f},\boldsymbol{g})).$$

Similarly one defines a morphism

$$p_g^- : H^1_{\mathrm{Iw,bal}}(\mathbf{Q}_p(\mu_{p^\infty}), V(\boldsymbol{f},\boldsymbol{g})) \longrightarrow H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), \mathscr{F}^{+-}D(\boldsymbol{f},\boldsymbol{g})).$$

As explained in Theorem 7.1.4 of [LZ16], the work of Nakamura [Nak14] yields a Perrin-Riou logarithm map

$$\mathcal{L}^{-+} : H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), \mathscr{F}^{-+}D(\boldsymbol{f},\boldsymbol{g})) \longrightarrow \mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{f})\hat{\otimes}_L \mathrm{tg}_{\mathrm{dR}}(\boldsymbol{g})\hat{\otimes}_{\mathbf{Q}_p}\mathcal{O}(\mathcal{W}),$$

which is an injective morphism of $\mathcal{O}(U_{\boldsymbol{f}}\otimes U_{\boldsymbol{g}}\times\mathcal{W})$-modules. (We refer to Sections 6 and 7 of [LZ16] for the precise definition and the interpolation property which characterises $\mathcal{L}^{-+}$, denoted $\mathcal{L}$ there.) Define

$$\mathscr{L}_{\boldsymbol{f}} = \left\langle \mathcal{L}^{-+}\circ p_f^-(\cdot), \eta_{\boldsymbol{f}}\otimes\omega_{\boldsymbol{g}}\right\rangle_{\boldsymbol{fg}} : H^1_{\mathrm{Iw,bal}}(\mathbf{Q}_p(\mu_{p^\infty}), V(\boldsymbol{f},\boldsymbol{g})) \longrightarrow \mathscr{O}_{\boldsymbol{fg}}\hat{\otimes}_{\mathbf{Q}_p}\mathcal{O}(\mathcal{W}).$$

Switching the roles of $\boldsymbol{f}$ and $\boldsymbol{g}$, one similarly defines

$$\mathscr{L}_{\boldsymbol{g}} = \left\langle \mathcal{L}^{+-}\circ p_g^-(\cdot), \omega_{\boldsymbol{f}}\otimes\eta_{\boldsymbol{g}}\right\rangle_{\boldsymbol{fg}} : H^1_{\mathrm{Iw,bal}}(\mathbf{Q}_p(\mu_{p^\infty}), V(\boldsymbol{f},\boldsymbol{g})) \longrightarrow \mathscr{O}_{\boldsymbol{fg}}\hat{\otimes}_{\mathbf{Q}_p}\mathcal{O}(\mathcal{W}).$$

2.5. Beilinson–Flach elements and reciprocity laws. — The proof of the main result of this paper grounds on the following result, which extends and refines the explicit reciprocity laws for Beilinson–Flach elements of Bertolini–Darmon–Rotger and Kings–Loeffler–Zerbes [BDR15, KLZ17, LZ16] to the case where one of the Coleman families $\boldsymbol{f}$ and $\boldsymbol{g}$ specialises to a p-irregular weight-one Eisenstein series (i.e., satisfies condition $\mathbf{E}_3$ in Assumption 2.1). Denote by

$$L_p(\boldsymbol{f},\boldsymbol{g}) = L_p(\boldsymbol{f},\boldsymbol{g},\boldsymbol{s}) \quad \text{and} \quad L_p(\boldsymbol{g},\boldsymbol{f}) = L_p(\boldsymbol{g},\boldsymbol{f},\boldsymbol{s})$$

the three-variable p-adic Rankin–Selberg convolutions associated by Hida, Panchishkin and Urban to the ordered pairs of Coleman families $(\boldsymbol{f}, \boldsymbol{g})$ and $(\boldsymbol{g}, \boldsymbol{f})$ respectively. We refer to [**Urb14**] and [**AI21**, Appendix II] by Urban for the construction of these p-adic L-functions. (See also Theorem 2.7.4 of [**KLZ17**] for a description of the interpolation properties which characterise them.) Let

$$H^1_{\mathrm{Iw,bal}}(\mathbf{Q}(\mu_{p^\infty}), V(\boldsymbol{f}, \boldsymbol{g})) \hookrightarrow H^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), V(\boldsymbol{f}, \boldsymbol{g})) \otimes_{\Lambda_\infty} \mathcal{O}(\mathcal{W})$$

be the submodule of global Iwasawa classes whose restriction at p belong to the balanced local condition $H^1_{\mathrm{Iw,bal}}(\mathbf{Q}_p(\mu_{p^\infty}), V(\boldsymbol{f}, \boldsymbol{g}))$ and which are unramified at each rational prime not dividing pN, where N is the least common multiple of $N_{\boldsymbol{f}}$ and $N_{\boldsymbol{g}}$.

Proposition 2.3. — *Assume that the following conditions are satisfied.*

1. *The family $\boldsymbol{f}$ satisfies condition $\mathbf{E}_1$ in Assumption 2.1.*
2. *The family $\boldsymbol{g}$ satisfies condition $\mathbf{E}_3$ in Assumption 2.1.*

Then, for each integer $c \geqslant 2$ coprime to $6Np$, there exists a Beilinson–Flach element

$${}_c\mathbf{BF}(\boldsymbol{f} \otimes \boldsymbol{g}) \in H^1_{\mathrm{Iw,bal}}(\mathbf{Q}(\mu_{p^\infty}), V(\boldsymbol{f}, \boldsymbol{g}))$$

satisfying the explicit reciprocity laws

$$\mathscr{L}_{\boldsymbol{\xi}}\big(\mathrm{res}_p\big({}_c\mathbf{BF}(\boldsymbol{f} \otimes \boldsymbol{g})\big)\big) = \mathscr{N}_{\boldsymbol{\xi},c} \cdot L_p(\boldsymbol{\xi}, \boldsymbol{\xi}', 1 + \boldsymbol{s}).$$

Here $(\boldsymbol{\xi}, \boldsymbol{\xi}')$ is equal to either $(\boldsymbol{f}, \boldsymbol{g})$ or $(\boldsymbol{g}, \boldsymbol{f})$ and

$$\mathscr{N}_{\boldsymbol{\xi},c} = (-1)^{1+\boldsymbol{s}} \cdot w_{\boldsymbol{\xi}} \cdot \big(c^2 - c^{2\boldsymbol{s}-\boldsymbol{k}-+4} \cdot \chi_{\boldsymbol{f}}(c)^{-1}\chi_{\boldsymbol{g}}(c)^{-1}\big),$$

where $w_{\boldsymbol{\xi}}$ a unit in $\mathscr{O}^*_{\boldsymbol{\xi}}$ satisfying $w_{\boldsymbol{\xi}}(u)^2 = (-N_{\boldsymbol{\xi}})^{2-u}$ for each u in $U_{\boldsymbol{\xi}}$.

Proof. — Shrinking $U_{\boldsymbol{f}}$ if necessary, assume that the composition of $a_p(\boldsymbol{f})$ with the p-adic valuation (normalised by $\mathrm{ord}_p(p) = 1$) is constant with value $\lambda = \lambda_{\boldsymbol{\xi}} \geqslant 0$. Let $(\boldsymbol{\xi}, \lambda_\xi)$ denote one of the pairs $(\boldsymbol{f}, \lambda)$ or $(\boldsymbol{g}, 0)$. For each integer $s \geqslant 3$, let $Y_1(s)$ be the affine modular curve of level $\Gamma_1(s)$ over $\mathbf{Z}[1/sp]$, and let $\pi_s : E_1(s) \longrightarrow Y_1(s)$ be the universal elliptic curve over it. For each $u \geqslant \lambda_\xi$ in $U_{\boldsymbol{\xi}} \cap \mathbf{Z}_{\geqslant 2}$ set

$$V(u)^{\leqslant \lambda_\xi} = H^1_{\mathrm{par}}(Y_\xi, \mathscr{L}_{u-2})^{\leqslant \lambda_\xi} \otimes_{\mathbf{Z}_p} L(1),$$

where $Y_\xi = Y_1(N_{\boldsymbol{\xi}}p) \otimes_{\mathbf{Z}[1/N_{\boldsymbol{\xi}}p]} \bar{\mathbf{Q}}$, $\mathscr{L}_{u-2} = \mathrm{TSym}^{u-2} R^1\pi_{N_{\boldsymbol{\xi}}p}\mathbf{Z}_p(1)$, $H^1_{\mathrm{par}} = H^1_{\text{ét},\mathrm{par}}$ and $\cdot^{\leqslant \lambda_\xi}$ is the subspace of $\cdot$ on which the dual Hecke operator U'_p acts with slope less or equal to λ_ξ. Moreover, with the notation introduced in Section 2.3, set

$$V(U_{\boldsymbol{\xi}})^{\leqslant \lambda_\xi} = H^1_{\mathrm{par}}(\Gamma_{\boldsymbol{\xi}}, \boldsymbol{\mathcal{L}}_{\boldsymbol{\xi}})^{\leqslant \lambda_\xi}(1) \otimes_{\Lambda_{\boldsymbol{\xi}}} \mathscr{O}_{\boldsymbol{\xi}},$$

where $\leqslant \lambda_\xi$ refers to the slope decomposition with respect to U'_p (cf. Proposition 4.2 of [**BSV21b**]). By construction there is a natural $\boldsymbol{\xi}$-isotypic projection

$$\mathrm{pr}_{\boldsymbol{\xi}} : V(U_{\boldsymbol{\xi}})^{\leqslant \lambda_\xi} \twoheadrightarrow V(\boldsymbol{\xi}).$$

Evaluation at u on $\mathscr{O}_{\boldsymbol{\xi}}$ then induces natural isomorphisms of $L[G_{\mathbf{Q}}]$-modules

$$\rho_u : V(U_{\boldsymbol{\xi}})^{\leqslant \lambda_\xi} \otimes_u L \simeq V(u)^{\leqslant \lambda_\xi} \quad \text{and} \quad \rho_u : V(\boldsymbol{\xi}) \otimes_u L \simeq V(\boldsymbol{\xi}_u), \tag{12}$$

where $\mathrm{pr}_{\boldsymbol{\xi}_u} : V(u)^{\leqslant h_{\boldsymbol{\xi}}} \longrightarrow\!\!\!\!\rightarrow V(\boldsymbol{\xi}_u)$ is the maximal quotient on which T'_n acts as multiplication by $a_n(\boldsymbol{\xi}_u) = a_n(\boldsymbol{\xi})(u)$ for each $n \geqslant 1$. (See Sections 4.1.3 and 4.1.4 of [**BSV21b**] for more details.) Define similarly

$$\mathrm{pr}_{\boldsymbol{\xi}} : \tilde{V}(U_{\boldsymbol{\xi}})^{\leqslant \lambda_\xi} \twoheadrightarrow \tilde{V}(\boldsymbol{\xi}) \quad \text{and} \quad \mathrm{pr}_{\boldsymbol{\xi}_u} : \tilde{V}(u)^{\leqslant \lambda_\xi} \twoheadrightarrow \tilde{V}(\boldsymbol{\xi}_u)$$

after replacing the parabolic cohomology groups $H^1_{\mathrm{par}}(\Gamma_{\boldsymbol{\xi}}, \cdot)$ and $H^1_{\mathrm{par}}(Y_\xi, \cdot)$ with the full cohomology groups $H^1(\Gamma_{\boldsymbol{\xi}}, \cdot)$ and $H^1(Y_\xi, \cdot)$ in the definitions of $V(U_{\boldsymbol{\xi}})^{\leqslant \lambda_\xi}$ and $V(u)^{\leqslant \lambda_\xi}$ respectively. The specialisation maps ρ_u extend to isomorphisms

$$\rho_u : \tilde{V}(U_{\boldsymbol{\xi}})^{\leqslant \lambda_\xi} \otimes_u L \simeq \tilde{V}(u)^{\leqslant \lambda_\xi} \quad \text{and} \quad \rho_u : \tilde{V}(\boldsymbol{\xi}) \otimes_u L \simeq \tilde{V}(\boldsymbol{\xi}_u). \tag{13}$$

By assumption 1 in the statement, the inclusion $V(U_{\boldsymbol{f}})^{\leqslant \lambda} \hookrightarrow \tilde{V}(U_{\boldsymbol{f}})^{\leqslant \lambda}$ induces on the $\boldsymbol{f}$-isotypic quotients an isomorphism of $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}}]$-modules

$$V(\boldsymbol{f}) \simeq \tilde{V}(\boldsymbol{f}), \tag{14}$$

which we consider as equality. As $\boldsymbol{\xi}_u$ (for $\boldsymbol{\xi}$ and u as above) is cuspidal, the inclusion $V(u)^{\leqslant \lambda_\xi} \hookrightarrow \tilde{V}(u)^{\leqslant \lambda_\xi}$ similarly yields an isomorphism of $L[G_{\mathbf{Q}}]$-modules

$$V(\boldsymbol{\xi}_u) \simeq \tilde{V}(\boldsymbol{\xi}_u). \tag{15}$$

Let $\mathcal{X}^{\mathrm{geom}}$ be the set of triples of integers (k, l, m) in $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times \mathcal{W}$ such that

$$k \geqslant 2, \quad l \geqslant 3 \quad \text{and} \quad 0 \leqslant m \leqslant \min\{k-2, l-2\}.$$

For each $x = (k, l, m)$ in $\mathcal{X}^{\mathrm{geom}}$ and each positive integer $r \geqslant 0$, denote by

$$\mathrm{Eis}(x) \in H^3(Y(p^r, Np^{r+1})^2, \mathscr{L}_{k-2} \boxtimes \mathscr{L}_{l-2}(2-m))$$

the pull-black of the étale Rankin–Eisenstein class $\mathrm{Eis}^{[k,l,m]}_{\text{ét},1,Np^{r+1}}$ introduced in [**KLZ17**, Definition 3.3.1] to the affine modular curve $Y(p^r, Np^{r+1})$ over $\mathbf{Z}[1/Np]$ classifying elliptic curves E with embeddings $i_E : \mathbf{Z}/p^r\mathbf{Z} \times \mathbf{Z}/Np^{r+1}\mathbf{Z} \hookrightarrow E$. Following Kato [**Kat04**, Equation (5.1.2)], denote by $t_r : Y(p^r, Np^{r+1}) \longrightarrow Y_1(Np) \otimes_{\mathbf{Z}} \mathbf{Z}[\mu_{p^r}]$ the map sending (E, i_E) to $\big((E/\mathbf{Z}\cdot P, Q + \mathbf{Z}\cdot P), \langle P, Np\cdot Q\rangle_{E[p^r]}\big)$, where $P = i_E(1,0)$, $Q = i_E(0,1)$ and $\langle \cdot, \cdot \rangle_{E[p^r]}$ is the Weil pairing on $E[p^r]$. The push-forward of $\mathrm{Eis}(x)$ along $t_r \times t_r$, together with the Hochschild–Serre spectral sequence, the Künneth decomposition and the natural projection $Y_1(Np)^2 \longrightarrow Y_f \times Y_g$ (sending $(E,P) \times (E', P')$ to $(E, (N/N_{\boldsymbol{f}}) \cdot P) \times (E', (N/N_{\boldsymbol{g}}) \cdot P')$), yields a Beilinson–Flach element

$$\widetilde{\mathrm{BF}}_r(x) \in H^1(G_r, \tilde{V}(k)^{\leqslant \lambda} \otimes_{\mathbf{Q}_p} \tilde{V}(l)^{\leqslant 0}(-m)),$$

where $G_r = G_{\mathbf{Q}(\mu_{p^r}), Np}$ is the Galois group of the maximal algebraic extension of $\mathbf{Q}(\mu_{p^r})$ unramified outside $Np\infty$. For each integer $c \geqslant 2$ coprime to $6Np$ set

$${}_c\widetilde{\mathrm{BF}}_r(x) = \big(c^2 - c^{2m-k-l+4} \cdot \langle c\rangle_f \otimes \langle c\rangle_g\big) \cdot \widetilde{\mathrm{BF}}_r(x),$$

where $\langle c \rangle_\xi$ is the diamond operator acting on $\tilde{V}(u)^{\leqslant \lambda_\xi}$.

Let $m \geqslant 0$ be a nonnegative integer and let $\mathcal{X}^{\mathrm{geom}}_m$ be the set of triples in $\mathcal{X}^{\mathrm{geom}}$ having m as third component. The work of Kings–Loeffler–Zerbes yields a class

$${}_c\widetilde{\mathbf{BF}}_{m,r}(\boldsymbol{f} \otimes U_{\boldsymbol{g}}) \in H^1(G_r, V(\boldsymbol{f}) \hat{\otimes}_{\mathbf{Q}_p} \tilde{V}(U_{\boldsymbol{g}})^{\leqslant 0}(-m))$$

such that, for each triple $x = (k, l, m)$ in $\mathcal{X}_m^{\mathrm{geom}}$, one has

$$\binom{k-2}{m}\binom{l-2}{m} \cdot \varrho_{k,l}\big({}_c\widetilde{\mathbf{BF}}_{m,r}(\boldsymbol{f} \otimes U_{\boldsymbol{g}})\big) = {}_c\widetilde{\mathrm{BF}}_r(\boldsymbol{f}_k, l, m), \tag{16}$$

where $\varrho_{k,l}$ is the morphism induced by $\varrho_k \hat{\otimes} \varrho_l$ (cf. Equations (12) and (13)) and

$${}_c\widetilde{\mathrm{BF}}_r(\boldsymbol{f}_k, l, m) = (\mathrm{pr}_{\boldsymbol{f}_k} \otimes \mathrm{id})\big({}_c\widetilde{\mathrm{BF}}_r(x)\big) \in H^1(G_r, V(\boldsymbol{f}_k) \otimes \tilde{V}(l)^{\leqslant 0}(-m))$$

is the image of ${}_c\widetilde{\mathrm{BF}}_r(x)$ under the map induced in cohomology by the $\boldsymbol{f}_k$-isotypic projection $\mathrm{pr}_{\boldsymbol{f}_k} : \tilde{V}(k)^{\leqslant \lambda} \twoheadrightarrow \tilde{V}(\boldsymbol{f}_k) \simeq V(\boldsymbol{f}_k)$ (cf. Equation (14)). With the notations of [LZ16, Section 5.3] (and identifying $V(\boldsymbol{f})$ with $\tilde{V}(\boldsymbol{f})$) one has

$$(\mathrm{pr}_{\boldsymbol{f}} \otimes \mathrm{pr}^{\leqslant 0})_*\big({}_c\mathcal{BF}^{[U_{\boldsymbol{f}}, U_{\boldsymbol{g}}, m]}_{p^r, N_f, N_g, 1}\big) = \binom{\nabla_{\boldsymbol{f}}}{m}\binom{\nabla_{\boldsymbol{g}}}{m} \cdot {}_c\widetilde{\mathbf{BF}}_{m,r}(\boldsymbol{f} \otimes U_{\boldsymbol{g}}),$$

where ($\nabla_{\boldsymbol{f}}$ and $\nabla_{\boldsymbol{g}}$ are the functions denoted by ∇_1 and ∇_2 in loc. cit. and)

$$\mathrm{pr}^{\leqslant 0} : H^1(\Gamma_{\boldsymbol{g}}, \boldsymbol{\mathcal{L}_g})(1) \otimes_{\Lambda_{\boldsymbol{g}}} \mathscr{O}_{\boldsymbol{g}} \longrightarrow \tilde{V}(U_{\boldsymbol{g}})^{\leqslant 0}$$

is the projection onto the ordinary part. (Cf. [LZ16, Proposition 5.3.4]).

