ON p-ADIC ANALOGUES OF THE BIRCH AND SWINNERTON-DYER CONJECTURE FOR GARRETT L-FUNCTIONS

by

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Introduction

Let A be an elliptic curve over the field $\mathbf{Q}$ of rational numbers and let ϱ_1, ϱ_2 be a pair of two-dimensional odd Artin representations of the absolute Galois group of $\mathbf{Q}$. Set $\varrho = \varrho_1 \otimes \varrho_2$ and denote by K_ϱ the extension of $\mathbf{Q}$ cut out by ϱ. Assume the self-duality hypothesis $\det(\varrho_1) = \det(\varrho_2)^{-1}$. The equivariant Birch and Swinnerton-Dyer conjecture aims at understanding the ϱ-component $A(K_\varrho)^\varrho$ of the Mordell-Weil group of A/K_ϱ in terms of the complex L-function $L(A, \varrho, s)$ of A twisted by ϱ.

The purpose of this article is twofold. The first objective is to formulate a p-adic analogue of this equivariant Birch and Swinnerton-Dyer conjecture. Assume for simplicity (but see Section 1.1 for generalisations) that p is an ordinary prime for A and that ϱ_1 and ϱ_2 are irreducible. Let (f, g, h) be the triple of cuspidal modular forms associated to $(A, \varrho_1, \varrho_2)$ by the modularity theorems. Hida's theory associates to (f, g, h) a triple $(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ of p-adic families of ordinary cuspidal modular forms, where $\boldsymbol{f}$ specialises in weight 2 to the unique ordinary p-stabilisation of f, while $\boldsymbol{g}_\alpha$ and $\boldsymbol{h}_\alpha$ specialise in weight 1 to a choice of p-stabilisations g_α and h_α of g and h respectively. Our conjecture replaces $L(A, \varrho, s)$ with a p-adic L-function $L_p^{\alpha\alpha}(A, \varrho)$ arising from the triple of p-adic families $(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$. The L-function $L_p^{\alpha\alpha}(A, \varrho)$ interpolates the central critical values of the complex L-functions of $f_k \otimes g_l \otimes h_m$ at triples of classical weights (k, l, m) such that $k \geqslant l + m$, where f_k, g_l and h_m denotes the specialisation of $\boldsymbol{f}$, $\boldsymbol{g}_\alpha$ and $\boldsymbol{h}_\alpha$ at k, l and m respectively. A p-adic avatar of the Birch and Swinnerton-Dyer conjecture suggests that the behaviour of $L_p^{\alpha\alpha}(A,\varrho)$ at the triple of weights $(2,1,1)$ should reflect the arithmetic of A over K_ϱ. This is the content of our Conjecture 1.1, which states that the order of vanishing of $L_p^{\alpha\alpha}(A,\varrho)$ at $(2,1,1)$ is equal to the rank of the ϱ-component $A^\dagger(K_\varrho)^\varrho$ of the extended Mordell–Weil group of A/K_ϱ. Furthermore, it relates the leading term of $L_p^{\alpha\alpha}(A,\varrho)$ to the regulator $R_p^{\alpha\alpha}(A,\varrho)$ of a p-adic height pairing on this extended Mordell–Weil group, constructed in Section 2 by exploiting Nekovář's theory of Selmer complexes associated to Hida's deformation of the Galois representations of (f, g_α, h_α).

The second objective of this article is to understand the Elliptic Stark Conjectures of Darmon, Lauder and Rotger [**DLR15, DR16**] within the conceptual framework of the p-adic variants of the Birch and Swinnerton-Dyer conjecture. Under the assumption that the Mordell–Weil rank is equal to 2, the above mentioned works obtained experimentally a relation between an iterated p-adic integral associated to the triple (f, g_α, h_α) and certain combinations of p-adic logarithms of rational points in the ϱ-component of the Mordell–Weil group of A. Section 3 (see in particular Conjecture 3.4 and Remarks 3.5) shows that these conjectural relations are a consequence of Conjecture 1.1, combined with a formula *à la* Rubin–Perrin-Riou established in Theorem 3.2 for the derivatives of a *big* diagonal class encoding $L_p(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)$ by an explicit reciprocity law.

1. The p-adic Birch and Swinnerton-Dyer conjecture

This section states the main conjecture of this paper, assuming the precise definition of the Garrett–Nekovář p-adic height pairings given in Section 2 below. To ease the exposition we state our conjecture for p-ordinary elliptic curves over $\mathbf{Q}$, i.e. p-stabilised ordinary weight-two newforms with trivial character and rational Fourier coefficients. See Section 1.1 below for possible generalisations.

Fix a rational prime $p > 3$, algebraic closures $\bar{\mathbf{Q}}$ and $\bar{\mathbf{Q}}_p$ of $\mathbf{Q}$ and $\mathbf{Q}_p$ respectively and an embedding of $\bar{\mathbf{Q}}$ into $\bar{\mathbf{Q}}_p$. For positive integers k and m, a Dirichlet character $\chi : (\mathbf{Z}/m\mathbf{Z})^* \longrightarrow \bar{\mathbf{Q}}^*$ and a subfield F of $\bar{\mathbf{Q}}_p$, denote by $M_k(m,\chi)_F$ the F-module of modular forms of weight k, level $\Gamma_1(m)$, character χ and Fourier coefficients contained in F, and by $S_k(m,\chi)_F$ its subspace of cuspidal modular forms. When χ is the trivial character, we omit it from the notation.

Let A be an elliptic curve defined over $\mathbf{Q}$ and let

$$\varrho = \varrho_1 \otimes \varrho_2$$

be the tensor product of two odd, two-dimensional Artin representations

$$\varrho_i : G_{\mathbf{Q}} \longrightarrow \mathrm{GL}(V_{\varrho_i}) \simeq \mathrm{GL}_2(\mathbf{Q}(\varrho))$$

of $G_{\mathbf{Q}} = \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ with coefficients in a number field $\mathbf{Q}(\varrho)$ (contained in $\bar{\mathbf{Q}}$), satisfying the *self-duality condition*

$$\det(\varrho_1) = \det(\varrho_2)^{-1}. \tag{1}$$

According to the modularity theorem of Wiles, Taylor–Wiles et al., the p-adic Tate module of $A/\mathbf{Q}$ with $\mathbf{Q}_p$-coefficients is isomorphic to the dual of the p-adic Deligne representation of the weight-two cuspidal newform

$$f = \sum_{n \geqslant 1} a_n(f) \cdot q^n \in S_2(N_f)_{\mathbf{Q}},$$

where N_f is the conductor of $A/\mathbf{Q}$ and $a_\ell(f) = 1 + \ell - |A(\mathbf{Z}/\ell\mathbf{Z})|$ for each prime $\ell \nmid N_f$. Similarly, the Serre conjecture, proved by Khare and Wintenberger, implies that ϱ_1 and ϱ_2 are isomorphic respectively to the duals of the Deligne–Serre representations associated with weight-one normalised Hecke eigenforms

$$g = \sum_{n \geqslant 0} a_n(g) \cdot q^n \in M_1(N_g, \chi_g)_{\mathbf{Q}(\varrho)}$$

and

$$h = \sum_{n \geqslant 0} a_n(h) \cdot q^n \in M_1(N_h, \chi_h)_{\mathbf{Q}(\varrho)}$$

of conductors N_g and N_h equal to those of ϱ_1 and ϱ_2 respectively and characters χ_g and $\chi_h = \chi_g^{-1}$ (cf. Equation (1)). The form g (resp., h) is cupidal precisely if the Artin representation ϱ_1 (resp., ϱ_2) is irreducible.

Assume that A has good ordinary or multiplicative reduction at p, so that N_f is of the form $M_f \cdot p^{r_f}$ with $r_f \leqslant 1$ and M_f coprime with p. The p-th Hecke polynomial $X^2 - a_p(f) \cdot X + 1_{N_f}(p) \cdot p$ of f has a unique root α_f which is a p-adic unit, the other root being $\beta_f = 1_{N_f}(p)p/\alpha_f$. (Here 1_{N_f} is the trivial character modulo N_f.) By Hida theory, the ordinary p-stabilisation

$$f_\alpha(q) = f(q) - \beta_f \cdot f(q^p) \in S_2(M_f p)_{\mathbf{Q}(\alpha_f)}$$

is the specialisation at weight two of a unique cuspidal Hida family

$$\boldsymbol{f} = \boldsymbol{f}_\alpha = \sum_{n \geqslant 1} a_n(\boldsymbol{f}) \cdot q^n \in \mathcal{O}(U_{\boldsymbol{f}})[\![q]\!],$$

for a suitable connected open disc $U_{\boldsymbol{f}}$ centred at 2 in the weight space $\mathcal{W}$ over $\mathbf{Q}_p$. Here $\mathcal{O}(U_{\boldsymbol{f}})$ is the ring of analytic functions on $U_{\boldsymbol{f}}$. For each classical weight k in $U_{\boldsymbol{f}} \cap \mathbf{Z}_{>2}$, the weight-$k$ specialisation $\boldsymbol{f}_k = \sum_{n \geqslant 1} a_n(\boldsymbol{f})(k) \cdot q^n$ of $\boldsymbol{f}$ is (the q-expansion of) the ordinary p-stabilisation of a p-ordinary newform f_k of weight k and level $\Gamma_0(M_f)$.

Let ξ denote either g or h, and let α_ξ and $\beta_\xi = \chi_\xi(p)/\alpha_\xi$ be the roots of its pth Hecke polynomial $X^2 - a_p(\xi) \cdot X + \chi_\xi(p)$. Fix a finite extension L of $\mathbf{Q}_p$ which contains the Fourier coefficients of ξ, the roots α_f and α_ξ (for $\xi = g, h$), and the N-th roots of unity, where N is the least common multiple of N_f, N_g and N_h. We assume that p does not divide N_ξ and that ξ is *cuspidal* and *p-regular* (viz. the roots α_ξ and β_ξ are distinct). Moreover we assume that ξ is not the theta series associated with a ray class character of a *real* quadratic field in which *p splits*. Under these assumptions the p-stabilisation

$$\xi_\alpha(q) = \xi(q) - \beta_\xi \cdot \xi(q^p) \in S_1(N_\xi p, \chi_\xi)_L$$

is the weight-one specialisation of a unique cuspidal Hida family

$$\boldsymbol{\xi}_\alpha = \sum_{n\geqslant 1} a_n(\boldsymbol{\xi}_\alpha)\cdot q^n \in \mathcal{O}(U_{\boldsymbol{\xi}})[\![q]\!],$$

where $U_{\boldsymbol{\xi}}$ is a (small) connected open neighbourhood of 1 in $\mathcal{W}\otimes_{\mathbf{Q}_p} L$. For each classical weight u in $U_{\boldsymbol{\xi}}\cap \mathbf{Z}_{\geqslant 1}$, the weight-$u$ specialisation $\boldsymbol{\xi}_{\alpha,u} = \sum_{n\geqslant 1} a_n(\boldsymbol{\xi})(u)\cdot q^n$ is the ordinary p-stabilisation of a p-ordinary newform ξ_u of weight u, level $\Gamma_1(N_\xi)$ and character χ_ξ. We refer the reader to [**BSV20**] (especially the discussion following Assumption 1.1, Remark 1.4 and Section 5) and the references therein for more details.

Let Σ^{cl} denote the set of classical triples, namely the intersection of $U_{\boldsymbol{f}}\times U_{\boldsymbol{g}}\times U_{\boldsymbol{h}}$ with $\mathbf{Z}^3_{\geqslant 1}$. Under the self-duality assumption (1), for each (k,l,m) in Σ^{cl} the complex Garrett L-function $L(f_k\otimes g_l\otimes h_m, s)$ admits an analytic continuation to all of $\mathbf{C}$ and satisfies a functional equation with sign $+1$ or -1 relating its values at s and $k+l+m-2-s$. Assume from now on that the conductors N_g and N_h of g and h are coprime to the conductor N_f of the elliptic curve A:

$$(N_g\cdot N_h, N_f) = 1. \tag{2}$$

Assumption (2) guarantees that the signs in the above functional equations are equal to $+1$ for all classical triples (k,l,m) in the *f-unbalanced region*, id est triples (k,l,m) in Σ^{cl} such that $k\geqslant l+m$. In particular the complex Garrett L-function

$$L(A,\varrho,s) = L(f\otimes g\otimes h, s)$$

vanishes to *even* order at the central critical point $s=1$. Set $\mathscr{O}_{\boldsymbol{fgh}} = \mathscr{O}_{\boldsymbol{f}}\hat{\otimes}_{\mathbf{Q}_p}\mathscr{O}_{\boldsymbol{g}}\hat{\otimes}_L\mathscr{O}_{\boldsymbol{h}}$, where $\mathscr{O}_\cdot$ denotes the ring of bounded functions on $U_\cdot$. The article [**Hsi20**] associates to the triple of Hida families $(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)$ a *square-root Garrett–Hida p-adic L-function*

$$\mathscr{L}_p^{\alpha\alpha}(A,\varrho) = \mathscr{L}_p(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)\in\mathscr{O}_{\boldsymbol{fgh}},$$

whose square, the *Garrett–Hida p-adic L-function of* (A,ϱ),

$$L_p^{\alpha\alpha}(A,\varrho) = \mathscr{L}_p^{\alpha\alpha}(A,\varrho)^2$$

interpolates the central critical values

$$L\left(f_k\otimes g_l\otimes h_m, \frac{k+l+m-2}{2}\right)$$

of the complex Garrett L-functions $L(f_k\otimes g_l\otimes h_m, s)$ at classical triples (k,l,m) in the f-unbalanced region. We refer to [**BSV20**, Section 6.1] (where $L_p^{\alpha\alpha}(A,\varrho)$ is denoted by $L_p(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)$) for the precise interpolation property (see in particular Equation (132) of loc. cit.). The L-function $L_p^{\alpha\alpha}(A,\varrho)$ is symmetric in the families $\boldsymbol{g}_\alpha$ and $\boldsymbol{h}_\alpha$.

Enlarging $\mathbf{Q}(\varrho)$ if necessary, we assume it contains α_ξ for ξ equal to f, g and h. The weight-one specialisation (cf. Section 2.1 below)

$$V(\xi) = V(\boldsymbol{\xi}_\alpha)\otimes_1 L$$

of the Galois representation $V(\boldsymbol{\xi}_\alpha)$ associated with $\boldsymbol{\xi}_\alpha$ affords the dual of the p-adic Deligne–Serre representation of ξ with coefficients in L. The $G_{\mathbf{Q}}$-representation $V(\boldsymbol{\xi}_\alpha)$ is a free rank-two $\mathscr{O}_{\boldsymbol{\xi}}$-module and the tensor product $\cdot\otimes_1 L = \cdot\otimes_{\mathscr{O}_{\boldsymbol{\xi}},1} L$ is taken with respect to evaluation at 1 in $U_{\boldsymbol{\xi}}$. The global p-adic representation $V(\xi)$ is equipped with a canonical, $G_{\mathbf{Q}}$-equivariant, perfect, skew-symmetric pairing

$$\pi_\xi : V(\xi) \otimes_L V(\xi) \longrightarrow L(\chi_\xi), \tag{3}$$

arising as the weight-one specialisation of a suitably twisted Poincaré duality on $V(\boldsymbol{\xi}_\alpha)$ (cf. Section 2.1). Enlarging L if necessary, fix isomorphisms

$$\gamma_g : V_{\varrho_1} \otimes_{\mathbf{Q}(\varrho)} L \simeq V(g) \quad \text{and} \quad \gamma_h : V_{\varrho_2} \otimes_{\mathbf{Q}(\varrho)} L \simeq V(h) \tag{4}$$

of $L[G_{\mathbf{Q}}]$-modules such that the perfect dualities $\pi_g \circ \gamma_g \otimes \gamma_g$ and $\pi_h \circ \gamma_h \otimes \gamma_h$ map the $\mathbf{Q}(\varrho)$-structures $V_{\varrho_1} \otimes_{\mathbf{Q}(\varrho)} V_{\varrho_1}$ and $V_{\varrho_2} \otimes_{\mathbf{Q}(\varrho)} V_{\varrho_2}$ into the $\mathbf{Q}(\varrho)$-structures $\mathbf{Q}(\varrho)(\chi_g)$ and $\mathbf{Q}(\varrho)(\chi_h)$ of $L(\chi_g)$ and $L(\chi_h)$ respectively.

