# HEEGNER POINTS AND EXCEPTIONAL ZEROS OF GARRETT $p$-ADIC $L$-FUNCTIONS

*by*

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***Abstract.*** — This article proves a case of the $p$-adic Birch and Swinnerton-Dyer conjecture for Garrett $p$-adic $L$-functions of [BSV21c], in the imaginary dihedral exceptional zero setting of extended analytic rank 4.

## 1. Statement of the main result

Let $A$ be an elliptic curve defined over the field $\mathbf{Q}$ of rational numbers, having multiplicative reduction at a rational prime $p > 3$. Let $K$ be a quadratic imaginary field of discriminant $d_K$ coprime to the conductor $N_A$ of $A$, and let

$$\nu_g : G_K \longrightarrow \bar{\mathbf{Q}}^* \quad \text{and} \quad \nu_h : G_K \longrightarrow \bar{\mathbf{Q}}^*$$

be finite order characters of the absolute Galois group $G_K = \mathrm{Gal}(\bar{\mathbf{Q}}/K)$ of $K$, where $\bar{\mathbf{Q}}$ is the field of algebraic complex numbers. Write $N_A = N_A^+ \cdot N_A^-$, where each prime divisor of $N_A^+$ (resp., $N_A^-$) splits (resp., is inert) in $K$. We make the following

***Assumption 1.1.*** —

1. *(Heegner assumption) The prime $p$ is* inert *in $K$ (id est divides $N_A^-$) and $N_A^-$ is a square-free product of an* even *number of primes.*
2. *(Self-duality) The central characters of $\nu_g$ and $\nu_h$ are inverse to each other.*
3. *(Cuspidality) The characters $\nu_g$ and $\nu_h$ are not induced by Dirichlet characters.*
4. *(Local signs) The conductors of $\nu_g$ and $\nu_h$ are coprime to $d_K \cdot N_A$.*

Let $f = \sum_{n\geqslant 1} a_n(f) \cdot q^n$ in $S_2(\Gamma_0(N_f))$ be the newform of conductor $N_f = N_A$ attached to $A$ by the modularity theorem. For $\nu_\xi = \nu_g, \nu_h$, let $\varrho_\xi : G_{\mathbf{Q}} \longrightarrow \mathrm{GL}_2(\mathbf{C})$ be the odd irreducible (cf. Assumption 1.1.(3)) Artin representation of $G_{\mathbf{Q}}$ induced by $\nu_\xi$, corresponding by modularity to the cuspidal weight one theta series

$$\xi = \sum_{(\mathfrak{a},\mathfrak{f}_\xi)=\mathcal{O}_K} \nu_\xi(\mathfrak{a}) \cdot q^{\mathbf{N}\mathfrak{a}} \in S_1(N_\xi, \chi_\xi).$$

Here $\mathfrak{a}$ runs the set of non-zero ideals of $\mathcal{O}_K$ coprime to the conductor $\mathfrak{f}_\xi$ of $\nu_\xi$, $\mathbf{N}\mathfrak{a} = |\mathcal{O}_K/\mathfrak{a}|$, $N_\xi = d_K \cdot \mathbf{N}\mathfrak{f}_\xi$ and $\chi_\xi = \varepsilon_K \cdot \nu_\xi^{\mathrm{cen}}$, where $\varepsilon_K : (\mathbf{Z}/d_K\mathbf{Z})^* \longrightarrow \mu_2$ is the quadratic character of $K$ and $\nu_\xi^{\mathrm{cen}} : G_{\mathbf{Q}} \longrightarrow \bar{\mathbf{Q}}^*$ is the central character of $\nu_\xi$. Since $p$ is inert in $K$ by Assumption 1.1.(1), the $p$-th Hecke polynomial of $\xi$ equals $X^2 + \chi_\xi(p)$ (id est the $p$-th Fourier coefficient of $\xi$ is equal to zero). In addition $\chi_\xi(p)$ is non-zero by Assumption 1.1.(4), hence $X^2 + \chi_\xi(p) = (X - \alpha_\xi) \cdot (X - \beta_\xi)$ has distinct roots $\alpha_\xi$ and $\beta_\xi = -\alpha_\xi$. According to Assumption 1.1.(2) one has $\alpha_g \cdot \alpha_h = \beta_g \cdot \beta_h = \pm 1$ and $\alpha_g \cdot \beta_h = \beta_g \cdot \alpha_h = -\alpha_g \cdot \alpha_h$, hence we can, and will, assume

$$(1) \qquad \alpha_f = \alpha_g \cdot \alpha_h = \beta_g \cdot \beta_h \quad \text{and} \quad -\alpha_f = \beta_g \cdot \alpha_h = \alpha_g \cdot \beta_h$$

by reordering the roots $(\alpha_\xi, \beta_\xi)$ of $X^2 + \chi_\xi(p)$ if necessary, where $\alpha_f = a_p(f) = \pm 1$.

Fix an algebraic closure $\bar{\mathbf{Q}}_p$ of $\mathbf{Q}_p$, an embedding $i_p : \bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$, and a finite extension $L$ of $\mathbf{Q}_p$ containing (the images under $i_p$ of) the values of $\nu_\xi$ and $\alpha_\xi$, for $\xi = g, h$. Denote by $\xi_\alpha$ in $S_1(pN_\xi, \chi_\xi)$ the $p$-stabilisation of $\xi$ with $U_p$-eigenvalue $\alpha_\xi$. According to [**Hid86, BD16**], there exist unique Hida families

$$\boldsymbol{f} = \sum_{n \geqslant 1} a_n(\boldsymbol{f}) \cdot q^n \in \mathscr{O}_{\boldsymbol{f}}[\![q]\!] \quad \text{and} \quad \boldsymbol{\xi}_\alpha = \sum_{n \geqslant 1} a_n(\boldsymbol{\xi}_\alpha) \cdot q^n \in \mathscr{O}_{\boldsymbol{\xi}_\alpha}$$

specialising to $f = \boldsymbol{f}_2$ and $\xi_\alpha = \boldsymbol{\xi}_{\alpha,1}$ in weights two and one respectively. Here $\mathscr{O}_{\boldsymbol{f}}$ is the ring of bounded analytic functions on a (small) connected open disc $U_{\boldsymbol{f}}$ centred at 2 in the weight space $\mathcal{W} = \mathrm{Hom}_{\mathrm{cont}}(\mathbf{Z}_p^*, \mathbf{C}_p^*)$ over $\mathbf{Q}_p$. For each $k$ in $U_{\boldsymbol{f}} \cap \mathbf{Z}_{\geqslant 4}$, the weight-$k$ specialisation $\boldsymbol{f}_k$ of $\boldsymbol{f}$ is the ordinary $p$-stabilisation of a $p$-ordinary newform $f_k$ of weight $k$ and level $\Gamma_0(N_f/p)$. Similarly $\mathscr{O}_{\boldsymbol{\xi}_\alpha}$ is the ring of bounded analytic functions on a connected open disc $U_{\boldsymbol{\xi}_\alpha}$ centred at 1 in $\mathcal{W}_L = \mathcal{W} \otimes_{\mathbf{Q}_p} L$, and $\boldsymbol{\xi}_{\alpha,u}$ is the $p$-stabilisation of a newform $\xi_u$ of weight $u$ and level $\Gamma_1(N_\xi)$ for each $l$ in $U_{\boldsymbol{\xi}_\alpha} \cap \mathbf{Z}_{\geqslant 1}$, with $\xi_1 = \xi$. In order to lighten the notation, we write $U_{\boldsymbol{\xi}} = U_{\boldsymbol{\xi}_\alpha}$ and $\mathscr{O}_{\boldsymbol{\xi}} = \mathscr{O}_{\boldsymbol{\xi}_\alpha}$.

Set $\varrho = \varrho_g \otimes \varrho_h$ and $\mathscr{O}_{\boldsymbol{fgh}} = \mathscr{O}_{\boldsymbol{f}} \hat{\otimes}_{\mathbf{Q}_p} \mathscr{O}_{\boldsymbol{g}} \hat{\otimes}_L \mathscr{O}_{\boldsymbol{h}}$. Under Assumption 1.1, Theorem A of [**Hsi21**] associates with $(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ a Garrett–Hida square root $p$-adic $L$-function

$$\mathscr{L}_p^{\alpha\alpha}(A, \varrho) = \mathscr{L}_p(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha) \in \mathscr{O}_{\boldsymbol{fgh}}$$

(denoted $\mathcal{L}_{\boldsymbol{F}}^{\boldsymbol{f}}$ in loc. cit., where $\boldsymbol{F} = (\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$), whose square

$$L_p^{\alpha\alpha}(A, \varrho) = L_p(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha) = \mathscr{L}_p(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)^2$$

interpolates the central critical values $L(f_k \otimes g_l \otimes h_m, (k+l+m-2)/2)$ of the Garrett $L$-functions attached to $(f_k, g_l, h_m)$ for classical triples $(k, l, m)$ in the $f$-*unbalanced* region, viz. triples $(k, l, m)$ in $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}} \cap \mathbf{Z}_{\geqslant 1}^3$ satisfying $k \geqslant l + m$. The first equality in (1) implies that $L_p^{\alpha\alpha}(A, \varrho)$ has an exceptional zero in the sense of [**MTT86**] at the "Birch and Swinnerton-Dyer point" $w_o = (2, 1, 1)$ (cf. [**BSV21d**, Section 1.2]).

Fix a number field $\mathbf{Q}(\varrho)$ containing the values of $\nu_g$ and $\nu_h$, and for $\xi = g, h$ fix a $\mathbf{Q}(\varrho)[G_{\mathbf{Q}}]$-module $V_\xi$, two-dimensional over $\mathbf{Q}(\varrho)$, affording the Artin representation $\varrho_\xi$. Define $A(K_\varrho)^\varrho = H^0(\mathrm{Gal}(K_\varrho/\mathbf{Q}), A(K_\varrho) \otimes_{\mathbf{Z}} V_{gh})$, where $V_{gh} = V_g \otimes_{\mathbf{Q}(\varrho)} V_h$ and $K_\varrho$ is the number field cut-out by $\varrho = \varrho_g \otimes \varrho_h$. Following [**MTT86**] one exploits Tate's $p$-adic uniformisation to define an extended Mordell–Weil group

$$A^\dagger(K_\varrho)^\varrho = A(K_\varrho)^\varrho \oplus \mathcal{Q}_p(A, \varrho),$$

where $\mathcal{Q}_p(A,\varrho)$ is a two-dimensional $\mathbf{Q}(\varrho)$-vector space depending only on the base change of $A$ to $\mathbf{Q}_p$ and on the restriction of $V_{gh}$ to $G_{\mathbf{Q}_p}$ (cf. Section 2.1.3 below). Moreover, Section 2 of [BSV21c] constructs a *Garrett–Nekovář* height-pairing

$$\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} : A^\dagger(K_\varrho)^\varrho \otimes_{\mathbf{Q}(\varrho)} A^\dagger(K_\varrho)^\varrho \longrightarrow \mathscr{I}/\mathscr{I}^2,$$

where $\mathscr{I}$ is the kernel of evaluation at $w_o$ on $\mathscr{O}_{\boldsymbol{fgh}}$. It is a skew-symmetric bilinear form, arising from an application of Nekovář's theory of Selmer complexes to the big self-dual Galois representation associated with $(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)$. After setting $r^\dagger = \dim_{\mathbf{Q}(\varrho)} A(K_\varrho)^\varrho$, Conjecture 1.1 of [BSV21c] predicts that $L_p^{\alpha\alpha}(A,\varrho)$ belongs to $\mathscr{I}^{r^\dagger} - \mathscr{I}^{r^\dagger+1}$, and that its image in $(\mathscr{I}^{r^\dagger}/\mathscr{I}^{r^\dagger+1})/\mathbf{Q}(\varrho)^{*2}$ is equal to the discriminant

$$R_p^{\alpha\alpha}(A,\varrho) = \det\left(\langle\!\langle P_i, P_j \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}\right)_{1\leqslant i,j\leqslant r^\dagger}$$

of the $p$-adic height $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$, where $P_1,\dots,P_{r^\dagger}$ is any $\mathbf{Q}(\varrho)$-basis of $A^\dagger(K_\varrho)^\varrho$.

The following theorem is the main result of this note.

***Theorem.*** — *Assume that Assumption 1.1 and Assumption 1.2 (stated below) are satisfied. If $L(f\otimes g\otimes h, s)$ has order of vanishing 2 at $s=1$, then*

$$\dim_{\mathbf{Q}(\varrho)} A^\dagger(K_\varrho)^\varrho = 4, \qquad L_p^{\alpha\alpha}(A,\varrho) \in \mathscr{I}^4 - \mathscr{I}^5$$

*and the equality*

$$L_p^{\alpha\alpha}(A,\varrho) \pmod{\mathscr{I}^5} = R_p^{\alpha\alpha}(A,\varrho)$$

*holds in the quotient of $\mathscr{I}^4/\mathscr{I}^5$ by the multiplicative action of $\mathbf{Q}(\varrho)^{*2}$.*

In the present setting, the Garrett $L$-function $L(f\otimes g\otimes h,s)$ factors as the product of the Rankin–Selberg $L$-functions $L(A/K,\varphi,s)$ and $L(A/K,\psi,s)$, where $\varphi = \nu_g\cdot\nu_h$ and $\psi = \nu_g\cdot\nu_h^c$, and $\nu_h^c$ is the conjugate of $\nu_h$ by the nontrivial element of $\mathrm{Gal}(K/\mathbf{Q})$. Note that $\varphi$ and $\psi$ are *dihedral* by Assumption 1.1.(2), and that both $L(A/K,\varphi,s)$ and $L(A/K,\psi,s)$ have sign $-1$ in their functional equation by Assumption 1.1.(1). In particular the assumptions of the Theorem imply that $L(A/K,\chi,s)$ has a simple zero at $s=1$ for $\chi=\varphi$ and $\chi=\psi$, hence $A(K_\varrho)^\varrho$ is two-dimensional over $\mathbf{Q}(\varrho)$ and generated by Heegner points by the Kolyvagin–Gross–Zagier–Zhang theorem.