The proof of the proposition rests on the following

Lemma 2.4. — *The class ${}_c\widetilde{\mathbf{BF}}_{m,r}(\boldsymbol{f} \otimes U_{\boldsymbol{g}})$ admits a unique lift*

$${}_c\mathbf{BF}_{m,r}(\boldsymbol{f} \otimes U_{\boldsymbol{g}}) \in H^1(G_{\mathbf{Q}(\mu_{p^r}),N}, V(\boldsymbol{f}) \hat{\otimes}_L V(U_{\boldsymbol{g}})^{\leqslant 0}(-m)).$$

Proof. — Set $E = \tilde{V}(U_{\boldsymbol{g}})^{\leqslant 0}/V(U_{\boldsymbol{g}})^{\leqslant 0}$. It is a free $\mathscr{O}_{\boldsymbol{g}}$-module of finite rank (cf. [Oht00]), and the absolute Galois group G_K of the cyclotomic field $K = \mathbf{Q}(\mu_{Np})$ acts trivially on it. Indeed its base change $E_l = E \otimes_l L$ along evaluation at l in $U_{\boldsymbol{g}} \cap \mathbf{Z}_{\geqslant 3}$ is isomorphic to the ordinary part of $H^0(C_g \otimes_{\mathbf{Q}} \bar{\mathbf{Q}}, \mathbf{Q}_p)$, where C_g is the set of cusps of $X_g = X_1(N_g p)_{\mathbf{Q}}$. (Cf. [Sch90, Theorem 1.2.1] and the discussion preceding it.) Since C_g is the union of a finite number of $\mathbf{Q}(\mu_{N_g p})$-rational points of X_g, it follows that G_K acts trivially on E_l for each l in $U_{\boldsymbol{g}} \cap \mathbf{Z}_{\geqslant 2}$. As E is free over $\mathscr{O}_{\boldsymbol{g}}$, this implies that G_K acts trivially on E. One deduces the equalities

$$H^i(G_r, V(\boldsymbol{f}) \hat{\otimes}_L E(-m)) = \big(H^i(G_{K,r}, V(\boldsymbol{f})(-m)) \hat{\otimes}_L E\big)^{\mathrm{Gal}(K(\mu_{p^r})/\mathbf{Q}(\mu_{p^r}))}$$

for $i \geqslant 0$, where $G_{K,r}$ is the Galois group of the maximal algebraic extension of $K(\mu_{p^r})$ unramified outside $Np\infty$. Because $V(\boldsymbol{f}_{k_o})(-m) = V(\boldsymbol{f})(-m) \otimes_{k_o} L$ has no nontrivial $G_{K,r}$-invariant, the modules $H^0(G_{K,r}, V(\boldsymbol{f})(-m))$ and $H^1(G_{K,r}, V(\boldsymbol{f})(-m))[\mathfrak{m}_{k_o}]$ vanish, where $\mathfrak{m}_{k_o}$ is the kernel of evaluation at k_o on $\mathscr{O}_{\boldsymbol{f}}$ and $\cdot[\mathfrak{m}_{k_o}]$ is the $\mathfrak{m}_{k_o}$-torsion submodule of $\cdot$. Shrinking $U_{\boldsymbol{f}}$ if necessary, one deduces by the previous equation that $H^1(G_r, V(\boldsymbol{f}) \hat{\otimes} E(-m))$ is a torsion-free $\mathscr{O}_{\boldsymbol{fg}}$-module and that the natural map

$$H^1(G_r, V(\boldsymbol{f}) \hat{\otimes} V(U_{\boldsymbol{g}})^{\leqslant 0}(-m)) \longrightarrow H^1(G_r, V(\boldsymbol{f}) \hat{\otimes} \tilde{V}(U_{\boldsymbol{g}})^{\leqslant 0}(-m))$$

is injective. To prove the lemma it is then sufficient to show that

$$\varrho_{k,l}({}_c\widetilde{\mathbf{BF}}_{m,r}(\boldsymbol{f} \otimes U_{\boldsymbol{g}}))$$

belongs to the image of

$$H^1(G_r, V(\boldsymbol{f}_k) \otimes_{\mathbf{Q}_p} V(l)^{\leqslant 0}(-m)) \longrightarrow H^1(G_r, V(\boldsymbol{f}_k) \otimes \tilde{V}(l)^{\leqslant 0}(-m))$$

for each triple $x=(k,l,m)$ in the Zariski-dense subset $\mathcal{X}_m^{\mathrm{geom}}$ of $U_{\boldsymbol{f}}\times U_{\boldsymbol{g}}\times\{m\}$. In light of Equation (16), this follows from Section 9 of [BC16] and Theorem 1.2.1 of [Sch90], which prove that the Beilinson–Flach element

$$\widetilde{\mathrm{BF}}_r(x)\in H^1(\mathbf{Q}(\mu_{p^r}),\tilde{V}(k)^{\leqslant\lambda}\otimes_{\mathbf{Q}_p}\tilde{V}(l)^{\leqslant 0}(-m))$$

admits a (canonical) lift to $H^1(\mathbf{Q}(\mu_{p^r}),V(k)^{\leqslant\lambda}\otimes_{\mathbf{Q}_p}V(l)^{\leqslant 0}(-m))$. □

Resuming the proof of the proposition, for each $m\geqslant 0$ and $r\geqslant 1$ define

$${}_c\mathbf{BF}_{m,r}(\boldsymbol{f}\otimes\boldsymbol{g})\in H^1(G_r,V(\boldsymbol{f},\boldsymbol{g})(-m))$$

to be the image of ${}_c\mathbf{BF}_{m,r}(\boldsymbol{f}\otimes U_{\boldsymbol{g}})$ under the map induced in cohomology by the projection $\mathrm{pr}_{\boldsymbol{g}}:V(U_{\boldsymbol{g}})^{\leqslant 0}\longrightarrow V(\boldsymbol{g})$ onto the $\boldsymbol{g}$-isotypic component. The proof of Theorem 5.4.2 of [LZ16] shows that there exists a unique Iwasawa class

$${}_c\mathbf{BF}(\boldsymbol{f}\otimes\boldsymbol{g})\in H^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}),V(\boldsymbol{f},\boldsymbol{g}))\otimes_{\Lambda_\infty}\mathcal{O}(\mathcal{W})$$

interpolating the elements $(a_p(\boldsymbol{f})\cdot a_p(\boldsymbol{g}))^{-r}\cdot m!^{-1}\cdot{}_c\mathbf{BF}_{m,r}(\boldsymbol{f}\otimes\boldsymbol{g})$ for all $m\geqslant 0$ and $r\geqslant 1$. Moreover, for each $x=(k,l,m)$ in $\mathcal{X}^{\mathrm{geom}}$ one has the equality

$$\varrho_x\big({}_c\mathbf{BF}(\boldsymbol{f}\otimes\boldsymbol{g})\big)=\frac{1}{m!\binom{k-2}{m}\binom{l-2}{m}}\left(1-\frac{p^m}{a_p(\boldsymbol{f}_k)\cdot a_p(\boldsymbol{g}_l)}\right)\cdot{}_c\mathrm{BF}(\boldsymbol{f}_k,\boldsymbol{g}_l,m)$$

in $H^1(\mathbf{Q},V(\boldsymbol{f}_k,\boldsymbol{g}_l)(-m))$, where the specialisation map

$$\varrho_x:H^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}),V(\boldsymbol{f},\boldsymbol{g}))\otimes_{\Lambda_\infty}\mathcal{O}(\mathcal{W})\longrightarrow H^1(\mathbf{Q},V(\boldsymbol{f}_k,\boldsymbol{g}_l)(-m))$$

arises from $\varrho_k\hat{\otimes}\varrho_l:V(\boldsymbol{f},\boldsymbol{g})\longrightarrow V(\boldsymbol{f}_k,\boldsymbol{g}_l)$ and evaluation at m on $\mathcal{O}(\mathcal{W})$, and where

$${}_c\mathrm{BF}(\boldsymbol{f}_k,\boldsymbol{g}_l,m)\in H^1(\mathbf{Q},V(\boldsymbol{f}_k,\boldsymbol{g}_l)(-m))$$

is the image of ${}_c\widetilde{\mathrm{BF}}_0(x)$ under the map induced by the projection (cf. Equation (15))

$$\mathrm{pr}_{\boldsymbol{f}_k}\otimes\mathrm{pr}_{\boldsymbol{g}_l}:\tilde{V}(k)^{\leqslant h}\otimes\tilde{V}(l)^{\leqslant 0}\longrightarrow\tilde{V}(\boldsymbol{f}_k)\otimes\tilde{V}(\boldsymbol{g}_l)\simeq V(\boldsymbol{f}_k,\boldsymbol{g}_l)$$

onto the $\boldsymbol{f}_k\otimes\boldsymbol{g}_l$-isotypic component. The proofs of Theorems 7.12 and 7.15 of [LZ16] show respectively that the Beilinson–Flach element ${}_c\mathbf{BF}(\boldsymbol{f}\otimes\boldsymbol{g})$ belongs to the balanced Selmer group $H^1_{\mathrm{Iw,bal}}(\mathbf{Q}(\mu_{p^\infty}),V(\boldsymbol{f},\boldsymbol{g}))$ and satisfies the reciprocity laws

$$\mathscr{L}_{\boldsymbol{\xi}}(\mathrm{res}_p({}_c\mathbf{BF}(\boldsymbol{f}\otimes\boldsymbol{g})))=\mathscr{N}_{\boldsymbol{\xi},c}\cdot L_p(\boldsymbol{\xi},\boldsymbol{\xi}',1+\boldsymbol{s})$$

for $(\boldsymbol{\xi},\boldsymbol{\xi}')=(\boldsymbol{f},\boldsymbol{g})$ and $(\boldsymbol{\xi},\boldsymbol{\xi}')=(\boldsymbol{g},\boldsymbol{f})$, concluding the proof of the proposition. □

3. Proof of Theorem B: p-ordinary canonical Hecke characters

Let $\mathscr{K}$ be a quadratic imaginary extension of $\mathbf{Q}$ with discriminant $d_{\mathscr{K}}$ congruent to five modulo eight:

$$d_{\mathscr{K}}\equiv 5\pmod 8.$$

Let χ be a canonical Hecke character of $\mathscr{K}$ in the sense of [Roh80], viz. $\chi\cdot\chi^c=\mathbf{N}$, the values of χ on principal ideals lie in $\mathscr{K}$ and the conductor of χ is equal to $\sqrt{d_{\mathscr{K}}}\cdot\mathcal{O}_{\mathscr{K}}$. Here χ^c is the conjugate of χ by the non-trivial element c of $\mathrm{Gal}(\mathscr{K}/\mathbf{Q})$ and $\mathbf{N}=\mathbf{N}_K$ is the norm character (so that $\chi^c(\mathfrak{a})=\chi(c(\mathfrak{a}))$ and $\mathbf{N}(\mathfrak{a})=|\mathcal{O}_{\mathscr{K}}/\mathfrak{a}|$ for each non-zero ideal $\mathfrak{a}$ of $\mathcal{O}_{\mathscr{K}}$). The Hecke L-function $L(\chi, s)$ of χ is equal to that $L(\vartheta_\chi, s)$ of the weight-two newform

$$\vartheta_\chi = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) \cdot q^{\mathbf{N}\mathfrak{a}} \in S_2(\Gamma_0(d_{\mathscr{K}}^2))$$

(where $\mathfrak{a}$ runs over the non-zero ideals of $\mathcal{O}_{\mathscr{K}}$ coprime to $d_{\mathscr{K}}$). The congruence condition imposed on $d_{\mathscr{K}}$ implies that $L(\vartheta_\chi, s)$ has sign -1 in its functional equation. Lying deeper, Theorem 1.1 of [**MY00**] yields

$$\operatorname{ord}_{s=1} L(\vartheta_\chi, s) = 1. \tag{17}$$

Let A_χ be the modular abelian variety of GL_2-type associated with ϑ_χ, viz. the quotient of the Jacobian of $X_1(d_{\mathscr{K}}^2)$ on which the Hecke operator T_n acts as multiplication by $a_n(\vartheta_\chi)$ for each positive integer n. It is an abelian variety defined over $\mathbf{Q}$ of dimension the class number $h_{\mathscr{K}}$ of $\mathscr{K}$. The totally real number field

$$F_\chi = \mathbf{Q}(\chi(\mathfrak{a}) + \chi(\bar{\mathfrak{a}}); \ \mathfrak{a} \text{ non-zero ideal of } \mathcal{O}_{\mathscr{K}})$$

generated by the Fourier coefficients of ϑ_χ has degree $h_{\mathscr{K}}$ and the endomorphism ring $\mathrm{End}_{\mathbf{Q}}(A_\chi)$ is naturally isomorphic to an order $\mathcal{O}_\chi$ in F_χ. In particular, the Mordell–Weil group $A_\chi(\mathbf{Q}) \otimes_{\mathbf{Z}} \mathbf{Q}$ is equipped with a natural structure of F_χ-vector space. Equation (17) and the theorem of Gross–Zagier–Kolyvagin imply that $A_\chi(\mathbf{Q}) \otimes_{\mathbf{Z}} \mathbf{Q}$ has dimension *one* over F_χ and that the Shafarevich–Tate group of A_χ over $\mathbf{Q}$ is finite.

The p-adic representation $V(A_\chi) = \mathrm{Ta}_p(A_\chi) \otimes_{\mathcal{O}_\chi \otimes_{\mathbf{Z}} \mathbf{Z}_p} L$ (where $L = i_p(F_\chi) \cdot \mathbf{Q}_p$) is canonically isomorphic to $V(\vartheta_\chi)$, hence the p-adic Beilinson–Kato element $\zeta_{\vartheta_\chi}^{\mathrm{Kato}}$ associated with ϑ_χ yields an element

$$\zeta_{A_\chi}^{\mathrm{Kato}} \in H^1(\mathbf{Q}, V(A_\chi)).$$

Write $\log_{\omega_\chi}$ as a shothand for $\langle \log_p(\cdot), \omega_{\vartheta_\chi} \rangle$, where $\log_p$ is the Bloch–Kato p-adic logarithm on the finite subspace of $H^1(\mathbf{Q}_p, V(A_\chi))$. For each global point P in $A_\chi(\mathbf{Q}) \otimes_{\mathbf{Z}} \mathbf{Q}$ denote by $\log_{\omega_\chi}(P)$ the value of $\log_{\omega_\chi}$ at the image of $i_p(P)$ under the composition $A_\chi(\mathbf{Q}_p) \hat{\otimes}_{\mathbf{Z}_p} \mathbf{Q}_p \longrightarrow H^1(\mathbf{Q}_p, V_p(A_\chi)) \longrightarrow H^1(\mathbf{Q}_p, V(A_\chi))$. Here $V_p(A_\chi) = \mathrm{Ta}_p(A_\chi) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ is the p-adic Tate module of A_χ with $\mathbf{Q}_p$-coefficients, the first arrow is the local Kummer map and the second arrow is induced by the natural projection of $G_{\mathbf{Q}}$-modules $V_p(A_\chi) \longrightarrow V(A_\chi)$. Set finally $E_\chi = \mathscr{K} \cdot F_\chi$.

The following result verifies Theorem B for $f = \vartheta_\chi$, under the assumption that p splits in $\mathscr{K}$. Its proof heavily relies on the work of Kato, Perrin-Riou and Bertolini–Darmon–Prasanna [**Kat04, PR93, BDP12**].

Theorem 3.1. — *Assume that p splits in $\mathscr{K}/\mathbf{Q}$. Then the Beilinson–Kato element $\zeta_{A_\chi}^{\mathrm{Kato}}$ belongs to the Selmer group $\mathrm{Sel}(\mathbf{Q}, V(A_\chi))$ and there exists a generator $\boldsymbol{P}_\chi$ of the E_χ-vector space $A_\chi(\mathbf{Q}) \otimes_{\mathbf{Z}} \mathscr{K}$ such that*

$$\log_{\omega_\chi}\big(\mathrm{res}_p(\zeta_{A_\chi}^{\mathrm{Kato}})\big) = \log_{\omega_\chi}^2(\boldsymbol{P}_\chi).$$

In particular the Selmer group $\mathrm{Sel}(\mathbf{Q}, V(A_\chi))$ is generated over L by the Beilinson–Kato element $\zeta_{A_\chi}^{\mathrm{Kato}}$.

The proof of Theorem 3.1 occupies the rest of this section. Write $p\cdot\mathcal{O}_{\mathscr{K}} = \wp\cdot\bar{\wp}$ with ($\wp \neq \bar{\wp}$ and) $\wp$ the prime corresponding to the fixed embedding i_p. Set $f = \vartheta_\chi$, so that the p-th Hecke polynomial of f has roots $\alpha_f = \chi(\bar{\wp})$ in $\mathcal{O}_L^*$ and $\beta_f = \chi(\wp) = p/\alpha_f$. Let $f_\alpha = \vartheta_\chi(q) - \chi(\wp)\cdot\vartheta_\chi(q^p)$ be the ordinary p-stabilisation of f.