Let $V(f) = \mathrm{Ta}_p(A/\mathbf{Q}) \otimes_{\mathbf{Z}_p} L$ be the p-adic Tate module $A/\mathbf{Q}$ with coefficients in L, and let $V(f)^-$ be the maximal unramified quotient of the restriction of $V(f)$ to $G_{\mathbf{Q}_p}$. It is a 1-dimensional L-module, on which an arithmetic Frobenius in $G_{\mathbf{Q}_p}$ acts as multiplication by α_f. Set $V_\varrho = V_{\varrho_1} \otimes_{\mathbf{Q}(\varrho)} V_{\varrho_2}$, $V(f,\varrho) = V(f) \otimes_{\mathbf{Q}(\varrho)} V_\varrho$ and $V(f,\varrho)^- = V(f)^- \otimes_{\mathbf{Q}(\varrho)} V_\varrho$, so that $V(f,\varrho)^-$ is the maximal $G_{\mathbf{Q}_p}$-unramified quotient of $V(f,\varrho)$, on which an arithmetic Frobenius acts with eigenvalues $\alpha_f\alpha_g\alpha_h$, $\alpha_f\beta_g\alpha_h$, $\alpha_f\alpha_g\beta_h$ and $\alpha_f\beta_g\beta_h$. Define the module of *p-adic periods* of (A,ϱ):

$$\mathscr{Q}_p(A,\varrho)_L = H^0(\mathbf{Q}_p, V(f,\varrho)^-)$$

to be the space of $G_{\mathbf{Q}_p}$-invariants of $V(f,\varrho)^-$. As suggested by the notation

$$\mathscr{Q}_p(A,\varrho)_L = \mathscr{Q}_p(A,\varrho) \otimes_{\mathbf{Q}(\varrho)} L$$

for a canonical $\mathbf{Q}(\varrho)$-submodule $\mathscr{Q}_p(A,\varrho)$ defined as follows. Note first that $\mathscr{Q}_p(A,\varrho)_L$ is zero if A has good reduction at p. In this case set $\mathscr{Q}_p(A,\varrho) = 0$. If A has multiplicative reduction at p, Tate's theory gives a rigid analytic isomorphism

$$\wp_{\mathrm{Tate}} : \mathbf{G}^{\mathrm{an}}_{m,\mathbf{Q}_{p^2}}/q_A^{\mathbf{Z}} \simeq A_{\mathbf{Q}_{p^2}},$$

unique up to sign. Here $A_{\mathbf{Q}_{p^2}}$ is the base change of A to the quadratic unramified extension $\mathbf{Q}_{p^2}$ of $\mathbf{Q}_p$ and q_A in $p\mathbf{Z}_p$ is the Tate period of A. Taking the p-adic Tate modules $\wp_{\mathrm{Tate}}$ induces a (canonical up to sign) isomorphism of $G_{\mathbf{Q}_{p^2}}$-modules $V(f)^- \simeq L$. Write $q(A)$ in $V(f)^-$ for the element corresponding to the identity of L under this isomorhism and define

$$\mathscr{Q}_p(A,\varrho) = \big(\mathbf{Q}(\varrho)\cdot q(A) \otimes_{\mathbf{Q}(\varrho)} V_\varrho\big)^{G_{\mathbf{Q}_p}}.$$

Let $X_1(N_f,p)$ be the compact modular curve of level $\Gamma_1(N_f,p) = \Gamma_1(N_f)\cap\Gamma_0(p)$ over $\mathbf{Q}$. Fix a modular parametrisation (viz. a non-constant map of $\mathbf{Q}$-schemes)

$$\wp_\infty : X_1(N_f,p) \longrightarrow A.$$

Let K_ϱ be a finite Galois extension of $\mathbf{Q}$ such that ϱ_1 and ϱ_2 factor through $\mathrm{Gal}(K_\varrho/\mathbf{Q})$. Define the *$p$-extended Mordell–Weil group* of (A,ϱ) by

$$A^\dagger(K_\varrho)^\varrho = \big(A(K_\varrho)\otimes_{\mathbf{Z}} V_\varrho\big)^{\mathrm{Gal}(K_\varrho/\mathbf{Q})} \oplus \mathscr{Q}_p(A,\varrho).$$

Section 2 below associates with the triple (f, g_α, h_α), the modular parametrisation $\wp_\infty$, and the isomorphisms γ_g and γ_h a *Garrett–Nekovář p-adic height pairing*

$$\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} : A^\dagger(K_\varrho)^\varrho \times A^\dagger(K_\varrho)^\varrho \longrightarrow \mathscr{I}/\mathscr{I}^2, \tag{5}$$

where $\mathscr{I}$ is the ideal of analytic functions in $\mathscr{O}_{\boldsymbol{fgh}}$ vanishing at $w_o = (2,1,1)$. The pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ is skew-symmetric and associated by cohomological means to an appropriate self-dual twist of the representation $V(\boldsymbol{f})\hat{\otimes}_{\mathbf{Q}_p}V(\boldsymbol{g}_\alpha)\hat{\otimes}_L V(\boldsymbol{h}_\alpha)$, viewed as a p-adic deformation of $V(f,g,h) = V(f) \otimes_L V(g) \otimes_L V(h)$. Its construction grounds on Nekovář's theory of Selmer complexes and generalised Poitou–Tate duality [Nek06]. More precisely, after identifying $V(f)$ with the f_α-isotypic component of the cohomology group $H^1_{\text{ét}}(X_1(N_f,p)_{\bar{\mathbf{Q}}}, L(1))$ via the fixed modular parametrisation $\wp_\infty$, Section 2 below defines a *canonical* Garrett–Nekovář p-adic height pairing

$$\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} : \mathrm{Sel}^\dagger(\mathbf{Q}, V(f,g,h)) \otimes_L \mathrm{Sel}^\dagger(\mathbf{Q}, V(f,g,h)) \longrightarrow \mathscr{I}/\mathscr{I}^2, \tag{6}$$

where the *(naive) extended Selmer group*

$$\mathrm{Sel}^\dagger(\mathbf{Q}, V(f,g,h)) = \mathrm{Sel}(\mathbf{Q}, V(f,g,h)) \oplus H^0(\mathbf{Q}_p, V(f,g,h)^-) \tag{7}$$

is the direct sum of the Bloch–Kato Selmer group of $V(f,g,h)$ over $\mathbf{Q}$ and the module of $G_{\mathbf{Q}_p}$-invariants of the maximal p-unramified quotient $V(f,g,h)^-$ of $V(f,g,h)$. The global Kummer map $A(K_\varrho) \longrightarrow H^1(K_\varrho, V(f))$ and the fixed isomorphisms γ_g and γ_h give rise to an embedding $\gamma_{gh} : A^\dagger(K_\varrho)^\varrho \hookrightarrow \mathrm{Sel}^\dagger(\mathbf{Q}, V(f,g,h))$, and one defines (5) as the restriction of the canonical height pairing (6) along γ_{gh}.

Set

$$r^\dagger(A,\varrho) = \dim_{\mathbf{Q}(\varrho)} A^\dagger(K_\varrho)^\varrho$$

and define the *Garrett–Nekovář regulator*

$$R_p^{\alpha\alpha}(A,\varrho) \in \left(\mathscr{I}^{r^\dagger(A,\varrho)}/\mathscr{I}^{r^\dagger(A,\varrho)+1}\right)/\mathbf{Q}(\varrho)^{*2}$$

to be the discriminant of the Garrett–Nekovář p-adic height pairing:

$$R_p^{\alpha\alpha}(A,\varrho) = \det\left(\langle\!\langle P_i, P_j \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}\right)_{1\leqslant i,j\leqslant r^\dagger(A,\varrho)},$$

where $P_1, \ldots, P_{r^\dagger(A,\varrho)}$ is a $\mathbf{Q}(\varrho)$-basis of the p-extended Mordell–Weil group $A^\dagger(K_\varrho)^\varrho$. In view of the normalisation of the isomorphisms γ_g and γ_h fixed in (4), the regulator $R_p^{\alpha\alpha}(A,\varrho)$ is independent of the choice of γ_g and γ_h. Moreover, it does not depend on the modular parametrisation $\wp_\infty$.

If $\mathcal{Q}_p(A,\varrho)$ is non-zero –the *exceptional case*– assume that either

$$\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} \neq \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{g}_\alpha} \quad \text{or} \quad \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} \neq \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{h}_\alpha}, \tag{8}$$

where the *analytic $\mathscr{L}$-invariants* of $\boldsymbol{f}$ and $\boldsymbol{\xi}_\alpha = \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha$ are defined respectively as the logarithmic derivatives

$$\mathfrak{L}^{\mathrm{an}}_{\boldsymbol{f}} = -2 \cdot d\log(a_p(\boldsymbol{f}))_{\boldsymbol{k}=2} \quad \text{and} \quad \mathfrak{L}^{\mathrm{an}}_{\boldsymbol{\xi}_\alpha} = -2 \cdot d\log(a_p(\boldsymbol{\xi}_\alpha))_{\boldsymbol{u}=1} \tag{9}$$

of -2 times the pth Fourier coefficients of $\boldsymbol{f}$ and $\boldsymbol{\xi}_\alpha$ at $\boldsymbol{k}=2$ and $\boldsymbol{u}=1$. Here $\mathscr{O}_{\boldsymbol{f}}$ and $\mathscr{O}_{\boldsymbol{\xi}}$ are identified with subrings of the power series rings $L[\![\boldsymbol{k}-2]\!]$ and $L[\![\boldsymbol{u}-1]\!]$, where $\boldsymbol{k}-2$ and $\boldsymbol{u}-1$ are uniformisers at the centres 2 and 1 of $U_{\boldsymbol{f}}$ and $U_{\boldsymbol{\xi}}$ respectively.

We say that a non-zero element F of $\mathscr{O}_{\boldsymbol{fgh}}$ has order of vanishing $r \in \mathbf{Z}_{\geqslant 0}$ at $w_o = (2,1,1)$ if it belongs to $\mathscr{I}^r - \mathscr{I}^{r+1}$, and denote by F^* its leading term in the Taylor expansion at w_o, namely its image in the quotient $\mathscr{I}^r/\mathscr{I}^{r+1}$.

Conjecture 1.1. — *The Garrett–Hida p-adic L-function $L_p^{\alpha\alpha}(A,\varrho)$ has order of vanishing $r^\dagger(A,\varrho)$ at $w_o=(2,1,1)$, and the following equality holds in the quotient of $\mathscr{I}^{r^\dagger(A,\varrho)}/\mathscr{I}^{r^\dagger(A,\varrho)+1}$ by the multiplicative action of $\mathbf{Q}(\varrho)^{*2}$.*

$$L_p^{\alpha\alpha}(A,\varrho)^* = R_p^{\alpha\alpha}(A,\varrho)$$

In particular, the Garrett–Nekovář p-adic height pairing $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ is non-degenerate.

Remarks 1.2. —

1. Under the current assumptions, the module $\mathcal{Q}_p(A,\varrho)$ is non-zero precisely if
$$\alpha_f=\alpha_g\cdot\alpha_h \quad\text{or}\quad \alpha_f=\beta_g\cdot\alpha_h,$$
in which case $\dim_{\mathbf{Q}(\varrho)}\mathcal{Q}_p(A,\varrho)=2$ and one says that (A,ϱ) is *exceptional at p*. Since by assumption g is p-regular, only one of the displayed equalities can be satisfied. Moreover, as α_ξ and β_ξ are roots of unity for $\xi=g,h$, if (A,ϱ) is exceptional at p, then $\alpha_f^2=1$ and either $\alpha_g\cdot\alpha_h=\alpha_f=\beta_g\cdot\beta_h$ or $\alpha_g\cdot\beta_h=\alpha_f=\beta_g\cdot\alpha_h$ by the self-duality assumption (1).
2. The value of $L_p^{\alpha\alpha}(A,\varrho)$ at $w_0=(2,1,1)$ is a non-zero complex multiple of
$$\left(1-\frac{\alpha_g\alpha_h}{\alpha_f}\right)^2\left(1-\frac{\beta_g\alpha_h}{\alpha_f}\right)^2\left(1-\frac{\alpha_g\beta_h}{\alpha_f}\right)^2\left(1-\frac{\beta_g\beta_h}{\alpha_f}\right)^2\cdot L(A,\varrho,1).$$
It follows that (A,ϱ) is exceptional at p precisely if $L_p^{\alpha\alpha}(A,\varrho)$ has an exceptional zero in the sense of [**MTT86**], viz. one of the Euler factors which appear in the previous expression is equal to zero. In this case $r^\dagger(A,\varrho)=\dim_{\mathbf{Q}(\varrho)}A(K_\varrho)^\varrho+2$, hence Conjecture 1.1 and the classical Birch and Swinnerton-Dyer conjecture predict that the order of vanishing of $L_p^{\alpha\alpha}(A,\varrho)$ at w_o equals $\mathrm{ord}_{s=1}L(A,\varrho,s)+2$.
3. Since $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ is skew-symmetric, the regulator $R_p^{\alpha\alpha}(A,\varrho)$ vanishes if $r^\dagger(A,\varrho)$ is odd. On the other hand, the assumption (2) implies that the order of vanishing of $L(A,\varrho,s)$ at $s=1$ is even, hence $r^\dagger(A,\varrho)$ should also be even by the classical Birch and Swinnerton-Dyer conjecture (and the first remark).
4. If $L(A,\varrho,s)$ does not vanish at $s=1$ and (A,ϱ) is not exceptional at p, then $L_p^{\alpha\alpha}(A,\varrho)(w_o)$ is the square of a non-zero element of $\mathbf{Q}(\varrho)^*$. In this case Conjecture 1.1 is a consequence of the classical Birch and Swinnerton-Dyer conjecture.
5. Assume that (A,ϱ) is exceptional at p. The article [**BSV21b**] proves Conjecture 1.1 when $L(A,\varrho,s)$ does not vanish at $s=1$. It also shows the equality
$$\langle\!\langle q,q'\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}=(\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}}-\mathfrak{L}_{\boldsymbol{g}}^{\mathrm{an}})\cdot(\boldsymbol{l}-1)+\varepsilon\cdot(\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}}-\mathfrak{L}_{\boldsymbol{h}}^{\mathrm{an}})\cdot(\boldsymbol{m}-1)$$
in $(\mathscr{I}/\mathscr{I}^2)/\mathbf{Q}(\varrho)^*$ (cf. Equation (9)), where (q,q') is a $\mathbf{Q}(\varrho)$-basis of $\mathcal{Q}_p(A,\varrho)$ and $\varepsilon=+1$ if $\alpha_f=\alpha_g\cdot\alpha_h$ while $\varepsilon=-1$ if $\alpha_f=\beta_g\cdot\alpha_h$. (Recall that $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ is skew-symmetric, and that by assumption either $\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}}\neq\mathfrak{L}_{\boldsymbol{g}}^{\mathrm{an}}$ or $\mathfrak{L}_{\boldsymbol{f}}^{\mathrm{an}}\neq\mathfrak{L}_{\boldsymbol{h}}^{\mathrm{an}}$, hence $\langle\!\langle q,q'\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ is a non-zero square root of $R_p^{\alpha\alpha}(A,\varrho)$.)
6. Assume that (A,ϱ) is exceptional and that $L(A,\varrho,s)$ vanishes at $s=1$. Let (q,q') be a $\mathbf{Q}(\varrho)$-basis of $\mathcal{Q}_p(A,\varrho)$. Conjecture 1.1 predicts the equality
$$\frac{\partial^2\mathscr{L}_p^{\alpha\alpha}(A,\varrho)}{\partial\boldsymbol{k}^2}(w_o)=\log_q(P)\cdot\log_{q'}(Q)-\log_{q'}(P)\cdot\log_q(Q)$$

in $L/\mathbf{Q}(\varrho)^*$ for two rational points P and Q in $A(K_\varrho)^\varrho$, where $\log_{q^\cdot}(\cdot)$ is the evaluation at $q^\cdot$ of the Bloch–Kato p-adic logarithm for $q^\cdot = q, q'$. The reader is referred to Section 2.2 of [BSV21b] for details.

