RESEARCH SEMINAR ON CONDENSED MATHEMATICS

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This is a seminar aiming to introduce the new theory of condensed mathematics of Clausen and Scholze.

Introduction. Condensed mathematics offers a new way to develop a theory of analytic geometry, which unifies both the geometry of complex analytic spaces and the p-adic theories of rigid/Berkovich/adic spaces. Moreover it is better suited to face the topological issues that one encounters in these theories. More generally, it is a good theory to do algebra when there is a topology floating around.

A hurdle one encounters when attempting to do analytic geometry over a topological field K (e.g. $\mathbb{Q}_p$ or $\mathbb{C}$) is that one needs to work with various topological rings over K, such as rings of converging power series. Their modules do not a priori live in a nice abelian category, and so for instance techniques from homological algebra are sometimes unavailable. It is also hard to define a good notion of quasi-coherent sheaves. Indeed, if $A \to B$ is a morphism of topological K-algebras and if M is a topological A-module, one wants to form a base-change $B \widehat{\otimes}_A M$ but it is not always clear how to complete the tensor product, and even when it can be done the resulting construction does not behave well in general.

To explain how the theory of condensed mathematics tries to fix this, we first explain how one may view the construction of schemes in algebraic geometry. Namely, one must first start with a nice symmetric monoidal abelian category (namely the category `Ab` of abelian groups), then consider ring objects (which are just rings here), and given a ring R one can consider the module objects over it (which are just R-modules here). To any such ring R we can associate an affine scheme $\mathrm{Spec}(R)$, and to any R-module M we can associate a quasi-coherent sheaf. By gluing we then obtain general schemes and quasi-coherent sheaves over them.

When trying to do analytic geometry, the first step is already difficult: finding a nice symmetric monoidal abelian category which includes some/all topological abelian groups is not a priori obvious. Indeed the category of topological abelian groups is not an abelian category: for instance the identity map

$$(\mathbb{R}^{\mathrm{disc}}, +) \to (\mathbb{R}^{\mathrm{eucl}}, +)$$

has trivial kernel and cokernel, but is not an isomorphism. We want a framework which includes discrete abelian groups and groups such as $(\mathbb{R}^{\mathrm{eucl}}, +)$, so we need to construct a bigger category which is abelian and where the above map would have a non-trivial kernel or cokernel. The category of condensed abelian group will turn out to work for this, and more generally it will be a good fit for our purposes.

Once a nice theory of condensed mathematics is developed, one must try to use it to write down a suitable definition of analytic spaces and study it. This is also done by Clausen-Scholze ([4]), but it goes beyond what this seminar aims to cover.

Content of the seminar. The material will be taken from Scholze's lectures [3]. It can be broken into three segments. Talks 1-5 will be an introduction to condensed sets and condensed abelian groups. After defining the main objects of study (condensed sets/abelian groups/rings), the first main result will be that the category of condensed abelian groups is a nice abelian category satisfying many of Grothendieck's AB conditions [3, Theorem 2.2]. Moreover, for any compact Hausdorff space S, the sheaf cohomology on S of the constant sheaf $\mathbb{Z}$ agrees with the condensed cohomology with integral coefficients [3, Theorem 3.2]. We will also see that the embedding of locally compact abelian groups into condensed abelian groups extends to the derived level [3, Corollary 4.9].

Talks 6-9 will then study the notion of solid objects. We will introduce solid abelian groups which are in some sense analogues of complete objects (in the topological sense) in this world. The main result [3, Theorem 5.8] says that they form a nice abelian category, and moreover it provides a description of the derived category. After introducing the notion of analytic rings, we will investigate solid modules over analytic rings. The main result there is a construction of a lower shriek functor in this setting [3, Theorem 8.1] which as a consequence will allow us to define 'coherent cohomology with compact support' (in the affine case) [3, Theorem 8.2].

Talks 10-12 will globalise the results from Talk 8 in order to obtain a proof of coherent duality for schemes. In order to glue various categories of solid modules, one has to work at the derived level where the language of ∞-categories becomes necessary. The main technical result is [3, Theorem 9.8], which says that for a discrete adic space X, one can define a certain sheaf of ∞-categories on it. However, the proof itself doesn't use the formalism of ∞-categories that much. The important point to take from this result will be that one can then define a derived category of solid modules over the analytic ring $(\mathcal{O}_X, \mathcal{O}_X^+)_\blacksquare$ corresponding to X. In the final talk, we will use these constructions and tools to prove coherent duality [3, Theorem 11.1], and build a whole 6-functors formalism.

List of talks. We now give details on what each of the talks should cover. We will roughly follow along Scholze's lectures [3] at the rhythm of one talk per lecture. We will sometimes omit some of the material (in particular we will not focus much on the results over $\mathbb{R}$) and also sometimes add background material. The speakers are encouraged to contact one of the organisers if they have any questions.