Recall that the global Iwasawa class $\boldsymbol{\zeta}_f^{\mathrm{Kato}}$ (and then $\zeta_{A_\chi}^{\mathrm{Kato}}$) depends on the choice of complex Shimura periods $\Omega_f^{\pm}$. In the present weight-two CM setting we can, and will, assume that Ω_f^+ and Ω_f^- are both equal to the complex CM period $\Omega(\chi^c)$ associated with the Hecke character χ^c in Section 2C of [**BDP12**].

3.1. — Let $L_\wp(\mathscr{K}) = L_{\wp,\sqrt{d_{\mathscr{K}}}\cdot\mathcal{O}_{\mathscr{K}}}(\mathscr{K},\cdot)$ be the Katz p-adic L-function associated with $(K,\wp,\sqrt{d_{\mathscr{K}}}\cdot\mathcal{O}_{\mathscr{K}})$ and normalised as in Theorem 3.1 of [**BDP12**] (where it is denoted by $\mathscr{L}_{p,\sqrt{d_{\mathscr{K}}}\cdot\mathcal{O}_{\mathscr{K}}}$.) It is an element of the completed group ring $\hat{\mathbf{Z}}_p^{\mathrm{un}}[\![G(\mathfrak{f}p^\infty)]\!]$, where $\hat{\mathbf{Z}}_p^{\mathrm{un}}$ is the ring of Witt vectors of $\bar{\mathbf{F}}_p$, $\mathfrak{f} = \sqrt{d_{\mathscr{K}}}\cdot\mathcal{O}_{\mathscr{K}}$ and $G(\mathfrak{f}p^\infty)$ is the Galois group of the union of the ray class fields of $\mathscr{K}$ of conductors $\mathfrak{f}p^n$ for $n \geqslant 1$. For $\chi^\cdot = \chi, \chi^c$ and σ in $\mathcal{W}$ define

$$L_\wp(\chi^\cdot,\sigma) = L_\wp(\mathscr{K},\hat{\chi}^\cdot\sigma_K),$$

where σ_K is the restriction to G_K of $\sigma\circ\chi_{\mathrm{cyc}}$ and $\hat{\chi}^\cdot$ is the p-adic character of G_K corresponding to $\chi^\cdot$ via class field theory. Then $L_\wp(\chi^\cdot) = L(\chi^\cdot,\cdot)$ is a bounded analytic function in $\mathcal{O}(\mathcal{W})\hat{\otimes}_{\mathbf{Q}_p}\hat{\mathbf{Q}}_p^{nr}$, where $\hat{\mathbf{Q}}_p^{\mathrm{nr}}$ is the maximal unramified extension of $\mathbf{Q}_p$. Since $L_p(f_\alpha)$ is also a bounded analytic function on $\mathcal{W}$, a direct comparison between the interpolation formulae satisfied by $L_\wp(\chi)$ and $L_p(f_\alpha, 1+\boldsymbol{s})$ at finite order characters yields the identity

$$a_\chi\cdot L_p(f_\alpha, 1+\boldsymbol{s}) = \Omega_p(\chi^c)^{-1}\cdot L_\wp(\chi)$$

for a non-zero algebraic constant a_χ in E_χ^*, where $\Omega_p(\chi^c)$ in $\hat{\mathbf{Z}}_p^{\mathrm{nr}}$ is the non-zero p-adic period associated with χ^c in Section 2D of [**BDP12**]. The main result of [**Roh84**] implies that $L_\wp(\chi)$ is non-zero.

The previous equation and Kato's explicit reciprocity law Equation (1) yield

$$a_\chi\cdot\big\langle\mathrm{Log}_f\big(\mathrm{res}_p\big(\boldsymbol{\zeta}_f^{\mathrm{Kato}}\big)\big),\eta_f^\alpha\big\rangle_f = \Omega_p(\chi^c)^{-1}\cdot L_\wp(\chi). \tag{18}$$

3.2. — A direct comparison between Beilinson–Kato elements and the Euler system of elliptic units, carried out by Kato in [**Kat04**, Section 12.5] and further exploited by Lei et al. in [**LLZ13**], gives

$$b_\chi\cdot\big\langle\mathrm{Log}_f\big(\mathrm{res}_p\big(\boldsymbol{\zeta}_f^{\mathrm{Kato}}\big)\big),\omega_f\big\rangle_f = \Omega_p(\chi^c)\cdot\ell_o\cdot L_\wp(\chi^c). \tag{19}$$

for a non-zero algebraic constant b_χ in E_χ^*, where $\ell_o(\sigma) = \log_p(\sigma(1+p))/\log_p(1+p)$ for each σ in $\mathcal{W}$. The rest of this section explains how to deduce Equation (19) above from the results of [**LLZ13**] and [**Kat04**, Section 15].

Denote by $V_{E_\chi}(f)$ the maximal E_χ-quotient of the Betti chomology group $H^1(Y_1(d_{\mathscr{K}}^2)(\mathbf{C}),\mathbf{Z})\otimes_{\mathbf{Z}}E_\chi$ on which the dual Hecke operator T_n' acts as multiplication by $a_n(f)$ for each positive integer n. The comparison isomorphism between Betti and étale cohomology gives a natural isomorphism $V_{E_\chi}(f)\otimes_{E_\chi,i_p}L \cong V(f)$, under which we consider $V_{E_\chi}(f)$ as an E_χ-structure on $V(f)$. Theorem 3.2 of [LLZ13] (cf. [Kat04, Section 15.16]) proves that the identity

$$\mathrm{Log}_f\big(\mathrm{res}_p(\boldsymbol{\zeta}_f^{\mathrm{Kato}})\big) = L_\wp(\chi)\cdot 1\otimes\xi + \ell_o \cdot L_\wp(\chi^c)\cdot t^{-1}\otimes c(\xi) \tag{20}$$

holds in $\hat{\mathbf{Q}}_p^{\mathrm{nr}}\otimes_{\mathbf{Q}_p} V_{\mathrm{cris}}(f)\otimes_{\mathbf{Q}_p}\mathcal{O}(\mathcal{W})$ for an element ξ in $V_{E_\chi}(f)$ satisfying the identity $g(\xi)=\chi^c(g)\cdot\xi$ for each g in $G_{\mathscr{K}}$. Note that the elements $1\otimes\xi$ and $t^{-1}\otimes c(\xi)$ of $B_{\mathrm{cris}}\otimes_{\mathbf{Q}_p} V(f) = B_{\mathrm{cris}}\otimes_{\mathbf{Q}_p} V_{\mathrm{cris}}(f)$ are invariant under the action of the inertia subgroup $I_{\mathbf{Q}_p}$ of $G_{\mathbf{Q}_p}$, hence can naturally be viewed as elements of $\hat{\mathbf{Q}}_p^{\mathrm{nr}}\otimes_{\mathbf{Q}_p} V_{\mathrm{cris}}(f)$.

Remark 3.2. — The statement of Theorem 3.2 of [LLZ13], which applies more generally to CM modular forms ϑ_ψ associated with Hecke characters ψ of infinity type $(k-1,0)$ with $k\geqslant 2$, requires the choice of isomorphisms between the Betti, de Rham and p-adic étale realisations of the motives of ϑ_ψ and ψ (cf. Lemma 2.26 of loco citato). For $k\geqslant 3$, these motives are not known to be isomorphic and it is unclear how to choose the isomorphisms compatibly with the comparison isomorphisms. By contrast, when $k=2$, the motives of f and χ are naturally isomorphic (cf. [Sch88, Chapter V]), making Equation (20) a direct consequence of [LLZ13, Theorem 3.2]. Here the crucial point is to guarantee that the element ξ, satisfying Equation (20) and $g(\xi)=\chi^c(g)\cdot\xi$ for each g in $G_{\mathscr{K}}$, belongs to the Betti E_χ-structure $V_{E_\chi}(f)$ on the p-adic étale realisation $V(f)$ of the motive of f.

In the present weight-two setting, $V(f)$ is equal to $V^*(f)(1)$ and the elements $\omega_f(1)=\omega_f\otimes t^{-1}\otimes\zeta_{p^\infty}$ and $\eta_f^\alpha(1)=\eta_f^\alpha\otimes t^{-1}\otimes\zeta_{p^\infty}$ give the dual basis of η_f^α and $-\omega_f$ under the duality $\langle\cdot,\cdot\rangle_f$ (cf. Section 2.2). Write

$$1\otimes\xi = \mho_p\otimes\omega_f(1) \quad\text{and}\quad t^{-1}\otimes c(\xi) = \Omega_p\otimes\eta_f^\alpha(1)$$

with $\mho_p$ and Ω_p in $\hat{\mathbf{Q}}_p^{\mathrm{nr}}$. Because (as recalled above) $L_\wp(\chi)$ is non-zero, Equations (18) and (20) give

$$\mho_p \sim_{E_\chi^*} \Omega_p(\chi^c)^{-1},$$

where $\sim_{E_\chi^*}$ denotes equality up to multiplication by a non-zero element of E_χ. Moreover by construction

$$\mho_p\cdot\Omega_p \sim_{E_\chi^*} 1\otimes\,\langle c(\xi),\xi(-1)\rangle_f$$

in $B_{\mathrm{cris}}\otimes_{\mathbf{Q}_p} L$, with $\xi=\xi(-1)\otimes\zeta_{p^\infty}$ (and $\langle\cdot,\cdot\rangle_f$ the Poincaré duality pairing). As ξ belongs to the E_χ-structure $V_{E_\chi}(f)$ of $V(f)$, so do $c(\xi)$ and $\xi(-1)$. Since $\langle\cdot,\cdot\rangle_f$ maps $V_{E_\chi}(f)^{\otimes 2}$ into E_χ, the previous two equations yield

$$\Omega_p \sim_{E_\chi^*} \Omega_p(\chi^c).$$

Together with Equation (20), this yields Equation (19).

3.3. — We conclude the proof of Theorem 3.1. To ease notation set

$$\mathcal{L}_f = \mathrm{Log}_f\big(\mathrm{res}_p(\boldsymbol{\zeta}_f^{\mathrm{Kato}})\big).$$

The point $s=0$ lies in the interpolation domain of $L_\wp(\chi)$, hence

$$L_\wp(\chi,0) = L_\wp(\mathscr{K},\chi)$$

is a non-zero multiple of the complex value $L(\chi^{-1},0) = L(\vartheta_\chi,1)$. Equations (17) and (18) then imply that $\zeta^{\rm Kato}_{A_\chi}$ is crystalline at p, hence belongs to the Bloch–Kato Selmer group $\mathrm{Sel}(\mathbf{Q}, V(A_\chi))$. Proposition 2.2.2 of [PR93] then yields

$$\log_{\omega_\chi}\big(\mathrm{res}_p(\zeta^{\rm Kato}_{A_\chi})\big) = \big(1-p^{-1}\chi(\wp)^{-1}\big)\,\big(1-\chi(\wp)^{-1}\big)^{-1}\cdot\big\langle \mathcal{L}_f'(0),\omega_f\big\rangle_f.$$

On the other hand, Equation (19) (and the identities $\ell_o(0)=0$ and $\ell_o'(0)=1$) give

$$b_\chi\cdot\big\langle \mathcal{L}_f'(0),\omega_f\big\rangle_f = \Omega_p(\chi^c)\cdot L_\wp(\chi^c,0).$$

Finally, according to Theorem 2 of [BDP12, Theorem 2] one has

$$\Omega_p(\chi^c)\cdot L_\wp(\chi^c,0) = d_\chi\cdot \log^2_{\omega_\chi}(\boldsymbol{P}_\chi)$$

for a non-zero algebraic constant d_χ in E_χ^* and a generator $\boldsymbol{P}_\chi$ of the E_χ-vector space $A_\chi(\mathbf{Q})\otimes_{\mathbf{Z}}\mathscr{K}$. Theorem 3.1 is a direct consequence of the previous three equations.

4. Proof of Theorem B: the p-non-exceptional case

Let f and $K/\mathbf{Q}$ be as in Section 1.1. This section proves Theorem B stated in loc. cit. under the assumption that f is not p-exceptional (cf. [MTT86]), viz. its p-th Fourier coefficient $a_p(f)$ is different from $p^{k_o/2-1}$.

4.1. The Coleman family $\boldsymbol{f}=\boldsymbol{f}_\alpha$. — The assumptions $\mathrm{ord}_p(\alpha) < k_o-1$ and $\alpha\neq\beta$ guarantee that f_α is an étale point of the Coleman–Mazur eigencurve (cf. the discussion following Assumption 2.1). As a consequence, if $U_{\boldsymbol{f}}$ is a sufficiently small connected affinoid disc in $\mathcal{W}_L$ centred at k_o, there exists a unique (up to conjugation) Coleman family $\boldsymbol{f}=\sum_{n\geqslant 1} a_n(\boldsymbol{f})\cdot q^n$ in $\mathscr{O}_{\boldsymbol{f}}[\![q]\!]$ of tame level N_f, trivial tame character and slope $\lambda_{\boldsymbol{f}} = \mathrm{ord}_p(\alpha)$ which specialises to $\boldsymbol{f}_{k_o} = f_\alpha$ at weight k_o.

The formal q-expansion $\boldsymbol{f}\otimes\varepsilon_K = \sum_{n\geqslant 1}\varepsilon_K(n)a_n(\boldsymbol{f})\cdot q^n$ in $\mathscr{O}_{\boldsymbol{f}}[\![q]\!]$ defines a primitive Coleman family of tame level $N_f d_K^2$, trivial tame character and slope $\lambda_{\boldsymbol{f}}$.

4.2. Theta series and the Hida family $\boldsymbol{g}$. — To prove Theorem B, we apply the results described in Section 2 to a pair of Coleman families $(\boldsymbol{f},\boldsymbol{g})$, where $\boldsymbol{f}=\boldsymbol{f}_\alpha$ is the Coleman family introduced in Section 4.1 and $\boldsymbol{g}$ is an auxiliary ordinary CM family associated with K. This section defines $\boldsymbol{g}$ and discusses its main properties.

Consider the weight-one Eisenstein series

$$\mathrm{Eis}_1(\varepsilon_K) = \frac{1}{2}L(\varepsilon_K,0) + \sum_{n\geqslant 1} q^n\sum_{d|n}\varepsilon_K(d) \in M_1(-d_K,\varepsilon_K).$$

of level $\Gamma_1(-d_K)$ and character ε_K. Because p splits in $K/\mathbf{Q}$, the eigenform $\mathrm{Eis}_1(\varepsilon_K)$ is p-irregular, viz. its p-th Hecke polynomial $X^2 - a_p(\mathrm{Eis}_1(\varepsilon_K))\cdot X + \varepsilon_K(p) = (X-1)^2$ has a double root (cf. Assumption 2.1.3). Define

$$g = \mathrm{Eis}_1(\varepsilon_K)(q) - \mathrm{Eis}_1(\varepsilon_K)(q^p) \in M_1(-pd_K,\varepsilon_K)$$

to be its unique p-stabilisation. As recalled in Section 2.3, the article [BDP21] proves that g is an étale point of the cuspidal Coleman–Mazur eigencurve. In particular, if the local field L is large enough and $U_{\boldsymbol{g}}$ is a sufficiently small connected affinoid disc in $\mathcal{W}_L$ centred at $l_o = 1$, there exists a unique (up to conjugation) Hida family

$$\boldsymbol{g} = \sum_{n \geqslant 1} a_n(\boldsymbol{g}) \cdot q^n \in \mathscr{O}_{\boldsymbol{g}}[\![q]\!]$$

of tame level $-d_K$ and tame character $\chi_{\boldsymbol{g}} = \varepsilon_K$ which specialises to $\boldsymbol{g}_1 = g$ at weight one, and thus satisfies condition $\mathbf{E}_3$ in Assumption 2.1. In the present setting the family $\boldsymbol{g}$ has complex multiplication by K and can be explicitly described as follows.