1.1. Generalisations. —

1.1.1. *The semi-stable case.* — Assume that A has semi-stable reduction at p, and let α_f be a non-zero root of the p-th Hecke polynomial $h_{f,p} = X^2 - a_p(f)\cdot X + 1_{N_f}(p)\cdot p$ of f. If A has good ordinary reduction at p and α_f is the root of $h_{f,p}$ with positive p-adic valuation, assume in addition that A does not have complex multiplication. Under these assumptions, there exists a unique Coleman family (of slope $\mathrm{ord}_p(\alpha_f)$) which specialises to $f_\alpha = f(q) - \beta_f \cdot f(q^p)$ in weight 2, where $\beta_f \cdot \alpha_f = 1_{N_f}(p)\cdot p$. By combining the results of [Hsi20] and [AI20], one should be able to associate to the triple $(\boldsymbol{f}_\alpha, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ a canonical p-adic L-function $L_p^{\alpha\alpha}(f_\alpha,\varrho) = \mathscr{L}_p(\boldsymbol{f}_\alpha, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)^2$ (generalising the construction of $L_p^{\alpha\alpha}(A,\varrho) = L_p^{\alpha\alpha}(f_\alpha,\varrho)$ when A is p-ordinary and α_f is the unit root of $h_{f,p}$). On the algebraic side of the matter, (while not necessarily ordinary) the Galois representation $V(\boldsymbol{f}_\alpha)$ associated with $\boldsymbol{f}_\alpha$ is trianguline at p. In light of the extension of Nekovář's theory to families of trianguline representations obtained in [Pot13], the construction of $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\boldsymbol{f}_\alpha\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$, given in Section 2 below when A is p-ordinary and α_f is the unit root of $h_{f,p}$, easily generalises to the present setting. Conjecture 1.1 should then extend to the semi-stable setting.

1.1.2. *The reducible case.* — The formalism leading to the definition of the p-adic regulator $R_p^{\alpha\alpha}(A,\varrho)$ extends to the case in which one or both the Artin representations ϱ_1 and ϱ_2 is reducible and p-irregular, i.e. of the form $\chi \oplus \chi'$ for Dirichlet characters satisfying $\chi(p) = \chi'(p)$. Let $\xi = g$ or h be the associated weight-one Eisenstein series $\mathrm{Eis}_1(\chi,\chi')$. According to the main result of [BDP19] there exists a unique cuspidal Hida family $\boldsymbol{\xi}_\alpha$ specialising in weight one to the (unique) p-stabilisation ξ_α of ξ. The construction of $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ given in Section 2 carries over to this setting, if $V(\boldsymbol{\xi}_\alpha)$ is replaced by its parabolic counterpart. This guarantees the freeness of $V(\boldsymbol{\xi}_\alpha)$ and of its maximal p-unramified quotient. Note that the p-regular reducible cases would involve the Hida–Rankin p-adic L-functions associated to $\boldsymbol{f}$ and one or two families of Eisenstein series.

1.1.3. *The higher-weight case.* — One can formulate a higher-weight analogue of Conjecture 1.1, in which the weight-2 newform associated with A is replaced by a newform

$$f = \sum_{n\geqslant 1} a_n(f)\cdot q^n \in S_k(N_f)_L$$

of even weight $k \geqslant 2$ and trivial character. Assume for simplicity that p does not divide the conductor N_f of f, and that $a_p(f)$ is a p-adic unit (under the embedding $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$ fixed at the outset). Let $\boldsymbol{f} = \boldsymbol{f}_\alpha$ be the unique Hida family specialising to the ordinary p-stabilisation f_α of f at weight k. The article [Hsi20] associates to $(\boldsymbol{f}_\alpha, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ a p-adic L-function $L_p^{\alpha\alpha}(f_\alpha,\varrho) = \mathscr{L}_p^f(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)^2$. Let $\mathcal{E}^{k-2}$ be the $(k-2)$-fold fibre product of the universal generalised elliptic curve $\mathcal{E} \longrightarrow X_1(N_f)$

over the modular curve $X_1(N_f)$ of level $\Gamma_1(N_f)$ over $\mathbf{Q}$. The self-dual twist V_f of the Deligne representation of f is a direct summand of $H^{k-1}_{\text{ét}}(\mathscr{E}^{k-2}\otimes_{\mathbf{Q}}\bar{\mathbf{Q}}, L(k/2))$, hence the p-adic Abel–Jacobi map yields a morphism (cf. [NN16])

$$r_{\text{ét}} : \left(\mathrm{CH}^{k/2}(\mathscr{E}^{k-2}\otimes_{\mathbf{Q}} K_\varrho)_0 \otimes_{\mathbf{Q}} V_\varrho)\right)^{\mathrm{Gal}(K_\varrho/\mathbf{Q})} \longrightarrow \mathrm{Sel}(\mathbf{Q}, V_f \otimes_{\mathbf{Q}(\varrho)} V_\varrho),$$

where $\mathrm{CH}^i(\cdot)_0$ is the Chow group of homologically trivial codimension i cycles in $\cdot$ with $\mathbf{Q}$-coefficients, and $\mathrm{Sel}(\mathbf{Q},\cdot)$ is the Bloch–Kato Selmer group of $\cdot$ over $\mathbf{Q}$. Define $A_f(K_\varrho)^\varrho$ to be the image of the Abel–Jacobi map $r_{\text{ét}}$:

$$A_f(K_\varrho)^\varrho = \mathrm{Image}(r_{\text{ét}}).$$

The constructions of Section 2 below readily generalise to give a pairing

$$\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} : A_f(K_\varrho)^\varrho \otimes_{\mathbf{Q}(\varrho)} A_f(K_\varrho)^\varrho \longrightarrow \mathscr{I}_k/\mathscr{I}_k^2,$$

where $\mathscr{I}_k$ is the ideal of functions in $\mathscr{O}_{\boldsymbol{fgh}}$ which vanish at $(k,1,1)$. The pairing $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ is skew-symmetric, and canonical up to the choice of the isomorphisms γ_g and γ_h fixed in (4). The Bloch–Kato conjecture predicts that $r_{\text{ét}}$ is injective, and that the dimension $r(f_\alpha,\varrho)$ of $A_f(K_\varrho)^\varrho$ over $\mathbf{Q}(\varrho)$ is finite. Generalising Conjecture 1.1, we expect that $L_p^{\alpha\alpha}(f_\alpha,\varrho)$ belongs to $\mathscr{I}_k^{r(f_\alpha,\varrho)} - \mathscr{I}_k^{r(f_\alpha,\varrho)+1}$, and that its image in $(\mathscr{I}^{r(f_\alpha,\varrho)}/\mathscr{I}^{r(f_\alpha,\varrho)+1})/\mathbf{Q}(\varrho)^{*2}$ is equal to the discriminant of $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$, computed with respect to any $\mathbf{Q}(\varrho)$-basis of $A_f(K_\varrho)^\varrho$.

2. Garrett–Nekovář p-adic height pairings

Notation. In this section we set $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = (\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)$. We denote by G_{Np} the Galois group of the maximal algebraic extension of $\mathbf{Q}$ which is uramified at all the rational primes not dividing Np.

2.1. Galois representations (cf. [BSV20]). — Let $\boldsymbol{\xi}$ be one of $\boldsymbol{f},\boldsymbol{g}$ and $\boldsymbol{h}$, and let $V(\boldsymbol{\xi})$ be the Galois representation introduced in [BSV20, Section 5]. Under the current assumptions it is a free $\mathscr{O}_{\boldsymbol{\xi}}$-module of rank two, equipped with a linear action of G_{Np}. (Recall that $\mathscr{O}_{\boldsymbol{\xi}}$ denotes the ring of bounded functions on $U_{\boldsymbol{\xi}}$, cf. Section 1.) For each classical point u in $U_{\boldsymbol{\xi}}\cap \mathbf{Z}_{\geqslant 2}$, there is a natural specialisation isomorphism

$$\rho_u : V(\boldsymbol{\xi})\otimes_u L \simeq V(\boldsymbol{\xi}_u)$$

between the base change of $V(\boldsymbol{\xi})$ along evaluation at u on $\mathscr{O}_{\boldsymbol{\xi}}$ and the homological Deligne representation $V(\boldsymbol{\xi}_u)$ of $\boldsymbol{\xi}_u$. (We refer to Equation (106) of loc. cit. for more details.) Moreover, if $\boldsymbol{\xi} = \boldsymbol{g},\boldsymbol{h}$, the base change of $V(\boldsymbol{\xi})$ along evaluation at 1 on $U_{\boldsymbol{\xi}}$ yields a canonical model of the (homological) Deligne–Serre representation associated with the weight-one cuspidal eigenform $\boldsymbol{\xi}_1$. In this case we set (cf. Section 1) $V(\xi) = V(\boldsymbol{\xi}_1) = V(\boldsymbol{\xi})\otimes_1 L$ and denote by $\rho_1 : V(\boldsymbol{\xi})\otimes_1 L \simeq V(\boldsymbol{\xi}_1)$ the identity.

The representation $V(\boldsymbol{f}_2)$ is the f-isotypic component of $H^1_{\text{ét}}(X_1(N_f,p), L(1))$ and the modular parametrisation $\wp_\infty : X_1(N_f,p) \longrightarrow A$ fixed in Section 1 induces an isomorphism $\wp_{\infty *} : V(\boldsymbol{f}_2) \simeq V(f)$. With a slight abuse of notation we write again

$$\rho_2 : V(\boldsymbol{f})\otimes_2 L \simeq V(f) \tag{10}$$

for the composition of $\wp_{\infty*}$ with the specialisation isomorphism ρ_2.

The restriction of $V(\boldsymbol{\xi})$ to $G_{\mathbf{Q}_p}$ is nearly-ordinary: let $\chi_{\mathrm{cyc}}^{\boldsymbol{u}-1} : G_{\mathbf{Q}} \longrightarrow \mathscr{O}_{\boldsymbol{\xi}}^*$ be the character whose composition with evaluation at u in $U_{\boldsymbol{\xi}} \cap \mathbf{Z}$ is the $(u-1)$-th power of the p-adic cyclotomic character $\chi_{\mathrm{cyc}} : G_{\mathbf{Q}} \longrightarrow \mathbf{Z}_p^*$, and let $\breve{a}_p(\boldsymbol{\xi}) : G_{\mathbf{Q}_p} \longrightarrow \mathscr{O}_{\boldsymbol{\xi}}^*$ be the unramified character sending an arithmetic Frobenius to the p-th Fourier coefficient $a_p(\boldsymbol{\xi})$ of $\boldsymbol{\xi}$. Then there exists a natural short exact sequence of $\mathscr{O}_{\boldsymbol{\xi}}[G_{\mathbf{Q}_p}]$-modules

$$0 \longrightarrow V(\boldsymbol{\xi})^+ \xrightarrow{i^+} V(\boldsymbol{\xi}) \xrightarrow{p^-} V(\boldsymbol{\xi})^- \longrightarrow 0$$

with

$$V(\boldsymbol{\xi})^+ \simeq \mathscr{O}_{\boldsymbol{\xi}}(\chi_{\mathrm{cyc}}^{\boldsymbol{u}-1} \cdot \chi_\xi \cdot \breve{a}_p(\boldsymbol{\xi})^{-1}) \quad \text{and} \quad V(\boldsymbol{\xi})^- \simeq \mathscr{O}_{\boldsymbol{\xi}}(\breve{a}_p(\boldsymbol{\xi})). \tag{11}$$

According to Equations (103) and (114) of [BSV20], there exists a natural skew-symmetric $G_{\mathbf{Q}}$-equivariant perfect pairing

$$\pi_{\boldsymbol{\xi}} : V(\boldsymbol{\xi}) \otimes_{\mathscr{O}_{\boldsymbol{\xi}}} V(\boldsymbol{\xi}) \longrightarrow \mathscr{O}_{\boldsymbol{\xi}}(\chi_\xi \cdot \chi_{\mathrm{cyc}}^{\boldsymbol{u}-1}).$$

For each u in $U_{\boldsymbol{\xi}} \cap \mathbf{Z}_{\geqslant 2}$, the base change of $\pi_{\boldsymbol{\xi}}$ along evaluation at u and the specialisation isomorphism ρ_u yield a perfect pairing $\pi_{\boldsymbol{\xi}_u} : V(\boldsymbol{\xi}_u) \otimes_E V(\boldsymbol{\xi}_u) \longrightarrow L(\chi_\xi + u - 1)$. If $\boldsymbol{\xi} = \boldsymbol{f}$ and $u = 2$, then $\pi_{\boldsymbol{f}_2}$ is equal, up to sign, to the pairing arising from the Poincaré duality $H^1(X_1(N_f, p), \mathbf{Q}_p(1))^{\otimes 2} \longrightarrow \mathbf{Q}_p(1)$ (cf. loco citato), hence its composition

$$\pi_f : V(f) \otimes_L V(f) \longrightarrow L(1)$$

with the inverse of $\wp_{\infty*}^{\otimes 2}$ is a rational multiple of the Weil pairing. If $\boldsymbol{\xi}$ equals either $\boldsymbol{g}$ or $\boldsymbol{h}$, then the base change of $\pi_{\boldsymbol{\xi}}$ along evaluation at $u = 1$ on $\mathscr{O}_{\boldsymbol{\xi}}$ yields the perfect pairing $\pi_\xi : V(\xi) \otimes_L V(\xi) \longrightarrow L(\chi_\xi)$ introduced in Equation (3).

As in Section 1, set $\mathscr{O}_{\boldsymbol{fgh}} = \mathscr{O}_{\boldsymbol{f}} \hat{\otimes}_{\mathbf{Q}_p} \mathscr{O}_{\boldsymbol{g}} \hat{\otimes}_L \mathscr{O}_{\boldsymbol{h}}$ and define

$$V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) = V(\boldsymbol{f}) \hat{\otimes}_{\mathbf{Q}_p} V(\boldsymbol{g}) \hat{\otimes}_L V(\boldsymbol{h}) \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \Xi_{\boldsymbol{fgh}},$$

where $\Xi_{\boldsymbol{fgh}} : G_{\mathbf{Q}} \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}^*$ is the character satisfying

$$\Xi_{\boldsymbol{fgh}}(g)(w) = \chi_{\mathrm{cyc}}(g)^{(4-k-l-m)/2}$$

for each g in $G_{\mathbf{Q}}$ and $w = (k, l, m)$ in $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}} \cap \mathbf{Z}^3$. The $G_{\mathbf{Q}}$-representation $V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ is a free $\mathscr{O}_{\boldsymbol{fgh}}$-module of rank eight. Moreover $V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ is Kummer self-dual: because $\chi_g = \chi_h^{-1}$ (cf. Equation (1)), the product of the perfect pairings $\pi_{\boldsymbol{\xi}}$ (for $\boldsymbol{\xi} = \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}$) define a $G_{\mathbf{Q}}$-equivariant and skew-symmetric perfect pairing

$$\pi_{\boldsymbol{fgh}} : V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}(1). \tag{12}$$

Set $w_o = (2, 1, 1)$. Then the specialisation map (10) induces an isomorphism

$$\rho_{w_o} : V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \otimes_{w_o} L \simeq V(f, g, h) \tag{13}$$

between the base change of $V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ along evaluation at w_o on $\mathscr{O}_{\boldsymbol{fgh}}$ and

$$V(f, g, h) = V(f) \otimes_L V(g) \otimes_L V(h).$$

The pairing $\pi_{\boldsymbol{fgh}}$ and ρ_{w_o} yield a $G_{\mathbf{Q}}$-equivariant, skew-symmetric and perfect duality

$$\pi_{fgh} : V(f, g, h) \otimes_L V(f, g, h) \longrightarrow L(1),$$

which by construction equals the product of the dualities π_f, π_g and π_h.