If $\chi=\varphi,\psi$ is quadratic, $\bar{\mathbf{Q}}^{\ker(\chi)} = \mathbf{Q}(\sqrt{cd_1},\sqrt{cd_2})$, where $c$, $d_1$ and $d_2$ are fundamental discriminants such that $d_K = d_1\cdot d_2$. (We consider 1 as a fundamental discriminant). In this case $L(A/K,\chi,s)$ further factors as the product of the Hasse-Weil $L$-functions $L(A/\mathbf{Q},\chi_1,s)$ and $L(A/\mathbf{Q},\chi_2,s)$ of the twists of $A$ by the quadratic characters $\chi_i$ of $\mathbf{Q}(\sqrt{cd_i})$. By Assumptions 1.1.(1) and 1.1.(4), we can order $\chi_1$ and $\chi_2$ in such a way that $\mathrm{sign}(A,\chi_1) = -1$ and $\mathrm{sign}(A,\chi_2)=+1$, where $\mathrm{sign}(A,\chi_i)$ is the sign in the functional equation satisfied by $L(A/\mathbf{Q},\chi_i,s)$.

***Assumption 1.2.*** — *If $\chi=\varphi$ or $\chi=\psi$ is quadratic, then $\chi_1(p)=\alpha_f$.*

Under the assumptions of the Theorem, the results of [BSV21d, BSV21a] imply that $L_p^{\alpha\alpha}(A,\varrho)$ belongs to $\mathscr{I}^4-\mathscr{I}^5$. The actual contribution of this note is the proof of the identity $L_p^{\alpha\alpha}(A,\varrho) \pmod{\mathscr{I}^5} = R_p^{\alpha\alpha}(A,\varrho)$, which grounds on the results of loc. cit. and an extension of the techniques of [Ven13, Ven16a, Ven16b].

## 2. Proof of the main result

### 2.1. Preliminaries. —

**2.1.1.** *Galois representations.* — To lighten the notation, set $(\boldsymbol{g}, \boldsymbol{h}) = (\boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$. For $\boldsymbol{\xi} = \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}$ let $V(\boldsymbol{\xi})$ be the big Galois representation attached to $\boldsymbol{\xi}$ (cf. Section 5 of [**BSV21d**]). Under the current assumptions, it is a free $\mathscr{O}_{\boldsymbol{\xi}}$-module of rank two, equipped with a continuous $\mathscr{O}_{\boldsymbol{\xi}}$-linear action of $G_{\mathbf{Q}}$. For each $u$ in $U_{\boldsymbol{\xi}} \cap \mathbf{Z}_{\geqslant 2}$, evaluation at $u$ on $U_{\boldsymbol{\xi}}$ induces a natural specialisation isomorphism

$$\rho_u : V(\boldsymbol{\xi}) \otimes_u E \simeq V(\boldsymbol{\xi}_u),$$

where $E = \mathbf{Q}_p$ if $\boldsymbol{\xi} = \boldsymbol{f}$ and $E = L$ if $\boldsymbol{\xi} = \boldsymbol{g}, \boldsymbol{h}$, where $\cdot \otimes_u E$ denotes the base change along evaluation at $u$ on $\mathscr{O}_{\boldsymbol{\xi}}$, and where $V(\boldsymbol{\xi}_u)$ is the homological $p$-adic Deligne representation of $\boldsymbol{\xi}_u$ with coefficients in $E$ (cf. Section 2.4 of [**BSV21d**]).

When $\boldsymbol{\xi} = \boldsymbol{f}$ and $u = 2$, the representation $V(f) = V(\boldsymbol{f}_2)$ is equal to the $f$-isotypic component of the cohomology group $H^1_{\text{ét}}(X_1(N_f)_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(1))$, where $X_1(N_f)_{\bar{\mathbf{Q}}}$ is the base change to $\bar{\mathbf{Q}}$ of the compact modular curve $X_1(N_f)$ of level $\Gamma_1(N_f)$ defined over $\mathbf{Q}$. Fix a modular parametrisation (viz. a non-constant morphism of $\mathbf{Q}$-curves)

$$\wp_\infty : X_1(N_f) \longrightarrow A,$$

which induces an isomorphism of $\mathbf{Q}_p[G_{\mathbf{Q}}]$-modules between $V(f)$ and the $p$-adic Tate module $V_p(A) = H^1_{\text{ét}}(A_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(1))$ of $A$ with $\mathbf{Q}_p$-coefficients.

When $\boldsymbol{\xi} = \boldsymbol{g}, \boldsymbol{h}$ and $u = 1$, the $L[G_{\mathbf{Q}}]$-module

$$V(\xi) = V(\boldsymbol{\xi}) \otimes_1 L$$

affords the dual of the Deligne–Serre representation of $\xi$, id est the induced from $G_K$ to $G_{\mathbf{Q}}$ of the character $\nu_\xi$ with coefficients in $L$. (Recall that $\boldsymbol{\xi}_1 = \xi_\alpha$. Here we favour the lighter notation $V(\xi)$ for $V(\boldsymbol{\xi}) \otimes_1 L$ over the more consistent one $V(\xi_\alpha)$.)

There exists a perfect $G_{\mathbf{Q}}$-equivariant and skew-symmetric pairing

$$\pi_{\boldsymbol{\xi}} : V(\boldsymbol{\xi}) \otimes_{\mathscr{O}_{\boldsymbol{\xi}}} V(\boldsymbol{\xi}) \longrightarrow \mathscr{O}_{\boldsymbol{\xi}}(\chi_\xi \cdot \chi_{\text{cyc}}^{\boldsymbol{u}-1}),$$

where $\chi_{\text{cyc}} : G_{\mathbf{Q}} \longrightarrow \mathbf{Z}_p^*$ is the $p$-adic cyclotomic character and $\chi_{\text{cyc}}^{\boldsymbol{u}-1} : G_{\mathbf{Q}} \longrightarrow \mathscr{O}_{\boldsymbol{\xi}}^*$ satisfies $\chi_{\text{cyc}}^{\boldsymbol{u}-1}(\sigma)(u) = \chi_{\text{cyc}}(\sigma)^{u-1}$ for each $\sigma$ in $G_{\mathbf{Q}}$ and each $u$ in $U_{\boldsymbol{\xi}} \cap \mathbf{Z}$. (With the notations of [**BSV21d**, Section 5], the pairing $\pi_{\boldsymbol{\xi}}$ is the composition of the twist by $\chi_\xi \cdot \chi_{\text{cyc}}^{\boldsymbol{u}-1}$ of the $\mathscr{O}_{\boldsymbol{\xi}}$-adic Poincaré duality $\langle \cdot, \cdot \rangle_{\boldsymbol{f}} : V(\boldsymbol{\xi}) \otimes_{\mathscr{O}_{\boldsymbol{\xi}}} V^*(\boldsymbol{\xi}) \longrightarrow \mathscr{O}_{\boldsymbol{\xi}}$ defined in [**BSV21d**, Equation (103)] with $\mathrm{id}_{V(\boldsymbol{\xi})} \otimes w_{N_\xi p}^{-1}$, where $w_{N_\xi p} : V^*(\boldsymbol{\xi})(\chi_\xi \cdot \chi_{\text{cyc}}^{\boldsymbol{u}-1}) \simeq V(\boldsymbol{\xi})$ is the $\mathscr{O}_{\boldsymbol{\xi}}$-adic Atkin–Lehner isomorphism defined in [**BSV21d**, Equation (114)].) Up to sign, the pairing $\pi_f : V(f) \otimes_{\mathbf{Q}_p} V(f) \longrightarrow \mathbf{Q}_p(1)$ arising from the base change of $\pi_{\boldsymbol{f}}$ along evaluation at $k = 2$ on $\mathscr{O}_{\boldsymbol{f}}$ and the specialisation isomorphism $\rho_2$ is the one induced on the $f$-isotypic components by the Poincaré duality on $H^1_{\text{ét}}(X_1(N_f)_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(1))$. If $\boldsymbol{\xi} = \boldsymbol{g}, \boldsymbol{h}$, the weight-one specialisation of $\pi_{\boldsymbol{\xi}}$ yields a perfect skew-symmetric duality

$$\pi_\xi : V(\xi) \otimes_L V(\xi) \longrightarrow L(\chi_\xi).$$

Identify $G_{\mathbf{Q}_p}$ with a subgroup of $G_{\mathbf{Q}}$ via the embedding $i_p : \bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$ fixed at the outset, and let $\check{a}_p(\boldsymbol{\xi}) : G_{\mathbf{Q}_p} \longrightarrow \mathscr{O}_{\boldsymbol{\xi}}^*$ be the unramified character sending an arithmetic Frobenius to the $p$-th Fourier coefficient $a_p(\boldsymbol{\xi})$ of $\boldsymbol{\xi}$. In the present setting there is a natural short exact sequence of $\mathscr{O}_{\boldsymbol{\xi}}[G_{\mathbf{Q}_p}]$-modules $V(\boldsymbol{\xi})^+ \hookrightarrow V(\boldsymbol{\xi}) \twoheadrightarrow V(\boldsymbol{\xi})^-$, where $V(\boldsymbol{\xi})^+$ and $V(\boldsymbol{\xi})^-$ are free $\mathscr{O}_{\boldsymbol{\xi}}$-modules of rank one and $G_{\mathbf{Q}_p}$ acts on them via the characters $\chi_\xi \cdot \chi_{\mathrm{cyc}}^{\boldsymbol{u}-1} \cdot \breve{a}_p(\boldsymbol{\xi})^{-1}$ and $\breve{a}_p(\boldsymbol{\xi})$ respectively (cf. Section 5 of [**BSV21d**]). If $\boldsymbol{\xi} = \boldsymbol{f}$, the specialisation isomorphism $\rho_2 : V(\boldsymbol{f}) \otimes_2 \mathbf{Q}_p \simeq V(f)$ identifies $V(\boldsymbol{f})^- \otimes_2 \mathbf{Q}_p$ with the maximal $p$-unramified quotient of $V(f)$ and $V(\boldsymbol{\xi})^+ \otimes_2 \mathbf{Q}_p$ with the kernel $V(f)^+$ of the projection $V(f) \longrightarrow V(f)^-$. If $\boldsymbol{\xi} = \boldsymbol{g}, \boldsymbol{h}$ define

$$V(\xi)_\alpha = V(\boldsymbol{\xi})^- \otimes_1 L \quad \text{and} \quad V(\xi)_\beta = V(\boldsymbol{\xi})^+ \otimes_1 L,$$

so that $V(\xi)_\gamma$ (for $\gamma = \alpha, \beta$) is the submodule of $V(\xi)$ on which an arithmetic Frobenius in $G_{\mathbf{Q}_p}$ acts as multiplication by $\gamma_\xi$, and (as $L[G_{\mathbf{Q}_p}]$-modules)

$$V(\xi) = V(\xi)_\alpha \oplus V(\xi)_\beta.$$

Define

$$\boldsymbol{V} = V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) = V(\boldsymbol{f}) \hat{\otimes}_{\mathbf{Q}_p} V(\boldsymbol{g}) \hat{\otimes}_L V(\boldsymbol{h})(\Xi_{\boldsymbol{fgh}}),$$

where $\Xi_{\boldsymbol{fgh}} = \chi_{\mathrm{cyc}}^{(4-\boldsymbol{k}-\boldsymbol{l}-\boldsymbol{m})/2} : G_{\mathbf{Q}} \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}^*$ satisfies $\Xi_{\boldsymbol{fgh}}(\sigma)(w) = \chi_{\mathrm{cyc}}(\sigma)^{\frac{4-k-l-m}{2}}$ for each $\sigma$ in $G_{\mathbf{Q}}$ and each $w = (k, l, m)$ in $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}} \cap \mathbf{Z}^3$, and

$$V = V(f, g, h) = V(f) \otimes_{\mathbf{Q}_p} V(g) \otimes_L V(h).$$

Evaluation at $w_o = (2, 1, 1)$ on $\mathscr{O}_{\boldsymbol{fgh}}$ induces a specialisation isomorphism

$$\rho_{w_o} : \boldsymbol{V} \otimes_{w_o} L \simeq V.$$

The product of the pairing $\pi_{\boldsymbol{\xi}}$ for $\boldsymbol{\xi} = \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}$ yields a perfect, $G_{\mathbf{Q}}$-equivariant and skew-symmetric duality (cf. Assumption 1.1.(2))

$$\pi_{\boldsymbol{fgh}} : \boldsymbol{V} \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \boldsymbol{V} \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}(1),$$

whose base change along evaluation at $w_o$ on $\mathscr{O}_{\boldsymbol{fgh}}$ recasts (via $\rho_{w_o}$) the perfect duality

$$\pi_{fgh} : V \otimes_L V \longrightarrow L(1)$$

defined by the product of the perfect pairings $\pi_\xi$ for $\xi = f, g, h$.