1. Introduction, 15/04. Introduction and overview.

2. Condensed sets, 22/04. In this talk, we will introduce condensed sets and give some topological background. In more details:

(1) Recall briefly basic facts about profinite sets: how they are compact totally disconnected spaces, or equivalently inverse limits of finite discrete spaces [6, 08ZW].
(2) Give the definition of the pro-étale site of a point and of a condensed set/abelian group/ring, and spell out the notion of a sheaf in this context [3, Definition 1.2]. Discuss briefly the set theoretic issues[1] in [3, Remarks 1.3 & 1.4].
(3) Give in more details the example of the condensed set associated to any topological space [3, Example 1.5].
(4) Recall the definition of the Stone-Čech compactification and its universal property.
(5) Introduce compactly generated spaces and explain how first-countable spaces are compactly generated [3, Remark 1.6], see also [7]. State and prove [3, Proposition 1.7]. Conclude with [3, Example 1.9].

3. Condensed abelian groups, 06/05. In this talk, we will prove that the category of condensed abelian groups is an abelian category which satisfies all sorts of nice axioms. This will require to introduce a further topological notion, namely that of extremally disconnected spaces.

[1] We will mostly ignore such issues.

(1) Define extremally disconnected spaces and explain why the Stone-Čech compactification of a discrete space is an example of such a space [3, 2.4 & 2.5]. Show the equivalent definitions of condensed sets via sheaves on the compact Hausdorff site [3, 2.3] or on the site of extremally disconnected spaces [3, 2.7]
(2) Recall the notions of compact objects and generators in a category (include examples). Recall roughly what the Grothendieck AB axioms are (see [6, 079B]). State and prove [3, Theorem 2.2].
(3) Finish by explaining what the symmetric monoidal tensor product and the internal Hom are in this category. Explain why if T is a condensed set, then the corresponding condensed abelian group $\mathbb{Z}[T]$ is flat.

4. Condensed cohomology, 20/05. In this talk, we will investigate cohomology in the category of condensed abelian groups. More specifically, we show that for compact Hausdorff spaces, the condensed cohomology with integral coefficients agrees with sheaf cohomology.

(1) Briefly recall the notions of Grothendieck topologies, sheaves on a site and toposes, and recall what sheaf cohomology is in this context [6, 03NF, 00VL, 00X9, 01FT]. Also recall the three classical cohomology theories on compact Hausdorff spaces (singular cohomology, Čech cohomology and sheaf cohomology) and explain how they are related. State [3, Proposition 3.1] without proof.
(2) Explain in details how to construct a simplicial hypercover of a compact Hausdorff space using extremally disconnected spaces and how this helps compute the cohomology of sheaves on the compact Hausdorff site [5] (see also [1, Sections 1 & 2] for an introduction to simplicial methods, but there is no need here to explain what simplicial hypercovers are in general).
(3) State and prove [3, Theorem 3.2] which says that the condensed cohomology of the constant sheaf on $\mathbb{Z}$ agrees with sheaf cohomology on any compact Hausdorff space.
(4) State [3, Theorem 3.3] without proof, which computes the condensed cohomology of compact Hausdorff spaces with real coefficients.

5. Locally compact abelian groups, 27/05. In this talk, we will study locally compact abelian groups as condensed abelian groups. The aim of this talk is to compute $R\underline{\mathrm{Hom}}(A, B)$ for certain locally compact abelian groups A, B and to get the 'correct' result.

(1) Recall the notion of locally compact abelian group, with examples. State the structure theorem [3, Theorem 4.1].
(2) State and prove [3, Proposition 4.2]. State [3, 4.5 & 4.8] which give a resolution which will be useful for computations.
(3) State [3, Theorem 4.3]. Give the proof of part (i) and, if time permits, explain the idea of the proof of (ii).
(4) Explain what the Hoffman-Spitzweck bounded derived category of locally compact abelian groups is [2] (the notion of strict exactness should be explained, but no need to give general definitions in the setup of quasi-abelian categories). State and prove [3, Corollary 4.9].

6. Solid abelian groups I, 10/06. The next two talks will introduce the solid abelian groups[2]. Our aim will be to understand the statement of [3, Theorem 5.8], which roughly says that solid abelian groups form a nice abelian subcategory of condensed abelian groups, and that there is a nice description of the derived category.

[2]It might be useful for the two speakers to coordinate

(1) Give some motivation for solid abelian groups [4, p.9-10]. Then define solid abelian groups and solid complexes [3, 5.1-5.3]. Explain the relation to measures.
(2) State [3, Theorem 5.4] without proof (you can explain how to construct the basis if you want) and use it to explain the structure of the solid abelian group $\mathbb{Z}[S]^{\blacksquare}$ [3, 5.5-5.7].
(3) State [3, Theorem 5.8]. Explain what conditions need to be verified in order to show it by using [3, Lemmas 5.9-5.10] (it might be enlightening to look at the start of Lecture 6 in [3] to explicitly say what we want to apply these Lemmas to and what the conditions mean there). Give the proof of [3, Lemmas 5.10]. Time permitting, give an idea of the proof of [3, Lemmas 5.9].