Write $p \cdot \mathcal{O}_K = \mathfrak{p} \cdot \bar{\mathfrak{p}}$ with $\mathfrak{p}$ the prime of $\mathcal{O}_K$ of norm p corresponding to the embedding $i_p : \bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$ fixed at the outset. Let $\mathbf{A}_K^*$ be the group of idèles of K and set $U_{\mathfrak{p}} = K^* \cdot \mathbf{C}^* \cdot \prod_{\mathfrak{q} \neq \mathfrak{p}} \mathcal{O}_{\mathfrak{q}}^* \cdot \mu_{\mathfrak{p}}$, where $\mathcal{O}_{\mathfrak{q}}$ is the ring of integers of the completion of K at the prime ideal $\mathfrak{q}$ and $\mu_{\mathfrak{p}} = \mu_{p-1}$ is the torsion subgroup of $\mathcal{O}_{\mathfrak{p}}^*$. The kernel of the ideal map $G_{\mathfrak{p}} = \mathbf{A}_K^*/U_{\mathfrak{p}} \longrightarrow \mathrm{Pic}(\mathcal{O}_K)$ is equal to the group $1 + p\mathcal{O}_{\mathfrak{p}} = 1 + p\mathbf{Z}_p$ of principal units of $K_{\mathfrak{p}} \hookrightarrow \bar{\mathbf{Q}}_p^*$. Fix an extension

$$\varphi_{\mathfrak{p}} : \mathbf{A}_K^*/K^* \twoheadrightarrow G_{\mathfrak{p}} \longrightarrow \bar{\mathbf{Q}}_p^*$$

of the character $1 + p\mathcal{O}_{\mathfrak{p}} \longrightarrow \bar{\mathbf{Q}}_p^*$ sending the principal unit u to its inverse u^{-1}. By construction $\varphi_{\mathfrak{p}}$ is an algebraic p-adic Hecke character of weights $(1, 0)$, conductor $\mathfrak{p}$ and central character the Teichmüller lift $\omega : \mathbf{F}_p^* \simeq \mu_{p-1}$. The character

$$\psi_{\mathfrak{p}} : \mathbf{A}_K^*/K^* \longrightarrow \mathbf{C}^*$$

which on the class of the idèle $x = (x_v)_v$ takes the value

$$\psi_{\mathfrak{p}}(x) = i_\infty \circ i_p^{-1}\big(\varphi_{\mathfrak{p}}(x) \cdot x_{\mathfrak{p}}\big) \cdot x_\infty^{-1}$$

(where $i_\infty : \bar{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $i_p : \bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$ are the field embeddings fixed at the outset) is then an algebraic Hecke character of infinity type $(1, 0)$ and conductor $\mathfrak{p}$. Let I_K (resp., $I_K(\mathfrak{p})$) be the group of fractional ideals of K (resp., coprime with $\mathfrak{p}$). With a slight abuse of notation, we denote again by $\psi_{\mathfrak{p}} : I_K(\mathfrak{p}) \longrightarrow \bar{\mathbf{Q}}^*$ the character sending $\mathfrak{a}$ to (the image under i_∞^{-1} of) $\prod_{\mathfrak{q}|\mathfrak{a}} \psi_{\mathfrak{p}}(\pi_{\mathfrak{q}})^{\mathrm{ord}_{\mathfrak{q}}(\mathfrak{a})}$, where $\pi_{\mathfrak{q}}$ is a uniformiser of the completion of K at the prime $\mathfrak{q}$. Enlarging L if necessary, assume it contains the values of (the composition of i_p with) $\psi_{\mathfrak{p}}$ and write $\langle \psi_{\mathfrak{p}} \rangle$ for the composition of $\psi_{\mathfrak{p}}$ with projection onto the group of principal units of $\mathcal{O}_L$. For $U_{\boldsymbol{g}}$ as above, let

$$\boldsymbol{\psi} : I_K(\mathfrak{p}) \longrightarrow \mathscr{O}_{\boldsymbol{g}}^*$$

be the unique character satisfying $\boldsymbol{\psi}(\mathfrak{a})(l) = \langle \psi_{\mathfrak{p}} \rangle (\mathfrak{a})^{l-1}$ for each $\mathfrak{a}$ in I_K and each l in $U_{\boldsymbol{g}} \cap \mathbf{Z}_{\geqslant 1}$. The sought for Hida family $\boldsymbol{g}$ is then given by

$$\boldsymbol{g} = \sum \boldsymbol{\psi}(\mathfrak{a}) \cdot q^{\mathbf{N}\mathfrak{a}},$$

where the sum is over the non-zero ideals $\mathfrak{a}$ of $\mathcal{O}_K$ coprime to $\mathfrak{p}$ and $\mathbf{N}\mathfrak{a} = |\mathcal{O}_K/\mathfrak{a}|$. In particular, for m in $(p-1) \cdot \mathbf{Z}_{\geqslant 1}$, extend the m-th power of $\psi_{\mathfrak{p}}$ to a Hecke character

$$\psi_m : I_K \longrightarrow \bar{\mathbf{Q}}^* \tag{21}$$

of weights $(m, 0)$ and trivial conductor by setting $\psi_m(\mathfrak{p}) = \psi_{\mathfrak{p}}(\bar{\mathfrak{p}})^{-m} \cdot p^m$, so that the theta series (cf. Theorem 4.8.3 of [Miy89])

$$\vartheta(\psi_m) = \sum_{\mathfrak{a} \text{ non-zero ideal of } \mathcal{O}_K} \psi_m(\mathfrak{a}) \cdot q^{\mathbf{N}\mathfrak{a}} \in S_{m+1}(-d_K, \varepsilon_K)$$

is a cuspidal primitive form of weight $m+1$, level $\Gamma_1(-d_K)$ and character ε_K. Then for each integer l in $U_{\boldsymbol{g}} \cap \mathbf{Z}_{>1}$ which is congruent to one modulo $q_L - 1$, with q_L the cardinality of the residue field of L, the weight-l specialisation of $\boldsymbol{g}$ is equal to the ordinary p-stabilisation of $\vartheta(\psi_{l-1})$, viz. $\boldsymbol{g}_l = \vartheta(\psi_{l-1})(q) - \psi_{l-1}(\mathfrak{p}) \cdot \vartheta(\psi_{l-1})(q^p)$.

For each m in $(p-1) \cdot \mathbf{Z}$ write $\varphi_m : G_K \longrightarrow \bar{\mathbf{Q}}_p^*$ for the p-adic Galois character corresponding to ψ_m by global class field theory, so that the dual Deligne representation $V(\vartheta(\psi_m))$ associated with $\vartheta(\psi_m)$ (cf. Section 2.2) is isomorphic to the induced $\mathrm{Ind}_K^{\mathbf{Q}} \varphi_m$ from G_K to $G_{\mathbf{Q}}$ of $L(\varphi_m)$. As above, there exists a unique character

$$\boldsymbol{\varphi} : G_K \longrightarrow \mathscr{O}_{\boldsymbol{g}}^*$$

specialising to φ_{l-1} at each integer l in $U_{\boldsymbol{g}}$ which is congruent to one modulo $q_L - 1$. Denote by $\mathrm{Ind}_K^{\mathbf{Q}} \boldsymbol{\varphi}$ the induced from G_K to $G_{\mathbf{Q}}$ of $\boldsymbol{\varphi}$, viz. the free rank-two $\mathscr{O}_{\boldsymbol{g}}$-module of maps $\xi : G_{\mathbf{Q}} \longrightarrow \mathscr{O}_{\boldsymbol{g}}$ satisfying $\xi(\tau\sigma) = \boldsymbol{\varphi}(\tau) \cdot \xi(\sigma)$ for each τ in G_K and each σ in $G_{\mathbf{Q}}$, equipped with the $G_{\mathbf{Q}}$-action defined by $(\sigma \cdot \xi)(\sigma') = \xi(\sigma'\sigma)$ for each σ and σ' in $G_{\mathbf{Q}}$. The $\mathscr{O}_{\boldsymbol{g}}$-adic representations $V(\boldsymbol{g})$ (cf. Section 2.3) and $\mathrm{Ind}_K^{\mathbf{Q}} \boldsymbol{\varphi}$ are irreducible and unramified outside $d_K p$. Moreover, for each prime ℓ not dividing $d_K p$, an arithmetic Frobenius at ℓ acts on them with trace $a_\ell(\boldsymbol{g})$. It follows that $V(\boldsymbol{g})$ and $\mathrm{Ind}_K^{\mathbf{Q}} \boldsymbol{\varphi}$ become isomorphic after base change to the fraction field of $\mathscr{O}_{\boldsymbol{g}}$. Shrinking $U_{\boldsymbol{g}}$ if necessary this implies the existence of an isomorphism of $\mathscr{O}_{\boldsymbol{g}}[\pi^{-1}][G_{\mathbf{Q}}]$-modules

$$V(\boldsymbol{g})[\pi^{-1}] \simeq \mathrm{Ind}_K^{\mathbf{Q}} \boldsymbol{\varphi}[\pi^{-1}], \tag{22}$$

where π is a generator of the ideal of functions in $\mathscr{O}_{\boldsymbol{g}}$ which vanish at $= 1$. Actually one has the following consequence of Proposition 2.2.

Proposition 4.1. — *The $\mathscr{O}_{\boldsymbol{g}}[G_{\mathbf{Q}}]$-modules $V(\boldsymbol{g})$ and $\mathrm{Ind}_K^{\mathbf{Q}} \boldsymbol{\varphi}$ are isomorphic.*

Proof. — Let c in $G_{\mathbf{Q}}$ denote complex conjugation, and let $\boldsymbol{\varphi}^c$ be the conjugate of $\boldsymbol{\varphi}$ by c (so that $\boldsymbol{\varphi}^c(\sigma) = \boldsymbol{\varphi}(c \cdot \sigma \cdot c)$ for each σ in G_K).

It is sufficient to prove that the restriction of $V(\boldsymbol{g})$ to G_K is isomorphic to the direct sum of $\mathscr{O}_{\boldsymbol{g}}(\boldsymbol{\varphi})$ and $\mathscr{O}_{\boldsymbol{g}}(\boldsymbol{\varphi}^c)$. (Indeed, if this the case, c maps $V(\boldsymbol{g})^{G_K = \boldsymbol{\varphi}}$ isomorphically onto $V(\boldsymbol{g})^{G_K = \boldsymbol{\varphi}^c}$, i.e. $V(\boldsymbol{g}) = \mathscr{O}_{\boldsymbol{g}} \cdot \boldsymbol{v} \oplus \mathscr{O}_{\boldsymbol{g}} \cdot c(\boldsymbol{v})$ for any $\mathscr{O}_{\boldsymbol{g}}$-basis $\boldsymbol{v}$ of $V(\boldsymbol{g})^{G_K = \boldsymbol{\varphi}}$.) This in turn follows from the existence of an isomorphism of $\mathscr{O}_{\boldsymbol{g}}[G_{\mathbf{Q}_p}]$-modules between $V(\boldsymbol{g})$ and $V(\boldsymbol{g})^+ \oplus V(\boldsymbol{g})^-$. Indeed, assume that $V(\boldsymbol{g})$ is equal to $\mathscr{O}_{\boldsymbol{g}} \cdot \boldsymbol{v}^+ \oplus \mathscr{O}_{\boldsymbol{g}} \cdot \boldsymbol{v}^-$, with $G_{\mathbf{Q}_p}$ acting on $\boldsymbol{v}^+$ and $\boldsymbol{v}^-$ via the characters $\chi_{\mathrm{cyc}}^{-1} \cdot \breve{a}_p(\boldsymbol{g})^{-1}$ and $\breve{a}_p(\boldsymbol{g})$ respectively (cf. Equation (5)). For each integer $l \geqslant 3$ in $U_{\boldsymbol{g}}$ congruent to 1 modulo $q_L - 1$, the weight-l specialisation of $\mathscr{O}_{\boldsymbol{g}} \cdot \boldsymbol{v}^-$ is the maximal $G_{\mathbf{Q}_p}$-unramified quotient of the representation $V(\boldsymbol{g}_l)$, which is isomorphic to $\mathrm{Ind}_K^{\mathbf{Q}} \varphi_l$ as an $L[G_{\mathbf{Q}}]$-module. It follows that the specialisation at l of $\mathscr{O}_{\boldsymbol{g}} \cdot \boldsymbol{v}^-$ is isomorphic to the G_K-invariant line $L(\varphi_l^c)$ of $\mathrm{Ind}_K^{\mathbf{Q}} \varphi_l$. As a consequence $\mathscr{O}_{\boldsymbol{g}} \cdot \boldsymbol{v}^-$ is a G_K-invariant direct summand of $V(\boldsymbol{g})$

isomorphic to $\mathscr{O}_{\boldsymbol{g}}(\boldsymbol{\varphi}^c)$. Similarly one shows that $\mathscr{O}_{\boldsymbol{g}} \cdot \boldsymbol{v}^+$ is a G_K-invariant submodule of $V(\boldsymbol{g})$ isomorphic to $\mathscr{O}_{\boldsymbol{g}}(\boldsymbol{\varphi})$.

For $\cdot = \emptyset, \pm$, set $W^\cdot = V(\boldsymbol{g})^\cdot \otimes_{\mathscr{O}_{\boldsymbol{g}}} \mathrm{Hom}_{\mathscr{O}_{\boldsymbol{g}}}(V(\boldsymbol{g})^-, \mathscr{O}_{\boldsymbol{g}})$, so that W^- is naturally isomorphic to $\mathscr{O}_{\boldsymbol{g}}$. The short exact sequence $V(\boldsymbol{g})^+ \hookrightarrow V(\boldsymbol{g}) \twoheadrightarrow V(\boldsymbol{g})^-$ yields a short exact sequence $W^+ \hookrightarrow W \twoheadrightarrow \mathscr{O}_{\boldsymbol{g}}$, which corresponds to an element

$$w \in H^1(\mathbf{Q}_p, W^+)[\pi^\infty]$$

by Equation (22), where $\cdot[\pi^\infty]$ is the set of elements of the $\mathscr{O}_{\boldsymbol{g}}$-module $\cdot$ which are killed by a power of π. We have to prove that w is zero.

Set $W_1^+ = W^+ \otimes_1 L$ and consider the composition

$$\partial : W_1^+ = H^0(\mathbf{Q}_p, W_1^+) \simeq H^1(\mathbf{Q}_p, W^+)[\pi] \longrightarrow H^1(\mathbf{Q}_p, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} W_1^+,$$

where the isomorphism is the connecting morphism arising from multiplication by π on W^+ and the arrow is induced by specialisation at weight one (i.e. reduction modulo π). Identify $H^1(\mathbf{Q}_p, \mathbf{Q}_p)$ with the group of continuous $\mathbf{Q}_p$-valued morphisms on $\mathbf{Q}_p^*$ via the local Artin map sending p^{-1} to an arithmetic Frobenius. A direct computation shows that for each x in W_1^+, the restriction of $\partial(x)$ to $\mathbf{Z}_p^*$ is equal to $\log_p \otimes x$. In particular the map ∂ is non-zero, so that

$$H^1(\mathbf{Q}_p, W^+)[\pi^\infty] = H^1(\mathbf{Q}_p, W^+)[\pi] \simeq W_1^+$$

is killed by π, and w is zero precisely if its weight one specialisation $w(1)$ in $H^1(\mathbf{Q}_p, W_1^+)$ is. On the other hand, Proposition 2.2 proves that $G_{\mathbf{Q}_p}$ acts trivially on $W \otimes_1 L \simeq V(g)$, i.e. $w(1)$ is zero, thus concluding the proof of the proposition. □

Fix an isomorphism of $\mathscr{O}_{\boldsymbol{g}}[G_{\mathbf{Q}}]$-modules

$$\gamma : V(\boldsymbol{g}) \cong \mathrm{Ind}_K^{\mathbf{Q}} \boldsymbol{\varphi}. \tag{23}$$

Since p splits in K, the restrictions of $\mathrm{Ind}_K^{\mathbf{Q}} \boldsymbol{\varphi}$ to G_K and $G_{\mathbf{Q}_p}$ both decompose as the direct sum of $\boldsymbol{\varphi}$ and its complex conjugate $\boldsymbol{\varphi}^c$, with $\boldsymbol{\varphi}^c|_{G_{\mathbf{Q}_p}}$ unramified and mapping an arithmetic Frobenius to the p-th Fourier coefficient $a_p(\boldsymbol{g}) = \boldsymbol{\psi}(\bar{\mathfrak{p}})$ of $\boldsymbol{g}$. Accordingly the restriction of $V(\boldsymbol{g})$ to $G_{\mathbf{Q}_p}$ decomposes as the direct sum (cf. the previous proof)

$$V(\boldsymbol{g}) = V(\boldsymbol{g})^+ \oplus V(\boldsymbol{g})^-, \quad \text{with} \quad \gamma(V(\boldsymbol{g})^+) = \boldsymbol{\varphi}|_{G_{\mathbf{Q}_p}} \quad \text{and} \quad \gamma(V(\boldsymbol{g})^-) = \boldsymbol{\varphi}^c|_{G_{\mathbf{Q}_p}}.$$

With the notations of Section 2.3, the rank-one (φ, Γ)-modules $D(\boldsymbol{g})^\pm$ over the ring $\mathscr{R}_{\boldsymbol{g}} = \mathscr{R} \hat{\otimes}_{\mathbf{Q}_p} \mathscr{O}_{\boldsymbol{g}}$ are the images of the $\mathscr{O}_{\boldsymbol{g}}$-adic representations $V(\boldsymbol{g})^\pm$ under the Berger–Colmez functor $\mathbf{D}^\dagger_{\mathrm{rig}, \mathscr{O}_{\boldsymbol{g}}}$.

Write $V(g) = V(\boldsymbol{g}) \otimes_1 \mathbf{Q}_p$ for the base change of $V(\boldsymbol{g})$ along evaluation at $= 1$ on $\mathscr{O}_{\boldsymbol{g}}$. Similarly define the $G_{\mathbf{Q}_p}$-submodules

$$V(g)^+ = V(\boldsymbol{g})^+ \otimes_1 \mathbf{Q}_p \quad \text{and} \quad V(g)^- = V(\boldsymbol{g})^- \otimes_1 \mathbf{Q}_p$$

of $V(g) = V(g)^+ \oplus V(g)^-$. The isomorphism (23) specialises to an isomorphism of $G_{\mathbf{Q}}$-modules (denoted by the same symbol)

$$\gamma : V(g) \cong \big(\mathbf{1} \oplus \varepsilon_K\big) \otimes_{\mathbf{Q}} L,$$

where $\mathbf{1}$ and ε_K are shorthands for the trivial $G_{\mathbf{Q}}$-representation $\mathbf{Q}$ and its twist by ε_K respectively. Let $\boldsymbol{v}^+$ and $\boldsymbol{v}^-$ be the canonical $\mathscr{O}_{\boldsymbol{g}}$-bases of the G_K-submodules $\boldsymbol{\varphi}$

and $\boldsymbol{\varphi}^c$ of $\mathrm{Ind}_K^{\mathbf{Q}}\boldsymbol{\varphi}$, viz. maps $\boldsymbol{v}^{\pm} : G_{\mathbf{Q}} \longrightarrow \mathscr{O}_{\boldsymbol{g}}$ defined by $(\boldsymbol{v}^+(1), \boldsymbol{v}^+(c)) = (1, 0)$ and $(\boldsymbol{v}^-(1), \boldsymbol{v}^-(c)) = (0, 1)$, where c is complex conjugation. Set $\boldsymbol{v}_g^{\pm} = \gamma^{-1}(\boldsymbol{v}^{\pm})$ in $V(\boldsymbol{g})^{\pm}$, let $v_g^{\pm}$ in $V(g)^{\pm}$ be their weight-one specialisations and define

$$v_{g,\mathbf{1}} = v_g^+ + v_g^- \quad \text{and} \quad v_{g,\varepsilon_K} = v_g^+ - v_g^-. \tag{24}$$

By construction c exchanges the vectors $\boldsymbol{v}^+$ and $\boldsymbol{v}^-$, hence the elements $\gamma(v_{g,\mathbf{1}})$ and $\gamma(v_{g,\varepsilon_K})$ give $\mathbf{Q}$-bases of the $G_{\mathbf{Q}}$-representations $\mathbf{1}$ and ε_K respectively.