For $\boldsymbol{\xi} = \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}$ let $\mathscr{F}^\bullet V(\boldsymbol{\xi})$ be the decreasing filtration on the $\mathscr{O}_{\boldsymbol{fgh}}[G_{\mathbf{Q}_p}]$-module $V(\boldsymbol{\xi})$ defined by $\mathscr{F}^1 V(\boldsymbol{\xi}) = V(\boldsymbol{\xi})^+$, $\mathscr{F}^i V(\boldsymbol{\xi}) = V(\boldsymbol{\xi})$ for each $i \leqslant 0$ and $\mathscr{F}^i V(\boldsymbol{\xi}) = 0$ for each $i \geqslant 2$. Define the *balanced* submodule $\mathscr{F}^2 \boldsymbol{V}$ of $\boldsymbol{V}$ by

$$\mathscr{F}^2 \boldsymbol{V} = \left[ \sum_{a+b+c=2} \mathscr{F}^a V(\boldsymbol{f}) \hat{\otimes}_{\mathbf{Q}_p} \mathscr{F}^b V(\boldsymbol{g}) \hat{\otimes}_L \mathscr{F}^c V(\boldsymbol{h}) \right] \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \Xi_{\boldsymbol{fgh}},$$

and the $f$-*unbalanced* submodule $\boldsymbol{V}^+$ of $\boldsymbol{V}$ by

$$\boldsymbol{V}^+ = V(\boldsymbol{f})^+ \hat{\otimes}_{\mathbf{Q}_p} V(\boldsymbol{g}) \hat{\otimes}_L V(\boldsymbol{h}) \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \Xi_{\boldsymbol{fgh}}.$$

These are $G_{\mathbf{Q}_p}$-invariant free $\mathscr{O}_{\boldsymbol{fgh}}$-submodules of $\boldsymbol{V}$ of rank $4 = \frac{1}{2}\mathrm{rank}_{\mathscr{O}_{\boldsymbol{fgh}}} \boldsymbol{V}$, which are maximal isotropic with respect to the skew-symmetric duality $\pi_{\boldsymbol{fgh}}$. After setting

$$\boldsymbol{V}^- = \boldsymbol{V}/\boldsymbol{V}^+ \quad \text{and} \quad \boldsymbol{V}_f = V(\boldsymbol{f})^- \hat{\otimes}_{\mathbf{Q}_p} V(\boldsymbol{g})^+ \hat{\otimes}_L V(\boldsymbol{h})^+ \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \Xi_{\boldsymbol{fgh}},$$

one has a commutative diagram of $\mathscr{O}_{\boldsymbol{fgh}}[G_{\mathbf{Q}_p}]$-modules

$$\begin{array}{ccc} \mathscr{F}^2\boldsymbol{V} & \xhookrightarrow{\ i_{\mathscr{F}}\ } & \boldsymbol{V} \\ {\scriptstyle p_f}\downarrow & & \downarrow{\scriptstyle p^-} \\ \boldsymbol{V}_f & \xhookrightarrow{\ i_f\ } & \boldsymbol{V}^- \end{array} \tag{2}$$

with $i_{\mathscr{F}}$ and $i_f$ the natural inclusions and $p^-$ the natural projection. Note that $p^- \circ i_{\mathscr{F}}$ and $i_f$ have the same image, hence the morphism $p_f$ is defined by the commutativity of the diagram. One defines the *balanced local subspace* $H^1_{\mathrm{bal}}(\mathbf{Q}_p, \boldsymbol{V})$ of $H^1(\mathbf{Q}_p, \boldsymbol{V})$ to be the image of the morphism induced in cohomology by $i_{\mathscr{F}}$. This morphism is injective (cf. Section 7.2 of [BSV21d]), hence gives a natural identification

$$H^1_{\mathrm{bal}}(\mathbf{Q}_p, \boldsymbol{V}) = H^1(\mathbf{Q}_p, \mathscr{F}^2\boldsymbol{V}) \tag{3}$$

Set $V^{\pm} = V(f)^{\pm} \otimes_{\mathbf{Q}_p} V(g) \otimes_L V(h)$. For each pair $(i,j)$ of elements of $\{\alpha, \beta\}$ define $V^{\cdot}_{ij} = V(f)^{\cdot} \otimes_{\mathbf{Q}_p} V(g)_i \otimes_L V(h)_j$, where $\cdot$ is one of symbols $\emptyset, +$ and $-$. Then

$$V^{\cdot} = V^{\cdot}_{\alpha\alpha} \oplus V^{\cdot}_{\alpha\beta} \oplus V^{\cdot}_{\beta\alpha} \oplus V^{\cdot}_{\beta\beta}$$

as $L[G_{\mathbf{Q}_p}]$-modules, and Equation (1) implies

$$H^0(\mathbf{Q}_p, V^-) = V^-_{\alpha\alpha} \oplus V^-_{\beta\beta} \quad \text{and} \quad H^0(\mathbf{Q}_p, V^+(-1)) = V^+_{\alpha\alpha}(-1) \oplus V^+_{\beta\beta}(-1). \tag{4}$$

The specialisation isomorphism $\rho_{w_o}$ identifies $\boldsymbol{V}^{\pm} \otimes_{w_o} L$, $\mathscr{F}^2\boldsymbol{V} \otimes_{w_o} L$ and $\boldsymbol{V}_f \otimes_{w_o} L$ with $V^{\pm}$, $\mathscr{F}^2 V = V_{\beta\beta} + V^+_{\alpha\beta} + V^+_{\beta\alpha}$ and $V^-_{\beta\beta}$ respectively. In particular the base change of the commutative diagram (2) along evaluation at $w_o$ on $\mathscr{O}_{\boldsymbol{fgh}}$ is equal to

$$\begin{array}{ccc} \mathscr{F}^2 V & \xhookrightarrow{\ i_{\mathscr{F}}\ } & V \\ {\scriptstyle p_f}\downarrow & & \downarrow{\scriptstyle p^-} \\ V^-_{\beta\beta} & \xhookrightarrow{\ i_f\ } & V^- \end{array} \tag{5}$$

with $i_{\mathscr{F}}$ and $i_f$ the natural inclusions and $p^-$ the natural projection.

The Bloch–Kato finite subspace of $H^1(\mathbf{Q}_p, V)$ is equal to the kernel of the map $p^- : H^1(\mathbf{Q}_p, V) \longrightarrow H^1(\mathbf{Q}_p, V^-)$, cf. Section 9.1 of [BSV21d]. (With a slight abuse of notation, we denote by the same symbol a morphism of $G_{\mathbf{Q}_p}$-modules and the maps it induces in cohomology.) By construction (cf. Equation (2) and (5)), the specialisation $\kappa = \rho_{w_o}(\boldsymbol{\kappa})$ in $H^1(\mathbf{Q}_p, V)$ at $w_o$ of a local balanced class $\boldsymbol{\kappa}$ in $H^1_{\mathrm{bal}}(\mathbf{Q}_p, \boldsymbol{V})$ belongs to the kernel of the map $H^1(\mathbf{Q}_p, V) \longrightarrow H^1(\mathbf{Q}_p, V^-_{ij})$ for $ij = \alpha\alpha, \alpha\beta, \beta\alpha$. Then $\kappa$ is crystalline precisely if it belongs to the kernel of $H^1(\mathbf{Q}_p, V) \longrightarrow H^1(\mathbf{Q}_p, V^-_{\beta\beta})$, id est if $p_f(\boldsymbol{\kappa})$ in $H^1(\mathbf{Q}_p, \boldsymbol{V}_f)$ (cf. Equation (3)) belongs to the kernel of the specialisation map $\rho_{w_o} : H^1(\mathbf{Q}_p, \boldsymbol{V}_f) \longrightarrow H^1(\mathbf{Q}_p, V^-_{\beta\beta})$. Since the ideal $\mathscr{I}$ of $\mathscr{O}_{\boldsymbol{fgh}}$ is generated by a regular sequence and $H^2(\mathbf{Q}_p, V^-_{\beta\beta}) = 0$, the specialisation map $\rho_{w_o}$ induces an isomorphism $H^1(\mathbf{Q}_p, \boldsymbol{V}_f) \otimes_{w_o} L \simeq H^1(\mathbf{Q}_p, V^-_{\beta\beta})$. We have proved the following

***Lemma 2.1.*** — *Let $\boldsymbol{\kappa}$ be a local balanced class in $H^1_{\mathrm{bal}}(\mathbf{Q}_p, \boldsymbol{V})$ and set $\kappa = \rho_{w_o}(\boldsymbol{\kappa})$ in $H^1(\mathbf{Q}_p, V)$. Then $\kappa$ is crystalline if and only if $p_f(\boldsymbol{\kappa})$ belongs to $\mathscr{I} \cdot H^1(\mathbf{Q}_p, \boldsymbol{V}_f)$.*

**2.1.2.** *$p$-adic periods.* — Let $\hat{\mathbf{Q}}_p^{\mathrm{nr}}$ be the $p$-adic completion of the maximal unramified extension of $\mathbf{Q}_p$, let $c = c(\chi_g)$ be the conductor of $\chi_g$, and for $\xi = g, h$ define

$$G(\chi_\xi) = (-c)^{i_\xi} \cdot \sum_{a \in (\mathbf{Z}/c\mathbf{Z})^*} \chi_\xi(a)^{-1} \otimes e^{2\pi i a/c} \in D_{\mathrm{cris}}(\chi_\xi),$$

where $i_g = 0$, $i_h = -1$ and $D_{\mathrm{cris}}(\chi_\xi)$ is a shorthand for $H^0(\mathbf{Q}_p, L(\chi_\xi) \otimes_{\mathbf{Q}_p} \hat{\mathbf{Q}}_p^{\mathrm{nr}})$.

As explained in Section 3.1 of [BSV21c], for $\boldsymbol{\xi} = \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}$ the module $D(\boldsymbol{\xi})^-$ of $G_{\mathbf{Q}_p}$-invariants of $V(\boldsymbol{\xi})^- \hat{\otimes}_{\mathbf{Q}_p} \hat{\mathbf{Q}}_p^{\mathrm{nr}}$ is free of rank one over $\mathscr{O}_{\boldsymbol{\xi}}$, and its base change $D(\boldsymbol{\xi})_u^- = D(\boldsymbol{\xi})^- \otimes_u L$ along evaluation at a classical weight $u$ in $U_{\boldsymbol{\xi}} \cap \mathbf{Z}_{\geqslant 2}$ on $\mathscr{O}_{\boldsymbol{\xi}}$ is canonically isomorphic to the $\boldsymbol{\xi}_u$-isotypic component $L \cdot \boldsymbol{\xi}_u$ of $S_u(pN_\xi, \chi_\xi)_L$. Moreover there exists an $\mathscr{O}_{\boldsymbol{\xi}}$-basis

$$\omega_{\boldsymbol{\xi}} \in D(\boldsymbol{\xi})^-$$

whose image $\omega_{\boldsymbol{\xi}_u}$ in $D(\boldsymbol{\xi})_u^-$ corresponds to $\boldsymbol{\xi}_u$ under the aforementioned isomorphism for each $u$ in $U_{\boldsymbol{\xi}} \cap \mathbf{Z}_{\geqslant 2}$. (We refer to loc. cit. and the references therein for the details.) The weight-two specialisation of $\omega_{\boldsymbol{f}}$ equals the de Rham class

$$\omega_f \in D_{\mathrm{cris}}(V(f)^-) \simeq \mathrm{Fil}^0 D_{\mathrm{dR}}(V(f))$$

associated with $f$ under the Faltings–Tsuji comparison isomorphism between the étale and de Rham cohomology of $X_1(N_f)_{\mathbf{Q}_p}$. (The isomorphism in the previous equation arises from the projection $V(f) \longrightarrow V(f)^-$.) Denote by

$$\langle \cdot, \cdot \rangle_f : D_{\mathrm{dR}}(V(f)) \otimes_L D_{\mathrm{dR}}(V(f)) \longrightarrow L$$

the perfect duality induced by $\pi_f$, and define $\eta_f$ in $D_{\mathrm{dR}}(V(f))/\mathrm{Fil}^0$ by the identity

$$\langle \eta_f, \omega_f \rangle_f = 1.$$

For $\boldsymbol{\xi} = \boldsymbol{g}, \boldsymbol{h}$, the weight-one specialisation of $\omega_{\boldsymbol{\xi}}$ yields a class

$$\omega_{\xi_\alpha} \in D_{\mathrm{cris}}(V(\xi)_\alpha) = D_{\mathrm{cris}}(V(\xi))^{\varphi = \alpha_\xi^{-1}}$$

(with $\varphi$ the crystalline Frobenius). The pairing $\pi_\xi = \pi_{\boldsymbol{\xi}} \otimes_1 L$ induces a perfect duality

$$\langle \cdot, \cdot \rangle_\xi : D_{\mathrm{cris}}(V(\xi)) \otimes_L D_{\mathrm{cris}}(V(\xi)) \longrightarrow D_{\mathrm{cris}}(\chi_\xi)$$

and one defines $\eta_{\xi_\alpha}$ in $D_{\mathrm{cris}}(V(\xi)_\beta) = D_{\mathrm{cris}}(V(\xi))^{\varphi = \beta_\xi^{-1}}$ by the identity

$$\langle \eta_{\xi_\alpha}, \omega_{\xi_\alpha} \rangle_\xi = G(\chi_\xi).$$

Along with $\omega_f$, it is important to consider another $p$-adic period

$$q(f) \in D_{\mathrm{cris}}(V(f)^-) = \mathrm{Fil}^0 D_{\mathrm{dR}}(V(f))$$

arising from the Tate uniformisation of $A_{\mathbf{Q}_p}$, cf. Section 2 of [BSV21b]. Let $K_p$ be the completion of $K$ at $p$ (namely the quadratic unramified extension of $\mathbf{Q}_p$). Tate's theory gives a rigid analytic uniformisation $\wp_{\mathrm{Tate}} : \mathbf{G}_{m,K_p}^{\mathrm{rig}} \longrightarrow A_{K_p}$, unique up to sign, with kernel the lattice generated by the Tate period $q_A$ in $p\mathbf{Z}_p$ of $A_{\mathbf{Q}_p}$. One sets

$$q(A) = p^-\big(\wp_{\mathrm{Tate}}(\sqrt[p^\infty]{q_A})\big) \in V_p(A)^- \quad \text{and} \quad q(f) = \sqrt{m_p} \cdot \wp_\infty^{-1}(q(A)), \tag{6}$$

where $\sqrt[p^\infty]{q_A}$ is any compatible system of $p^n$-th roots of $q_A$, $\wp_\infty : V(f)^- \simeq V_p(A)^-$ is the isomorphism arising from the fixed modular parametrisation $\wp_\infty$, $m_p = 1$ if $\alpha_f = 1$ and $m_p = d_K$ if $\alpha_f = -1$. As in loc. cit., define the generators

$$q_{\alpha\alpha} = q(f) \otimes \omega_{g_\alpha} \otimes \omega_{h_\alpha} \quad \text{and} \quad q_{\beta\beta} = q(f) \otimes \eta_{g_\alpha} \otimes \eta_{h_\alpha}$$

of the subspaces $V^-_{\alpha\alpha}$ and $V^-_{\beta\beta}$ respectively of $H^0(\mathbf{Q}_p, V^-) = D_{\mathrm{cris}}(V^-)^{\varphi=1}$.