2.2. Selmer complexes (cf. [Nek06]). — For $\boldsymbol{\xi} = \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}$, denote by $\Lambda_{\boldsymbol{\xi}}$ the ring of analytic functions on $U_{\boldsymbol{\xi}}$ bounded by 1, and set $\Lambda_{\boldsymbol{fgh}} = \Lambda_{\boldsymbol{f}} \hat{\otimes}_{\mathbf{Z}_p} \Lambda_{\boldsymbol{g}} \hat{\otimes}_{\mathcal{O}_L} \Lambda_{\boldsymbol{h}}$, so that $\mathscr{O}_{\boldsymbol{fgh}} = \Lambda_{\boldsymbol{fgh}}[1/p]$. The $\mathcal{O}_L$-algebra $\Lambda_{\boldsymbol{fgh}}$ is isomorphic to a three-variable power series ring with coefficients in $\mathcal{O}_L$. In particular it is a regular local complete Noetherian ring with finite residue field. Let G denote either G_{Np} or $G_{\mathbf{Q}_\ell}$, for a rational prime ℓ dividing Np, and let $(\mathtt{B}, \mathtt{M})$ denote one of the pairs $(\mathcal{O}_L, \mathtt{V}(f,g,h))$ and $(\Lambda_{\boldsymbol{fgh}}, \mathtt{V}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$, where $\mathtt{V}(f,g,h)$ (resp., $\mathtt{V}(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$) is an $\mathcal{O}_L$-lattice (resp., a $\Lambda_{\boldsymbol{fgh}}$-lattice) in $V(f,g,h)$ (resp., $V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$) preserved by the action of G_{Np}. Equip G with the profinite topology and $\mathtt{M}$ with the $m_{\mathtt{B}}$-adic topology, where $m_{\mathtt{B}}$ is the maximal ideal of $\mathtt{B}$. Set $(B, M) = (\mathtt{B}[1/p], \mathtt{M}[1/p])$ and

$$\mathrm{C}^{\bullet}_{\mathrm{cont}}(G, M) = \mathrm{C}^{\bullet}_{\mathrm{cont}}(G, \mathtt{M}) \otimes_{\mathtt{B}} B,$$

where $\mathrm{C}^{\bullet}_{\mathrm{cont}}(G, \mathtt{M})$ is the complex of non-homogeneous continuous cochains of G with values in $\mathtt{M}$. If $G = G_{\mathbf{Q}_\ell}$, we also write $\mathrm{C}^{\bullet}_{\mathrm{cont}}(\mathbf{Q}_\ell, M)$ as a shorthand for $\mathrm{C}^{\bullet}_{\mathrm{cont}}(G_{\mathbf{Q}_\ell}, M)$.

Recall the $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}_p}]$-submodule $V(\boldsymbol{f})^+$ of $V(\boldsymbol{f})$ introduced in Section 2.1 and set

$$V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})^+ = V(\boldsymbol{f})^+ \hat{\otimes}_{\mathbf{Q}_p} V(\boldsymbol{g}) \hat{\otimes}_L V(\boldsymbol{h}) \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \Xi_{\boldsymbol{fgh}}.$$

Define $V(f)^+$ to be the image of $V(\boldsymbol{f})^+ \otimes_2 L$ under the specialisation isomorphism $\rho_2 : V(\boldsymbol{f}) \otimes_2 L \simeq V(f)$ (cf. Equation (10)), and set

$$V(f,g,h)^+ = V(f)^+ \otimes_L V(g) \otimes_L V(h).$$

Denote by $i^+ : M^+ \hookrightarrow M$ the natural inclusion, fix a $G_{\mathbf{Q}_p}$-stable $\mathtt{B}$-lattice $\mathtt{M}^+$ mapping into $\mathtt{M}$ under i^+, and define $\mathrm{C}^{\bullet}_{\mathrm{cont}}(G_{\mathbf{Q}_p}, M^+) = \mathrm{C}^{\bullet}_{\mathrm{cont}}(\mathbf{Q}_p, M^+)$ to be the base change to B of the complex $\mathrm{C}^{\bullet}_{\mathrm{cont}}(G_{\mathbf{Q}_p}, \mathtt{M}^+)$ of continuous non-homogeneous cochains of $G_{\mathbf{Q}_p}$ with values in $\mathtt{M}^+$. The inclusion i^+ induces a morphism of complexes

$$i^+ : \mathrm{C}^{\bullet}_{\mathrm{cont}}(\mathbf{Q}_p, M^+) \longrightarrow \mathrm{C}^{\bullet}_{\mathrm{cont}}(\mathbf{Q}_p, M),$$

which we call the *f-Greenberg local condition* on the $G_{\mathbf{Q}_p}$-representation M.

The *f-Nekovář Selmer complex*

$$\tilde{\mathrm{C}}^{\bullet}_f(G_{Np}, M)$$

of the G_{Np}-representation M is the complex of B-modules

$$\mathrm{Cone}\left(\mathrm{C}^{\bullet}_{\mathrm{cont}}(G_{Np}, M) \oplus \mathrm{C}^{\bullet}_{\mathrm{cont}}(\mathbf{Q}_p, M^+) \overset{\mathrm{res}_{Np} - i^+}{\longrightarrow} \bigoplus_{\ell | Np} \mathrm{C}^{\bullet}_{\mathrm{cont}}(\mathbf{Q}_\ell, M)\right)[-1],$$

where $\mathrm{res}_{Np} = \oplus_{\ell|Np} \mathrm{res}_\ell$ is the direct sum over the primes dividing Np of the restriction morphisms $\mathrm{res}_\ell : \mathbf{R}\Gamma_{\mathrm{cont}}(G_{Np}, M) \longrightarrow \mathbf{R}\Gamma_{\mathrm{cont}}(\mathbf{Q}_\ell, M)$ associated with fixed embeddings $i_\ell : \bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_\ell$ (with i_p the embedding fixed at the outset.) Denote by

$$\mathbf{R}\tilde{\Gamma}_f(\mathbf{Q}, M) \in \mathrm{D}^b_{\mathrm{ft}}(B)$$

the image of $\tilde{\mathrm{C}}^{\bullet}_f(G_{Np}, M)$ in the derived category $\mathrm{D}^b_{\mathrm{ft}}(B)$ of bounded complexes of B-modules with cohomology of finite type over B and by

$$\tilde{H}^{\cdot}_f(\mathbf{Q}, M) = H^{\cdot}\left(\mathbf{R}\tilde{\Gamma}_f(\mathbf{Q}, M)\right)$$

its cohomology. (The complex $\mathbf{R}\tilde{\Gamma}_f(\mathbf{Q}, M)$ is indeed a perfect complex of perfect amplitude contained in $[0,3]$, cf. [Nek06].) Similarly denote by

$$\mathbf{R}\Gamma_{\mathrm{cont}}(G_{Np}, M), \quad \mathbf{R}\Gamma_{\mathrm{cont}}(\mathbf{Q}_\ell, M) \quad \text{and} \quad \mathbf{R}\Gamma_{\mathrm{cont}}(\mathbf{Q}_p, M^+)$$

the images in $\mathrm{D}^b_{\mathrm{ft}}(B)$ of $\mathrm{C}^\bullet_{\mathrm{cont}}(G_{Np}, M)$, $\mathrm{C}^\bullet_{\mathrm{cont}}(\mathbf{Q}_\ell, M)$ and $\mathrm{C}^\bullet_{\mathrm{cont}}(\mathbf{Q}_p, M^+)$, and by

$$H^\cdot(G_{Np}, M), \quad H^\cdot(\mathbf{Q}_\ell, M) \quad \text{and} \quad H^\cdot(\mathbf{Q}_p, M^+)$$

their cohomology.

The specialisation isomorphism (13) induces isomorphisms in $\mathrm{D}^b_{\mathrm{ft}}(L)$:

$$\rho_{w_o} : \mathbf{R}\Gamma_{\mathrm{cont}}(G, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) \otimes^{\mathbf{L}}_{\mathscr{O}_{\boldsymbol{fgh}}, w_o} L \simeq \mathbf{R}\Gamma_{\mathrm{cont}}(G, V(f, g, h)) \tag{14}$$

and

$$\rho_{w_o} : \mathbf{R}\Gamma_{\mathrm{cont}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})^+) \otimes^{\mathbf{L}}_{\mathscr{O}_{\boldsymbol{fgh}}, w_o} L \simeq \mathbf{R}\Gamma_{\mathrm{cont}}(\mathbf{Q}_p, V(f, g, h)^+),$$

which in turn induce on f-Selmer complexes an isomorphism

$$\rho_{w_o} : \mathbf{R}\tilde{\Gamma}_f(\mathbf{Q}, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})) \otimes^{\mathbf{L}}_{\mathscr{O}_{\boldsymbol{fgh}}, w_o} L \simeq \mathbf{R}\tilde{\Gamma}_f(\mathbf{Q}, V(f, g, h)). \tag{15}$$

(This follows easily by the fact the kernel of evaluation at w_o on $\mathscr{O}_{\boldsymbol{fgh}}$ is generated by an $\mathscr{O}_{\boldsymbol{fgh}}$-regular sequence.)

The local Tate duality implies that for each prime ℓ dividing N the complex $\mathbf{R}\Gamma_{\mathrm{cont}}(\mathbf{Q}_\ell, V(f, g, h))$ is isomorphic to zero, hence so is $\mathbf{R}\Gamma_{\mathrm{cont}}(\mathbf{Q}_\ell, V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))$ by Equation (14). It then follows from the definition of the Selmer complex $\tilde{\mathrm{C}}^\bullet_f(G_{Np}, M)$ that one has a distinguished triangle in $\mathrm{D}^b_{\mathrm{ft}}(R)$:

$$\mathbf{R}\tilde{\Gamma}_f(\mathbf{Q}, M) \longrightarrow \mathbf{R}\Gamma_{\mathrm{cont}}(G_{Np}, M) \stackrel{p^- \circ \mathrm{res}_p}{\longrightarrow} \mathbf{R}\Gamma_{\mathrm{cont}}(\mathbf{Q}_p, M^-), \tag{16}$$

where M^- is the quotient of M by M^+ and p^- is the map induced on complexes by the the projection $p^- : M \longrightarrow M^-$.

2.3. The extended Selmer group. — The exact triangle (16) gives rise to a long exact cohomology sequence

$$\tilde{H}^i_f(\mathbf{Q}, M) \longrightarrow H^i(G_{Np}, M) \longrightarrow H^i(\mathbf{Q}_p, M^-) \stackrel{\jmath}{\longrightarrow} \tilde{H}^{i+1}_f(\mathbf{Q}, M). \tag{17}$$

As easily checked

$$\mathrm{Sel}(\mathbf{Q}, V(f, g, h)) = \ker\left(H^1(G_{Np}, V(f, g, h)) \stackrel{p^- \circ \mathrm{res}_p}{\longrightarrow} H^1(\mathbf{Q}_p, V(f, g, h)^-)\right),$$

hence one can extract from the previous sequence the short exact sequence

$$0 \longrightarrow H^0(\mathbf{Q}_p, V(f, g, h)^-) \longrightarrow \tilde{H}^1_f(\mathbf{Q}, V(f, g, h)) \longrightarrow \mathrm{Sel}(\mathbf{Q}, V(f, g, h)) \longrightarrow 0. \tag{18}$$

The projection in the previous equation has a natural splitting

$$\imath_{\mathrm{ur}} : \mathrm{Sel}(\mathbf{Q}, V(f, g, h)) \hookrightarrow \tilde{H}^1_f(\mathbf{Q}, V(f, g, h)), \tag{19}$$

characterised by the following property. Denote by

$$\cdot^+ : \tilde{H}^1_f(\mathbf{Q}, V(f, g, h)) \longrightarrow H^1(\mathbf{Q}_p, V(f, g, h)^+) \tag{20}$$

the morphism induced by the natural map of complexes (i.e. projection)

$$\tilde{\mathrm{C}}^\bullet_f(G_{Np}, V(f, g, h)) \longrightarrow \mathrm{C}^\bullet_{\mathrm{cont}}(\mathbf{Q}_p, V(f, g, h)^+).$$

Then for any Selmer class $\mathfrak{x}$ in $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$ one has

$$\imath_{\mathrm{ur}}(\mathfrak{x})^+ \in H^1_{\mathrm{fin}}(\mathbf{Q}_p, V(f,g,h)^+).$$

We often identify the Bloch–Kato Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$ with a subgroup of the Nekovář extended Selmer group $\tilde{H}^1_f(\mathbf{Q}, V(f,g,h))$ via the splitting $\imath_{\mathrm{nr}}$. In other words, we use the splitting $\imath_{\mathrm{nr}}$ to identify the Nekovář extended Selmer group $\tilde{H}^1_f(\mathbf{Q}, V(f,g,h))$ with the naive extended Selmer group $\mathrm{Sel}^\dagger(\mathbf{Q}, V(f,g,h))$ introduced in Equation (7):

$$\tilde{H}^1_f(\mathbf{Q}, V(f,g,h)) = \mathrm{Sel}(\mathbf{Q}, V(f,g,h)) \oplus H^0(\mathbf{Q}_p, V(f,g,h)^-). \tag{21}$$

The Kummer map and the Shapiro isomorphism yield an injective morphism

$$(A(K_\varrho) \otimes_{\mathbf{Z}} V_\varrho)^{\mathrm{Gal}(K_\varrho/\mathbf{Q})} \otimes_{\mathbf{Q}(\varrho)} L \longrightarrow \mathrm{Sel}(\mathbf{Q}, V(f) \otimes_{\mathbf{Q}(\varrho)} V_\varrho).$$

Together with the isomorphism of $L[G_{\mathbf{Q}}]$-modules

$$\gamma_g \otimes \gamma_h : V_\varrho \otimes_{\mathbf{Q}(\varrho)} L \simeq V(g) \otimes_L V(h)$$

(cf. Equation (4)), it entails an injective morphism of L-vector spaces

$$\gamma_{gh} : A^\dagger(K_\varrho)^\varrho \otimes_{\mathbf{Q}(\varrho)} L \hookrightarrow \tilde{H}^1_f(\mathbf{Q}, V(f,g,h)), \tag{22}$$

which is an isomorphism precisely if the p-part of the ϱ-isotypic component of the Shafarevich–Tate group of A over K_ϱ is finite.

2.4. Generalised Poitou–Tate duality (cf. [Nek06]). — Section 6.3 of [Nek06] (see also Proposition 1.3.2) associates to the Kummer duality

$$\pi_{fgh} : V(f,g,h) \otimes_L V(f,g,h) \longrightarrow L(1)$$

(satisfying $\pi_{fgh}(V(f,g,h)^+ \otimes_L V(f,g,h)^+) = 0$) a global cup-product pairing

$$\cup_{\mathrm{Nek}} = \cup_{\mathrm{Nek}} : \mathbf{R}\tilde{\Gamma}_f(\mathbf{Q}, V(f,g,h)) \otimes^{\mathbf{L}}_L \mathbf{R}\tilde{\Gamma}_f(\mathbf{Q}, V(f,g,h)) \longrightarrow \mathbf{R}\tilde{\Gamma}_\emptyset(\mathbf{Q}, L(1)),$$

where $\mathbf{R}\tilde{\Gamma}_\emptyset(\mathbf{Q}, L(1))$ denotes the complex

$$\mathrm{Cone}\left(\mathbf{R}\Gamma_{\mathrm{cont}}(G_{Np}, L(1)) \xrightarrow{\mathrm{res}_{Np}} \bigoplus_{\ell | Np} \mathbf{R}\Gamma_{\mathrm{cont}}(\mathbf{Q}_\ell, L(1))\right)[-1].$$

Let $\tilde{H}^\cdot_\emptyset(\mathbf{Q}, L(1))$ be the cohomology of $\mathbf{R}\tilde{\Gamma}_\emptyset(\mathbf{Q}, L(1))$. The fundamental exact sequence of global class field theory yields a canonical isomorphism

$$\mathrm{Tr}_L : \tilde{H}^3_\emptyset(\mathbf{Q}, L(1)) \simeq \bigoplus_{\ell | Np} H^2(\mathbf{Q}_\ell, L(1))/\mathrm{res}_{Np}\big(H^2(G_{Np}, L(1))\big) \simeq L,$$

arising from the sum of the invariant maps $\mathrm{inv}_\ell : H^2(\mathbf{Q}_\ell, L(1)) \simeq L$ of local class field theory, for ℓ dividing Np (cf. Equation (5.3.1.3.2) of [Nek06]). Define

$$\langle \cdot, \cdot \rangle_{\mathrm{Nek}} : \tilde{H}^2_f(\mathbf{Q}, V(f,g,h)) \otimes_L \tilde{H}^1_f(\mathbf{Q}, V(f,g,h)) \longrightarrow \tilde{H}^3_\emptyset(\mathbf{Q}, L(1)) \simeq L. \tag{23}$$

to be the composition of the map $H^{2,1}(\cup_{\mathrm{Nek}})$ induced on $(2,1)$-cohomology by Nekovář's global cup-product $\cup_{\mathrm{Nek}}$ with the trace isomorphism Tr_L.