4.3. Comparison between Beilinson–Kato and Beilinson–Flach elements. — Let

$$\boldsymbol{\zeta}_f^{\mathrm{Kato}} \in H^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), V(f)) \quad \text{and} \quad \boldsymbol{\zeta}_{f\otimes\varepsilon_K}^{\mathrm{Kato}} \in H^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), V(f \otimes \varepsilon_K))$$

be the global Beilinson–Kato elements associated with f and its twist by ε_K respectively. They are characterised by Kato's explicit reciprocity law (1) and its analogue for $f \otimes \varepsilon_K$ respectively (with $(f \otimes \varepsilon_K)_\alpha = f_\alpha \otimes \varepsilon_K$). Note that the global representation $V(f \otimes \varepsilon_K)$ is isomorphic to the twist $V(f) \otimes \varepsilon_K$ of $V(f)$ by ε_K. Since p splits in $K/\mathbf{Q}$, the restriction to $G_{\mathbf{Q}_p}$ of $V(f) \otimes \varepsilon_K$ is equal to that of $V(f)$. An isomorphism of $L[G_{\mathbf{Q}}]$-modules $\imath : V(f \otimes \varepsilon_K) \longrightarrow V(f) \otimes \varepsilon_K$ then induces an isomorphism of filtered φ-modules between $V_{\mathrm{dR}}(f \otimes \varepsilon_K) = V_{\mathrm{st}}(f \otimes \varepsilon_K)$ and $V_{\mathrm{dR}}(f)$, which maps the canonical generator $\omega_{(f\otimes\varepsilon_K)^w}$ of $\mathrm{Fil}^0 V_{\mathrm{dR}}(f \otimes \varepsilon_K)$ to a non-zero multiple $u_\imath \cdot \omega_{f^w}$ of the generator ω_{f^w} of $\mathrm{Fil}^0 V_{\mathrm{dR}}(f)$ (cf. Section 2.2). Set

$$\boldsymbol{\zeta}_{f,\varepsilon_K}^{\mathrm{Kato}} = u_\imath^{-1} \cdot \imath_*(\boldsymbol{\zeta}_{f\otimes\varepsilon_K}^{\mathrm{Kato}}),$$

where

$$\imath_* : H^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), V(f \otimes \varepsilon_K)) \longrightarrow H^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), V(f) \otimes \varepsilon_K)$$

is the isomorphism induced by $\imath$, set $V(f,g) = V(f) \otimes_L V(g)$ and define

$$\mathbf{BK}^\alpha_{f\otimes g} = L_p(f_\alpha \otimes \varepsilon_K, 1 + \boldsymbol{s}) \cdot \boldsymbol{\zeta}_f^{\mathrm{Kato}} \otimes v_{g,\mathbf{1}} + L_p(f_\alpha, 1 + \boldsymbol{s}) \cdot \boldsymbol{\zeta}_{f,\varepsilon_K}^{\mathrm{Kato}} \otimes v_{g,\varepsilon_K}$$

in $H^1_{\mathrm{Iw}}(K(\mu_{p^\infty}), V(f,g)) \otimes_{\Lambda_\infty} \mathcal{O}(\mathcal{W})$. Since complex conjugation acts trivially on $\mathbf{BK}^\alpha_{f\otimes g}$, it descends to a class in $H^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), V(f,g)) \otimes_{\Lambda_\infty} \mathcal{O}(\mathcal{W})$.

Define the *balanced Iwasawa Selmer group*

$$H^1_{\mathrm{Iw,bal}}(\mathbf{Q}(\mu_{p^\infty}), V(f,g)) \hookrightarrow H^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}), V(f,g)) \otimes_{\Lambda_\infty} \mathcal{O}(\mathcal{W})$$

as in Section 2.4, after replacing $V(\boldsymbol{f}, \boldsymbol{g})$ and $\mathscr{F}^{ab}D(\boldsymbol{f}, \boldsymbol{g})$ with $V(f,g)$ and $\mathscr{F}^{ab}D(f,g) = D(f)^a_\alpha \otimes_L V(g)^b$ respectively in the definition of the local condition $H^1_{\mathrm{Iw,bal}}(\mathbf{Q}_p(\mu_{p^\infty}), V(\boldsymbol{f}, \boldsymbol{g}))$ (with $D(f)^\emptyset_\alpha = D(f)$ and $V(g)^\emptyset = V(g)$). Write

$$\varrho = \varrho_{f,g} : V(\boldsymbol{f}, \boldsymbol{g}) \longrightarrow V(f,g)$$

for the composition of the specialisation isomorphism (cf. Section 2.3)

$$\rho_{k_o} \hat{\otimes} \rho_1 : V(\boldsymbol{f}, \boldsymbol{g}) \otimes_{k_o,1} L \longrightarrow V(f_\alpha, g)$$

and the p-stabilisation isomorphism (cf. Section 2.2)

$$\Pi_{f_\alpha *} : V(f_\alpha) \longrightarrow V(f).$$

This induces a specialisation map

$$\varrho_* : H^1_{\mathrm{Iw,bal}}(\mathbf{Q}(\mu_{p^\infty}), V(\boldsymbol{f}, \boldsymbol{g})) \longrightarrow H^1_{\mathrm{Iw,bal}}(\mathbf{Q}(\mu_{p^\infty}), V(f,g)).$$

For each integer $c \geqslant 2$ coprime to $6N_f d_K p$, one defines the global Selmer class

$$_c\mathbf{BF}^{\alpha}_{f\otimes g} \in H^1_{\mathrm{Iw,bal}}(\mathbf{Q}(\mu_{p^\infty}), V(f,g))$$

by the identity (cf. Proposition 2.3)

$$\varrho_*\big({_c}\mathbf{BF}(\boldsymbol{f}\otimes\boldsymbol{g})\big) = \alpha(p-1)\left(1 - \frac{\mathbf{1}_{p^r}(p)\cdot p^{k_o-2}}{\alpha^2}\right)\left(1 - \frac{\mathbf{1}_{p^r}(p)\cdot p^{k_o-3}}{\alpha^2}\right)\cdot {_c}\mathbf{BF}^{\alpha}_{f\otimes g}.$$

Define finally the non-zero p-adic number $\Omega_{g,\gamma}$ in L^* (depending on the isomorphism γ fixed in Equation (23)) by the identity (cf. Equation (3))

$$\Omega_{g,\gamma} = 2\cdot \left\langle v_g^+, \omega_g\right\rangle_g . \tag{25}$$

The aim of this section is to prove the following result.

Theorem 4.2. — *The equality*

$$\Omega_{g,\gamma}\cdot {_c}\mathbf{BF}^{\alpha}_{f\otimes g} = \mathscr{A}_c\cdot \mathbf{BK}^{\alpha}_{f\otimes g}$$

holds in the balanced Iwasawa Selmer group $H^1_{\mathrm{Iw,bal}}(\mathbf{Q}(\mu_{p^\infty}), V(f,g))$ for an explicit element $\mathscr{A}_c = \mathscr{A}_{c,f_\alpha,K}$ in $\mathcal{O}(\mathcal{W})$ such that $\mathscr{A}_c(j)$ belongs to $K(\alpha)^*$ for each integer j.

Proof. — If χ denotes either ε_K or the trivial Dirichler character $\mathbf{1}$ and one sets $\boldsymbol{\zeta}^{\mathrm{Kato}}_{f,\mathbf{1}} = \boldsymbol{\zeta}^{\mathrm{Kato}}_{f}$, Kato's explicit reciprocity law (1) yields

$$\left\langle \mathrm{Log}_f\big(\mathrm{res}_p\big(\boldsymbol{\zeta}^{\mathrm{Kato}}_{f,\chi}\big)\big), \eta_f^{\alpha}\right\rangle_f = L_p(f_\alpha\otimes\chi, 1+\boldsymbol{s}). \tag{26}$$

By definition (cf. Equation (24)) the image of $\mathrm{res}_p(\mathbf{BK}^{\alpha}_{f\otimes g})$ under the map

$$H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), V(f,g))\otimes_{\Lambda_\infty}\mathcal{O}(\mathcal{W}) \longrightarrow H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), V(f)\otimes_L V(g)^-)\otimes_{\Lambda_\infty}\mathcal{O}(\mathcal{W})$$

induced by the projection $V(g)\longrightarrow V(g)^-$ is equal to the product of v_g^- and

$$L_p(f_\alpha\otimes\varepsilon_K, 1+\boldsymbol{s})\cdot \mathrm{res}_p\big(\boldsymbol{\zeta}^{\mathrm{Kato}}_f\big) - L_p(f_\alpha, 1+\boldsymbol{s})\cdot\mathrm{res}_p\big(\boldsymbol{\zeta}^{\mathrm{Kato}}_{f,\varepsilon_K}\big),$$

which according to Equation (26) belongs to the kernel of the composition

$$\left\langle \mathrm{Log}_f, \eta_f^{\alpha}\right\rangle_f : H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), V(f))\otimes_{\Lambda_\infty}\mathcal{O}(\mathcal{W}) \longrightarrow \mathcal{O}(\mathcal{W})$$

of the Perrin-Riou logarithm Log_f and the $\mathcal{O}(\mathcal{W})$-linear extension of the functional $\langle\cdot,\eta_f^{\alpha}\rangle$ on $V_{\mathrm{st}}(f)$. This composition factors through the morphism induced in cohomology by the projection $D(f)\longrightarrow D(f)^-_{\alpha}$, and the resulting map

$$\mathrm{Log}_f^- : H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), D(f)^-_{\alpha}) \longrightarrow \mathcal{O}(\mathcal{W})$$

is injective under the non-exceptionality assumption $a_p(f)\neq p^{k_o/2-1}$. (Indeed the kernel of Log_f^- equals the submodule of $D(f)^-_{\alpha}$ on which φ acts as multiplication by α_f^{-1}, which is zero unless α_f is a power of p. When p does not divide the conductor of f, this possibility is excluded by the Ramanujan–Petersson conjecture; when f is new at p one has $\alpha_f = a_p(f) = \pm p^{k_o/2-1}$, hence $\alpha_f = -p^{k_o/2-1}$ by assumption.) As a consequence the image of $\mathrm{res}_p(\mathbf{BK}^{\alpha}_{f\otimes g})$ in $H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), D(f)^-_{\alpha})\otimes_L V(g)^-$ is zero. In other words (cf. Equation (24))

$$p^-_{f,\alpha}\big(\mathrm{res}_p\big(\mathbf{BK}^{\alpha}_{f\otimes g}\big)\big) \in H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), D(f)^-_{\alpha})\otimes_L V(g)^+$$

is equal to

$$p_\alpha^-\Big(L_p(f_\alpha\otimes\varepsilon_K,1+\boldsymbol{s})\cdot\mathrm{res}_p\big(\boldsymbol{\zeta}_f^{\mathrm{Kato}}\big)+L_p(f_\alpha,1+\boldsymbol{s})\cdot\mathrm{res}_p\big(\boldsymbol{\zeta}_{f,\varepsilon_K}^{\mathrm{Kato}}\big)\Big)\otimes v_g^+, \tag{27}$$

where $p_{f,\alpha}^-$ and p_α^- are the maps induced by the projections $D(f,g)\longrightarrow D(f)_\alpha^-\otimes_L V(g)$ and $D(f)\longrightarrow D(f)_\alpha^-$ respectively. (Note that, since $G_{\mathbf{Q}_p}$ acts trivially on $V(g)$, the (φ,Γ)-module $D(f,g)=D(f)\otimes_{\mathscr{R}_L}D(g)$ is canonically isomorphic to $D(f)\otimes_L V(g)$.)

Let

$$\mathrm{Log}_{f\otimes g}^{-+}:H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),D(f)_\alpha^-)\otimes_L V(g)^+\longrightarrow\mathcal{O}(\mathcal{W})$$

be the morphism defined by the formulae

$$\mathrm{Log}_{f\otimes g}^{-+}(z\otimes v)=\langle v,\omega_g\rangle_g\cdot\mathrm{Log}_f^-(z)$$

for each z in $H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),D(f)_\alpha^-)$ and v in $V(g)^+$. Equations (26) and (27) yield

$$\mathrm{Log}_{f\otimes g}^{-+}\circ p_{f,\alpha}^-\circ\mathrm{res}_p\big(\mathbf{BK}_{f\otimes g}^\alpha\big)=\Omega_{g,\gamma}\cdot L_p(f_\alpha,1+\boldsymbol{s})\cdot L_p(f_\alpha\otimes\varepsilon_K,1+\boldsymbol{s}). \tag{28}$$

As above denote by

$$\varrho_*:H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),\mathscr{F}^{-+}D(\boldsymbol{f},\boldsymbol{g}))\longrightarrow H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),\mathscr{F}^{-+}D(f,g))$$

the map induced by the specialisation map $\rho_{k_o}\hat{\otimes}\rho_1$ and the p-stabilisation isomorphism $\Pi_{f_\alpha *}$. Lemma 8.4 of [BSV21b] and a direct comparison of the interpolation properties satisfied by Log_f and $\mathcal{L}^{-+}$ (cf. Section 2.4) show that the map

$$\mathrm{Log}_{f\otimes g}^{-+}\circ\varrho_*:H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}),\mathscr{F}^{-+}D(\boldsymbol{f},\boldsymbol{g}))\longrightarrow\mathcal{O}(\mathcal{W})$$

is equal to

$$(p-1)\alpha\left(1-\frac{\mathbf{1}_{p^r}(p)p^{k_o-2}}{\alpha^2}\right)\left(1-\frac{\mathbf{1}_{p^r}(p)p^{k_o-3}}{\alpha^2}\right)\cdot\mathrm{ev}_{k_o,1}\circ\big\langle\mathcal{L}^{-+},\eta_{\boldsymbol{f}}\otimes\omega_{\boldsymbol{g}}\big\rangle_{\boldsymbol{fg}},$$

where $\mathrm{ev}_{k_o,1}$ is evaluation at weights $(k_o,1)$ on $\mathscr{O}_{\boldsymbol{fg}}$. (Recall that $N_f p^r$ is the conductor of f and note that the Euler factors in the previous equation are non-zero.) The explicit reciprocity law Proposition 2.3 then gives

$$\mathrm{Log}_{f\otimes g}^{-+}\circ p_{f,\alpha}^-\circ\mathrm{res}_p\big({}_c\mathbf{BF}_{f\otimes g}^\alpha\big)=\mathscr{M}_{f,c}\cdot L_p(f_\alpha,g,1+\boldsymbol{s}), \tag{29}$$

where $L_p(f_\alpha,g)$ is the specialisation of $L_p(\boldsymbol{f},\boldsymbol{g})$ at weights $(k_o,1)$ and

$$\pm\mathscr{M}_{f,c}=N_f^{1-k_o/2}\cdot\big(c^2-c^{2\boldsymbol{s}-k_o+3}\cdot\varepsilon_K(c)\big).$$

(Since k_o is even, $\mathscr{M}_{f,c}(j)$ is a non-zero rational number for each integer j.)

We claim that one has the factorisation

$$L_p(f_\alpha,g)=\mathscr{A}\cdot L_p(f_\alpha)\cdot L_p(f_\alpha\otimes\varepsilon_K) \tag{30}$$

in $\mathcal{O}(\mathcal{W})$, where $\mathscr{A}=\mathscr{A}_{f_\alpha,K}$ is an explicit unit in $\mathcal{O}(\mathcal{W})^*$ such that $\mathscr{A}(j)$ belongs to $K(\alpha)^*$ for each j in $\mathbf{Z}$. Indeed, for χ equal to either $\mathbf{1}$ or ε_K, let $L_p(\boldsymbol{f}\otimes\chi)$ in $\mathcal{O}(U_{\boldsymbol{f}}\times\mathcal{W})$ be the two-variable Mazur–Kitagawa p-adic L-function attached to $\boldsymbol{f}$ (cf. [Bel12]). For each good classical point k in $U_{\boldsymbol{f}}$, each j in $\mathbf{Z}_{\geqslant 0}$ and each finite order character $\sigma:\mathbf{Z}_p^*\longrightarrow\bar{\mathbf{Q}}^*$, one has $L_p(\boldsymbol{f}\otimes\chi)(k,\sigma+j)=\lambda_k^\pm\cdot L_p(f_k\otimes\chi)(\sigma+j)$ with $\chi\sigma(-1)=\pm1$, where $\lambda_k^\pm$ are non-zero elements in L^* such that $\lambda_{k_o}^\pm=1$. These properties characterise $L_p(\boldsymbol{f}\otimes\chi)$ up to multiplication by a unit in $\mathcal{O}(U_{\boldsymbol{f}})$ taking the value one at $\boldsymbol{k} = k_o$. Define $L_p(\boldsymbol{f}, g)$ to be the restriction of $L_p(\boldsymbol{f}, \boldsymbol{g})$ to the plane $= 1$. Then the set $\mathscr{X}$ of pairs (k, j) in $U_{\boldsymbol{f}}^{\mathrm{cl}} \times \mathbf{Z}$ with k good and $1 \leqslant j \leqslant k-1$ is dense in $U_{\boldsymbol{f}} \times \mathcal{W}$ and contained in the interpolation domains of $L_p(\boldsymbol{f}, g)$ and $L_p(\boldsymbol{f} \otimes \chi)$. For each (k, j) in $\mathscr{X}$ one has

$$L_p(\boldsymbol{f}, g)(k, j) = \frac{a_K(k, j)}{\lambda_k^+ \lambda_k^- \left(1 - \frac{\beta_k}{\alpha_k}\right)\left(1 - \frac{\beta_k}{p\alpha_k}\right)} \cdot L(\boldsymbol{f})(k, j) \cdot L(\boldsymbol{f} \otimes \varepsilon_K)(k, j),$$

where a_K is a simple explicit unit in $\mathcal{O}(U_{\boldsymbol{f}} \times \mathcal{W})^*$ with $a_K(x)$ in K^* for x in $U_{\boldsymbol{f}}^{\mathrm{cl}} \times \mathbf{Z}$ and where one sets $\alpha_k = a_p(\boldsymbol{f}_k)$ and $\beta_k = p^{k_o - 1}/\alpha_k$. According to Theorem 3.4 of [**BD14**] and Section 5 of [**BSV21a**], the p-adic periods

$$\mathrm{Per}_p(k) = \lambda_k^+ \lambda_k^- (1 - \beta_k/\alpha_k)(1 - \beta_k/p\alpha_k)$$

are interpolated by a unit in $\mathcal{O}(U_{\boldsymbol{f}})^*$, whose value at k_o is equal to $\mathrm{Per}_p(k_o)$, respectively belongs to $\mathbf{Q}^*$, if p does not divide the conductor of f, respectively f is p-new. (In loco citato $\boldsymbol{f}$ is assumed to be ordinary, but the arguments readily generalise to the present setting.) One deduces that $L_p(\boldsymbol{f}, g)$ factors as the product of $L_p(\boldsymbol{f}) \cdot L_p(\boldsymbol{f} \otimes \varepsilon_K)$ and an explicit unit which takes values in $K(\alpha)^*$ on classical points. The weight-k_o specialisation of this factorisation yields Equation (30).