**2.1.3.** *The Garrett–Nekovář p-adic height pairing.* — Section 2 of [**BSV21c**] constructs a canonical skew-symmetric $p$-adic height pairing

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{fgh}} : \tilde{H}^1_f(\mathbf{Q}, V) \otimes_L \tilde{H}^1_f(\mathbf{Q}, V) \longrightarrow \mathscr{I}/\mathscr{I}^2$$

on the extended Selmer group $\tilde{H}^1_f(\mathbf{Q}, V)$ associated with the Greenberg local condition at $p$ arising from the inclusion $i^+ : V^+ \hookrightarrow V$. Let $\mathrm{Sel}(\mathbf{Q}, V)$ denote the Bloch–Kato Selmer group of $V$, which is equal to the kernel of $H^1(\mathbf{Q}, V) \longrightarrow H^1(\mathbf{Q}_p, V^-)$ in the present setting (cf. [**BSV21d**, Section 9.1]). One has a commutative exact diagram

$$(7) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & H^0(\mathbf{Q}_p, V^-) & \xrightarrow{\jmath} & \tilde{H}^1_f(\mathbf{Q}, V) & \xrightarrow{\pi} & \mathrm{Sel}(\mathbf{Q}, V) & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \cdot^+} & & \downarrow{\scriptstyle \mathrm{res}_p} & & \\ & & & & H^1(\mathbf{Q}_p, V^+) & \xrightarrow{i^+} & H^1(\mathbf{Q}_p, V) & & \end{array}$$

and there exists a unique section $\imath_{\mathrm{ur}} : \mathrm{Sel}(\mathbf{Q}, V) \hookrightarrow \tilde{H}^1_f(\mathbf{Q}, V)$ of $\pi$ such that the composition $\imath_{\mathrm{ur}}(\cdot)^+$ takes values in the finite subspace $H^1_{\mathrm{fin}}(\mathbf{Q}_p, V^+)$ of $H^1(\mathbf{Q}_p, V^+)$ (cf. Section 2.3 of [**BSV21c**]). As in loc. cit. we use the maps $\jmath$ and $\imath_{\mathrm{ur}}$ to identify Nekovář's extended Selmer group $\tilde{H}^1_f(\mathbf{Q}, V)$ with the *naive* extended Selmer group

$$\mathrm{Sel}^\dagger(\mathbf{Q}, V) = H^0(\mathbf{Q}_p, V^-) \oplus \mathrm{Sel}(\mathbf{Q}, V).$$

Enlarging $L$ if necessary, for $\xi = g, h$ fix an isomorphism of $L[G_{\mathbf{Q}}]$-modules

$$(8) \qquad \gamma_\xi : V_\xi \otimes_{\mathbf{Q}(\varrho)} L \simeq V(\xi) \ \text{ such that } \ \pi_\xi(\gamma_\xi(x) \otimes \gamma_\xi(y)) \in \mathbf{Q}(\varrho)(\chi_\xi)$$

for each $x$ and $y$ in $V_\xi$ (cf. Equation (4) of [**BSV21c**]). Set (cf. Equation (6))

$$(9) \qquad \mathscr{Q}_p(A, \varrho) = H^0(\mathbf{Q}_p, \mathbf{Q}(\varrho) \cdot q(A) \otimes_{\mathbf{Q}(\varrho)} V_{gh}).$$

The modular parametrisation $\wp_\infty : X_1(N_f) \longrightarrow A$ fixed in Section 2.1.1, the global Kummer map on $A(K_\varrho)\hat{\otimes}\mathbf{Q}_p$ and the isomorphisms $\gamma_g$ and $\gamma_h$ induce an embedding

$$(10) \qquad \gamma_{gh} : A^\dagger(K_\varrho)^\varrho \hookrightarrow \mathrm{Sel}^\dagger(\mathbf{Q}, V) = \tilde{H}^1_f(\mathbf{Q}, V),$$

and one defines the Garrett–Nekovář $p$-adic pairing (cf. Section 1)

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{fgh}} : A^\dagger(K_\varrho)^\varrho \otimes_{\mathbf{Q}(\varrho)} A^\dagger(K_\varrho)^\varrho \longrightarrow \mathscr{I}/\mathscr{I}^2$$

to be the restriction of the canonical height $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{fgh}}$ on $\tilde{H}^1_f(\mathbf{Q}, V)$ along $\gamma_{gh}$. Note that the discriminant $R^{\alpha\alpha}_p(A, \varrho)$ of $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{fgh}}$ on $A^\dagger(K_\varrho)^\varrho$ (cf. Section 1) is independent of the choice of the modular parametrisation $\wp_\infty$ and the isomorphisms $\gamma_g$ and $\gamma_h$.

**2.1.4.** *Logarithms.* — Let $V_{\mathrm{dR}} = D_{\mathrm{dR}}(V)$ be the de Rham module of $V = V(f,g,h)$. The duality $\pi_{fgh} : V \otimes_L V \longrightarrow L(1)$ induces a perfect pairing

$$\langle\cdot,\cdot\rangle_{fgh} : V_{\mathrm{dR}} \otimes_L V_{\mathrm{dR}} \longrightarrow L.$$

After identifying $V_{\mathrm{dR}}$ with $D_{\mathrm{dR}}(V(f)) \otimes_{\mathbf{Q}_p} D_{\mathrm{cris}}(V(g)) \otimes_L D_{\mathrm{cris}}(V(h))$ and $L$ with $D_{\mathrm{cris}}(\chi_g) \otimes_L D_{\mathrm{cris}}(\chi_h)$ under the natural isomorphisms (cf. Assumption 1.1.(2)), the pairing $\langle\cdot,\cdot\rangle_{fgh}$ agrees with the product of the pairings $\langle\cdot,\cdot\rangle_{\xi}$ for $\xi = f,g,h$.

The Bloch–Kato exponential map $\exp_p$ gives an isomorphism between the tangent space $V_{\mathrm{dR}}/\mathrm{Fil}^0$ of $V$ and the finite (viz. crystalline) subspace $H^1_{\mathrm{fin}}(\mathbf{Q}_p, V)$ of $H^1(\mathbf{Q}_p, V)$. Denote by $\log_p$ the inverse of $\exp_p$ and define the $\alpha\alpha$-logarithm

$$\log_{\alpha\alpha} = \big\langle \log_p, \omega_f \otimes \eta_{g_\alpha} \otimes \eta_{h_\alpha} \big\rangle_{fgh} : H^1_{\mathrm{fin}}(\mathbf{Q}_p, V) \longrightarrow L$$

to be the composition of $\log_p$ with evaluation at $\omega_f \otimes \eta_{g_\alpha} \otimes \eta_{h_\alpha}$ in $\mathrm{Fil}^0 V_{\mathrm{dR}}$ under the perfect duality $\langle\cdot,\cdot\rangle_{fgh}$. Similarly define the $\beta\beta$-logarithm

$$\log_{\beta\beta} = \big\langle \log_p, \omega_f \otimes \omega_{g_\alpha} \otimes \omega_{h_\alpha} \big\rangle : H^1_{\mathrm{fin}}(\mathbf{Q}_p, V) \longrightarrow L.$$

(Note that $\log_{ii}$ factors through the projection $H^1_{\mathrm{fin}}(\mathbf{Q}_p, V) \longrightarrow H^1(\mathbf{Q}_p, V_{ii})$.)

Set $\mathrm{tg}_{\mathrm{dR},K_p}(f) = H^0(K_p, V(f) \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})/\mathrm{Fil}^0$ and consider the composition

$$\log_{A,p} : A(K_p)\hat{\otimes}\mathbf{Q}_p \simeq H^1_{\mathrm{fin}}(K_p, V_p(A)) \simeq H^1_{\mathrm{fin}}(K_p, V(f)) \simeq \mathrm{tg}_{\mathrm{dR},K_p}(f),$$

where the first isomorphism is the local Kummer map, the second is induced by the fixed modular parametrisation $\wp_\infty : X_1(N_f) \longrightarrow A$ (cf. Section 2.1.1), and the third is the inverse of the Bloch–Kato exponential map. For $\chi = \varphi, \psi$ (cf. Section 1) define

$$\log_{\omega_f} = \langle \log_{A,p}, \omega_f \rangle_f : A(K_\chi) \longrightarrow K_p,$$

where $K_\chi$ is the ring class field of $K$ cut-out by $\chi$ and $A(K_\chi)$ is viewed as a subgroup of $A(K_p)$ via the embedding $i_p : \bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$ fixed at the outset. (Recall that $p$ is inert in $K$ and that $\chi$ is dihedral, hence $p\mathcal{O}_K$ splits completely in $K_\chi$.)

**2.2. Big logarithms and diagonal classes.** — Let

$$\mathscr{L}_{\boldsymbol{f}} : H^1(\mathbf{Q}_p, \boldsymbol{V}_f) \longrightarrow \mathscr{I}$$

be the big logarithm map constructed in Proposition 7.3 of [**BSV21d**] using the work of Coleman, Perrin-Riou et alii. (Note that the tame character $\chi_{\boldsymbol{f}}$ of $\boldsymbol{f}$ is trivial in the present setting and that the logarithm $\mathscr{L}_{\boldsymbol{f}}$ takes values in $\mathscr{I}$ by the exceptional zero condition $\alpha_f = \alpha_g \cdot \alpha_h$.) With a slight abuse of notation denote by

$$\mathscr{L}_{\boldsymbol{f}} : H^1_{\mathrm{bal}}(\mathbf{Q}_p, \boldsymbol{V}) \longrightarrow \mathscr{I}$$

also the composition $\mathscr{L}_{\boldsymbol{f}} \circ p_f$ (cf. Equation (3)).

Let $H^1_{\mathrm{bal}}(\mathbf{Q}, \boldsymbol{V})$ be the group of global classes in $H^1(\mathbf{Q}, \boldsymbol{V})$ whose restriction at $p$ belongs to the balanced local condition $H^1_{\mathrm{bal}}(\mathbf{Q}_p, \boldsymbol{V})$. According to Theorem A of [**BSV21d**] (cf. [**BSV21a**, Section 2.1]) there exists a canonical *big diagonal class*

$$\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) = \kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha) \in H^1_{\mathrm{bal}}(\mathbf{Q}, \boldsymbol{V})$$

such that

$$\mathscr{L}_{\boldsymbol{f}}(\mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))) = \mathscr{L}_p^{\alpha\alpha}(A, \varrho). \tag{11}$$

Define the *diagonal class*

$$\kappa(f, g_\alpha, h_\alpha) = \rho_{w_o}(\kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha))$$

to be the image in $H^1(\mathbf{Q}, V)$ of $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ under the map induced in cohomology by the specialisation isomorphism $\rho_{w_o} : \boldsymbol{V} \otimes_{w_o} L \simeq V$. Since by assumption the complex Garrett $L$-function $L(A, \varrho, s) = L(f \otimes g \otimes h, s)$ vanishes at $s = 1$, Theorem B of [**BSV21d**] implies that $\kappa(f, g_\alpha, h_\alpha)$ is crystalline at $p$, hence a Selmer class:

$$\kappa(f, g_\alpha, h_\alpha) \in \mathrm{Sel}(\mathbf{Q}, V). \tag{12}$$

Identify $\mathscr{O}_{\boldsymbol{fgh}}$ with a subring of the power series ring $L[\![\boldsymbol{k} - 2, \boldsymbol{l} - 1, \boldsymbol{m} - 1]\!]$, where $\boldsymbol{k} - 2$ in $\mathscr{O}_{\boldsymbol{f}}$ is a uniformiser at the centre 2 of $U_{\boldsymbol{f}}$, and $\boldsymbol{l} - 1$ and $\boldsymbol{m} - 1$ are defined similarly. In light of Equation (12) and Lemma 2.1 there exist local classes $\mathfrak{Y}_{\boldsymbol{k}}, \mathfrak{Y}_{\boldsymbol{l}}$ and $\mathfrak{Y}_{\boldsymbol{m}}$ in $H^1(\mathbf{Q}_p, \boldsymbol{V}_f)$ satisfying the identity

$$p_f(\mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))) = \sum_{\boldsymbol{u}} \mathfrak{Y}_{\boldsymbol{u}} \cdot (\boldsymbol{u} - u_o). \tag{13}$$

Equation (11) gives

$$\mathscr{L}_p^{\alpha\alpha}(A, \varrho) = \sum_{\boldsymbol{u}} \mathscr{L}_{\boldsymbol{f}}(\mathfrak{Y}_{\boldsymbol{u}}) \cdot (\boldsymbol{u} - u_o) \in \mathscr{I}^2. \tag{14}$$

The following key lemma, proved in Part 1 of Proposition 9.3 of [**BSV21d**], gives an explicit description of the linear term of $\mathscr{L}_{\boldsymbol{f}}(\mathfrak{Y}_{\boldsymbol{u}})$ at $w_o$. Identify the $p$-adic completion of the Galois group of the maximal abelian extension of $\mathbf{Q}_p$ with that of $\mathbf{Q}_p^*$ via the local Artin map, normalised in such a way that $p^{-1}$ corresponds to the arithmetic Frobenius. This identifies $H^1(\mathbf{Q}_p, \mathbf{Q}_p)$ with $\mathrm{Hom}_{\mathrm{cont}}(\mathbf{Q}_p^*, \mathbf{Q}_p)$, hence (recalling that $G_{\mathbf{Q}_p}$ acts trivially on $V_{\beta\beta}^-$, cf. Equation (4))

$$H^1(\mathbf{Q}_p, V_{\beta\beta}^-) = \mathrm{Hom}_{\mathrm{cont}}(\mathbf{Q}_p^*, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} V_{\beta\beta}^-, \tag{15}$$

and the Bloch–Kato dual exponential $\exp_p^*$ on $H^1(\mathbf{Q}_p, V_{\beta\beta}^-)$ satisfies

$$\exp_p^*(\varphi \otimes v) = \varphi(e(1)) \cdot v$$

in $D_{\mathrm{cris}}(V_{\beta\beta}^-) = V_{\beta\beta}^-$ for each $\varphi$ in $\mathrm{Hom}_{\mathrm{cont}}(\mathbf{Q}_p^*, \mathbf{Q}_p)$ and $v$ in $V_{\beta\beta}^-$, where

$$e(1) = (1 + p) \hat{\otimes} \log_p(1 + p)^{-1} \in \mathbf{Z}_p^* \hat{\otimes} \mathbf{Q}_p.$$

For $x = \varphi \otimes v$ in $H^1(\mathbf{Q}_p, V_{\beta\beta}^-)$ (with $\varphi$ and $v$ as above) and $q$ in $\mathbf{Q}_p^* \hat{\otimes} \mathbf{Q}_p$, set

$$x(q) = \varphi(q) \cdot v \quad \text{and} \quad x(q)_f = \langle x(q), \eta_f \otimes \omega_{g_\alpha} \otimes \omega_{h_\alpha} \rangle_{fgh}.$$

If $(\boldsymbol{\xi}, \boldsymbol{u})$ denotes one of the pairs $(\boldsymbol{f}, \boldsymbol{k}), (\boldsymbol{g}, \boldsymbol{l})$ and $(\boldsymbol{h}, \boldsymbol{m})$, define

$$\tilde{D}_{\boldsymbol{u}} : H^1(\mathbf{Q}_p, \boldsymbol{V}_f) \longrightarrow L$$

to be the linear map which on $\mathfrak{Y}$ in $H^1(\mathbf{Q}_p, \boldsymbol{V}_f)$ takes the value

$$\tilde{D}_{\boldsymbol{u}}(\mathfrak{Y}) = \frac{(-1)^{u_o}}{2\,(1 - p^{-1})} \cdot \big(\mathfrak{y}(p^{-1})_f - \mathfrak{L}_{\boldsymbol{\xi}}^{\mathrm{an}} \cdot \mathfrak{y}(e(1))_f\big). \tag{16}$$

Here $\mathfrak{y} = \rho_{w_o}(\mathfrak{Y})$ in $H^1(\mathbf{Q}_p, V_{\beta\beta}^-)$ is the $w_o$-specialisation of $\mathfrak{Y}$, $u_o = 2$ if $\boldsymbol{u} = \boldsymbol{k}$ and $u_o = 1$ if $\boldsymbol{u} = \boldsymbol{l}, \boldsymbol{m}$, and $\mathfrak{L}_{\boldsymbol{\xi}}^{\mathrm{an}}$ in $L$ is the *analytic $\mathscr{L}$-invariant of $\boldsymbol{\xi}$*, defined by

$$\mathfrak{L}_{\boldsymbol{\xi}}^{\mathrm{an}} = -2 \cdot d\log a_p(\boldsymbol{\xi})(u_o)$$

(where $d\log a = a'/a$ for $a$ in $\mathscr{O}_{\boldsymbol{\xi}}^*$). We can finally state the aforementioned key lemma.