2.5. The p-adic height pairing. — To lighten the notation, we abbreviate $V(f,g,h), V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}), \mathbf{R}\tilde{\Gamma}_f(\mathbf{Q},\cdot)$ and $\tilde{H}(\mathbf{Q},\cdot)$ with $V, \boldsymbol{V}, \mathbf{R}\tilde{\Gamma}_f(\cdot)$ and $\tilde{H}_f(\cdot)$ respectively.

Applying $\mathbf{R}\tilde{\Gamma}_f(\boldsymbol{V}) \otimes^{\mathbf{L}}_{\mathscr{O}_{\boldsymbol{fgh}}} \cdot$ to the exact triangle

$$\mathscr{I}/\mathscr{I}^2 \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}/\mathscr{I}^2 \longrightarrow L \xrightarrow{\delta} \mathscr{I}/\mathscr{I}^2[1] \tag{24}$$

arising from evaluation at w_o on $\mathscr{O}_{\boldsymbol{fgh}}$, yields a morphism in $\mathrm{D}^b_{\mathrm{ft}}(\mathscr{O}_{\boldsymbol{fgh}})$:

$$\mathbf{R}\tilde{\Gamma}_f(\boldsymbol{V}) \otimes^{\mathbf{L}}_{\mathscr{O}_{\boldsymbol{fgh}},w_o} L \longrightarrow \mathbf{R}\tilde{\Gamma}_f(\boldsymbol{V}) \otimes^{\mathbf{L}}_{\mathscr{O}_{\boldsymbol{fgh}}} \mathscr{I}/\mathscr{I}^2[1]. \tag{25}$$

The specialisation map ρ_{w_o} gives rise to isomorphisms (cf. Equation (15))

$$\rho_{w_o} : \mathbf{R}\tilde{\Gamma}_f(\boldsymbol{V}) \otimes^{\mathbf{L}}_{\mathscr{O}_{\boldsymbol{fgh}},w_o} L \simeq \mathbf{R}\tilde{\Gamma}_f(V)$$

and

$$\rho_{w_o} \otimes \mathrm{id} : \mathbf{R}\tilde{\Gamma}_f(\boldsymbol{V}) \otimes^{\mathbf{L}}_{\mathscr{O}_{\boldsymbol{fgh}}} \mathscr{I}/\mathscr{I}^2 \simeq \mathbf{R}\tilde{\Gamma}_f(V) \otimes_L \mathscr{I}/\mathscr{I}^2,$$

which together with (25) induce a *derived Bockstein map*

$$\tilde{\boldsymbol{\beta}}_{\boldsymbol{fgh}} : \mathbf{R}\tilde{\Gamma}_f(\mathbf{Q}, V(f,g,h)) \longrightarrow \mathbf{R}\tilde{\Gamma}_f(\mathbf{Q}, V(f,g,h))[1] \otimes_L \mathscr{I}/\mathscr{I}^2.$$

The *Garrett–Nekovář canonical p-adic height pairing*

$$\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\boldsymbol{fgh}} : \tilde{H}^1_f(\mathbf{Q}, V(f,g,h)) \otimes_L \tilde{H}^1_f(\mathbf{Q}, V(f,g,h)) \longrightarrow \mathscr{I}/\mathscr{I}^2$$

is the composition of the Nekovář cup-product pairing (cf. Equation (23))

$$\langle \cdot,\cdot \rangle_{\mathrm{Nek}} \otimes \mathscr{I}/\mathscr{I}^2 : \tilde{H}^2_f(V) \otimes_L \tilde{H}^1_f(V) \otimes_L \mathscr{I}/\mathscr{I}^2 \longrightarrow \mathscr{I}/\mathscr{I}^2$$

with the morphism

$$\tilde{\beta}_{\boldsymbol{fgh}} \otimes \mathrm{id} : \tilde{H}^1_f(V) \otimes_L \tilde{H}^1_f(V) \longrightarrow \tilde{H}^2_f(V) \otimes_L \tilde{H}^1_f(V) \otimes_L \mathscr{I}/\mathscr{I}^2,$$

where the *Bockstein map*

$$\tilde{\beta}_{\boldsymbol{fgh}} : \tilde{H}^1_f(\mathbf{Q}, V(f,g,h)) \longrightarrow \tilde{H}^2_f(\mathbf{Q}, V(f,g,h)) \otimes_L \mathscr{I}/\mathscr{I}^2 \tag{26}$$

is the map $H^1(\tilde{\boldsymbol{\beta}}_{\boldsymbol{fgh}})$ induced on the first cohomology groups by $\tilde{\boldsymbol{\beta}}_{\boldsymbol{fgh}}$.

Proposition 2.1. — *The p-adic height $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\boldsymbol{fgh}}$ is skew-symmetric.*

Proof. — As explained in Section 2.1, the Kummer self-duality π_{fgh} on $V(f,g,h)$ lifts (under ρ_{w_o}) to a skew-symmetric, $G_{\mathbf{Q}}$-equivariant perfect pairing

$$\pi_{\boldsymbol{fgh}} : V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}(1),$$

under which the $G_{\mathbf{Q}_p}$-submodule $V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})^+$ of $V(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ is its own orthogonal complement. The proposition then follows from the results of [**Ven13**, Appendix C]. □

The p-adic height pairing (cf. Equation (5))

$$\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\boldsymbol{fg}_\alpha\boldsymbol{h}_\alpha} : A^\dagger(K_\varrho)^\varrho \otimes_{\mathbf{Q}(\varrho)} A^\dagger(K_\varrho)^\varrho \longrightarrow \mathscr{I}/\mathscr{I}^2$$

which appears in Conjecture 1.1 is defined to be restriction of the canonical height pairing $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{\boldsymbol{fg}_\alpha\boldsymbol{h}_\alpha} : \tilde{H}^1_f(\mathbf{Q}, V(f,g,h))^{\otimes 2} \longrightarrow \mathscr{I}/\mathscr{I}^2$ to the p-extended Mordell–Weil group $A^\dagger(K_\varrho)^\varrho$ along the injective morphism γ_{gh} introduced in Equation (22).

3. Diagonal classes and rational points

As proved in [**BSV20**, Theorem A] and [**DR20**, Theorem 5.1], the square root p-adic L-function $\mathscr{L}_p^{\alpha\alpha}(A,\varrho)$ is the image of a *big diagonal class* $\kappa(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)$ in $H^1(\mathbf{Q},V(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha))$ under an appropriate branch of the Perrin-Riou big logarithm. The leading term of $L_p^{\alpha\alpha}(A,\varrho)$ at $w_o=(2,1,1)$ is then intimately connected to the derivatives of the class $\kappa(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)$ at w_o. This section exploits this connection and its relation with Conjecture 1.1.

To simplify the exposition, we assume in this section that

$$\alpha_f \neq \alpha_g \cdot \alpha_h \quad \text{and} \quad \alpha_f \neq \beta_g \cdot \alpha_h. \tag{27}$$

This condition is equivalent to the vanishing of the module of p-adic periods $\mathcal{Q}_p(A,\varrho)$ of (A,ϱ) (or equivalently of the module $H^0(\mathbf{Q}_p,V(f,g,h)^-)$), and is satisfied when A has good (ordinary) reduction at p (cf. Remark 1.2.1). In particular, in this section, the Nekovář extended Selmer group and the Bloch–Kato Selmer group of $V(f,g,h)$ over $\mathbf{Q}$ are equal to each other (cf. Equation (21)):

$$\tilde{H}^1_f(\mathbf{Q},V(f,g,h)) = \mathrm{Sel}(\mathbf{Q},V(f,g,h)).$$

3.1. Differentials and logarithms. — Let $\boldsymbol{\xi}$ denote one of $\boldsymbol{f}$, $\boldsymbol{g}_\alpha$ or $\boldsymbol{h}_\alpha$, and recall the short exact sequence of $\mathscr{O}_{\boldsymbol{\xi}}$-modules $V(\boldsymbol{\xi})^+ \hookrightarrow V(\boldsymbol{\xi}) \twoheadrightarrow V(\boldsymbol{\xi})^-$ (cf. Section 2.1).

If $\boldsymbol{\xi}=\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha$ define

$$V(\xi)_\alpha = V(\boldsymbol{\xi})^- \otimes_1 L \quad \text{and} \quad V(\xi)_\beta = V(\boldsymbol{\xi})^+ \otimes_1 L.$$

Equation (11) implies that $V(\xi)_\alpha = V(\xi)^{\mathrm{Frob}_p=\alpha_\xi}$ and $V(\xi)_\beta = V(\xi)^{\mathrm{Frob}_p=\beta_\xi}$ are the subspaces of $V(\xi)$ on which an arithmetic Frobenius Frob_p in $G_{\mathbf{Q}_p}$ acts as multiplication by α_ξ and β_ξ respectively. In particular one has the decomposition

$$V(\xi) = V(\xi)_\alpha \oplus V(\xi)_\beta$$

of $L[G_{\mathbf{Q}_p}]$-modules. (Recall that by assumption the roots α_ξ and $\beta_\xi=\chi_\xi(p)\cdot\alpha_\xi^{-1}$ of the p-th Hecke polynomial of ξ are distinct, cf. Section 1.)

Set $D(\boldsymbol{\xi})^- = H^0(\mathbf{Q}_p,V(\boldsymbol{\xi})^-\hat{\otimes}_{\mathbf{Q}_p}\hat{\mathbf{Q}}_p^{\mathrm{nr}})$, where $\hat{\mathbf{Q}}_p^{\mathrm{nr}}$ is the p-adic completion of the maximal unramified extension of $\mathbf{Q}_p$ (equipped with its natural $G_{\mathbf{Q}_p}$-action). As explained in [**BSV20**, Section 5], the $\mathscr{O}_{\boldsymbol{\xi}}$-module $D(\boldsymbol{\xi})^-$ is free of rank one, and its base change $D(\boldsymbol{\xi})^-_u = D(\boldsymbol{\xi})^-\otimes_u L$ along evaluation at a classical weight u in $U_{\boldsymbol{\xi}}\cap\mathbf{Z}_{\geqslant 2}$ on $\mathscr{O}_{\boldsymbol{\xi}}$ is canonically isomorphic to the $\boldsymbol{\xi}_u$-isotypic component $L\cdot\boldsymbol{\xi}_u$ of $S_u(pN_\xi,\chi_\xi)_L$. Moreover, there exists an $\mathscr{O}_{\boldsymbol{\xi}}$-basis

$$\omega_{\boldsymbol{\xi}} \in D(\boldsymbol{\xi})^-$$

whose image $\omega_{\boldsymbol{\xi}_u}$ in $D(\boldsymbol{\xi})^-_u$ corresponds to $\boldsymbol{\xi}_u$ under the aforementioned isomorphism for each classical weight u in $U_{\boldsymbol{\xi}}\cap\mathbf{Z}_{\geqslant 2}$ (cf. Equations (117)–(119) of [**BSV20**]).

Remark 3.1. — We caution the reader that the notation used here differ from that of [**BSV20**]. Precisely, Section 5 of loc. cit. introduces a differential $\omega_{\boldsymbol{\xi}}=\omega_{\boldsymbol{\xi}}^{\mathrm{BSV}}$ in a suitable dual $D^*(\boldsymbol{\xi})^-$ of $D(\boldsymbol{\xi})^-$. Here we denote by $\omega_{\boldsymbol{\xi}}$ the image of $\omega_{\boldsymbol{\xi}}^{\mathrm{BSV}}$ under the isomorphism $w_{Np}^- : D^*(\boldsymbol{\xi})^- \simeq D(\boldsymbol{\xi})^-$ induced by the Atkin–Lehner isomorphism $w_{Np}^- : V^*(\boldsymbol{\xi})^-(1+\boldsymbol{\kappa}_{U_f}) \simeq V(\boldsymbol{\xi})^-$ defined in [**BSV20**, Equation (114)]. Accordingly the canonical isomorphism $D(\boldsymbol{\xi})^-_u \simeq L \cdot \boldsymbol{\xi}_u$ mentioned above arises from the specialisation isomorphism $D^*(\boldsymbol{\xi})^- \otimes_u L \simeq \mathrm{Fil}^1 V^*_{\mathrm{dR}}(\boldsymbol{\xi}_u)$ defined in [**BSV20**, Equation (116)] and the Atkin–Lehner operator (cf. Equation (29) of loc. cit.).

If $\boldsymbol{\xi}$ is either $\boldsymbol{g}_\alpha$ or $\boldsymbol{h}_\alpha$, the weight-one specialisation of $\omega_{\boldsymbol{\xi}}$ yields canonical elements

$$\omega_{\xi_\alpha} \in D(\boldsymbol{\xi})^-_1 = D_{\mathrm{cris}}(V(\xi)_\alpha).$$

In this case, let η_{ξ_α} in $D_{\mathrm{cris}}(V(\xi)_\beta)$ be the class satisfying

$$\langle \eta_{\xi_\alpha}, \omega_{\xi_\alpha} \rangle_\xi = 1,$$

where

$$\langle \cdot,\cdot \rangle_\xi : D_{\mathrm{cris}}(V(\xi)_\alpha) \otimes_L D_{\mathrm{cris}}(V(\xi)_\beta) \longrightarrow D_{\mathrm{cris}}(L(\chi_\xi)) \simeq L$$

is the perfect pairing induced by the duality π_ξ introduced in Equation (3). (The crystalline module $D_{\mathrm{cris}}(\chi_\xi) = H^0(\mathbf{Q}_p, L(\chi_\xi) \otimes_{\mathbf{Q}_p} B_{\mathrm{cris}})$ of the one-dimensional representation $L(\chi_\xi)$ is generated over L by the Gauß sum

$$G(\chi_\xi) = \sum_{a \in (\mathbf{Z}/c(\chi_\xi)\mathbf{Z})^*} \chi_\xi(a) \otimes e^{2\pi i a/c(\chi_\xi)}$$

in $L \otimes_{\mathbf{Q}_p} \mathbf{Q}_p(\mu_{N_\xi})$ of the primitive character $\chi_\xi : (\mathbf{Z}/c(\chi_\xi)\mathbf{Z})^* \longrightarrow L^*$ associated with χ_ξ. Since by assumption L contains $\mathbf{Q}(\mu_{N_\xi})$, here we identify $G(\chi_\xi)$ with the element $\sum_a \chi_\xi(a) \cdot e^{2\pi i a/c(\chi_\xi)}$ of L, hence $D_{\mathrm{cris}}(\chi_\xi)$ with L.)