Set $\mathscr{A}_c = \mathscr{A} \cdot \mathscr{M}_{f,c}$. Equations (28)–(30) show that the difference between the classes $\Omega_{g,\gamma} \cdot {}_c\mathbf{BF}^{\alpha}_{f \otimes g}$ and $\mathscr{A}_c \cdot \mathbf{BK}^{\alpha}_{f \otimes g}$ is killed by the linear form $\mathrm{Log}^{-+}_{f \otimes g} \circ p^-_{f,\alpha} \circ \mathrm{res}_p$, hence by $p^-_{f,\alpha} \circ \mathrm{res}_p$ (since as observed above Log^-_f, and then $\mathrm{Log}^{-+}_{f \otimes g}$, is injective in the present non-exceptional setting). In other words this difference defines an element of the triangulinе Selmer group $\mathrm{Sel}_{\mathrm{Iw}}(K(\mu_{p^\infty}), V(f))$ of classes in $H^1_{\mathrm{Iw}}(K(\mu_{p^\infty}), V(f)) \otimes_{\Lambda_\infty} \mathcal{O}(\mathcal{W})$ which are unramified at each prime different from p and which map to zero in the semi-local cohomology group $H^1_{\mathrm{Iw}}(K_p(\mu_{p^\infty}), D(f)^-_\alpha)$. For each finite order character μ of G_∞, the base change of the finite torsion-free module $\mathrm{Sel}_{\mathrm{Iw}}(K(\mu_{p^\infty}), V(f))$ along the morphism $\mu \cdot \chi_{\mathrm{cyc}}^{1-k_o/2} : \Lambda_\infty \longrightarrow \mathbf{Q}_p(\mu)$ is isomorphic to a submodule of the Bloch–Kato Selmer group $\mathrm{Sel}(K, \mathcal{V}(f \otimes \mu^{-1}))$ of $\mathcal{V}(f \otimes \mu^{-1}) = V(f)(1 - k_o/2) \otimes \mu^{-1}$ over K. According to the main results of [**Roh84, Roh88**], for each $0 \leqslant i \leqslant p-1$ there exists μ such that the complex L-values $L(f \otimes \mu, k_o/2)$ and $L(f \otimes \mu\varepsilon_K, k_o/2)$ are non-zero and $\mu|_{\mathbf{F}_p^*} = \omega^i$, where we identify G_∞ with $\mathbf{Z}_p^*$ via χ_{cyc} and $\omega : \mathbf{F}_p^* \longrightarrow \mathbf{Z}_p^*$ is the Teichmüller character. For such characters, Kato's theorem [**Kat04**, Introduction] implies that the Bloch–Kato Selmer group $\mathrm{Sel}(K, \mathcal{V}(f \otimes \mu^{-1}))$ vanishes. As a consequence $\mathrm{Sel}_{\mathrm{Iw}}(K(\mu_{p^\infty}), V(f))$ is trivial, thus concluding the proof of the theorem. □

4.4. Heegner classes. — Let $n \geqslant 4$ be an integer such that (K, n) satisfies the Heegner condition, let $\mathfrak{n}$ be an ideal of K of norm n and let H be the Hilbert class field of K. Fix an elliptic curve E over H with complex multiplication by the maximal order $\mathcal{O}_K$ of K and good reduction at the prime of H associated with the embedding $i_p : \bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$ fixed at the outset. We identify $\mathcal{O}_K$ with $\mathrm{End}_H(E)$ via the isomorphism $[\cdot]$ satisfying $[\lambda]^*\omega = \lambda \cdot \omega$ for each ω in $\Gamma(E, \Omega^1_{E/H})$. Choose a generator $t_{\mathfrak{n}}$ of the $\mathfrak{n}$-torsion subgroup $E_{\mathfrak{n}}$ of E. Then the isomorphism class of the pair $(E, t_{\mathfrak{n}})$ defines a closed point $i_E : \mathrm{Spec}(F) \longrightarrow Y_1(n)_F$ of the modular curve $Y_1(n)_F = Y_1(n) \otimes_{\mathbf{Z}[1/n]} F$ of level $\Gamma_1(n)$ over a finite abelian extension F of H.

For each positive integer r define the p-adic étale sheaves

$$\mathscr{S}_r = \mathrm{Symm}^r_{\mathbf{Z}_p} R^1(E_1(n) \longrightarrow Y_1(n))_* \mathbf{Z}_p \quad \text{and} \quad \mathscr{H}_r(E) = \mathrm{Symm}^r_{\mathbf{Z}_p} H^1_{\text{ét}}(E_{\bar{\mathbf{Q}}}, \mathbf{Z}_p)$$

on $Y_1(n)$ and $\mathrm{Spec}(H)$ respectively, where $E_1(n) \longrightarrow Y_1(n)$ is the universal elliptic curve. Then (the restriction to $\mathrm{Spec}(F)$ of) $\mathscr{H}_r(E)$ is canonically isomorphic to the pull-back $i_E^*(\mathscr{S}_r)$ of (the restriction to $Y_1(n)_F$ of) $\mathscr{S}_r$ along the closed immersion i_E. This yields a push-forward

$$i_{E*} : H^0_{\text{ét}}\big(F, \mathscr{H}_{2r}(E)(r)\big) \longrightarrow H^2_{\text{ét}}\big(Y_1(n)_F, \mathscr{S}_{2r}(r+1)\big).$$

The p-adic Tate module $T_p(E) = H^1_{\text{ét}}(E_{\bar{\mathbf{Q}}}, \mathbf{Z}_p(1))$ of E decomposes as the direct sum of the one-dimensional p-adic representations χ_E and $\bar{\chi}_E$ for a Hecke character $\chi_E : G_H \longrightarrow \mathbf{Z}_p^*$. Let x_E and y_E be any generators of the lines $\chi_E(-1)$ and $\bar{\chi}_E(-1)$ of $\mathscr{H}_1(E)$ respectively, which pair to one under the Weil pairing. Then

$$H^0_{\text{ét}}(H, \mathscr{H}_r(E)(r)) = \mathbf{Z}_p \cdot x_E^r y_E^r,$$

where the *canonical* invariant $x_E^r y_E^r$ is the image of $x_E^{\otimes r} \otimes y_E^{\otimes r}$ in $\mathscr{H}_1(E)^{\otimes 2r}$ in the symmetric quotient $\mathscr{H}_r(E)$.

Let $\xi = \sum_{n \geqslant 1} a_n(\xi) \cdot q^n$ in $S_{2r+2}(\Gamma_0(n))_L$ be a normalised cuspidal eigenform of weight $2r+2$, level $\Gamma_0(n)$ and Fourier coefficients in L. Recall the p-adic sheaf $\mathscr{L}_i = \mathrm{Tsym}^i(E_1(n) \longrightarrow Y_1(n))_* \mathbf{Z}_p(1)$, so that the dual Deligne representation $V(\xi)$ of ξ is the maximal L-quotient of $H^1_{\text{ét}}(Y_1(n)_{\bar{\mathbf{Q}}}, \mathscr{L}_{2r}(1)) \otimes_{\mathbf{Z}_p} L$ on which the dual Hecke operator T'_ℓ acts as $a_\ell(\xi)$ for each prime ℓ (cf. Section 2.2). As explained in [**BSV21b**, Section 3], there is a natural isomorphism $\mathtt{s}_i$ between the $\mathbf{Q}_p$-linear extension of $\mathscr{S}_i(i)$ and that of $\mathscr{L}_i$ and one writes

$$\mathrm{pr}_\xi : H^1_{\text{ét}}(Y_1(n)_{\bar{\mathbf{Q}}}, \mathscr{S}_{2r}(r+1))_{\mathbf{Q}_p} \longrightarrow V(\xi) \otimes \chi_{\mathrm{cyc}}^{-r} = \mathcal{V}(\xi)$$

for the composition of the ξ-isotypic projection with the map induced by $\mathtt{s}_{2r}$. Define

$$z_E(\xi) = \mathrm{pr}_{\xi *} \circ \mathtt{HS}_{\text{ét}} \circ i_{E*}(x_E^r y_E^r) \in \mathrm{Sel}(H, \mathcal{V}(\xi))$$

to be the image of the invariant $x_E^r y_E^r$ under the composition $\mathrm{pr}_{\xi *} \circ \mathtt{HS}_{\text{ét}} \circ i_{E*}$, where $\mathrm{pr}_{\xi *}$ is the map induced in G_F-cohomology by pr_ξ and

$$\mathtt{HS}_{\text{ét}} : H^2_{\text{ét}}(Y_1(n)_F, \mathscr{S}_{2r}(r+1)) \longrightarrow H^1(G_F, H^1_{\text{ét}}(Y_1(n)_{\bar{\mathbf{Q}}}, \mathscr{S}_{2r})(r+1))$$

is the morphism arising from the Hochschild–Serre spectral sequence. The fact that $z_E(\xi)$ belongs to the Bloch–Kato Selmer group $\mathrm{Sel}(F, \mathcal{V}(\xi))$ is a consequence of [**NN16**, Theorem 5.9]. Moreover, because ξ is a form of level $\Gamma_0(n)$ and the isomorphism class of the pair $(E, \mathbf{Z} \cdot t_{\mathfrak{n}})$ defines an H-rational point of the modular curve $Y_0(n)$, the class $z_E(\xi)$ is fixed by the action of $\mathrm{Gal}(F/H)$ on $\mathrm{Sel}(F, \mathcal{V}(\xi))$, hence can naturally be viewed as an element of the Selmer group of $\mathcal{V}(\xi)$ over the Hilbert class field H of K. Define finally the *Heegner class* of (ξ, K) by

$$z_K(\xi) = \mathrm{Trace}_{H/K}\big(z_E(\xi)\big) \in \mathrm{Sel}(K, \mathcal{V}(\xi)).$$

4.5. Comparison between Beilinson–Flach and Heegner classes. — Set $\mathcal{V}(f,g) = V(f,g)(1-k_o/2)$. As explained in Section 1.1, evaluation at an integer i in $\mathcal{W}$ induces a morphism χ_{cyc}^i from $H^1_{\mathrm{Iw,bal}}(\mathbf{Q}(\mu_{p^\infty}), V(f,g))$ to $H^1(\mathbf{Q}, V(f,g)(-j))$ (cf. the definition of the character $[\cdot] : G_\infty \longrightarrow \mathcal{O}(\mathcal{W})^*$ in Section 2.4). Recall the balanced Iwasawa class ${}_c\mathbf{BF}^\alpha_{f\otimes g}$ introduced in Section 4.3 and define

$$ {}_c\mathrm{BF}^\alpha_{f\otimes g} = \chi_{\mathrm{cyc}}^{k_o/2-1}\big({}_c\mathbf{BF}^\alpha_{f\otimes g}\big) \in H^1(\mathbf{Q}, \mathcal{V}(f,g)). $$

Let $u_{\mathfrak{p}}$ in $\mathcal{O}_K[1/p]^*$ be a generator of $\mathfrak{p}^{h_K}$, with h_K the class number of K.

Theorem 4.3. — *Assume that the complex Hecke L-series $L(f,s)$ vanishes at the central critical point $s = k_o/2$. Then the class ${}_c\mathrm{BF}^\alpha_{f\otimes g}$ belongs to the Bloch–Kato Selmer group* $\mathrm{Sel}(\mathbf{Q}, \mathcal{V}(f,g))$ *and the equality*

$$ \log_p(u_{\mathfrak{p}})\cdot \left\langle \log_p\big(\mathrm{res}_p\big({}_c\mathrm{BF}^\alpha_{f\otimes g}\big)\big), \omega_f\otimes\eta_g\right\rangle_{fg} = \log^2_{\omega_f}\big(\mathrm{res}_{\mathfrak{p}}\big(z_K(f)\big)\big). $$

holds in L up to multiplication by an explicit non-zero constant in the number field $K(a_n(f_\alpha);\, n \geqslant 1)$.

The proof of Theorem 4.3 occupies the rest of this section.

4.5.1. — This subsection briefly describes the main result of [**BDP13**]. With the notations of Section 4.4, set $n = N_f$, $\xi = f$ and write $\mathfrak{N}_f = \mathfrak{n}$.

Denote by $\mathscr{L}_{\mathfrak{p}}(f)$ the square-root anticyclotomic p-adic L-function associated in Section 5 of [**BDP13**] to the level-$\Gamma_0(N_f p^r)$ newform f, the prime $\mathfrak{p}$ of K and the data $(\mathfrak{N}_f, E, \omega_E)$, where ω_E is a non-zero invariant differential in $\Gamma(E, \Omega^1_{E/H})$. It is a continuous $\mathbf{C}_p$-valued function defined on a suitable p-adic completion $\hat{\Sigma}_{\mathrm{cc}}(f)$ of the set $\Sigma_{\mathrm{cc}}(f)$ of algebraic Hecke characters of K with conductor dividing $\mathfrak{N}_f$, trivial central character and infinity type $(k_o+a, -a)$ with a in $\mathbf{Z}$. For each character χ in $\Sigma_{\mathrm{cc}}(f)$ of infinity type $(k_o+j, -j)$ with $j \geqslant 0$, the square $\mathscr{L}_p(f,\chi)^2$ of the value of $\mathscr{L}_{\mathfrak{p}}(f)$ at χ is a non-zero explicit multiple of the central critical value $L(f, \bar{\chi}^{-1}, 0)$ of the Rankin–Selberg convolution of f and the theta series of weight k_o+1+2j associated with $\mathbf{N}^{k_o+j}\cdot\bar{\chi}^{-1}$. We refer to loc. cit. for the precise interpolation property satisfied by $\mathscr{L}_{\mathfrak{p}}(f)$, whose square is denoted there by $L_p(f)$. (Note that Section 5 of [**BDP13**] assumes that p does not divide the conductor of f, but the constructions and results readily generalise to the present semistable setting. More generally, one can easily define a $\mathbf{C}_p$-valued continuous function $\mathscr{L}_{\mathfrak{p}}(\boldsymbol{f})$ on $U_{\boldsymbol{f}} \times \hat{\Sigma}_{\mathrm{cc}}(f)$ which restricts to $\mathscr{L}_{\mathfrak{p}}(f_k)$ at each classical point k in $U^{\mathrm{cl}}_{\boldsymbol{f}}$.)

Note that the character $\mathbf{N}^{k_o/2}$ does *not* belong to the interpolation domain of $\mathscr{L}_{\mathfrak{p}}(f)$. The main result Theorem 5.13 of [**BDP13**] and its extension [**Cas18**, Theorem 2.11] to the p-semistable setting yield the identity

$$ (k_o/2-1)!\cdot\mathscr{L}_{\mathfrak{p}}(f, \mathbf{N}^{k_o/2}) = \left(1-\frac{\alpha}{p^{k_o/2}}\right)\left(1-\frac{\beta}{p^{k_o/2}}\right)\cdot\log_{\omega_f}\big(\mathrm{res}_{\mathfrak{p}}(z_K(f))\big). \tag{31} $$

Recall that α and β are the roots of the p-th Hecke polynomial of f, ordered in such a way that $\mathrm{ord}_p(\alpha) \leqslant \mathrm{ord}_p(\beta)$. In particular β is zero if f is p-new (i.e. if $r=1$) and the Euler factors which appear in the previous equation are non-zero.

4.5.2. — The aim of this subsection is to prove the following

Lemma 4.4. — *One has the equality*

$$\log_p(u_{\mathfrak{p}}) \cdot L_p(\boldsymbol{g},\boldsymbol{f})(k_o,1,k_o/2) = \mathscr{B} \cdot \mathscr{L}_{\mathfrak{p}}(f,\mathbf{N}^{k_o/2})^2,$$

where $\mathscr{B} = \mathscr{B}(f,K)$ is an explicit non-zero element of $K(a_n(f);\, n \geqslant 1)$.

Proof. — In the proof write $U_{\boldsymbol{g}}^{\mathrm{cl}}$ for the set of integers in $U_{\boldsymbol{g}}$ which are congruent to one modulo $q_L - 1$ (where q_L is the cardinality of the residue field of L, cf. Section 4.2). Set $\mathscr{X}^{\mathrm{cl}} = \{k_o\} \times U_{\boldsymbol{g}}^{\mathrm{cl}}$ and let $\mathscr{X}_\infty^{\mathrm{cl}}$ be the set of pairs (k_o,l) in $\mathscr{X}^{\mathrm{cl}}$ such that $l \geqslant k_o/2+1$. For each $x=(k_o,l)$ in $\mathscr{X}^{\mathrm{cl}}$ set (cf. Equation (21))

$$\nu_x = \mathbf{N}^{k_o/2-l+1} \cdot \psi_{2l-2} : I_K \longrightarrow \mathbf{C}^*.$$

Note that ν_x has infinity type $(k_o+j_x,-j_x)$ with $j_x = l-(k_o/2+1)$, so that $j_x \geqslant 0$ precisely if x belongs to $\mathscr{X}_\infty^{\mathrm{cl}}$.

For each $x=(k_o,l)$ in $\mathscr{X}_\infty^{\mathrm{cl}}$ the character ν_x belongs to the interpolation domain of $\mathscr{L}_{\mathfrak{p}}(f_k)$. According to [**BDP13**, Section 5] (and the functional equation satisfied by Rankin L-series) one has

$$\mathscr{L}_{\mathfrak{p}}(f,\nu_x)^2 = \mathscr{C}_1(l)\left(\frac{\Omega_p}{\Omega_\infty}\right)^{4l-4}\pi^{2l-3}\Gamma(l-k_o/2)\Gamma(k_o/2+l-1)\cdot$$

$$\cdot\left(1-\frac{\alpha}{\nu_x(\bar{\mathfrak{p}})}\right)^2\left(1-\frac{\beta}{\nu_x(\bar{\mathfrak{p}})}\right)^2 L\big(f\otimes\vartheta(\psi_{2l-2}),k_o/2+l-1\big). \tag{32}$$

Here $\Omega_p = \Omega_p(E,\omega_E)$ in $\mathbf{C}_p^*$ and $\Omega_\infty = \Omega_\infty(E,\omega_E)$ in $\mathbf{C}^*$ are the p-adic and complex periods associated in [**BDP13**] with the fixed pair (E,ω_E) and $\mathscr{C}_1 = \mathscr{C}_1(f,K)$ is a unit in $\mathscr{O}_{\boldsymbol{g}}$ such that, for each l in $U_{\boldsymbol{g}}\cap\mathbf{Z}$, the value $\mathscr{C}_1(l)$ is a non-zero explicit element of the number field $K(a_n(f);\, n\geqslant 1)$.