***Lemma 2.2.*** — *For each local class $\mathfrak{Y}$ in $H^1(\mathbf{Q}_p, \boldsymbol{V}_f)$ one has*

$$\mathscr{L}_{\boldsymbol{f}}(\mathfrak{Y}) \pmod{\mathscr{I}^2} = \sum_{\boldsymbol{u}} \tilde{D}_{\boldsymbol{u}}(\mathfrak{Y}) \cdot (\boldsymbol{u} - u_o).$$

For each pair $(\boldsymbol{u}, \boldsymbol{v})$ of distinct elements of $\{\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}\}$, define (cf. Equation (13))

$$\tilde{D}_{\boldsymbol{u},\boldsymbol{u}}(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) = \tilde{D}_{\boldsymbol{u}}(\mathfrak{Y}_{\boldsymbol{u}}) \quad \text{and} \quad \tilde{D}_{\boldsymbol{u},\boldsymbol{v}}(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) = \tilde{D}_{\boldsymbol{u}}(\mathfrak{Y}_{\boldsymbol{v}}) + \tilde{D}_{\boldsymbol{v}}(\mathfrak{Y}_{\boldsymbol{u}}).$$

Equation (14) and Lemma 2.2 give the following lemma (which implies that the *derivatives* $\tilde{D}_{\cdot}(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$ are independent of the choice of the classes $\mathfrak{Y}_{\boldsymbol{u}}$ satisfying (13)).

***Lemma 2.3.*** — *One has the following equality in $\mathscr{I}^2/\mathscr{I}^3$.*

$$\mathscr{L}_p^{\alpha\alpha}(A, \varrho) \pmod{\mathscr{I}^3} = \sum_{\boldsymbol{u},\boldsymbol{v}} \tilde{D}_{\boldsymbol{u},\boldsymbol{v}}(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) \cdot (\boldsymbol{u} - u_o)(\boldsymbol{v} - v_o)$$

### 2.3. An exceptional zero formula *à la* Rubin–Perrin-Riou.

— For a positive integer $n$ and each $2n$-tuple $\boldsymbol{y} = (y_1, \ldots, y_{2n})$ of elements of $\tilde{H}_f^1(\mathbf{Q}, V)$ denote by

$$\mathscr{R}_p^{\alpha\alpha}(\boldsymbol{y}) = \mathrm{Pf}\Big(\big(\langle\!\langle y_i, y_j \rangle\!\rangle_{\boldsymbol{fgh}}\big)_{1 \leqslant i,j \leqslant 2n}\Big) \in \mathscr{I}^n/\mathscr{I}^{n+1}$$

the Pfaffian of the skew-symmetric $2n \times 2n$ matrix whose $ij$-entry is $\langle\!\langle y_i, y_j \rangle\!\rangle_{\boldsymbol{fgh}}$, and define the *extended Garrett–Nekovář $p$-adic height pairing*

$$\tilde{h}_p^{\alpha\alpha} : \mathrm{Sel}(\mathbf{Q}, V) \otimes_L \mathrm{Sel}(\mathbf{Q}, V) \longrightarrow \mathscr{I}^2/\mathscr{I}^3$$

to be the bilinear form which on $y \otimes y'$ in $\mathrm{Sel}(\mathbf{Q}, V)^{\otimes 2}$ takes the value

$$\tilde{h}_p^{\alpha\alpha}(y \otimes y') = \mathscr{R}_p^{\alpha\alpha}(q_{\alpha\alpha}, q_{\beta\beta}, y, y').$$

The aim of this section is to prove the following proposition.

***Proposition 2.4.*** — *Up to sign, one has the equality*

$$\tilde{h}_p^{\alpha\alpha}(\kappa(f, g_\alpha, h_\alpha) \otimes \cdot) = c_A \cdot \log_{\alpha\alpha}(\mathrm{res}_p(\cdot)) \cdot \mathscr{L}_p^{\alpha\alpha}(A, \varrho) \pmod{\mathscr{I}^3}$$

*of $\mathscr{I}^2/\mathscr{I}^3$-valued $L$-linear forms on* $\mathrm{Sel}(\mathbf{Q}, V)$*, where* $c_A = \frac{m_p \cdot (1 - p^{-1}) \cdot \mathrm{ord}_p(q_A)}{\deg(\wp_\infty)}$.

We divide the proof of Proposition 2.4 in a series of lemmas. Define

$$c_p(f) = \langle q(f), \eta_f \rangle_f$$

in $L^*$ (cf. Section 2.1.2). As in Section 2.2, identify $H^1(\mathbf{Q}_p, \mathbf{Q}_p)$ with $\mathrm{Hom}_{\mathrm{cont}}(\mathbf{Q}_p^*, \mathbf{Q}_p)$ via the local Artin map (sending $p^{-1}$ to an arithmetic Frobenius), and set

$$\log_{\boldsymbol{\xi}} = \log_p - \mathfrak{L}_{\boldsymbol{\xi}}^{\mathrm{an}} \cdot \mathrm{ord}_p \in H^1(\mathbf{Q}_p, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} L,$$

where $\log_p : \mathbf{Q}_p^* \longrightarrow \mathbf{Q}_p$ is the (branch of the) $p$-adic logarithm (vanishing at $p$) and $\mathrm{ord}_p : \mathbf{Q}_p^* \longrightarrow \mathbf{Z}$ is the $p$-adic valuation normalised by $\mathrm{ord}_p(p) = 1$.

***Lemma 2.5.*** — *For each Selmer class $y$ in* $\mathrm{Sel}(\mathbf{Q}, V)$ *one has*

$$-2 \cdot \langle\!\langle q_{\beta\beta}, y \rangle\!\rangle_{\boldsymbol{fgh}} = c_p(f) \cdot \log_{\alpha\alpha}(\mathrm{res}_p(y)) \cdot (\boldsymbol{k} - \boldsymbol{l} - \boldsymbol{m})$$

*and*

$$-\frac{2 \cdot \deg(\wp_\infty)}{m_p \cdot \mathrm{ord}_p(q_A)} \cdot \langle\!\langle q_{\beta\beta}, q_{\alpha\alpha} \rangle\!\rangle_{\boldsymbol{fgh}} = (\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{g}}) \cdot (\boldsymbol{l} - 1) + (\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{h}}) \cdot (\boldsymbol{m} - 1)$$

*Proof.* — See Equations (17) and (27) of [**BSV21b**]. (Note that the $p$-adic logarithm denoted by $\log_{\alpha\alpha}$ in [**BSV21b**] is equal to $\langle \log_p, q_{\beta\beta} \rangle_{fgh} = -c_p(f) \cdot \log_{\alpha\alpha}$.) □

Let $\mathrm{C}^\bullet_{\mathrm{cont}}(\mathbf{Q}_p, \boldsymbol{V}^-)$ be the complex of (inhomogeneous) continuous cochains of $G_{\mathbf{Q}_p}$ with values in the quotient $p^- : \boldsymbol{V} \longrightarrow \boldsymbol{V}^-$ of $\boldsymbol{V}$ (cf. Section 2.1.1), and let

$$\langle \cdot, \cdot \rangle_{\mathrm{Tate}} : H^1(\mathbf{Q}_p, V^-) \otimes_L H^1(\mathbf{Q}_p, V^+) \longrightarrow L$$

the local Tate pairing arising from the perfect duality $\pi_{fgh} : V \otimes_L V \longrightarrow L(1)$. Recall the morphism $\cdot^+ : \tilde{H}^1_f(\mathbf{Q}, V) \longrightarrow H^1(\mathbf{Q}_p, V^+)$ introduced in Diagram (7).

***Lemma 2.6.*** — *There exist 1-cochains* $X_{\boldsymbol{k}}, X_{\boldsymbol{l}}$ *and* $X_{\boldsymbol{m}}$ *in* $\mathrm{C}^1_{\mathrm{cont}}(\mathbf{Q}_p, \boldsymbol{V}^-)$ *such that*

$$p^-(\mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))) = \mathrm{cl}\Big(\sum_{\boldsymbol{u}} X_{\boldsymbol{u}} \cdot (\boldsymbol{u} - u_o)\Big), \tag{17}$$

*id est* $\sum_{\boldsymbol{u}} X_{\boldsymbol{u}} \cdot (\boldsymbol{u} - u_o)$ *is a 1-cocycle representing* $p^-(\mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})))$, *and*

$$\langle\!\langle \kappa(f, g_\alpha, h_\alpha), y \rangle\!\rangle_{\boldsymbol{fgh}} = \sum_{\boldsymbol{u}} \langle \mathfrak{x}_{\boldsymbol{u}}, y^+ \rangle_{\mathrm{Tate}} \cdot (\boldsymbol{u} - u_o)$$

*for each extended Selmer class $y$ in* $\tilde{H}^1_f(\mathbf{Q}, V)$, *where*

$$\mathfrak{x}_{\boldsymbol{u}} = \mathrm{cl}(\rho_{w_o}(X_{\boldsymbol{u}}))$$

*is the local class in* $H^1(\mathbf{Q}_p, V^-)$ *represented by the 1-cocycle* $\rho_{w_o}(X_{\boldsymbol{u}})$.

*Proof.* — This follows from Equations (30)–(37) in Section 3.4 of [**BSV21c**]. (The paragraphs containing the aforementioned equations do not use the non-exceptionality assumption [**BSV21c**, Equation (26)] imposed in [**BSV21c**, Section 3].) □

Fix in what follows 1-cochains $X_{\boldsymbol{k}}, X_{\boldsymbol{l}}$ and $X_{\boldsymbol{m}}$ satisfying the conclusions of Lemma 2.6. For $i = \alpha\alpha, \beta\beta$ let $\mathrm{pr}_i : H^1(\mathbf{Q}_p, V^-) \longrightarrow H^1(\mathbf{Q}_p, V^-_i)$ be the natural projection.

***Lemma 2.7.*** — *For $\boldsymbol{u}$ equal to one of $\boldsymbol{k}$, $\boldsymbol{l}$ and $\boldsymbol{m}$, one has*

$$\mathrm{pr}_{\alpha\alpha}(\mathfrak{x}_{\boldsymbol{u}}) = \mu_{\boldsymbol{u}} \cdot \log_{\boldsymbol{f}} \otimes q_{\alpha\alpha}$$

*in* $H^1(\mathbf{Q}_p, V^-_{\alpha\alpha}) = H^1(\mathbf{Q}_p, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} V^-_{\alpha\alpha}$ *for some* $\mu_{\boldsymbol{u}}$ *in* $L$.

*Proof.* — Set $\kappa_{\alpha\alpha} = \kappa(f, g_\alpha, h_\alpha)$. As explained in Section 3.3 of [**BSV21b**] (cf. Equation (15) of loc. cit.) one has (cf. Diagram (7))

$$q^+_{\beta\beta} = \frac{m_p}{\deg(\wp_\infty)} \cdot (q_A \hat{\otimes} 1) \otimes q^*_{\alpha\alpha}$$

in the direct summand

$$H^1(\mathbf{Q}_p, V^+_{\beta\beta}) = H^1(\mathbf{Q}_p, \mathbf{Q}_p(1)) \otimes_{\mathbf{Q}_p} V^+_{\beta\beta}(-1)$$

of $H^1(\mathbf{Q}_p, V^+)$, where $q^*_{\alpha\alpha}$ in $V^+_{\beta\beta}(-1)$ is the dual basis of $q_{\alpha\alpha}$ under the pairing $\pi_{fgh}(-1)$. It then follows from Lemma 2.6 and local class field theory that

$$\langle\!\langle \kappa_{\alpha\alpha}, q_{\beta\beta} \rangle\!\rangle_{\boldsymbol{fgh}} = \sum_{\boldsymbol{u}} \mathfrak{r}^{\alpha\alpha}_{\boldsymbol{u}}(q_A) \cdot (\boldsymbol{u} - u_o),$$

where the class $\mathfrak{r}^{\alpha\alpha}_{\boldsymbol{u}}$ in $H^1(\mathbf{Q}_p, \mathbf{Q}_p)$ is defined by the identity

$$\mathrm{pr}_{\alpha\alpha}(\mathfrak{r}_{\boldsymbol{u}}) = \mathfrak{r}^{\alpha\alpha}_{\boldsymbol{u}} \otimes q_{\alpha\alpha}.$$

On the other hand, since $\log_{\alpha\alpha}(\mathrm{res}_p(\kappa_{\alpha\alpha})) = 0$ (because $\kappa_{\alpha\alpha}$ is a balanced class, cf. Section 9.1 of [**BSV21d**]), Lemma 2.5 and the skew-symmetry of $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{fgh}}$ yield

$$\langle\!\langle \kappa_{\alpha\alpha}, q_{\beta\beta} \rangle\!\rangle_{\boldsymbol{fgh}} = -\langle\!\langle q_{\beta\beta}, \kappa_{\alpha\alpha} \rangle\!\rangle_{\boldsymbol{fgh}} = 0,$$

hence $\mathfrak{r}^{\alpha\alpha}_{\boldsymbol{u}}(q_A) = 0$, id est $\mathfrak{r}^{\alpha\alpha}_{\boldsymbol{u}}$ is a multiple of $\log_{q_A}$. The lemma follows from this and Theorem 3.18 of [**GS93**], according to which $\log_{q_A}$ equals $\log_{\boldsymbol{f}}$. □