Identify $V(f) = \mathrm{Ta}_p(A) \otimes_{\mathbf{Z}_p} L$ with the f-isotypic component of the étale cohomology group $H^1_{\mathrm{\acute{e}t}}(X_1(N_f,p)_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(1)) \otimes_{\mathbf{Q}_p} L$ under the modular parametrisation $\wp_\infty$ fixed in Section 1. The modular form f in $\mathrm{Fil}^0 H^1_{\mathrm{dR}}(X_1(N_f,p)_{\mathbf{Q}_p}, \mathbf{Q}_p(1))$ then defines (via the comparison isomorphism between étale and de Rham cohomology) a class

$$\omega_f \in \mathrm{Fil}^0 D_{\mathrm{dR}}(V(f))$$

(where $D_{\mathrm{dR}}(\cdot) = H^0(\mathbf{Q}_p, \cdot \otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})$ is Fontaine's de Rham functor). Define η_f in $D_{\mathrm{dR}}(V(f))/\mathrm{Fil}^0$ to be the de Rham class satisfying

$$\langle \eta_f, \omega_f \rangle_f = 1,$$

where $\langle \cdot,\cdot \rangle_f : D_{\mathrm{dR}}(V(f)) \otimes_L D_{\mathrm{dR}}(V(f)) \longrightarrow L$ is the perfect pairing induced on the de Rham modules by the Weil pairing on $V(f)$.

Set $V_{\mathrm{dR}}(f,g,h) = D_{\mathrm{dR}}(V(f,g,h))$. The Bloch–Kato exponential map gives an isomorphism between $V_{\mathrm{dR}}(f,g,h)/\mathrm{Fil}^0$ and the finite subspace $H^1_{\mathrm{fin}}(\mathbf{Q}_p, V(f,g,h))$ of $H^1(\mathbf{Q}_p, V(f,g,h))$ (cf. Lemma [**BSV20**, 9.1]). Denote by

$$\log_p : H^1_{\mathrm{fin}}(\mathbf{Q}_p, V(f,g,h)) \longrightarrow V_{\mathrm{dR}}(f,g,h)/\mathrm{Fil}^0$$

the inverse of the Bloch–Kato exponential. Under the self-duality assumption (1), the product of the pairings $\langle \cdot,\cdot \rangle_\xi$, for $\xi = f,g,h$, yields a perfect duality

$$\langle \cdot,\cdot \rangle_{fgh} : V_{\mathrm{dR}}(f,g,h) \otimes_L V_{\mathrm{dR}}(f,g,h) \longrightarrow D_{\mathrm{dR}}(\mathbf{Q}_p(1)) \otimes_{\mathbf{Q}_p} L = L.$$

(Here one identifies $V_{\mathrm{dR}}(f,g,h)$ with the tensor product of $D_{\mathrm{dR}}(V(f))$, $D_{\mathrm{cris}}(V(g))$ and $D_{\mathrm{cris}}(V(h))$ under the natural isomorphism.) Define the *$\alpha\alpha$-logarithm*

$$\log_{\alpha\alpha} = \big\langle \log_p(\cdot), \omega_f \otimes \eta_{g_\alpha} \otimes \eta_{h_\alpha} \big\rangle_{fgh} : H^1_{\mathrm{fin}}(\mathbf{Q}_p, V(f,g,h)) \longrightarrow L.$$

to be the composition of the Bloch–Kato p-adic logarithm with evaluation on the class $\omega_f \otimes \eta_{g_\alpha} \otimes \eta_{h_\alpha}$ in $\mathrm{Fil}^0 V_{\mathrm{dR}}(f,g,h)$ under the duality $\langle\cdot,\cdot\rangle_{fgh}$. If κ is a global Selmer class in $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$, we often write $\log_{\alpha\alpha}(\kappa)$ as a shorthand for $\log_{\alpha\alpha}(\mathrm{res}_p(\kappa))$.

3.2. Diagonal classes. — Following [BSV20, Section 7.2] define (cf. Section 2.1)

$$\mathscr{F}^2 V(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha) = \left[\sum_{p+q+r=2} \mathscr{F}^p V(\boldsymbol{f}) \hat{\otimes}_L \mathscr{F}^q V(\boldsymbol{g}_\alpha) \hat{\otimes}_L \mathscr{F}^r V(\boldsymbol{h}_\alpha)\right] \otimes_{\mathscr{O}_{\boldsymbol{fgh}}} \Xi_{\boldsymbol{fgh}},$$

where for $\boldsymbol{\xi} = \boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha$ one sets $\mathscr{F}^i V(\boldsymbol{\xi}) = V(\boldsymbol{\xi})$ for $i \leqslant 0$, $\mathscr{F}^1 V(\boldsymbol{\xi}) = V(\boldsymbol{\xi})^+$ and $\mathscr{F}^j V(\boldsymbol{\xi}) = 0$ for $j \geqslant 2$. It is an $\mathscr{O}_{\boldsymbol{fgh}}[G_{\mathbf{Q}_p}]$-submodule of $V(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)$, free of rank four over $\mathscr{O}_{\boldsymbol{fgh}}$. We call the image of the injective natural map

$$H^1(\mathbf{Q}_p, \mathscr{F}^2 V(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)) \longrightarrow H^1(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha))$$

the *balanced local condition*, and denote it by $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha))$. The *balanced Selmer group* $H^1_{\mathrm{bal}}(\mathbf{Q}, V(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha))$ is the module of global cohomology classes in $H^1(\mathbf{Q}, V(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha))$ which are unramified at every prime $\ell \neq p$ and whose restriction at p belongs to the balanced local condition. For each classical triple $w=(k,l,m)$ in $U_{\boldsymbol{f}} \times U_{\boldsymbol{g}} \times U_{\boldsymbol{h}} \cap \mathbf{Z}^3_{\geqslant 2}$, one defines similarly the balanced local condition $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V_w)$, where $V_w = V(\boldsymbol{f}_k, \boldsymbol{g}_{\alpha,l}, \boldsymbol{h}_{\alpha,m})$ is the self-dual Tate twist of the tensor product of the homological Deligne representations $V(\boldsymbol{\xi}_u)$ of $\boldsymbol{\xi}_u = \boldsymbol{f}_k, \boldsymbol{g}_{\alpha,l}, \boldsymbol{h}_{\alpha,m}$. If w is balanced (id est $k < l+m$, $l < k+m$ and $m < k+l$), then $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V_w)$ equals the Bloch–Kato finite subspace of $H^1(\mathbf{Q}_p, V_w)$ (cf. [BSV20, Lemma 7.2]). The work of Perrin-Riou et alii yields a big logarithm map

$$\mathscr{L}_{\boldsymbol{f}} : H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)) \longrightarrow \mathscr{O}_{\boldsymbol{fgh}},$$

satisfying the following interpolation property. Let $\mathfrak{Z}$ be a local balanced class in $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha))$, and let $w=(k,l,m)$ be a balanced classical triple. Denote by $\mathfrak{Z}_w$ in $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V_w)$ the image of $\mathfrak{Z}$ under the map induced in cohomology by the specialisation isomorphism $\rho_w : V(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha) \otimes_w L \simeq V_w$ (the latter being defined as the tensor product of the specialisation isomorphisms $\rho_u : V(\boldsymbol{\xi}) \otimes_u L \simeq V(\boldsymbol{\xi}_u)$, for $\boldsymbol{\xi}_u = \boldsymbol{f}_k, \boldsymbol{g}_{\alpha,l}, \boldsymbol{h}_{\alpha,m}$, cf. Section 2.1). Set $c_w = (k+l+m-2)/2$, $\alpha_k = a_p(\boldsymbol{f})(k)$, $\alpha_l = a_p(\boldsymbol{g}_\alpha)(l)$, $\alpha_m = a_p(\boldsymbol{h}_\alpha)(m)$, and define β_ξ by the identities $\alpha_k \cdot \beta_k = p^{k-1}$, $\alpha_l \cdot \beta_l = \chi_g(p) \cdot p^{l-1}$ and $\alpha_m \cdot \beta_m = \chi_h(p) \cdot p^{m-1}$. Then one has

$$\mathscr{L}_{\boldsymbol{f}}(\mathfrak{Z})(w) = \frac{(-1)^{c_w-k}}{(c_w-k)!} \cdot \frac{\left(1 - \frac{\beta_k \alpha_l \alpha_m}{p^{c_w}}\right)}{\left(1 - \frac{\alpha_k \beta_l \beta_m}{p^{c_w}}\right)} \cdot \left\langle \log_p(\mathfrak{Z}_w), \mho_u \right\rangle_w,$$

where $\log_p$ is the Bloch–Kato logarithm map, $\mho_w$ in $\mathrm{Fil}^0 D_{\mathrm{dR}}(V_w)$ denotes the differential $\eta_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_{\alpha,l}} \otimes \omega_{\boldsymbol{h}_{\alpha,m}}$ (defined similarly as in Section 3.1), and the pairing $\langle\cdot,\cdot\rangle_w : D_{\mathrm{dR}}(V_w)/\mathrm{Fil}^0 \otimes_L \mathrm{Fil}^0 D_{\mathrm{dR}}(V_w) \longrightarrow L$ is the one induced by the specialisation at w_o of the perfect duality $\pi_{\boldsymbol{fgh}}$ (cf. Equation (12)). We refer to Proposition 7.3 of [BSV20] for a proof of the existence of $\mathscr{L}_{\boldsymbol{f}}$.

Theorem A of [BSV20] constructs a canonical *big balanced diagonal class*

$$\kappa(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha) \in H^1_{\mathrm{bal}}(\mathbf{Q}, V(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha))$$

such that

$$\mathscr{L}_{\boldsymbol{f}}\big(\mathrm{res}_p\big(\kappa(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)\big)\big) = \mathscr{L}_p^{\alpha\alpha}(A,\varrho). \tag{28}$$

One defines the *(balanced) diagonal class*

$$\kappa(f,g_\alpha,h_\alpha) \in H^1(\mathbf{Q},V(f,g,h))$$

to be the image of $\kappa(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)$ under the map induced in cohomology by the specialisation isomorphism ρ_{w_o} defined in Equation (13). Note that w_o lies outside the balanced region, hence the class $\kappa(f,g_\alpha,h_\alpha)$ is not necessarily crystalline at p. Indeed, under the current assumption (27), it follows from the explicit reciprocity law (28) and Perrin-Riou's reciprocity law for big dual exponentials that $\kappa(f,g_\alpha,h_\alpha)$ is crystalline at p (hence a Selmer class) precisely if the the complex Garrett L-function $L(A,\varrho,s) = L(f\otimes g\otimes h,s)$ vanishes at the central point $s=1$. (Cf. Theorem B of [**BSV20**], proved in the present setting in Section 9.1 of loco citato.) In this case

$$\log_{\alpha\alpha}(\kappa(f,g_\alpha,h_\alpha)) = 0, \tag{29}$$

as follows from the fact that $\kappa(f,g_\alpha,h_\alpha)$ is by construction (the specialisation at w_o of) a balanced class (cf. the discussion following Diagram (193) of [**BSV20**]).

When $L(f\otimes g\otimes h,s)$ vanishes at $s=1$, the following proposition relates the linear form $\langle\!\langle \kappa(f,g_\alpha h_\alpha),\cdot\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ on $\mathrm{Sel}(\mathbf{Q},V(f,g,h))$ and the derivative of $\mathscr{L}_p^{\alpha\alpha}(A,\varrho)$.

Theorem 3.2. — *Assume that the complex Garrett L-function $L(A,\varrho,s)$ vanishes at $s=1$, so that $\kappa(f,g_\alpha,h_\alpha)$ is a Selmer class. Then*

$$\frac{\left(1-\frac{\alpha_g\alpha_h}{\alpha_f}\right)}{\left(1-\frac{\alpha_f}{p\alpha_g\alpha_h}\right)}\cdot \langle\!\langle \kappa(f,g_\alpha,h_\alpha),\cdot\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} = \log_{\alpha\alpha}\big(\mathrm{res}_p(\cdot)\big)\cdot\mathscr{L}_p^{\alpha\alpha}(A,\varrho) \pmod{\mathscr{I}^2}$$

as $\mathscr{I}/\mathscr{I}^2$-valued linear maps on the Selmer group $\mathrm{Sel}(\mathbf{Q},V(f,g,h))$.

Theorem 3.2 is proved in Section 3.4 below.

Remark 3.3. — The construction of the class $\kappa(\boldsymbol{f},\boldsymbol{g}_\alpha,\boldsymbol{h}_\alpha)$ and the proof of the reciprocity law (28) given in [**BSV20**] work also when the assumption (27) is not satisfied, id est if A has multiplicative reduction at p and α_f equals either $\alpha_g\cdot\alpha_h$ or $\beta_g\cdot\alpha_h$. (Since g is p-regular by an assumption of Section 1, one has $\alpha_g\cdot\alpha_h\neq\beta_g\cdot\alpha_h$.) Assume that $\alpha_f=\alpha_g\cdot\alpha_h$ and that $L(A,\varrho,s)$ vanishes at $s=1$, so that $\kappa(f,g_\alpha,h_\alpha)$ is crystalline at p by Theorem B of [**BSV20**]. Let q and q' be generators of $\mathcal{Q}_p(A,\varrho)$. For Selmer classes x and y in $\mathrm{Sel}(\mathbf{Q},V(f,g,h))$, denote by $\tilde{h}_p^{\alpha\alpha}(x\otimes y)$ the square-root of the discriminant of $\langle\!\langle\cdot,\cdot\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ computed on the $\mathbf{Q}(\varrho)$-submodule of $\tilde{H}^1_f(\mathbf{Q},V(f,g,h))$ generated by x, y, q and q'. The article [**BSV21a**] proves the equality

$$\tilde{h}_p^{\alpha\alpha}\big(\kappa(f,g_\alpha,h_\alpha)\otimes y\big) = \log_{\alpha\alpha}\big(\mathrm{res}_p(y)\big)\cdot\mathscr{L}_p^{\alpha\alpha}(A,\varrho) \pmod{\mathscr{I}^3}$$

in $(\mathscr{I}^2/\mathscr{I}^3)/\mathbf{Q}(\varrho)^*$ for each Selmer class y.

3.3. Perrin-Riou conjecture for diagonal classes. — Recall the map

$$\gamma_{gh} : A(K_\varrho)^\varrho \otimes_{\mathbf{Q}(\varrho)} L \hookrightarrow \mathrm{Sel}(\mathbf{Q}, V(f,g,h))$$

defined in Equation (22), arising from the Kummer map on $A(K_\varrho)$ and the isomorphisms γ_g and γ_h fixed in (4). Assume that $A(K_\varrho)^\varrho$ has dimension 2 over $\mathbf{Q}(\varrho)$. The classical Birch and Swinnerton-Dyer conjecture predicts that the Shafarevich–Tate group of A over K_ϱ is finite, hence that γ_{gh} is an isomorphism. In this case, if (P,Q) is a $\mathbf{Q}(\varrho)$-basis of $A(K_\varrho)^\varrho$, one has $\kappa(f, g_\alpha, h_\alpha) = a \cdot \gamma_{gh}(P) + b \cdot \gamma_{gh}(Q)$ with a and b in L. After setting

$$\mathscr{E} = \left(1 - \frac{\alpha_g \alpha_h}{\alpha_f}\right) \cdot \left(1 - \frac{\alpha_f}{p\alpha_g\alpha_h}\right)^{-1},$$

Theorem 3.2 and Proposition 2.1 yield the identities

$$\mathscr{E} \cdot a \cdot \langle\!\langle P, Q\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} = \log_{\alpha\alpha}(\gamma_{gh}(Q)) \cdot \mathscr{L}_p^{\alpha\alpha}(A,\varrho) \pmod{\mathscr{I}^2}$$

and

$$-\mathscr{E} \cdot b \cdot \langle\!\langle P, Q\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} = \log_{\alpha\alpha}(\gamma_{gh}(P)) \cdot \mathscr{L}_p^{\alpha\alpha}(A,\varrho) \pmod{\mathscr{I}^2}$$

Moreover, Conjecture 1.1 predicts that $\langle\!\langle P, Q\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ and $\mathscr{L}_p^{\alpha\alpha}(A,\varrho) \pmod{\mathscr{I}^2}$ are non-zero, and equal up to multiplication by a non-zero algebraic scalar in $\mathbf{Q}(\varrho)^*$. To sum up, when $\dim_{\mathbf{Q}(\varrho)} A(K_\varrho)^\varrho = 2$, one expects that $\kappa(f, g_\alpha, h_\alpha)$ is equal to $\log_{\alpha\alpha}(\gamma_{gh}(Q)) \cdot \gamma_{gh}(P) - \log_{\alpha\alpha}(\gamma_{gh}(P)) \cdot \gamma_{gh}(Q)$ up to multiplication by a non-zero scalar in $\mathbf{Q}(\varrho)^*$. When $\dim_{\mathbf{Q}(\varrho)} A(K_\varrho)^\varrho > 2$, Conjecture 1.1 predicts that $\mathscr{L}_p^{\alpha\alpha}(A,\varrho)$ belongs to $\mathscr{I}^2$ and that $\langle\!\langle \cdot, \cdot\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ is non-degenerate, hence that $\kappa(f, g_\alpha, h_\alpha)$ is zero by Theorem 3.2 and the conjectural finiteness of the relevant Shafarevich–Tate group. In light of the above discussion, the following conjecture is a direct consequence of Conjecture 1.1, the conjectural finiteness of the p-primary part of the ϱ-component of the Shafarevich–Tate group of A over K_ϱ, and Theorem 3.2.