If $x=(k_o,l)$ belongs to $\mathscr{X}_\infty^{\mathrm{cl}}$, then the classical triple

$$\varkappa = (k_o, 2l-1, k_o/2+l-1)$$

belongs to the interpolation domain of $L_p(\boldsymbol{g},\boldsymbol{f})$, and (cf. [**KLZ17**, Theorem 2.7.4])

$$L_p(\boldsymbol{g},\boldsymbol{f})(\varkappa) = \frac{\Gamma(l-k_o/2)\Gamma(k_o/2+l-1)}{\pi^{2l-1}(-i)^{2l-1-k_o}2^{4l-3}}\,\frac{\left(1-\frac{\alpha}{\nu_x(\bar{\mathfrak{p}})}\right)^2\left(1-\frac{\beta}{\nu_x(\bar{\mathfrak{p}})}\right)^2}{\big(1-\mu_l(\mathfrak{p})p^{-1}\big)\big(1-\mu_l(\bar{\mathfrak{p}})^{-1}\big)}\cdot$$

$$\cdot\frac{L\big(f\otimes\vartheta(\psi_{2l-2}),k_o/2+l-1\big)}{\big\langle\vartheta(\psi_{2l-2}),\vartheta(\psi_{2l-2})\big\rangle_{-d_K}}, \tag{33}$$

where μ_l denotes the inverse of the algebraic Hecke character $\psi_{4l-4}^c \cdot \mathbf{N}^{1-2l}$. After setting $\mathscr{C}_2(l) = \mathscr{C}_1(l)\cdot(-i)^{2l-1-k_o}\cdot 2^{4l-3}$, Equations (32) and (33) yield the identity

$$\mathscr{C}_2(l)^{-1}\cdot\mathscr{L}_{\mathfrak{p}}(f,\nu_x)^2 = L_p(\boldsymbol{g},\boldsymbol{f})(\varkappa)\cdot \tag{34}$$

$$\cdot\left(\frac{\pi\cdot\Omega_p}{\Omega_\infty}\right)^{4l-4}\big(1-\mu_l(\mathfrak{p})p^{-1}\big)\big(1-\mu_l(\bar{\mathfrak{p}})^{-1}\big)\big\langle\vartheta(\psi_{2l-2}),\vartheta(\psi_{2l-2})\big\rangle_{-d_K}.$$

Let $L_{\mathfrak{p}}(K)$ be the Katz p-adic L-function associated to $(K,\mathfrak{p})$ in [**Kat76**]. It is a continuous $\mathbf{C}_p$-valued function on a suitable p-adic completion $\hat{\Sigma}_K$ of the set Σ_K of algebraic Hecke characters of K of trivial conductor and infinity type (a,b) with $a \geqslant 1$ and $b \leqslant 0$. The value of $L_{\mathfrak{p}}(K)$ at χ in Σ_K is an explicit multiple of the algebraic part of the complex special value $L(\chi^{-1},0)$. We refer to Section 3.2 of [DLR15] for a description of the interpolation property which characterises $L_{\mathfrak{p}}(K)$. In particular, Lemmas 3.7 and 3.8 of loc. cit. yield the formula

$$\left(\frac{\pi\cdot\Omega_p}{\Omega_\infty}\right)^{4l-4}\big(1-\mu_l(\mathfrak{p})p^{-1}\big)\big(1-\mu_l(\bar{\mathfrak{p}})^{-1}\big)\big\langle\vartheta(\psi_{2l-2}),\vartheta(\psi_{2l-2})\big\rangle_{-d_K}=\mathscr{C}_3(l)\cdot L_{\mathfrak{p}}(K,\mu_l),$$

where $\mathscr{C}_3=\mathscr{C}_3(K)$ is a unit in $\mathscr{O}_{\boldsymbol{g}}$ such that $\mathscr{C}_3(l)$ is an elementary explicit scalar in K^* for each l in $U^{\mathrm{cl}}\cap\mathbf{Z}$. For $x=(k_o,l)$ in $\mathcal{X}_\infty^{\mathrm{cl}}$ and $\varkappa=(k_o,2l-1,k_o/2+l-1)$, Equation (34) can then be rewritten as

$$\mathscr{C}(l)\cdot\mathscr{L}_{\mathfrak{p}}(f,\nu_x)^2=L_{\mathfrak{p}}(K,\mu_l)\cdot L_p(\boldsymbol{g},\boldsymbol{f})(\varkappa),$$

where the unit $\mathscr{C}=\mathscr{C}(f,K)$ in $\mathscr{O}_{\boldsymbol{g}}$ is defined to be the product of the inverses of the units $\mathscr{C}_2$ and $\mathscr{C}_3$.

Define $\mathscr{B}=\mathscr{B}(f,K)$ in $K(a_n(f);\,n\geqslant 1)^*$ by the formula $(p-1)\cdot\mathscr{B}=2p\cdot\mathscr{C}(1)$. Let $x_n=(k_o,l_n)$ be any sequence in $\mathcal{X}_\infty^{\mathrm{cl}}$ which converges to $(k_o,1)$ in the p-adic topology (e.g. $l_n=1+(q_L-1)p^{c(n)}$ with $\lim_{n\to\infty}c(n)=+\infty$ in the archimedean topology). Then $\varkappa_n=(k_o,2l_n-1,k_o/2+l_n-1)$ (resp., ν_{x_n}, μ_{l_n}) is a sequence of classical points in the interpolation domain of $L_p(\boldsymbol{g},\boldsymbol{f})$ (resp., $\mathscr{L}_{\mathfrak{p}}(f)$, $L_{\mathfrak{p}}(K)$) converging to $(k_o,1,k_o/2)$ (resp., $\mathbf{N}^{k_o/2}$, $\mathbf{N}$). Taking $x=x_n$ in the previous displayed equation and then taking the limit for n tending to infinity yields

$$2\left(1-p^{-1}\right)^{-1}\cdot L_{\mathfrak{p}}(K,\mathbf{N})\cdot L_p(\boldsymbol{g},\boldsymbol{f})(k_o,1,k_o/2)=\mathscr{B}\cdot\mathscr{L}_{\mathfrak{p}}(f,\mathbf{N}^{k_o/2})^2.$$

Together with Katz's p-adic analogue of the Kronecker limit formula:

$$2\left(1-p^{-1}\right)^{-1}\cdot L_{\mathfrak{p}}(K,\mathbf{N})=\log_p(u_{\mathfrak{p}})$$

(cf. [Kat76, Sections 10.4 and 10.5]) this concludes the proof of the lemma. □

4.5.3. — Assume from now on that the Hecke L-series $L(f,s)$ vanishes at $s=k_o/2$.

Lemma 4.5. — *The Beilinson–Flach element* ${}_c\mathrm{BF}^{\alpha}_{f\otimes g}$ *belongs to the Bloch–Kato Selmer group* $\mathrm{Sel}(\mathbf{Q},\mathcal{V}(f,g))$*, and one has the identity*

$$L_p(\boldsymbol{g},\boldsymbol{f})(k_o,1,k_o/2)=\mathscr{C}\cdot\left\langle\log_p(\mathrm{res}_p({}_c\mathrm{BF}^{\alpha}_{f\otimes g})),\omega_f\otimes\eta_g\right\rangle_{fg}$$

for an explicit non-zero constant $\mathscr{C}$ *in the number field* $\mathbf{Q}(\alpha)$*.*

Proof. — Set

$$\mathcal{V}(f_\alpha,g)=\mathcal{V}(f_\alpha)\otimes_L V(g)\quad\text{and}\quad\mathcal{D}(f_\alpha,g)=\mathbf{D}^{\dagger}_{\mathrm{rig},L}(\mathcal{V}(f_\alpha,g)).$$

For a and b in $\{\emptyset,+,-\}$ define $\mathscr{F}^{ab}\mathcal{D}(f_\alpha,g)$ as in Section 2.4, using the triangulations on $D(f_\alpha)$ and $D(g)=\mathscr{R}_L\otimes_L V(g)$ defined in Equation (2). Denote by

$${}_c\mathrm{BF}(f_\alpha\otimes g)\in H^1(\mathbf{Q},\mathcal{V}(f_\alpha,g))$$

the specialisation of ${}_c\mathbf{BF}(\boldsymbol{f}\otimes\boldsymbol{g})$ at the classical triple

$$\varsigma=(k_o,1,k_o/2-1).$$

As the Beilinson–Flach element ${}_c\mathbf{BF}(\boldsymbol{f}\otimes\boldsymbol{g})$ belongs to the balanced Selmer group $H^1_{\mathrm{Iw,bal}}(\mathbf{Q}(\mu_{p^\infty}), V(\boldsymbol{f},\boldsymbol{g}))$, its image in $H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), \mathscr{F}^{\emptyset-}D(\boldsymbol{f},\boldsymbol{g}))$ under the composition $p_g^-\circ\mathrm{res}_p$ (cf. Section 2.4) arises from a unique element

$$ {}_c\mathbf{BF}(\boldsymbol{f}\otimes\boldsymbol{g})^{+-}\in H^1_{\mathrm{Iw}}(\mathbf{Q}_p(\mu_{p^\infty}), \mathscr{F}^{+-}V(\boldsymbol{f},\boldsymbol{g})). $$

Denote by

$$ {}_c\mathrm{BF}(f_\alpha\otimes g)^{+-}\in H^1(\mathbf{Q}_p, \mathscr{F}^{+-}\mathcal{D}(f_\alpha,g)) $$

the specialisation of ${}_c\mathbf{BF}(\boldsymbol{f}\otimes\boldsymbol{g})^{+-}$ at ς. Exchanging the roles of $\boldsymbol{f}$ and $\boldsymbol{g}$ in the previous discussion one defines similarly the local cohomology class

$$ {}_c\mathrm{BF}(f_\alpha\otimes g)^{-+}\in H^1(\mathbf{Q}_p, \mathscr{F}^{-+}\mathcal{D}(f_\alpha,g)). $$

Evaluating both sides of the explicit reciprocity laws (cf. Proposition 2.3)

$$ \mathscr{L}_{\boldsymbol{g}}(\mathrm{res}_p({}_c\mathbf{BF}(\boldsymbol{f}\otimes\boldsymbol{g}))) = \mathscr{N}_{\boldsymbol{g},c}\cdot L_p(\boldsymbol{g},\boldsymbol{f},1+\boldsymbol{s}) $$

and

$$ \mathscr{L}_{\boldsymbol{f}}(\mathrm{res}_p({}_c\mathbf{BF}(\boldsymbol{f}\otimes\boldsymbol{g}))) = \mathscr{N}_{\boldsymbol{f},c}\cdot L_p(\boldsymbol{f},\boldsymbol{g},1+\boldsymbol{s}) $$

at the classical triple $\varsigma=(k_o,1,k_o/2-1)$ yields respectively the formulae

$$ L_p(\boldsymbol{g},\boldsymbol{f})(k_o,1,k_o/2) = \mathscr{E}\cdot\left\langle \log_p({}_c\mathrm{BF}(f_\alpha\otimes g)^{+-}), \omega_{f_\alpha}\otimes\eta_g\right\rangle_{f_\alpha g} \tag{35} $$

and

$$ L_p(\boldsymbol{f},\boldsymbol{g})(k_o,1,k_o/2) = \mathscr{E}'\cdot\left\langle \exp_p^*({}_c\mathrm{BF}(f_\alpha\otimes g)^{-+}), \eta_{f_\alpha}\otimes\omega_g\right\rangle_{f_\alpha g} \tag{36} $$

where

$$ \mathscr{E} = \frac{\left(1-\frac{\alpha}{p^{k_o/2}}\right)}{\left(1-\frac{p^{k_o/2-1}}{\alpha}\right)\mathscr{N}_{\boldsymbol{g},c}(\varsigma)(k_o/2-1)!} \quad\text{and}\quad \mathscr{E}' = \frac{(k_o/2-1)!\left(1-\frac{p^{k_o/2-1}}{\alpha}\right)}{\mathscr{N}_{\boldsymbol{f},c}(\varsigma)\left(1-\frac{\alpha}{p^{k_o/2}}\right)}. $$

(Note that $\mathscr{N}_{\boldsymbol{f},c}(\varsigma)$, $\mathscr{N}_{\boldsymbol{g},c}(\varsigma)$ and the four Euler factors in the previous equation are all non-zero under the current non-exceptionality assumption $a_p(f)\neq p^{k_o/2-1}$.) The value of $L_p(\boldsymbol{f},\boldsymbol{g})$ at the classical triple $(k_o,1,k_o/2)$ is a multiple of the complex L-value $L(f\otimes g,k_o/2)$, which in turn is a multiple of $L(f,k_o/2)$. By assumption $L(f,k_o/2)$ is zero, hence so is ${}_c\mathrm{BF}(f_\alpha\otimes g)^{-+}$ by Equation (36). Since ${}_c\mathrm{BF}(f_\alpha\otimes g)^{--}$ is zero (because ${}_c\mathbf{BF}(\boldsymbol{f}\otimes\boldsymbol{g})$ is a balanced class), this implies that the global class ${}_c\mathrm{BF}(f_\alpha\otimes g)$ belongs to the Selmer group $\mathrm{Sel}(\mathbf{Q},\mathcal{V}(f_\alpha,g))$, hence

$$ \left\langle \log_p({}_c\mathrm{BF}(f_\alpha\otimes g)^{+-}),\omega_{f_\alpha}\otimes\eta_g\right\rangle_{f_\alpha g} = \left\langle \log_p\!\big(\mathrm{res}_p\big({}_c\mathrm{BF}(f_\alpha\otimes g)\big)\big),\omega_{f_\alpha}\otimes\eta_g\right\rangle_{f_\alpha g}. \tag{37} $$

By definition the class ${}_c\mathrm{BF}^{\alpha}_{f\otimes g}$ is an explicit non-zero multiple of the image of ${}_c\mathrm{BF}(f_\alpha\otimes g)$ under the map induced by the p-stabilisation isomorphism $\Pi_{f_\alpha *}: V(f_\alpha)\longrightarrow V(f)$. The lemma then follows from Equations (35) and (37). □

4.5.4. — Theorem 4.3 is a direct consequence of the Bertolini–Darmon–Prasanna p-adic Gross–Zagier formula (31), Lemma 4.4 and Lemma 4.5.

4.6. Conclusion of the proof. — This section concludes the proof of Theorem B (when f is not p-exceptional).

Recall the non-zero p-adic number $\Omega_{g,\gamma}$ introduced in Equation (25) and set

$$\mho_{g,\gamma} = 2 \cdot \left\langle v_g^-, \eta_g \right\rangle_g \quad \text{and} \quad \mathcal{L}(g) = \Omega_{g,\gamma}/\mho_{g,\gamma}.$$

Then $\mathcal{L}(g)$ is a non-zero element of L^* and is independent of the choice of the isomorphism γ made in Equation (23). Since f is not p-exceptional and p splits in $K/\mathbf{Q}$, the twist $f \otimes \varepsilon_K$ is not p-exceptional, hence $L_p(f_\alpha \otimes \varepsilon_K, k_o/2)$ is equal to $L(f, \varepsilon_K, k_o/2)_{\mathrm{alg}}$ (cf. Section 1.1) up to multiplication by non-zero explicit scalar in $\mathbf{Q}(\alpha)$. As by assumption $L(f,s)$, and hence $L_p(f_\alpha)$, vanishes at $s = k_o/2$, Theorems 4.2 and 4.3 prove that the identity

$$L(f, \varepsilon_K, k_o/2)_{\mathrm{alg}} \cdot \log_{\omega_f}\bigl(\mathrm{res}_p\bigl(\zeta_f^{\mathrm{Kato}}\bigr)\bigr) = \frac{\mathcal{L}(g)}{\log_p(u_{\mathfrak{p}})} \cdot \log^2_{\omega_f}\bigl(\mathrm{res}_{\mathfrak{p}}\bigl(z_K(f)\bigr)\bigr) \tag{38}$$

holds in L up to multiplication by a non-zero explicit scalar in the number field $K(a_n(f_\alpha); n \geqslant 1)$. Theorem B is a consequence of the the previous equation and the

Lemma 4.6. — *The ratio between $\mathcal{L}(g)$ and $\log_p(u_{\mathfrak{p}})$ belongs to $\mathbf{Q}^*$.*

Proof. — We give an indirect proof of Lemma 4.6 which uses Equation (38) and Theorem 3.1. Consider the set S_K of negative integers D satisfying the following properties.

1. D is a square-free negative integer congruent to 5 modulo 8.
2. Each prime divisor of D splits in K and p splits in $\mathbf{Q}(\sqrt{D})$.
3. There exists a canonical Hecke character χ_D of $\mathbf{Q}(\sqrt{D})$ such that $L(\chi_D \cdot \varepsilon_K, s)$ does not vanish at $s = 1$.

The set S_K is infinite. Indeed, the first two conditions are easily seen to be satisfied by infinitely many negative integers D. Moreover a theorem of Rohlrich [Roh80, Page 551] guarantees that the subtler condition 3 is satisfied by each square-free negative integer D congruent to 5 modulo 8 such that $-D$ is sufficiently large relative to d_K. (Recall from Section 3 that $L(\chi_D, s)$ has sign -1 in its functional equation, hence $L(\chi_D \cdot \varepsilon_K, s)$ has sign $+1$.)