***Lemma 2.8.*** — *Assume that either $\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} \neq \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{g}}$ or $\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} \neq \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{h}}$. Then the local classes $\mathfrak{r}_{\boldsymbol{k}}$, $\mathfrak{r}_{\boldsymbol{l}}$ and $\mathfrak{r}_{\boldsymbol{m}}$ belong to the direct summand $H^1(\mathbf{Q}_p, V^-_{\beta\beta})$ of $H^1(\mathbf{Q}_p, V^-)$.*

*Proof.* — The proof uses the main properties of the Bockstein map

$$\beta^-_{\boldsymbol{fgh}} : H^0(\mathbf{Q}_p, V^-) \longrightarrow H^1(\mathbf{Q}_p, V^-) \otimes_L \mathscr{I}/\mathscr{I}^2$$

introduced in [**BSV21b**, Section 3.1.1]. As $\kappa(f, g_\alpha, h_\alpha) = \rho_{w_o}(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$ is crystalline at $p$, Lemma 2.1 shows that there exist $\mathfrak{Z}_{\boldsymbol{k}}$, $\mathfrak{Z}_{\boldsymbol{l}}$ and $\mathfrak{Z}_{\boldsymbol{m}}$ in $H^1(\mathbf{Q}_p, \boldsymbol{V}_f)$ such that

$$p_f(\mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))) = \sum_{\boldsymbol{u}} \mathfrak{Z}_{\boldsymbol{u}} \cdot (\boldsymbol{u} - u_o). \tag{18}$$

Recall the specialisation isomorphism $\rho_{w_o} : \boldsymbol{V}_f \otimes_{w_o} L \simeq V^-_{\beta\beta}$ arising from evaluation at $w_o$ on $\mathscr{O}_{\boldsymbol{fgh}}$ (cf. Section 2.1.1), set $\mathfrak{z}_{\boldsymbol{u}} = \rho_{w_o}(\mathfrak{Z}_{\boldsymbol{u}})$ in $H^1(\mathbf{Q}_p, V^-_{\beta\beta})$ and define

$$\nabla_f = \sum_{\boldsymbol{u}} \mathfrak{z}_{\boldsymbol{u}} \cdot (\boldsymbol{u} - u_o)$$

in $H^1(\mathbf{Q}_p, V^-_{\beta\beta}) \otimes \mathscr{I}/\mathscr{I}^2$. It follows from Equations (17) and (18) and Lemma 3.2 of [**BSV21b**] that the difference $\sum_{\boldsymbol{u}} \mathfrak{r}_{\boldsymbol{u}} \cdot (\boldsymbol{u} - u_o) - \nabla_f$ belongs to the image of the Bockstein map $\beta^-_{\boldsymbol{fgh}}$. There exist then $\mu$ and $\nu$ in $L$ such that

$$\sum_{\boldsymbol{u}} \mathfrak{r}_{\boldsymbol{u}} \cdot (\boldsymbol{u} - u_o) - \nabla_f - \nu \cdot \beta^-_{\boldsymbol{fgh}}(q_{\beta\beta}) = \mu \cdot \beta^-_{\boldsymbol{fgh}}(q_{\alpha\alpha}). \tag{19}$$

Equation (8) of [**BSV21b**] shows that $\beta^-_{\boldsymbol{fgh}}(q_{\beta\beta})$ belongs to $H^1(\mathbf{Q}_p, V^-_{\beta\beta}) \otimes_L \mathscr{I}/\mathscr{I}^2$, hence Lemma 2.7 and the previous equation give

$$\sum_{\boldsymbol{u}} \mu_{\boldsymbol{u}} \cdot \log_{\boldsymbol{f}} \otimes q_{\alpha\alpha} \cdot (\boldsymbol{u} - u_o) = \sum_{\boldsymbol{u}} \mathrm{pr}_{\alpha\alpha}(\mathfrak{r}_{\boldsymbol{u}}) \cdot (\boldsymbol{u} - u_o) = \mu \cdot \mathrm{pr}_{\alpha\alpha}(\beta^-_{\boldsymbol{fgh}}(q_{\alpha\alpha})) \tag{20}$$

(where in the right-most term we write again $\mathrm{pr}_{\alpha\alpha}$ to denote the $\mathscr{I}/\mathscr{I}^2$-base change of the projection $\mathrm{pr}_{\alpha\alpha} : H^1(\mathbf{Q}_p, V^-) \longrightarrow H^1(\mathbf{Q}_p, V^-_{\alpha\alpha})$). The computations carried out in Sections 3.3 and 3.4 of [BSV21b] (see in particular Equation (30) of loc. cit. and the discussion preceding it) give the following equality in $H^1(\mathbf{Q}_p, V^-_{\alpha\alpha})\otimes_L \mathscr{I}/\mathscr{I}^2$:

$$2\cdot \mathrm{pr}_{\alpha\alpha}(\beta^-_{\boldsymbol{fgh}}(q_{\alpha\alpha})) = \sum_{\boldsymbol{u}} \log_{\boldsymbol{\xi}} \otimes q_{\alpha\alpha}\cdot(\boldsymbol{u}-u_o),$$

where $(\boldsymbol{\xi},\boldsymbol{u}) = (\boldsymbol{f},\boldsymbol{k}),(\boldsymbol{g},\boldsymbol{l}),(\boldsymbol{h},\boldsymbol{m})$. Together with Equation (20) this implies

$$2\mu_{\boldsymbol{k}} = \mu, \quad 2\mu_{\boldsymbol{l}}\cdot \log_{\boldsymbol{f}} = \mu\cdot\log_{\boldsymbol{g}} \quad \text{and} \quad 2\mu_{\boldsymbol{m}}\cdot\log_{\boldsymbol{f}} = \mu\cdot\log_{\boldsymbol{h}},$$

thus $\mu = \mu_{\boldsymbol{k}} = \mu_{\boldsymbol{l}} = \mu_{\boldsymbol{m}} = 0$ by the assumption on the analytic $\mathscr{L}$-invariants made in the statement. The lemma follows from this and Equation (19). □

Let $(\boldsymbol{u},\boldsymbol{\xi})$ denote one of $(\boldsymbol{k},\boldsymbol{f})$, $(\boldsymbol{l},\boldsymbol{g})$ and $(\boldsymbol{m},\boldsymbol{h})$. For each local class $x$ in $H^1(\mathbf{Q}_p, V^-)$, denote by $x_{\beta\beta} = \mathrm{pr}_{\beta\beta}(x)$ in $H^1(\mathbf{Q}_p, V^-_{\beta\beta})$ its $\beta\beta$-component and (with the notations introduced in Section 2.2) set

$$\ell_{\boldsymbol{u}}(x) = (-1)^{u_o}\cdot\left(x_{\beta\beta}(p^{-1})_f - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{\xi}}\cdot x_{\beta\beta}(e(1))_f\right).$$

For each pair $(\boldsymbol{u},\boldsymbol{v})$ of distinct elements of $\{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}\}$ define

$$\tilde{\mathtt{D}}_{\boldsymbol{u},\boldsymbol{u}} = \ell_{\boldsymbol{u}}(\mathfrak{x}_{\boldsymbol{u}}) \quad \text{and} \quad \tilde{\mathtt{D}}_{\boldsymbol{u},\boldsymbol{v}} = \ell_{\boldsymbol{u}}(\mathfrak{x}_{\boldsymbol{v}}) + \ell_{\boldsymbol{v}}(\mathfrak{x}_{\boldsymbol{u}}).$$

***Lemma 2.9.*** — *For each pair $(\boldsymbol{u},\boldsymbol{v})$ of elements of $\{\boldsymbol{k},\boldsymbol{l},\boldsymbol{m}\}$ one has*

$$2\left(1-p^{-1}\right)\cdot\tilde{D}_{\boldsymbol{u},\boldsymbol{v}}(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) = \tilde{\mathtt{D}}_{\boldsymbol{u},\boldsymbol{v}}.$$

*Proof.* — We give the proof for $(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{k},\boldsymbol{l})$ and $(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{k},\boldsymbol{k})$, the other cases being similar. We use the notations introduced in the proof of Lemma 2.8. Section 3 of [BSV21b] (see in particular Equations (8) and (30) of loc. cit.) gives the identities

$$2\cdot\beta^-_{\boldsymbol{fgh}}(q_{\beta\beta}) = \sum_{\boldsymbol{u}}(-1)^{u_o}\cdot\log_{\boldsymbol{\xi}}\otimes q_{\beta\beta}\cdot(\boldsymbol{u}-u_o) \quad \text{and} \quad \mathrm{pr}_{\beta\beta}(\beta^-_{\boldsymbol{fgh}}(q_{\alpha\alpha})) = 0.$$

Equation (19) (and the definition of derivatives $\tilde{D}_{\boldsymbol{u},\boldsymbol{v}}$) then yields

$$2\left(1-p^{-1}\right)\cdot\tilde{D}_{\boldsymbol{k},\boldsymbol{l}}(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) - \ell_{\boldsymbol{k}}(\mathfrak{x}_{\boldsymbol{l}}) - \ell_{\boldsymbol{l}}(\mathfrak{x}_{\boldsymbol{k}}) = \frac{\nu}{2}\left(\ell_{\boldsymbol{k}}(\log_{\boldsymbol{g}}\otimes q_{\beta\beta}) - \ell_{\boldsymbol{l}}(\log_{\boldsymbol{f}}\otimes q_{\beta\beta})\right) = 0$$

and

$$2\left(1-p^{-1}\right)\cdot\tilde{D}_{\boldsymbol{k},\boldsymbol{k}}(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})) - \ell_{\boldsymbol{k}}(\mathfrak{x}_{\boldsymbol{k}}) = -\frac{\nu}{2}\cdot\ell_{\boldsymbol{k}}(\log_{\boldsymbol{f}}\otimes q_{\beta\beta}) = 0,$$

quod erat demonstrandum. □

***Lemma 2.10.*** — *Assume that either $\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} \neq \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{g}}$ or $\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} \neq \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{h}}$. Then one has*

$$c_p(f)\cdot\langle\!\langle q_{\alpha\alpha}, \kappa(f,g_\alpha,h_\alpha)\rangle\!\rangle_{\boldsymbol{fgh}} = -\frac{m_p\cdot\mathrm{ord}_p(q_A)}{\deg(\wp_\infty)}\cdot\sum_{\boldsymbol{u}}\ell_{\boldsymbol{k}}(\mathfrak{x}_{\boldsymbol{u}})\cdot(\boldsymbol{u}-u_o).$$

*Proof.* — Under the assumption in the statement $\mathfrak{x}_{\boldsymbol{u}}$ belongs to $H^1(\mathbf{Q}_p, V^-_{\beta\beta})$ by Lemma 2.8. Together with the equality $\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} = \frac{\log_p(q_A)}{\mathrm{ord}_p(q_A)}$ (cf. [GS93]), this gives

$$\ell_{\boldsymbol{k}}(\mathfrak{x}_{\boldsymbol{u}}) = \mathfrak{x}_{\boldsymbol{u}}(p^{-1})_f - \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}}\cdot\mathfrak{x}_{\boldsymbol{u}}(e(1))_f = -\frac{1}{\mathrm{ord}_p(q_A)}\cdot\mathfrak{x}_{\boldsymbol{u}}(q_A)_f. \tag{21}$$

According to Equation (15) of [**BSV21b**], one has

$$q^+_{\alpha\alpha} = \frac{m_p}{\deg(\wp_\infty)} \cdot (q_A \hat{\otimes} 1) \otimes q^*_{\beta\beta},$$

where $q^*_{\beta\beta}$ in $V^+_{\alpha\alpha}$ is the dual basis of $q_{\beta\beta}$ under the perfect pairing $\pi_{fgh}(-1)$. Lemma 2.6, the skew-symmetry of $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{fgh}}$ and local class field theory then give

$$\langle\!\langle q_{\alpha\alpha}, \kappa(f, g_\alpha, h_\alpha) \rangle\!\rangle_{\boldsymbol{fgh}} = -\langle\!\langle \kappa(f, g_\alpha, h_\alpha), q_{\alpha\alpha} \rangle\!\rangle_{\boldsymbol{fgh}} = \frac{m_p}{\deg(\wp_\infty)} \cdot \sum_{\boldsymbol{u}} \mathfrak{x}^{\beta\beta}_{\boldsymbol{u}}(q_A) \cdot (\boldsymbol{u} - u_o),$$

where $\mathfrak{x}^{\beta\beta}_{\boldsymbol{u}}$ in $H^1(\mathbf{Q}_p, \mathbf{Q}_p)$ is defined by $\mathfrak{x}_{\boldsymbol{u}} = \mathfrak{x}^{\beta\beta}_{\boldsymbol{u}} \otimes q_{\beta\beta}$. The lemma follows from the previous equation, Equation (21) and the identity $\mathfrak{x}_{\boldsymbol{u}}(q_A)_f = \mathfrak{x}^{\beta\beta}_{\boldsymbol{u}}(q_A) \cdot c_p(f)$, □

***Lemma 2.11.*** — *Assume that either $\mathfrak{L}^{\rm an}_{\boldsymbol{f}} \neq \mathfrak{L}^{\rm an}_{\boldsymbol{g}}$ or $\mathfrak{L}^{\rm an}_{\boldsymbol{f}} \neq \mathfrak{L}^{\rm an}_{\boldsymbol{h}}$, so that $\mathfrak{x}_{\boldsymbol{u}}$ belongs to $H^1(\mathbf{Q}_p, V^-_{\beta\beta})$ for $\boldsymbol{u} = \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}$ by Lemma 2.8. Then*

$$\langle\!\langle \kappa(f, g_\alpha, h_\alpha), \cdot \rangle\!\rangle_{\boldsymbol{fgh}} = \log_{\alpha\alpha}\big(\mathrm{res}_p(\cdot)\big) \cdot \sum_{\boldsymbol{u}} \mathfrak{x}_{\boldsymbol{u}}(e(1))_f \cdot (\boldsymbol{u} - u_o)$$

*as $\mathscr{I}/\mathscr{I}^2$-valued $L$-linear forms on the Bloch–Kato Selmer group* $\mathrm{Sel}(\mathbf{Q}, V)$.