Conjecture 3.4. —

1. *Assume that the $\mathbf{Q}(\varrho)$-vector space $A(K_\varrho)^\varrho$ has dimension 2. Then, for each $\mathbf{Q}(\varrho)$-basis (P,Q) of $A(K_\varrho)^\varrho$, the equality*

   $$\kappa(f, g_\alpha, h_\alpha) = \log_{\alpha\alpha}(\gamma_{gh}(Q)) \cdot \gamma_{gh}(P) - \log_{\alpha\alpha}(\gamma_{gh}(P)) \cdot \gamma_{gh}(Q)$$

   holds in the Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$ up to multiplication by a non-zero element of $\mathbf{Q}(\varrho)^*$.
2. *If $A(K_\varrho)^\varrho$ has dimension greater than 2 over $\mathbf{Q}(\varrho)$, then the diagonal class $\kappa(f, g_\alpha, h_\alpha)$ is equal to zero.*

Remarks 3.5. —

1. The equality displayed in Part 1 of Conjecture 3.4 is independent of the choice of the isomorphisms γ_g and γ_h fixed in Equation (4).
2. Assume that both $r_{\mathrm{MW}} = \dim_{\mathbf{Q}(\varrho)} A(K_\varrho)^\varrho$ and $r_{\mathrm{S}} = \dim_L \mathrm{Sel}(\mathbf{Q}, V(f,g,h))$ are equal to 2, and let (P,Q) be a $\mathbf{Q}(\varrho)$-basis of $A(K_\varrho)^\varrho$. If $\log_{\alpha\alpha}$ is not identically zero on (the image under res_p of) $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$, then Equation (29) implies

$$\kappa(f, g_\alpha, h_\alpha) = \lambda \cdot \big(\log_{\alpha\alpha}(\gamma_{gh}(Q)) \cdot \gamma_{gh}(P) - \log_{\alpha\alpha}(\gamma_{gh}(P)) \cdot \gamma_{gh}(Q)\big) \tag{30}$$

for some constant λ in L. In this case, the actual content of Conjecture 3.4 is then the non-vanishing and rationality statement λ *belongs to* $\mathbf{Q}(\varrho)^*$.

3. Assume $r_{\mathrm{MW}} = r_{\mathrm{S}} = 2$ and that $\log_{\alpha\alpha}$ is not identically zero on the Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$. Fix a $\mathbf{Q}(\varrho)$-basis (P,Q) of $A(K_\varrho)^\varrho$. Equation (30), Proposition 2.1 and Theorem 3.2 and the non-triviality of $\log_{\alpha\alpha}$ give the identity
$$\mathscr{L}_p^{\alpha\alpha}(A,\varrho) \pmod{\mathscr{I}^2} = \lambda \cdot \langle\!\langle P, Q\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$$
in $(\mathscr{I}/\mathscr{I}^2)/\mathbf{Q}(\varrho)^*$. According to Proposition 2.1 and the current assumption (27) (which implies $A(K_\varrho)^\varrho = A^\dagger(K_\varrho)^\varrho$), the square of $\langle\!\langle P, Q\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ equals the regulator $R_p^{\alpha\alpha}(A,\varrho)$, hence the previous equation yields the equality
$$L_p^{\alpha\alpha}(A,\varrho) \pmod{\mathscr{I}^3} = \lambda^2 \cdot R_p^{\alpha\alpha}(A,\varrho)$$
in $(\mathscr{I}^2/\mathscr{I}^3)/\mathbf{Q}(\varrho)^{*2}$. As a consequence Conjecture 3.4, namely the statement λ belongs to $\mathbf{Q}(\varrho)^*$, and the non-degeneracy of $\langle\!\langle \cdot, \cdot\rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ on the Mordell–Weil group $A(K_\varrho)^\varrho$, is equivalent to Conjecture 1.1.
4. Since by assumption the forms g and h are p-regular (cf. Section 1), one can actually consider the four diagonal classes $\kappa(f, g_\alpha, h_\alpha)$ $\kappa(f, g_\alpha, h_\beta)$, $\kappa(f, g_\beta, h_\alpha)$ and $\kappa(f, g_\beta, h_\beta)$ arising from the different choices of the roots of the pth Hecke polynomials of g and h. Conjecture 3.4, combined with standard conjectures, predicts that these classes generate a non-trivial submodule of $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$ precisely when $r_{\mathrm{MW}} = 2$. Assuming $r_{\mathrm{MW}} = 2$, one has that res_p is not identically zero on $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$, hence one of the logarithms $\log_{\alpha\alpha}$, $\log_{\alpha\beta}$, $\log_{\beta\alpha}$ and $\log_{\beta\beta}$ (defined similarly as in Section 3.1) is not identically zero on $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$. Reordering the roots (α_g, β_g) and (α_h, β_h) if necessary, one can assume that $\log_{\alpha\alpha}$ is not identically zero. It follows from Conjecture 3.4 that the class $\kappa(f, g_\alpha, h_\alpha)$ is non-zero. Conversely, assume that $\kappa(f, g_\alpha, h_\alpha)$ is non-zero. According to the parity conjecture and the conjectural finiteness of the p-primary part of the ϱ-component of the Shafarevich–Tate group of A over K_ϱ one has that $r_{\mathrm{MW}} \geqslant 2$. Conjecture 3.4 implies the equality $r_{\mathrm{MW}} = 2$.
5. Conjecture 3.4 is a reformulation of [**DR16**, Conjecture 3.12], which (together with Conjecture 2.1 of loc. cit.) is a refinement of the Elliptic Stark Conjecture formulated in [**DLR15**] (cf. Proposition 3.13 and Remark 3.14 of [**DR16**]). The above discussion then gives a conceptual explanation of the conjectures formulated in [**DLR15, DR16**] in the framework of the p-adic analogues of the Birch and Swinnerton-Dyer conjecture.
6. Assume in this remark that (A,ϱ) is exceptional at p. When $\alpha_f = \alpha_g \cdot \alpha_h$ we expect that Conjecture 3.4 holds verbatim in light of Remark 3.3. By contrast, if $\alpha_f = \beta_g \cdot \alpha_h$, then the specialisation $\kappa(f, g_\alpha, h_\alpha)$ of $\kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ at $w_o = (2,1,1)$ is equal to zero, independently on whether $L(f \otimes g \otimes h, s)$ vanishes or not at $s = 1$. In this case, we expect that Conjecture 3.4 holds after replacing $\kappa(f, g_\alpha, h_\alpha)$ with the *improved diagonal class* $\kappa^*(f, g_\alpha, h_\alpha)$ defined in Section 1.2 of [**BSV20**] (cf. Theorem B of loco citato).

3.4. Proof of Theorem 3.2. — This section proves Theorem 3.2.

Under the running assumption (27), the module $H^0(\mathbf{Q}_p, V(f,g,h)^-)$ is equal to zero and we identify the Block–Kato Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$ with Nekovář's extended Selmer group $\tilde{H}^1_f(\mathbf{Q}, V(f,g,h))$ under the isomorphism (18). Fix a 1-cocycle

$$\tilde{z} = (z, z^+, a) \in \tilde{\mathrm{C}}^1_f(G_{Np}, V(f,g,h))$$

which represents the diagonal class $\kappa(f, g_\alpha, h_\alpha)$ in $\tilde{H}^1_f(\mathbf{Q}, V(f,g,h))$. Then

$$z \in \mathrm{C}^1_{\mathrm{cont}}(G_{Np}, V(f,g,h)), \quad z^+ \in \mathrm{C}^1_{\mathrm{cont}}(\mathbf{Q}_p, V(f,g,h)^+)$$

and

$$a = (a_v)_{v|Np} \in \bigoplus_{v|Np} V(f,g,h)$$

satisfy the relations

$$dz = 0, \quad \kappa(f, g_\alpha, h_\alpha) = \mathrm{cl}(z), \quad dz^+ = 0 \ \text{ and } \ \mathrm{res}_{Np}(z) = i^+(z^+) - da,$$

where d denotes the differentials of the complexes $\mathrm{C}^\bullet_{\mathrm{cont}}$ and $\mathrm{cl}(\cdot)$ denotes the cohomology class represented by $\cdot$. Let

$$Z \in \mathrm{C}^1_{\mathrm{cont}}(G_{Np}, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha))$$

be a 1-cocycle representing $\kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ and specialising to z at w_o:

$$dZ = 0, \quad \kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha) = \mathrm{cl}(Z) \quad \text{and} \quad \rho_{w_o}(Z) = z$$

(cf. Equation (14)). The 1-cocycle $\tilde{z}$ is then lifted by a 1-cochain of the form

$$\tilde{Z} = (Z, Z^+, A) \in \tilde{\mathrm{C}}^1_f(G_{Np}, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha))$$

under the morphism of complexes

$$\rho_{w_o} : \tilde{\mathrm{C}}^\bullet_f(G_{Np}, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)) \longrightarrow \tilde{\mathrm{C}}^\bullet_f(G_{Np}, V(f,g,h))$$

induced by ρ_{w_o} (cf. Equation (15)), where the cochains

$$Z^+ \in \mathrm{C}^1_{\mathrm{cont}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)) \quad \text{and} \quad A = (A_v)_{v|Np} \in \bigoplus_{v|Np} V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$$

are lifts of z^+ and a respectively under the map induced by ρ_{w_o}. As $\tilde{z}$ is a 1-cocycle, the differential $d\tilde{Z}$ of $\tilde{Z}$ in $\tilde{\mathrm{C}}^2_f(G_{Np}, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha))$ can be written as

$$d\tilde{Z} = (\boldsymbol{k}-2)\cdot \tilde{Z}_{\boldsymbol{k}} + (\boldsymbol{l}-1)\cdot \tilde{Z}_{\boldsymbol{l}} + (\boldsymbol{m}-1)\cdot \tilde{Z}_{\boldsymbol{m}} \tag{31}$$

with 2-cochains $\tilde{Z}_\cdot$ (for $\cdot = \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}$) in $\tilde{\mathrm{C}}^2_f(G_{Np}, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha))$ of the form

$$\tilde{Z}_\cdot = (Z_\cdot, Z_\cdot^+, W_\cdot), \tag{32}$$

where the 1-cochains $W_\cdot = (W_{\cdot,v})_{v|Np}$ in $\bigoplus_{v|Np} \mathrm{C}^1_{\mathrm{cont}}(\mathbf{Q}_v, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha))$ satisfy

$$(\boldsymbol{k}-2)\cdot W_{\boldsymbol{k}} + (\boldsymbol{l}-1)\cdot W_{\boldsymbol{l}} + (\boldsymbol{m}-1)\cdot W_{\boldsymbol{m}} = i^+(Z^+) - \mathrm{res}_{Np}(Z) - dA. \tag{33}$$

A slight extension of [Ven16a, Lemma 5.5] (cf. [Ven16b, Appendix C]) proves that

$$\tilde{z}_\cdot = \rho_{w_o}(\tilde{Z}_\cdot)$$

are 2-cocycles in $\tilde{\mathrm{C}}^2_f(G_{Np}, V(f,g,h))$ and (cf. Equation (26))

$$-\tilde{\beta}_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}(\kappa(f, g_\alpha, h_\alpha)) = (\boldsymbol{k}-2)\cdot \mathrm{cl}(\tilde{z}_{\boldsymbol{k}}) + (\boldsymbol{l}-1)\cdot \mathrm{cl}(\tilde{z}_{\boldsymbol{l}}) + (\boldsymbol{m}-1)\cdot \mathrm{cl}(\tilde{z}_{\boldsymbol{m}}). \tag{34}$$

For $V = V(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}), V(f, g, h)$, denote by

$$p^- : \mathrm{C}^\bullet_{\mathrm{cont}}(\mathbf{Q}_p, V) \longrightarrow \mathrm{C}^\bullet_{\mathrm{cont}}(\mathbf{Q}_p, V^-)$$

the morphism of complexes induced by the projection $p^- : V \longrightarrow V^-$. Define

$$(35)\qquad \begin{aligned} X_\cdot &= p^-(W_{\cdot,p}) \in \mathrm{C}^1_{\mathrm{cont}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)^-); \\ x_\cdot &= \rho_{w_o}(X_\cdot) = p^- \circ \rho_{w_o}(W_{\cdot,p}) \in \mathrm{C}^1_{\mathrm{cont}}(\mathbf{Q}_p, V(f, g, h)^-). \end{aligned}$$

After setting $A_p^- = p^-(A_p)$, Equation (33) yields

$$(36)\qquad (\boldsymbol{k} - 2) \cdot X_{\boldsymbol{k}} + (\boldsymbol{l} - 1) \cdot X_{\boldsymbol{l}} + (\boldsymbol{m} - 1) \cdot X_{\boldsymbol{m}} = -p^-\big(\mathrm{res}_p(Z)\big) - dA_p^-.$$

As Z is a 1-*cocycle*, this implies that the 1-cochains $x_\cdot$ are 1-cocycles, and one sets

$$\mathfrak{x}_\cdot = \mathrm{cl}(x_\cdot) \in H^1(\mathbf{Q}_p, V(f, g, h)^-).$$

Similarly, as Z is a 1-cocycle, Equations (31) and (32) imply that $\rho_{w_o}(Z_\cdot) = 0$, hence

$$\tilde{z}_\cdot = \big(0, \rho_{w_o}(Z_\cdot^+), \rho_{w_o}(W_\cdot)\big).$$

Because $\mathrm{C}^\bullet_{\mathrm{cont}}(\mathbf{Q}_v, V(f, g, h))$ is acyclic for $v \neq p$, this implies

$$(37)\qquad \mathrm{cl}(\tilde{z}_\cdot) = \jmath(\mathfrak{x}_\cdot)$$

(cf. Equations (17) and (35)). After recalling the definition of Garrett–Nekovář p-adic height $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}$ given in Section 2.5, Equations (34) and (37) yield

$$(38)\qquad \begin{aligned} \langle\!\langle \kappa(f, g_\alpha, h_\alpha), \cdot \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} &= \big\langle \tilde{\beta}_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha}(\kappa(f, g_\alpha, h_\alpha)), \cdot \big\rangle_{\mathrm{Nek}} \otimes \mathscr{I}/\mathscr{I}^2 \\ &= -\sum_{\boldsymbol{u}} \langle \jmath(\mathfrak{x}_{\boldsymbol{u}}), \cdot \rangle_{\mathrm{Nek}} \cdot (\boldsymbol{u} - u_o) \\ &= -\sum_{\boldsymbol{u}} \langle \mathfrak{x}_{\boldsymbol{u}}, \cdot^+ \rangle_{\mathrm{Tate}} \cdot (\boldsymbol{u} - u_o), \end{aligned}$$

where $(\boldsymbol{u}, u_o)$ denotes one of the pairs $(\boldsymbol{k}, 2)$, $(\boldsymbol{l}, 1)$ and $(\boldsymbol{m}, 1)$, where

$$\langle \cdot, \cdot \rangle_{\mathrm{Tate}} : H^1(\mathbf{Q}_p, V(f, g, h)^-) \otimes_L H^1(\mathbf{Q}_p, V(f, g, h)^+) \longrightarrow L$$

is the local Tate duality induced by the perfect pairing π_{fgh} (cf. Section 2.1), and where $\cdot^+$ is the morphism introduced in Equation (20). The last equality in Equation (38) follows from the adjointness of the maps $\jmath$ and $\cdot^+$ with respect to the pairings $\langle \cdot, \cdot \rangle_{\mathrm{Nek}}$ and $\langle \cdot, \cdot \rangle_{\mathrm{Tate}}$ (cf. Lemma 5.7 of [**Ven16a**].)