For each D in S_K write f_{χ_D} for the weight-two theta series of level $\Gamma_0(D^2)$ associated with a canonical Hecke character χ_D satisfying the above condition 3. Let A_{χ_D} and ω_{χ_D} be as in Section 3. Since $L(\chi_D \cdot \varepsilon_K, s)$ is equal to $L(f_{\chi_D}, \varepsilon_K, s)$, condition 2 implies that $L(f_{\chi_D}, \varepsilon_K, k_o/2)_{\mathrm{alg}}$ is a *non-zero* element of the number field E_{χ_D} generated by the values of χ_D, hence Equation (38) gives

$$\log_{\omega_{\chi_D}}\bigl(\mathrm{res}_p\bigl(\zeta^{\mathrm{Kato}}_{A_{\chi_D}}\bigr)\bigr) = c_{\chi_D} \cdot \frac{\mathcal{L}(g)}{\log_p(u_{\mathfrak{p}})} \cdot \log^2_{\omega_{\chi_D}}(z_K(f_{\chi_D}))$$

for a non-zero algebraic constant c_{χ_D} in E_{χ_D}. The $G_{\mathbf{Q}}$-representation $V(f_{\chi_D})$ is canonically isomorphic to $V(A_{\chi_D})$ and by construction $z_K(f_{\chi_D})$ is the image under the global Kummer map of the trace from H to K of a Heegner point in $A_{\chi_D}(H) \otimes_{\mathbf{Z}} \mathbf{Q}$. In addition, since $L(f_{\chi_D}, s) = L(\chi_D, s)$ has sign -1 in its functional equation, this

Heegner point is rational over $\mathbf{Q}$. In summary, we can rewrite the previous equation as

$$\log_{\omega_{\chi_D}}\big(\mathrm{res}_p\big(\zeta^{\mathrm{Kato}}_{A_{\chi_D}}\big)\big) = \frac{\mathcal{L}(g)}{\log_p(u_{\mathfrak{p}})} \cdot \log^2_{\omega_{\chi_D}}(P_{\chi_D})$$

for a global rational point P_{χ_D} in $A_{\chi_D}(\mathbf{Q}) \otimes_{\mathbf{Z}} \mathbf{Q}(\sqrt{D})$. On the other hand Theorem 3.1 yields the identity

$$\log_{\omega_{\chi_D}}\big(\mathrm{res}_p\big(\zeta^{\mathrm{Kato}}_{A_{\chi_D}}\big)\big) = \log^2_{\omega_{\chi_D}}\big(\boldsymbol{P}_{\chi_D}\big)$$

for a generator $\boldsymbol{P}_{\chi_D}$ of the E_{χ_D}-vector space $A_{\chi_D}(\mathbf{Q}) \otimes_{\mathbf{Z}} \mathbf{Q}(\sqrt{D})$. The previous two equations imply that the ratio between $\mathcal{L}(g)$ and $\log_p(u_{\mathfrak{p}})$ belongs to $E^*_{\chi_D}$. Assume that $|D|$ is prime. The definition of χ_D shows that the intersection of the fields $E_{\chi_D^\sigma}$ over the Galois orbit of χ_D is equal to K, so that

$$\mathbf{Q} = \bigcap_{D \text{ in } S_K} E_D$$

and Lemma 4.6 follows. □

5. Proof of Theorem B: the p-exceptional case

This section contains the proof of Theorem B in the p-exceptional case, viz. when $f = f_\alpha$ is new at p and its p-th Fourier coefficient $a_p(f) = \alpha$ is equal to $p^{k_o/2-1}$.

Throughout this section $\boldsymbol{f} = \boldsymbol{f}_\alpha$ and $\boldsymbol{g}$ denote the Coleman families introduced respectively in Sections 4.1 and 4.2. One fixes an integer $c \geqslant 2$ coprime to $pd_K N_f$ and denotes by $\mathbf{BF}(\boldsymbol{f} \otimes \boldsymbol{g})$ the Beilinson–Flach element ${}_c\mathbf{BF}(\boldsymbol{f} \otimes \boldsymbol{g})$ constructed in Proposition 2.3. (As in the previous section the choice of c is not relevant.)

5.1. Comparison between Beilinson–Flach and Beilinson–Kato elements. — Denote by

$$\mathrm{BF}(\boldsymbol{f} \otimes \boldsymbol{g}) = \chi_{\mathrm{cyc}}^{k_o/2-1}\big(\mathbf{BF}(\boldsymbol{f} \otimes \boldsymbol{g})\big) \in H^1(\mathbf{Q}, V(\boldsymbol{f}, \boldsymbol{g})(1-k_o/2))$$

the image of $\mathbf{BF}(\boldsymbol{f} \otimes \boldsymbol{g})$ under the morphism induced in cohomology by evaluation at $k_o/2 - 1$ on $\mathcal{O}(\mathcal{W})$. Proposition 5.3.4 and Theorem 5.4.2 of [LZ16] give

$$\mathrm{BF}(\boldsymbol{f} \otimes \boldsymbol{g}) = \left(1 - \frac{p^{k_o/2-1}}{a_p(\boldsymbol{f})a_p(\boldsymbol{g})}\right) \cdot \mathcal{BF}(\boldsymbol{f} \otimes \boldsymbol{g}) \tag{39}$$

for a canonical *improved* Beilinson–Flach class

$$\mathcal{BF}(\boldsymbol{f} \otimes \boldsymbol{g}) \in H^1(\mathbf{Q}, V(\boldsymbol{f}, \boldsymbol{g})(1-k_o/2))$$

unramified outside p. Define

$$\mathcal{BF}(f \otimes g) = \rho_{k_o,1}\big(\mathcal{BF}(\boldsymbol{f} \otimes \boldsymbol{g})\big) \in H^1(\mathbf{Q}, \mathcal{V}(f, g))$$

to be the specialisation of $\mathcal{BF}(\boldsymbol{f} \otimes \boldsymbol{g})$ at weights $(k_o, 1)$.

Theorem 5.1. — *Assume that $L(f,s)$ vanishes at $s = k_o/2$ and let $\mathcal{L}(g)$ in L^* be as in Section 4.6. Then $\mathcal{BF}(f\otimes g)$ and ζ_f^{Kato} belong to the Selmer groups $\mathrm{Sel}(\mathbf{Q},\mathcal{V}(f,g))$ and $\mathrm{Sel}(\mathbf{Q},\mathcal{V}(f))$ respectively and the equality*

$$\mathcal{L}(g)\cdot\big\langle \log_p\big(\mathrm{res}_p\big(\mathcal{BF}(f\otimes g)\big)\big),\omega_f\otimes\eta_g\big\rangle_{fg} = L(f,\varepsilon_K,k_o/2)_{\mathrm{alg}}\cdot \log_{\omega_f}\big(\mathrm{res}_p\big(\zeta_f^{\mathrm{Kato}}\big)\big)$$

holds in L up to multiplication by an explicit non-zero constant in the number field $K(a_n(f); n\geqslant 1)$.

Proof. — Using the techniques of [**BSV21b**] one can construct, for $\chi = \mathbf{1},\varepsilon_K$, an element

$$\boldsymbol{\zeta}_{\boldsymbol{f},\chi}^{\mathrm{Kato}}\in H^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}),V(\boldsymbol{f})\otimes\chi)$$

which specialise to $\lambda_k\cdot\boldsymbol{\zeta}_{f_k,\chi}^{\mathrm{Kato}}$ at each classical weight k in $U_{\boldsymbol{f}}^{\mathrm{cl}}$, where λ_k is a non-zero element of L with $\lambda_{k_o} = 1$. Here the classes $\boldsymbol{\zeta}_{f_k,\chi}^{\mathrm{Kato}}$ in $H^1_{\mathrm{Iw}}(\mathbf{Q}(\mu_{p^\infty}),V(f_k)\otimes\chi)$ are defined as in Section 4.3 and one identifies $V(\boldsymbol{f}_k)$ with $V(f_k)$ via the p-stabilisation isomorphism $\Pi_{\boldsymbol{f}_k *}$. (We remark that when f is p-ordinary, the existence of $\boldsymbol{\zeta}_{\boldsymbol{f},\chi}^{\mathrm{Kato}}$ is proved in [**Och06**].)

The restriction of the Mazur–Kitagawa p-adic L-function $L_p(\boldsymbol{f}\otimes\chi)$ (cf. Section 4.3) to the line $\boldsymbol{s} = k_o/2-1$ factors in $\mathscr{O}_{\boldsymbol{f}}$ as the product of the analytic Euler factor $1-\frac{p^{k_o/2-1}}{a_p(\boldsymbol{f})}$ and the *improved* p-adic L-function $\mathcal{L}_p(\boldsymbol{f}\otimes\chi)$ (cf. [**GS93, Bel12**]). If

$$\mathcal{BF}(\boldsymbol{f}\otimes g) = (\mathrm{id}\otimes\rho_1)_*(\mathcal{BF}(\boldsymbol{f}\otimes\boldsymbol{g}))\in H^1(\mathbf{Q},V(\boldsymbol{f},g)(1-k_o/2))$$

is the image of $\mathcal{BF}(\boldsymbol{f}\otimes\boldsymbol{g})$ under the map induced in cohomology by

$$\mathrm{id}\otimes\rho_1 : V(\boldsymbol{f},\boldsymbol{g})\longrightarrow V(\boldsymbol{f},g) = V(\boldsymbol{f})\otimes_L V(g),$$

then one has

$$\mathscr{C}^{-1}\cdot\Omega_{g,\gamma}\cdot\mathcal{BF}(\boldsymbol{f}\otimes g) = \mathcal{L}_p(\boldsymbol{f}\otimes\varepsilon_K)\cdot\boldsymbol{\zeta}_{\boldsymbol{f}}^{\mathrm{Kato}}\otimes v_{g,\mathbf{1}} + \mathcal{L}_p(\boldsymbol{f})\cdot\boldsymbol{\zeta}_{\boldsymbol{f},\varepsilon_K}^{\mathrm{Kato}}\otimes v_{g,\varepsilon_K}$$

for a unit $\mathscr{C}$ in $\mathscr{O}_{\boldsymbol{f}}$ with $\mathscr{C}(k_o)$ a non-zero explicit element of $K(a_n(f); n\geqslant 1)$. Since $H^1(\mathbf{Q},V(\boldsymbol{f},g)(1-k_o/2))$ is torsion free, this follows by applying Theorem 4.2 to f_k (in place of f) for each good classical point k in $U_{\boldsymbol{f}}^{\mathrm{cl}}$.

Since $\mathcal{L}_p(\boldsymbol{f}\otimes\chi)(k_o)$ is equal to the product of $L(f,\chi,k_o/2)_{\mathrm{alg}}$ and a non-zero explicit constant in $\mathbf{Q}(\alpha)$, evaluating the previous equation at $\boldsymbol{k} = k_o$ and using the assumption $L(f,k_o/2) = 0$ one gets the identity

$$\Omega_{g,\gamma}\cdot\mathcal{BF}(f\otimes g) = c_K\cdot L(f,\varepsilon_K,k_o/2)_{\mathrm{alg}}\cdot\zeta_f^{\mathrm{Kato}}\otimes v_{g,\mathbf{1}}$$

for an explicit c_K in $K(a_n(f); n\geqslant 1)^*$. Finally, the assumption $L(f,k_o/2) = 0$ and Kato's explicit reciprocity law imply that ζ_f^{Kato} is a Selmer class (cf. the proof of Theorem 16.6 of [**Kat04**]). The statement follows. □

5.2. Comparison between Beilinson–Flach and Heegner classes. — In the the present exceptional zero scenario, Theorem 4.3 admits the following variant.

Theorem 5.2. — *Assume that $L(f,s)$ vanishes at $s = k_o/2$, so that $\mathcal{BF}(f\otimes g)$ is a Selmer class. Then the equality*

$$\log_p(u_{\mathfrak{p}})\cdot\big\langle \log_p\big(\mathrm{res}_p\big(\mathcal{BF}(f\otimes g)\big)\big),\omega_f\otimes\eta_g\big\rangle_{fg} = \log^2_{\omega_f}\big(\mathrm{res}_{\mathfrak{p}}\big(z_K(f)\big)\big)$$

holds in L up to multiplication by an explicit non-zero constant in the number field $K(a_n(f_\alpha);\, n \geqslant 1)$.

Proof. — Equations (31) and Lemma 4.4 hold also in the present exceptional-zero setting. Moreover $\mathcal{BF}(f \otimes g)$ is crystalline at p by Theorem 5.1. As in the proof of Theorem 4.3, one is then reduced to show that the equality

$$\mathscr{L}_{\boldsymbol{g}}\big(\mathrm{res}_p\big(\mathbf{BF}(\boldsymbol{f} \otimes \boldsymbol{g})\big)\big)(k_o, 1, k_o/2-1) = \big\langle \log_p\big(\mathrm{res}_p\big(\mathcal{BF}(f \otimes g)\big)\big), \omega_f \otimes \eta_g \big\rangle_{fg} \tag{40}$$

holds up to multiplication by an explicit non-zero element of $K(a_n(f);\, n \geqslant 1)$.

Let $\varrho : \mathcal{O}(U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times \mathcal{W}) \longrightarrow \mathcal{O}(U_{\boldsymbol{f}} \times U_{\boldsymbol{g}})$ be the morphism sending the analytic function $F(\boldsymbol{k}, , \boldsymbol{s})$ to its restriction $F(\boldsymbol{k}, , \boldsymbol{k} - k_o/2-1)$ to the plane $\boldsymbol{s} = \boldsymbol{k} - k_o/2-1$. Let $V_\varrho(\boldsymbol{f}, \boldsymbol{g})$ be the base change of $V(\boldsymbol{f}, \boldsymbol{g}) \hat{\otimes}_{\mathbf{Q}_p} \mathcal{O}(\mathcal{W})(\varepsilon_\infty^{-1})$ along ϱ and let $\mathrm{BF}_\varrho(\boldsymbol{f} \otimes \boldsymbol{g})$ be the image of $\mathbf{BF}(\boldsymbol{f} \otimes \boldsymbol{g})$ under the morphism induced by ϱ. Using the techniques of [**BSV21b**, Section 8.3] one proves that

$$\mathrm{BF}_\varrho(\boldsymbol{f} \otimes \boldsymbol{g}) = \left(1 - \frac{a_p(\boldsymbol{g}) \cdot p^{k_o/2-1}}{a_p(\boldsymbol{f})}\right) \cdot \mathcal{BF}_\varrho(\boldsymbol{f} \otimes \boldsymbol{g}) \tag{41}$$

for a canonical improved class $\mathcal{BF}_\varrho(\boldsymbol{f} \otimes \boldsymbol{g})$ in $H^1(\mathbf{Q}, V_\varrho(\boldsymbol{f}, \boldsymbol{g}))$. This improved class is unramified outside p and belongs to the kernel of the composition

$$H^1(\mathbf{Q}, V_\varrho(\boldsymbol{f}, \boldsymbol{g})) \to H^1(\mathbf{Q}_p, V_\varrho(\boldsymbol{f}, \boldsymbol{g})) \simeq H^1(\mathbf{Q}_p, D_\varrho(\boldsymbol{f}, \boldsymbol{g})) \to H^1(\mathbf{Q}_p, \mathscr{F}^{--} D_\varrho(\boldsymbol{f}, \boldsymbol{g})),$$

where $\mathscr{F}^{\cdot\cdot} D_\varrho(\boldsymbol{f}, \boldsymbol{g})$ is the base change of $\mathscr{F}^{\cdot\cdot} D(\boldsymbol{f}, \boldsymbol{g})$ along ϱ, the first arrow is restriction at p and the second is induced by the projection $D_\varrho(\boldsymbol{f}, \boldsymbol{g}) \longrightarrow \mathscr{F}^{--} D_\varrho(\boldsymbol{f}, \boldsymbol{g})$. It follows that the image of $\mathrm{res}_p\big(\mathcal{BF}_\varrho(\boldsymbol{f} \otimes \boldsymbol{g})\big)$ in $H^1(\mathbf{Q}_p, \mathscr{F}^{\emptyset -} D_\varrho(\boldsymbol{f}, \boldsymbol{g}))$ arises from a unique element $\mathcal{BF}_\varrho(\boldsymbol{f} \otimes \boldsymbol{g})^{+-}$ in $H^1(\mathbf{Q}_p, \mathscr{F}^{+-} D_\varrho(\boldsymbol{f}, \boldsymbol{g}))$. Define

$$\mathcal{BF}_\varrho(f \otimes g) \in H^1(\mathbf{Q}, \mathcal{V}(g, h)) \quad \text{and} \quad \mathcal{BF}_\varrho(f \otimes g)^{+-} \in H^1(\mathbf{Q}_p, \mathscr{F}^{+-} \mathcal{D}(f, g))$$

to be the specialisations of $\mathcal{BF}_\varrho(\boldsymbol{f} \otimes \boldsymbol{g})$ and $\mathcal{BF}_\varrho(\boldsymbol{f} \otimes \boldsymbol{g})^{+-}$ respectively at weights $(k_o, 1, k_o/2-1)$. Equation (41) and the interpolation formula satisfied by $\mathscr{L}_{\boldsymbol{g}}$ (cf. Theorem 7.1.4 of [**LZ16**]) show that

$$\mathscr{L}_{\boldsymbol{g}}\big(\mathrm{res}_p\big(\mathbf{BF}(\boldsymbol{f} \otimes \boldsymbol{g})\big)\big)(k_o, 1, k_o/2-1)$$

is equal to

$$\frac{(-1)^{k_o/2-1}\left(1 - p^{-1}\right)}{(k_o/2-1)!} \cdot \big\langle \log_p\big(\mathcal{BF}_\varrho(f \otimes g)^{+-}\big), \omega_f \otimes \eta_g \big\rangle_{fg}.$$

Comparing the two factorisations of the restriction of $\mathbf{BF}(\boldsymbol{f} \otimes \boldsymbol{g})$ to the line $(\boldsymbol{k}, , \boldsymbol{s}) = (k_o, , k_o/2-1)$ arising from Equations (39) and (41) yields the identity

$$\mathcal{BF}(f \otimes g) = -\mathcal{BF}_\varrho(f \otimes g)$$

in $H^1(\mathbf{Q}, \mathcal{V}(f, g))$. In particular $\mathcal{BF}_\varrho(f \otimes g)$ is crystalline at p, and Equation (40) (and then the statement) follows from the previous two equations. □

5.3. Conclusion of the proof. — In the present p-exceptional setting, Theorem B is a direct consequence of Theorem 5.1, Theorem 5.2 and Lemma 4.6.

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