*Proof.* — Let $y$ be a Selmer class in $\mathrm{Sel}(\mathbf{Q}, V)$, and let $\tilde{y} = \imath_{\rm ur}(y)$ in $\tilde{H}^1_f(\mathbf{Q}, V)$ be the corresponding class in the extended Selmer group (cf. Section 2.3 of [**BSV21c**]). By construction $\tilde{y}^+$ belongs to the Bloch–Kato finite subspace of $H^1(\mathbf{Q}, V^+)$, and $\mathrm{res}_p(y) = i^+(\tilde{y}^+)$ is its image under the map $i^+$ induced in cohomology by the inclusion $V^+ \hookrightarrow V$. Define $\tilde{y}^+_{\alpha\alpha}$ in $\mathbf{Z}^*_p \otimes_{\mathbf{Z}_p} L$ by the identity

$$\mathrm{pr}_{\alpha\alpha}(\tilde{y}^+) = \tilde{y}^+_{\alpha\alpha} \otimes (\eta_f \otimes \omega_{g_\alpha} \otimes \omega_{h_\alpha}),$$

in $H^1_{\rm fin}(\mathbf{Q}_p, V^+_{\alpha\alpha}) = H^1_{\rm fin}(\mathbf{Q}_p, L(1)) \otimes_L V^+_{\alpha\alpha}(-1)$ (where as usual $H^1_{\rm fin}(\mathbf{Q}_p, L(1))$ is identified with $\mathbf{Z}^*_p \otimes_{\mathbf{Z}_p} L$ via the local Kummer map). Then one has

$$\log_{\alpha\alpha}(\mathrm{res}_p(y)) = \log_p(\tilde{y}^+_{\alpha\alpha})$$

where $\log_p$ is the $L$-linear extension of the $p$-adic logarithm on $\mathbf{Z}^*_p$. Write similarly

$$\mathfrak{x}_{\boldsymbol{u}} = \mathrm{pr}_{\beta\beta}(\mathfrak{x}_{\boldsymbol{u}}) = \mathfrak{x}^{\beta\beta}_{\boldsymbol{u}} \otimes (\omega_f \otimes \eta_{g_\alpha} \otimes \eta_{h_\alpha})$$

in $H^1(\mathbf{Q}_p, V^-_{\beta\beta}) = H^1(\mathbf{Q}_p, L) \otimes_L V^-_{\beta\beta}$ for some $\mathfrak{x}^{\beta\beta}_{\boldsymbol{u}}$ in $H^1(\mathbf{Q}_p, L)$, so that

$$\big\langle \mathfrak{x}_{\boldsymbol{u}}, \tilde{y}^+ \big\rangle_{\rm Tate} = -\mathfrak{x}^{\beta\beta}_{\rm u}(\tilde{y}^+_{\alpha\alpha}) = -\log_p(\tilde{y}^+_{\alpha\alpha}) \cdot \mathfrak{x}^{\beta\beta}_{\boldsymbol{u}}(e(1)) = \log_{\alpha\alpha}(\mathrm{res}_p(y)) \cdot \mathfrak{x}_{\boldsymbol{u}}(e(1))_f$$

by local class field theory. The statement then follows from Lemma 2.6. □

We can finally conclude the proof of Proposition 2.4.

*Proof of Proposition 2.4.* — To lighten the notation set $\kappa_{\alpha\alpha} = \kappa(f, g_\alpha, h_\alpha)$. By definition the extended height $\tilde{h}^{\alpha\alpha}_p(\kappa_{\alpha\alpha} \otimes y)$ is equal (up to sign) to

$$\langle\!\langle q_{\alpha\alpha}, q_{\beta\beta} \rangle\!\rangle_{\boldsymbol{fgh}} \cdot \langle\!\langle \kappa_{\alpha\alpha}, y \rangle\!\rangle_{\boldsymbol{fgh}} - \langle\!\langle q_{\alpha\alpha}, \kappa_{\alpha\alpha} \rangle\!\rangle_{\boldsymbol{fgh}} \cdot \langle\!\langle q_{\beta\beta}, y \rangle\!\rangle_{\boldsymbol{fgh}} + \langle\!\langle q_{\beta\beta}, \kappa_{\alpha\alpha} \rangle\!\rangle_{\boldsymbol{fgh}} \cdot \langle\!\langle q_{\alpha\alpha}, y \rangle\!\rangle_{\boldsymbol{fgh}}$$

for each Selmer class $y$ in $\mathrm{Sel}(\mathbf{Q}, V)$. Since $\kappa_{\alpha\alpha}$ is (the specialisation of) a balanced class, one has $\log_{\alpha\alpha}(\mathrm{res}_p(\kappa_{\alpha\alpha})) = 0$ (cf. Section 9.1 of [BSV21d]), hence $\langle\langle q_{\beta\beta}, \kappa_{\alpha\alpha}\rangle\rangle_{\boldsymbol{fgh}}$ is equal to zero by Lemma 2.5. As a consequence

$$\tilde{h}_p^{\alpha\alpha}(\kappa_{\alpha\alpha} \otimes y) = \det \begin{pmatrix} \langle\langle q_{\alpha\alpha}, q_{\beta\beta}\rangle\rangle_{\boldsymbol{fgh}} & \langle\langle q_{\alpha\alpha}, \kappa_{\alpha\alpha}\rangle\rangle_{\boldsymbol{fgh}} \\ \langle\langle q_{\beta\beta}, y\rangle\rangle_{\boldsymbol{fgh}} & \langle\langle \kappa_{\alpha\alpha}, y\rangle\rangle_{\boldsymbol{fgh}} \end{pmatrix}. \tag{22}$$

Assume first $\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}} = \mathfrak{L}_{\boldsymbol{g}}^{\mathrm{an}} = \mathfrak{L}_{\boldsymbol{h}}^{\mathrm{an}}$. Then $\langle\langle q_{\alpha\alpha}, q_{\beta\beta}\rangle\rangle_{\boldsymbol{fgh}}$ is equal to zero by Lemma 2.5, so that Equation (22) and Lemmas 2.5 and 2.10 yield the equality (up to sign)

$$\begin{aligned}\tilde{h}_p^{\alpha\alpha}(\kappa_{\alpha\alpha} \otimes y) &= \langle\langle q_{\alpha\alpha}, \kappa_{\alpha\alpha}\rangle\rangle_{\boldsymbol{fgh}} \cdot \langle\langle q_{\beta\beta}, y\rangle\rangle_{\boldsymbol{fgh}} \\ &= \frac{m_p \cdot \mathrm{ord}_p(q_A)}{2 \cdot \deg(\wp_\infty)} \cdot \log_{\alpha\alpha}(\mathrm{res}_p(y)) \cdot (\boldsymbol{k} - \boldsymbol{l} - \boldsymbol{m}) \cdot \sum_{\boldsymbol{u}} \ell_{\boldsymbol{k}}(\mathfrak{x}_{\boldsymbol{u}}) \cdot (\boldsymbol{u} - u_o).\end{aligned}$$

Moreover one has (by definition) $\ell_{\boldsymbol{k}} = -\ell_{\boldsymbol{l}} = -\ell_{\boldsymbol{m}}$, hence

$$(\boldsymbol{k} - \boldsymbol{l} - \boldsymbol{m}) \cdot \sum_{\boldsymbol{u}} \ell_{\boldsymbol{k}}(\mathfrak{x}_{\boldsymbol{u}}) \cdot (\boldsymbol{u} - u_o) = \sum_{\boldsymbol{u},\boldsymbol{v}} \tilde{\mathtt{D}}_{\boldsymbol{u},\boldsymbol{v}} \cdot (\boldsymbol{u} - u_o)(\boldsymbol{v} - v_o).$$

Proposition 2.4 follows from the previous two equations and Lemmas 2.3 and 2.9.

Assume from now on that the analytic $\mathscr{L}$-invariants $\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}}, \mathfrak{L}_{\boldsymbol{g}}^{\mathrm{an}}$ and $\mathfrak{L}_{\boldsymbol{h}}^{\mathrm{an}}$ are not all equal. Then Equation (22), Lemma 2.5, Lemma 2.10 and Lemma 2.11 yield

$$\tilde{h}_p^{\alpha\alpha}(\kappa_{\alpha\alpha} \otimes y) = \frac{m_p \cdot \mathrm{ord}_p(q_A)}{2 \cdot \deg(\wp_\infty)} \cdot \log_{\alpha\alpha}(\mathrm{res}_p(y)) \cdot \det(\mathtt{H}) \tag{23}$$

in $\mathscr{I}^2/\mathscr{I}^3$ for each Selmer class $y$ in $\mathrm{Sel}(\mathbf{Q}, V)$, where

$$\mathtt{H} = \begin{pmatrix} (\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}} - \mathfrak{L}_{\boldsymbol{g}}^{\mathrm{an}}) \cdot (\boldsymbol{l} - 1) + (\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}} - \mathfrak{L}_{\boldsymbol{h}}^{\mathrm{an}}) \cdot (\boldsymbol{m} - 1) & -\sum_{\boldsymbol{u}} \ell_{\boldsymbol{k}}(\mathfrak{x}_{\boldsymbol{u}}) \cdot (\boldsymbol{u} - u_o) \\ \boldsymbol{l} + \boldsymbol{m} - \boldsymbol{k} & \sum_{\boldsymbol{u}} \mathfrak{x}_{\boldsymbol{u}}(e(1))_f \cdot (\boldsymbol{u} - u_o) \end{pmatrix}.$$

A direct computation gives

$$\det(\mathtt{H}) = -\sum_{\boldsymbol{u},\boldsymbol{v}} \tilde{\mathtt{D}}_{\boldsymbol{u},\boldsymbol{v}} \cdot (\boldsymbol{u} - u_o)(\boldsymbol{v} - v_o). \tag{24}$$

Proposition 2.4 follows from Equations (23) and (24) and Lemmas 2.3 and 2.9. □

**2.4. Heegner points and diagonal classes.** — Assume from now on

$$\mathrm{ord}_{s=1} L(f \otimes g \otimes h, s) = 2 \tag{25}$$

and that Assumption 1.2 (stated in Section 1) is satisfied.

For each finite order character $\mu : G_K \longrightarrow \mathbf{Q}(\varrho)^*$, let $\mathrm{Ind}_K^{\mathbf{Q}}\mu$ be the $\mathbf{Q}(\varrho)$-module of functions $c : G_{\mathbf{Q}} \longrightarrow \mathbf{Q}(\varrho)$ satisfying $c(\tau\sigma) = \mu(\tau) \cdot c(\sigma)$ for each $\tau$ in $G_K$ and $\sigma$ in $G_{\mathbf{Q}}$, equipped with the action of $G_{\mathbf{Q}}$ defined by $(\sigma' \cdot c)(\sigma) = c(\sigma\sigma')$ for each $\sigma$ and $\sigma'$ in $G_{\mathbf{Q}}$. For $\xi = g, h$, the $\mathbf{Q}(\varrho)[G_{\mathbf{Q}}]$-module $\mathrm{Ind}_K^{\mathbf{Q}}\nu_\xi$ affords the representation $\varrho_\xi$. With the notations of Section 1 we can then take

$$V_\xi = \mathrm{Ind}_K^{\mathbf{Q}}\nu_\xi.$$

One has an isomorphism of $\mathbf{Q}(\varrho)[G_{\mathbf{Q}}]$-modules

$$V_{gh} = V_g \otimes_{\mathbf{Q}(\varrho)} V_h \simeq \mathrm{Ind}_K^{\mathbf{Q}}\varphi \oplus \mathrm{Ind}_K^{\mathbf{Q}}\psi, \tag{26}$$

where $\varphi = \nu_g \cdot \nu_h$ and $\psi = \nu_g \cdot \nu_h^c$ are dihedral characters of $K$ (cf. Section 1). The Artin formalism then yields the factorisation

$$L(f \otimes g \otimes h, s) = L(A/K, \varphi, s) \cdot L(A/K, \psi, s), \tag{27}$$

where $L(A/K, \chi, s) = L(f \otimes \vartheta_\chi, s)$ is the Hasse–Weil $L$-function of the base change of $A$ to $K$ twisted by $\chi = \varphi, \psi$ (viz. the Rankin–Selberg convolution of $f$ and the weight-one theta series $\vartheta_\chi$ associated with $\chi$).

Let $\chi$ denote either $\varphi$ or $\psi$, let $K_\chi$ be the ring class field of $K$ cut out by $\chi$, and let $A(K_\chi)^\chi$ be the submodule of $A(K_\chi) \otimes_{\mathbf{Z}} \mathbf{Q}(\varrho)$ on which $\mathrm{Gal}(K_\chi/K)$ acts via $\chi$. Fix a primitive Heegner point $P$ in $A(K_\chi)$ and set

$$P_\chi = \sum_{\sigma \in \mathrm{Gal}(K_\chi/K)} \chi(\sigma)^{-1} \cdot \sigma(P) \in A(K_\chi)^\chi.$$

Equations (25) and (27) and Assumption 1.1.(1) imply that $L(A/K, \chi, s)$ has a simple zero at $s = 1$, hence the Gross–Zagier–Kolyvagin–Zhang theorem yields

$$P_\chi \neq 0 \quad \text{and} \quad A(K_\chi)^\chi \otimes_{\mathbf{Q}(\varrho)} L = L \cdot P_\chi = \mathrm{Sel}(K_\chi, V_p(A))^\chi, \tag{28}$$

where $\mathrm{Sel}(K_\chi, V_p(A))$ is the Bloch–Kato Selmer group of the restriction of $V_p(A)$ to $G_{K_\chi}$, one denotes by $\mathrm{Sel}(K_\chi, V_p(A))^\chi$ the submodule of $\mathrm{Sel}(K_\chi, V_p(A)) \otimes_{\mathbf{Q}_p} L$ on which the Galois group of $K_\chi/K$ acts via the character $\chi$, and one considers $A(K_\chi)^\chi$ as a submodule of $\mathrm{Sel}(K_\chi, V_p(A))^\chi$ via the $K_\chi$-rational Kummer map.