To conclude the proof we need the following lemma. Set

$$V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)_f = V(\boldsymbol{f})^- \hat{\otimes}_L V(\boldsymbol{g}_\alpha)^+ \hat{\otimes}_L V(\boldsymbol{h}_\alpha)^+ \otimes_{\mathscr{O}_{\boldsymbol{f}\boldsymbol{g}\boldsymbol{h}}} \Xi_{\boldsymbol{f}\boldsymbol{g}\boldsymbol{h}}.$$

The projection $p^- : V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha) \longrightarrow V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)^-$ maps $\mathscr{F}^2 V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$ onto $V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)_f$, hence induces in cohomology a morphism

$$(39)\qquad p_f : H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)) \longrightarrow H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)_f).$$

(Recall that the natural map $H^1(\mathbf{Q}_p, \mathscr{F}^2 V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)) \longrightarrow H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha))$ is injective, hence identifies its source with $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha))$.)

Lemma 3.6. — *There exist* $\mathfrak{Y}_{\boldsymbol{k}}, \mathfrak{Y}_{\boldsymbol{l}}$ *and* $\mathfrak{Y}_{\boldsymbol{m}}$ *in* $H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)_f)$ *such that*

$$p_f(\mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha))) = (\boldsymbol{k}-2)\cdot\mathfrak{Y}_{\boldsymbol{k}} + (\boldsymbol{l}-1)\cdot\mathfrak{Y}_{\boldsymbol{l}} + (\boldsymbol{m}-1)\cdot\mathfrak{Y}_{\boldsymbol{m}}.$$

Moreover, if the previous equation is satisfied, then for $\boldsymbol{u} = \boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}$ *one has*

$$\mathfrak{x}_{\boldsymbol{u}} = -\rho_{w_o}(\mathfrak{Y}_{\boldsymbol{u}}).$$

Proof. — Set $V(f)^-_{\beta\beta} = V(f) \otimes_{\mathbf{Q}_p} V(g)_\beta \otimes_L V(h)_\beta$. It is an $L[G_{\mathbf{Q}_p}]$-direct summand of $V(f,g,h)^-$, and the specialisation map ρ_{w_o} induces an isomorphism

$$\rho_{w_o} : V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)_f \otimes_{w_o} \mathscr{O}_{\boldsymbol{fgh}} \simeq V(f)^-_{\beta\beta}.$$

Since the kernel of evaluation at w_o on $\mathscr{O}_{\boldsymbol{fgh}}$ is generated by a regular sequence and $H^2(\mathbf{Q}_p, V(f)^-_{\beta\beta})$ is equal to zero, the specialisation isomorphism ρ_{w_o} induces in cohomology an isomorphism (denoted by the same symbol)

$$\rho_{w_o} : H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)_f) \otimes_{w_o} L \simeq H^1(\mathbf{Q}_p, V(f)^-_{\beta\beta}). \tag{40}$$

As explained in Section 9.1 of [**BSV20**], the Bloch–Kato finite subspace of the local cohomology group $H^1(\mathbf{Q}_p, V(f,g,h))$ is equal to the kernel of

$$p^- : H^1(\mathbf{Q}_p, V(f,g,h)) \longrightarrow H^1(\mathbf{Q}_p, V(f,g,h)^-)$$

(cf. Section 9.1). Because $\kappa(f, g_\alpha, h_\alpha) = \rho_{w_o}(\kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha))$ is a Selmer class (under the current assumption $L(A, \varrho, 1) = 0$), it follows that the local class

$$\boldsymbol{\kappa}_f = p_f(\mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)))$$

belongs to the kernel of (40), thus proving the first statement.

Let $\mathfrak{Y}_{\boldsymbol{u}}$ in $H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)_f)$ be local classes satisfying

$$\boldsymbol{\kappa}_f = \sum_{\boldsymbol{u}} \mathfrak{Y}_{\boldsymbol{u}} \cdot (\boldsymbol{u} - u_o).$$

We prove that $\rho_{w_o}(\mathfrak{Y}_{\boldsymbol{u}})$ is equal to $-\mathfrak{x}_{\boldsymbol{u}}$ for $\boldsymbol{u} = \boldsymbol{k}$, the cases $\boldsymbol{u} = \boldsymbol{l}, \boldsymbol{m}$ being similar. Since by construction $\mathrm{cl}(Z) = \kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)$, according to Equation (36) one has

$$\mathrm{cl}\Big(\sum_{\boldsymbol{u}} X_{\boldsymbol{u}} \cdot (\boldsymbol{u} - u_o)\Big) = -\sum_{\boldsymbol{u}} i_f(\mathfrak{Y}_{\boldsymbol{u}}) \cdot (\boldsymbol{u} - u_o) \in H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)^-), \tag{41}$$

where i_f denotes both the inclusion $V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)_f \hookrightarrow V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)^-$ and the morphism it induces in cohomology. Let $\nu : \mathscr{O}_{\boldsymbol{fgh}} \longrightarrow \mathscr{O}_{\boldsymbol{f}}$ be the surjective morphism of rings sending the analytic function $F(\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m})$ to $F(\boldsymbol{k}, 1, 1)$, and set

$$V(\boldsymbol{f}, g, h)^- = V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)^- \otimes_\nu \mathscr{O}_{\boldsymbol{f}} \quad \text{and} \quad V(\boldsymbol{f})^-_{\beta\beta} = V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)_f \otimes_\nu \mathscr{O}_{\boldsymbol{f}}.$$

(Note that $V(\boldsymbol{f})^-_{\beta\beta} = V(\boldsymbol{f})^- \otimes_L V(g)_\beta \otimes_L V(h)_\beta \otimes_{\mathscr{O}_{\boldsymbol{f}}} \chi_{\mathrm{cyc}}^{1-\boldsymbol{k}/2}$ is an $\mathscr{O}_{\boldsymbol{f}}[G_{\mathbf{Q}_p}]$-direct summand of $V(\boldsymbol{f}, g, h)^-$ and $i_f \otimes_\nu \mathscr{O}_{\boldsymbol{f}}$ is the natural inclusion.) If one denotes by ν also the morphisms induced in cohomology (resp., on continuous cochains) by the projections $V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)^- \longrightarrow V(\boldsymbol{f}, g, h)^-$ and $V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)_f \longrightarrow V(\boldsymbol{f})^-_{\beta\beta}$, then $\nu(X_{\boldsymbol{k}})$ is a 1-cocycle in $\mathrm{C}^\bullet_{\mathrm{cont}}(\mathbf{Q}_p, V(\boldsymbol{f}, g, f)^-)$ (cf. Equation (36)) and Equation (41) gives

$$(\boldsymbol{k}-2)\cdot\big(\mathrm{cl}(\nu(X_{\boldsymbol{k}})) + \nu(\mathfrak{Y}_{\boldsymbol{k}})\big) = 0.$$

On the other hand, the $(\boldsymbol{k}-2)$-torsion of $H^1(\mathbf{Q}_p, V(\boldsymbol{f}, g, h)^-)$ is a quotient of $H^0(\mathbf{Q}_p, V(f, g, h)^-)$, which is zero by assumption (viz. (A, ϱ) is not exceptional at p). Then $\nu(\mathfrak{Y}_{\boldsymbol{k}}) = -\mathrm{cl}(\nu(X_{\boldsymbol{k}}))$, hence by construction $\rho_{w_o}(\mathfrak{Y}_{\boldsymbol{k}}) = -\mathfrak{x}_{\boldsymbol{k}}$. □

Let $\mathfrak{Y}_{\boldsymbol{u}}$ be as in the statement of Lemma 3.6, and let $\tilde{y}$ be an element of $\tilde{H}^1_f(\mathbf{Q}, V(f, g, h))$. Equation (38) and Lemma 3.6 give the identity

$$\langle\!\langle \kappa(f, g_\alpha, h_\alpha), \tilde{y} \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} = \sum_{\boldsymbol{u}} \big\langle \rho_{w_o}(\mathfrak{Y}_{\boldsymbol{u}}), \tilde{y}^+ \big\rangle_{\mathrm{Tate}} \cdot (\boldsymbol{u} - u_o). \tag{42}$$

If $\tilde{y} = \imath_{\mathrm{ur}}(y)$ corresponds to the Selmer class y in $\mathrm{Sel}(\mathbf{Q}_p, V(f, g, h))$, then the image of $\tilde{y}^+$ under the map induced in cohomology by the inclusion $V(f, g, h)^+ \hookrightarrow V(f, g, h)$ is equal to the restriction of y at p. In this case we claim that

$$\big\langle \rho_{w_o}(\mathfrak{Y}_{\boldsymbol{u}}), \tilde{y}^+ \big\rangle_{\mathrm{Tate}} = \log_{\alpha\alpha}(\mathrm{res}_p(y)) \cdot \big\langle \exp_p^*(\rho_{w_o}(\mathfrak{Y}_{\boldsymbol{u}})), \eta_f \otimes \omega_{g_\alpha} \otimes \omega_{h_\alpha} \big\rangle_{fgh}, \tag{43}$$

where $\exp_p^* : H^1(\mathbf{Q}_p, V(f, g, h)^-) \longrightarrow D_{\mathrm{dR}}(V(f, g, h)^-)$ is the Bloch–Kato dual exponential. Indeed, note that the projection $p^- : V(f, g, h) \twoheadrightarrow V(f, g, h)^-$ and the inclusion $i^+ : V(f, g, h)^+ \hookrightarrow V(f, g, h)$ induce natural isomorphisms

$$\mathrm{Fil}^0 V_{\mathrm{dR}}(f, g, h) \simeq D_{\mathrm{dR}}(V(f, g, h)^-) \quad \text{and} \quad D_{\mathrm{dR}}(V(f, g, h)^+) \simeq V_{\mathrm{dR}}(f, g, h)/\mathrm{Fil}^0,$$

which we consider as equalities. Moreover, since by assumption (A, ϱ) is not exceptional at p, the Bloch–Kato exponential map gives an isomorphism between $D_{\mathrm{dR}}(V(f, g, h)^+)$ and $H^1(\mathbf{Q}_p, V(f, g, h)^+)$. As $i^+(\tilde{y}^+) = \mathrm{res}_p(y)$, it follows that

$$\big\langle \rho_{w_o}(\mathfrak{Y}_{\boldsymbol{u}}), \tilde{y}^+ \big\rangle_{\mathrm{Tate}} = \big\langle \exp_p^*(\rho_{w_o}(\mathfrak{Y}_{\boldsymbol{u}})), \log_p(\mathrm{res}_p(y)) \big\rangle_{fgh}. \tag{44}$$

For (i, j) in $\{\alpha, \beta\}^2$ and $\cdot = \emptyset, \pm$, define

$$V(f)^{\cdot}_{ij} = V(f)^{\cdot} \otimes_{\mathbf{Q}_p} V(g)_i \otimes_L V(h)_j$$

(so that $V(f, g, h)^{\cdot}$ is the direct sum of the submodules $V(f)^{\cdot}_{ij}$). Then $\rho_{w_o}(\mathfrak{Y}_{\boldsymbol{u}})$ belongs to $H^1(\mathbf{Q}_p, V(f)^-_{\beta\beta})$ (cf. the proof of Lemma 3.6), hence the linear form

$$\big\langle \exp_p^*(\rho_{w_o}(\mathfrak{Y})_{\boldsymbol{u}}), \cdot \big\rangle_{fgh} : V_{\mathrm{dR}}(f, g, h)/\mathrm{Fil}^0 \longrightarrow L$$

factors through the map $\mathrm{pr}_{\alpha\alpha} : V_{\mathrm{dR}}(f, g, h)/\mathrm{Fil}^0 \longrightarrow D_{\mathrm{dR}}(V(f)_{\alpha\alpha})/\mathrm{Fil}^0$ induced by the projection $V(f, g, h) \twoheadrightarrow V(f)_{\alpha\alpha}$. Since by definition (cf. Section 3.1)

$$\mathrm{pr}_{\alpha\alpha}(\log_p(\mathrm{res}_p(y))) = \log_{\alpha\alpha}(\mathrm{res}_p(y)) \cdot \eta_f \otimes \omega_{g_\alpha} \otimes \omega_{h_\alpha}$$

the claim Equation (43) follows from Equation (44).

After setting

$$\exp_{\alpha\alpha}^*(\rho_{w_o}(\mathfrak{Y}_{\boldsymbol{u}})) = \big\langle \exp_p^*(\rho_{w_o}(\mathfrak{Y}_{\boldsymbol{u}})), \eta_f \otimes \omega_{g_\alpha} \otimes \omega_{h_\alpha} \big\rangle_{fgh},$$

Equations (42) and (43) prove the equality

$$\langle\!\langle \kappa(f, g_\alpha, h_\alpha), \cdot \rangle\!\rangle_{\boldsymbol{f}\boldsymbol{g}_\alpha\boldsymbol{h}_\alpha} = \log_{\alpha\alpha}(\mathrm{res}_p(\cdot)) \cdot \sum_{\boldsymbol{u}} \exp_{\alpha\alpha}^*(\rho_{w_o}(\mathfrak{Y})_{\boldsymbol{u}}) \cdot (\boldsymbol{u} - u_o) \tag{45}$$

of $\mathscr{I}/\mathscr{I}^2$-valued L-linear forms on the Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f, g, h))$.

By Proposition 7.3 of [BSV20], the Perrin-Riou logarithm $\mathscr{L}_{\boldsymbol{f}}$ introduced in Section 3.2 factors through the map p_f defined in Equation (39), and hence gives rise to a morphism (denoted again by the same symbol)

$$\mathscr{L}_{\boldsymbol{f}} : H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)_f) \longrightarrow \mathscr{O}_{\boldsymbol{fgh}}.$$

Moreover, for each local class $\mathfrak{Z}$ in $H^1(\mathbf{Q}_p, V(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)_f)$ one has (cf. loc. cit.)

$$\mathscr{L}_{\boldsymbol{f}}(\mathfrak{Z})(w_o) = \frac{\left(1 - \frac{\alpha_g \alpha_h}{\alpha_f}\right)}{\left(1 - \frac{\alpha_f}{p\alpha_g\alpha_h}\right)} \cdot \exp^*_{\alpha\alpha}(\rho_{w_o}(\mathfrak{Z})).$$

Applying $\mathscr{L}_{\boldsymbol{f}}$ to both sides of the identity

$$p_f\big(\mathrm{res}_p\big(\kappa(\boldsymbol{f}, \boldsymbol{g}_\alpha, \boldsymbol{h}_\alpha)\big)\big) = \sum_{\boldsymbol{u}} \mathfrak{Y}_{\boldsymbol{u}} \cdot (\boldsymbol{u} - u_o),$$

and using the explicit reciprocity law Equation (28), one then gets the identity

$$\mathscr{L}_p^{\alpha\alpha}(A, \varrho) \pmod{\mathscr{I}^2} = \frac{\left(1 - \frac{\alpha_g \alpha_h}{\alpha_f}\right)}{\left(1 - \frac{\alpha_f}{p\alpha_g\alpha_h}\right)} \cdot \sum_{\boldsymbol{u}} \exp^*_{\alpha\alpha}(\rho_{w_o}(\mathfrak{Y}_{\boldsymbol{u}})) \cdot (\boldsymbol{u} - u_o).$$

Theorem 3.2 follows from the previous equation and Equation (45).

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