7. Solid abelian groups II, 17/06. In this talk, we will explore the consequences of [3, Theorem 5.8] and give an idea of the proof.

(1) Sketch the proof of [3, Theorem 5.8]. It is too long to cover in full details. Focus on the main ideas and strategy. Then state and prove [3, Corollary 6.1].
(2) State and prove [3, 6.2-6.3] which describe the monoidal structure on solid abelian groups. Make sure to cover the examples in [3, Example 6.4].

8. Analytic rings, 24/06. In this talk, we will introduce the notion of analytic rings. The main aim is to construct examples of analytic rings.

(1) Define pre-analytic rings, give examples and define analytic rings [3, 7.1-7.4]. Then state and prove [3, Proposition 7.5] which describes the category of solid A-modules over an analytic ring A, its monoidal structure and the corresponding derived category.
(2) Explain the notions of morphisms of pre-analytic rings and morphisms of analytic rings. Then prove [3, Propositions 7.7 & 7.9]. Finish by explaining how to construct an analytic ring from the Huber pair $(\mathbb{Q}_p, \mathbb{Z}_p)$.[3]
(3) Time permitting, also talk about the criterion [3, 7.14-15] for a morphism between the underlying condensed rings to lift to a morphism of analytic rings.

9. Lower shriek functors for solid modules, 01/07. In this talk, we will introduce solid modules over an analytic ring and construct lower shriek functors.

(1) Recall the definition of pushforward with compact support $f_!$ and briefly mention the corresponding notion when when working with étale sheaves [6, 0F4W], including the fact that it equals pushforward for a proper morphism.
(2) State [3, Theorem 8.13 & Theorem 8.2]. Also explain the consequences [3, 8.3-8.5]. Then prove the theorems only in the case $R = \mathbb{Z}[T]$ as is done in the notes. The proof is long and can be sketched.

10. Discrete adic spaces, 08/07. In this talk, we will mostly provide background material on discrete adic spaces.

(1) Motivate the need to use adic spaces in order to globalise the results from the previous talk, define discrete Huber pairs, and explain how to associate an analytic ring to such a pair [3, 9.1].
(2) Explain the construction of the structure sheaf associated to a discrete Huber pair, and define discrete adic spaces [3, 9.2-9.5]. Explain the two different ways of embedding the category of schemes fully faithfully into

[3]Scholze says it can be done in general, and we will see how to do it for discrete Huber pairs in Talk 10

the category of adic spaces, and how these two functors are related [3, 9.6-9.7].

(3) Explain the need to work with derived categories and therefore with ∞-categories. Describe explicitly what the derived category of solid modules is without using the language of ∞-categories [3, p.70].

11. ∞-categories and globalisation, 15/07. The aim of this talk will be to prove [3, Theorem 9.8]. The ∞-category formalism only enters in one technical categorical result. The rest of the proof is more concrete, working with discrete adic spaces.

(1) Give a brief idea of what ∞-categories, the derived ∞-category, and sheaves of ∞-categories are.[4] State [3, Theorem 9.8]. Then state without proof [3, Proposition 10.5] and explain how we want to apply it to discrete adic spaces.
(2) Sketch the proof of the theorem [3, 10.1-10.4].

12. Coherent duality, 22/07. The aim of this talk will be to prove coherent duality for schemes.

(1) Recall the statements of Serre and Grothendieck duality. Also recall what the 6-functors formalism for schemes is and what its properties should be [3, p.72-73].
(2) State [3, Theorem 11.1] and explain how to construct the 6-functors in coherent duality, based on solid modules [3, 11.2-11.3].
(3) Prove [3, Theorem 11.1].

References

[1] B. Conrad, *Cohomological descent* , notes available at `https://math.stanford.edu/~conrad/papers/hypercover.pdf`.

[2] N. Hoffman and M. Spitzweck, *Homological algebra with locally compact abelian groups*, Adv. Math. **212** (2007), no. 2, 504-524.

[3] P. Scholze, *Lectures on Condensed Mathematics* , lecture notes available at `https://www.math.uni-bonn.de/people/scholze/Condensed.pdf`.

[4] P. Scholze, *Lectures on Analytic Geometry* , lecture notes available at `https://www.math.uni-bonn.de/people/scholze/Analytic.pdf`.

[5] F. Schremmer, *Condensed cohomology* , slides available at `http://www.mathematik.uni-regensburg.de/cisinski/SS20/talk-condensed-cohomology-slides.pdf?fbclid=IwAROD3OXseuwCOLOPhT3ubJbJSZMMymZ3N64Hi7SvfkoGvm2Zl5mrMUGBk9Y`

[6] The Stacks Project, `https://stacks.math.columbia.edu`.

[7] N. P. Strickland, *The Category of CGWH Spaces* , lecture notes available at `https://neil-strickland.staff.shef.ac.uk/courses/homotopy/cgwh.pdf`.

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[4]The speakers are encouraged to ask the organisers for advice/references