Let $\sigma_p$ in $G_{\mathbf{Q}} - G_K$ be an arithmetic Frobenius at $p$. For $\xi = g, h$ and each pair $(a, b)$ of elements of $\mathbf{Q}(\varrho)$, denote by $[a, b]_\xi$ in $V_\xi$ the $\mathbf{Q}(\varrho)$-valued function on $G_{\mathbf{Q}}$ sending the identity to $a$ and $\sigma_p$ to $b$. Then $G_K$ acts on the line $L \cdot [1, 0]_\xi$ via $\nu_\xi$, and on the line $L \cdot [0, 1]_\xi$ via the conjugate $\nu_\xi^c$ of $\nu_\xi$ by the nontrivial element $c = \sigma_p|_K$ of $\mathrm{Gal}(K/\mathbf{Q})$. Moreover, since $\nu_\xi(\sigma_p^2) = \nu_\xi^{\mathrm{cen}}(p) = \varepsilon_K(p) \cdot \chi_\xi(p) = -\chi_\xi(p) = \alpha_\xi^2$ (cf. Section 1), one has $\sigma_p \cdot [a, b]_\xi = [b, \alpha_\xi^2 \cdot a]_\xi$ for each $a$ and $b$ in $\mathbf{Q}(\varrho)$. Set

$$v_{\xi,\alpha} = [1, \alpha_\xi]_\xi \in V_\xi^{\sigma_p = \alpha_\xi} \quad \text{and} \quad v_{\xi,\beta} = [1, -\alpha_\xi]_\xi \in V_\xi^{\sigma_p = \beta_\xi}.$$

(recall that $\beta_\xi = -\alpha_\xi$), and for each pair $(i, j)$ of elements of $\{\alpha, \beta\}$ set

$$v_{ij} = v_{g,i} \otimes v_{h,j} \in V_g^{\sigma_p = i_g} \otimes_{\mathbf{Q}(\varrho)} V_h^{\sigma_p = j_h} \hookrightarrow V_{gh}^{\sigma_p = i_g \cdot j_h}.$$

A direct computation shows that the vectors

$$v_\varphi = v_{\alpha\alpha} + v_{\alpha\beta} + v_{\beta\alpha} + v_{\beta\beta} \quad \text{and} \quad v_\psi = v_{\alpha\alpha} - v_{\alpha\beta} + v_{\beta\alpha} - v_{\beta\beta}$$

of $V_{gh}$ are qual to $4 \cdot [1, 0]_g \otimes [1, 0]_h$ and $4\alpha_\xi \cdot [1, 0]_g \otimes [0, 1]_h$ respectively, hence $G_K$ acts on them via $\varphi = \nu_g \cdot \nu_h$ and $\psi = \nu_g \cdot \nu_h^c$ respectively. For $\chi = \varphi, \psi$ define

$$P(\chi) = \gamma_{gh}\big(P_\chi \otimes \sigma_p(v_\chi) + \sigma_p(P_\chi) \otimes v_\chi\big)$$

in $\mathrm{Sel}(\mathbf{Q}, V)$ to be image of $P_\chi \otimes \sigma_p(v_\chi) + \sigma_p(P_\chi) \otimes v_\chi$ in $A(K_\varrho)^\varrho$ under the embedding $\gamma_{gh}$ introduced in Equation (10), so that (cf. Equations (26) and (28))

$$\mathrm{Sel}(\mathbf{Q}, V) = L \cdot P(\varphi) \oplus L \cdot P(\psi). \tag{29}$$

Write $\varepsilon = \alpha_f$ and for $\chi$ equal to $\varphi$ or $\psi$ define

$$P_\chi^\varepsilon = P_\chi + \varepsilon \cdot \sigma_p(P_\chi).$$

The point $P_\chi^\varepsilon$ is non-zero. This follows from Equation (28) if $\chi$ is not quadratic. When $\chi$ is quadratic, one has $\sigma_p(P_\chi) = \chi_1(p) \cdot P_\chi$, hence $P_\chi^\varepsilon$ is non-zero by Equation (28) and Assumption 1.2. In order to lighten the notation, set $\kappa_{\alpha\alpha} = \kappa(f, g_\alpha, h_\alpha)$. The main result Theorem A of [BSV21a] proves the identity

$$\log_{\beta\beta}(\mathrm{res}_p(\kappa(f, g_\alpha, h_\alpha))) = \log_{\omega_f}(P_\varphi^\varepsilon) \cdot \log_{\omega_f}(P_\psi^\varepsilon) \in L^*/\mathbf{Q}(\varrho)^*. \tag{30}$$

Here $\log_{\omega_f} : A(K_\chi) \otimes_{\mathbf{Z}} L \longrightarrow L \otimes_{\mathbf{Q}_p} K_p$ denotes the $L$-linear extension of the logarithm $\log_{\omega_f}$ on $A(K_\chi)$ introduced in Section 2.1.4. (Note that the right hand side of the previous identity is an element of $L \otimes_{\mathbf{Q}_p} K_p$ fixed by the action of $\sigma_p$, id est of $L$.)

Recall that the roots $\alpha_\xi$ and $\beta_\xi = -\alpha_\xi$ of the $p$-th Hecke polynomial of $\xi = g, h$ are distinct, and that $\alpha_g \cdot \alpha_h = \alpha_f = \beta_g \cdot \beta_h$ (cf. Equation (1)). We can then replace in the above constructions the Hida family $\boldsymbol{\xi} = \boldsymbol{\xi}_\alpha$ with the one $\boldsymbol{\xi}_\beta$ specialising to the $p$-stabilisation $\xi_\beta(q) = \xi(q) - \alpha_\xi \cdot \xi(q^p)$ at weight one, for $\xi = g, h$. This produces a diagonal class $\kappa(f, g_\beta, h_\beta)$ in the Selmer group $\mathrm{Sel}(\mathbf{Q}, W)$ of the $p$-adic representation $W = V(\boldsymbol{f}, \boldsymbol{g}_\beta, \boldsymbol{h}_\beta) \otimes_{w_o} L$. Fix an isomorphism of $L[G_{\mathbf{Q}}]$-modules $\mu : W \simeq V$, and let

$$\kappa_{\beta\beta} = \mu(\kappa(f, g_\beta, h_\beta)) \in \mathrm{Sel}(\mathbf{Q}, V)$$

be the image of $\kappa(f, g_\beta, h_\beta)$ under the isomorphism it induces in cohomology. The analogue of Equation (30) proves that the $\alpha\alpha$-logarithm of $\kappa_{\beta\beta}$ is non-zero:

$$\log_{\alpha\alpha}(\mathrm{res}_p(\kappa_{\beta\beta})) \in L^*. \tag{31}$$

Since by the definition of the balanced local condition (cf. Section 2.1.1) one has

$$\log_{\alpha\alpha}(\mathrm{res}_p(\kappa_{\alpha\alpha})) = \log_{\beta\beta}(\mathrm{res}_p(\kappa_{\beta\beta})) = 0, \tag{32}$$

it follows that the diagonal classes $\kappa_{\alpha\alpha}$ and $\kappa_{\beta\beta}$ are linearly independent, hence

$$\mathrm{Sel}(\mathbf{Q}, V) = L \cdot \kappa_{\alpha\alpha} \oplus L \cdot \kappa_{\beta\beta}. \tag{33}$$

**2.4.1.** *Conclusion of the proof.* — Consider the $L$-basis (cf. Equations (6) and (8))

$$q_\flat = \frac{1}{\sqrt{m_p}} \cdot q(f) \otimes v_g^\alpha \otimes v_h^\alpha \quad \text{and} \quad q_\natural = \frac{1}{\sqrt{m_p}} \cdot q(f) \otimes v_g^\beta \otimes v_h^\beta$$

of $H^0(\mathbf{Q}_p, V^-)$, where $v_\xi^\cdot = \gamma_\xi(v_{\xi,\cdot})$ for $\xi = g, h$ and $\cdot = \alpha, \beta$. It is the image of the $\mathbf{Q}(\varrho)$-basis $\{q(A) \otimes v_{g,\alpha} \otimes v_{h,\alpha}, q(A) \otimes v_{g,\beta} \otimes v_{h,\beta}\}$ of $\mathcal{Q}_p(A, \varrho)$ (cf. Equation (9)) under the isomorphism $\mathcal{Q}_p(A, \varrho)_L \simeq H^0(\mathbf{Q}_p, V)$ arising from the modular parametrisation $\wp_\infty$ fixed in Section 2.1.1 and the embeddings $\gamma_g$ and $\gamma_h$ fixed in Equation (8). Define $\mathtt{M}$ and $\mathtt{N}$ in $\mathrm{GL}_2(L)$ by the identities (cf. Equations (29) and (33))

$$\begin{pmatrix} \kappa_{\alpha\alpha} \\ \kappa_{\beta\beta} \end{pmatrix} = \mathtt{M} \begin{pmatrix} P(\chi) \\ P(\psi) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} q_{\alpha\alpha} \\ q_{\beta\beta} \end{pmatrix} = \mathtt{N} \begin{pmatrix} q_\flat \\ q_\sharp \end{pmatrix}.$$

By the definition of the $p$-adic regulator $R_p^{\alpha\alpha}(A, \varrho)$ and Proposition 2.4 one has

$$R_p^{\alpha\alpha}(A, \varrho) = \frac{\log_{\alpha\alpha}^2(\mathrm{res}_p(\kappa_{\beta\beta}))}{\det(\mathtt{M})^2 \cdot \det(\mathtt{N})^2} \cdot L_p^{\alpha\alpha}(A, \varrho) \pmod{\mathscr{I}^5} \tag{34}$$

in the quotient of $\mathscr{I}^4/\mathscr{I}^5$ by the multiplicative action of $\mathbf{Q}(\varrho)^{*2}$.

Set $\hat{L} = L \otimes_{\mathbf{Q}_p} \hat{\mathbf{Q}}_p^{\mathrm{nr}}$ and for $\xi = g, h$ denote by

$$\hat{\pi}_\xi : V(\xi) \otimes_L V(\xi) \otimes_{\mathbf{Q}_p} \hat{\mathbf{Q}}_p^{\mathrm{nr}} \longrightarrow \hat{L}$$

the $\hat{\mathbf{Q}}_p^{\mathrm{nr}}$-base change of the perfect pairing $\pi_\xi$ introduced in Section 2.1.1. Since

$$\hat{\pi}_g(\eta_{g_\alpha} \otimes \omega_{g_\alpha}) \cdot \hat{\pi}_h(\eta_{h_\alpha} \otimes \omega_{h_\alpha}) = G(\chi_g) \cdot G(\chi_h) = 1$$

(cf. Assumption 1.1.(2) and the definitions introduced in Section 2.1.2), one has

$$\mathtt{N} = \frac{1}{\sqrt{m_p}} \cdot \begin{pmatrix} \hat{\pi}_g(v_g^\alpha \otimes \eta_{g_\alpha}) \cdot \hat{\pi}_h(v_h^\alpha \otimes \eta_{h_\alpha}) & 0 \\ 0 & \hat{\pi}_g(v_g^\beta \otimes \omega_{g_\alpha}) \cdot \hat{\pi}_h(v_h^\beta \otimes \omega_{h_\alpha}) \end{pmatrix}$$

(in $H^0(\sigma_p, \mathrm{GL}_2(\hat{L})) = \mathrm{GL}_2(L)$), hence

$$\det(\mathtt{N}) = m_p^{-1} \cdot \pi_g(v_g^\alpha \otimes v_g^\beta) \cdot \pi_h(v_h^\alpha \otimes v_h^\beta) \in \mathbf{Q}(\varrho)^* \tag{35}$$

by the normalisation imposed on the embeddings $\gamma_g$ and $\gamma_h$ (cf. Equation (8)).

According to Equations (30), (31) and (32) one has

$$\mathtt{M}^{-1} = \begin{pmatrix} \frac{\log_{\beta\beta}(P(\varphi))}{\log_{\beta\beta}(\kappa_{\alpha\alpha})} & \frac{\log_{\alpha\alpha}(P(\varphi))}{\log_{\alpha\alpha}(\kappa_{\beta\beta})} \\ \\ \frac{\log_{\beta\beta}(P(\psi))}{\log_{\beta\beta}(\kappa_{\alpha\alpha})} & \frac{\log_{\alpha\alpha}(P(\psi))}{\log_{\alpha\alpha}(\kappa_{\beta\beta})} \end{pmatrix}$$

(where $\log_{ii} : \mathrm{Sel}(\mathbf{Q}, V) \longrightarrow L$, for $i = \alpha, \beta$, is a shorthand for $\log_{ii} \circ \mathrm{res}_p$). After retracing the definitions given in Section 2.4, a direct computation yields

$$\log_{\alpha\alpha}(P(\chi)) = \varepsilon \cdot \log_{\omega_f}(P_\chi^\varepsilon) \cdot \hat{\pi}_g(v_g^\alpha \otimes \eta_{g_\alpha}) \cdot \hat{\pi}_h(v_h^\alpha \otimes \eta_{h_\alpha})$$

(in $H^0(\sigma_p, \hat{L}) = L$, where as usual $\chi$ denotes either $\varphi$ or $\psi$) and

$$\log_{\beta\beta}(P(\chi)) = \varepsilon_\chi \cdot \varepsilon \cdot \log_{\omega_f}(P_\chi^\varepsilon) \cdot \hat{\pi}_g(v_g^\beta \otimes \omega_{g_\alpha}) \cdot \hat{\pi}_h(v_h^\beta \otimes \omega_{h_\alpha}),$$

where $\varepsilon_\varphi = 1$ and $\varepsilon_\psi = -1$. As a consequence

$$\frac{\log_{\alpha\alpha}(\kappa_{\beta\beta})}{\det(\mathtt{M})} = 2 \cdot \frac{\log_{\omega_f}(P_\varphi^\varepsilon) \cdot \log_{\omega_f}(P_\psi^\varepsilon)}{\log_{\beta\beta}(\kappa_{\alpha\alpha})} \cdot \pi_g(v_g^\alpha \otimes v_g^\beta) \cdot \pi_h(v_h^\alpha \otimes v_h^\beta) \in \mathbf{Q}(\varrho)^* \tag{36}$$

by Equation (30) and Equation (8).

Equations (34), (35) and (36) give the identity

$$L_p^{\alpha\alpha}(A, \varrho) \pmod{\mathscr{I}^5} = R_p^{\alpha\alpha}(A, \varrho)$$

in the quotient of $\mathscr{I}^4/\mathscr{I}^5$ by the multiplicative action of $\mathbf{Q}(\varrho)^{*2}$. To conclude the proof of the Theorem stated in Section 1, it remains to prove that both sides of the previous identity are non-zero. This follows by combining Equation (30) with [**BSV21d**, Theorem A] and [**BSV21a**, Proposition 2.2], which prove the equality

$$\frac{\partial^2 \mathscr{L}_p^{\alpha\alpha}(A, \varrho)}{\partial \boldsymbol{k}^2}(w_o) = c_p(f) \cdot \frac{\deg(\wp_\infty)}{2m_p \mathrm{ord}_p(q_A)} \left(1 - \frac{1}{p}\right)^{-1} \cdot \log_{\beta\beta}(\kappa_{\alpha\alpha}).$$

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