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Research seminar on VGIT: Talk 2 (26/10/23)

Divisors and the ample cone

1h30

I Divisors. [Ch.4 of [CLS11]]

Throughout, X is a normal irreducible variety over $\mathbb{C}$.

1 Weil divisors.

a In general.

Def: A prime divisor D on X is an irreducible closed subvariety of codim 1 in X.

- $\mathcal{O}_{X,D}$ is the ring of rational functions on X which are defined on an open of non-empty intersection with D.

It is a discrete valuation ring, with $\nu_D : \underbrace{\mathbb{C}(X)}_{\text{field of rat. functions on } X}{}^* \to \mathbb{Z}$ onto group morphism units s.t. $\nu_D(f+g) \geq \min(\nu_D(f), \nu_D(g))$.

Ex: $X = \mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[x])$, $D \leftrightarrow \langle x \rangle$ (the origin).

$$\mathcal{O}_{X,D} = \mathbb{C}[x]_{\langle x\rangle}, \quad \nu_D\Big(\underbrace{\tfrac{f}{g}}_{\in \mathbb{C}(x)^*}\Big) = p - q \text{ where } f = x^p f_1, g = x^q g_1, \; x \nmid f_1, x \nmid g_1.$$

More gen. if $X = \mathrm{Spec}(R)$ with R an integral domain and $D \leftrightarrow$ prime ideal p in R:

$\mathcal{O}_{X,D} = R_p$ (localization) is a local ring of maximal ideal pR_p.

$\nu_D(\overset{R_p}{f}) = 0 \Leftrightarrow f \in R_p \setminus pR_p$

$\nu_D(\underset{\in R_p}{f}) = n \Leftrightarrow f \in p^n R_p \setminus p^{n+1} R_p$

Extend ν_D to $\mathrm{Frac}(R_p) = \mathrm{Frac}(R)$ by $\nu_D\left(\frac{f}{g}\right) = \nu_D(f) - \nu_D(g)$.

Def:
- A Weil divisor on X is a $\mathbb{Z}$-linear combination of prime divisors on X.
  $\mathrm{Div}(X)$ is the group (for +) of Weil divisors on X (free abelian group on ↑)

- An effective Weil divisor D on X is a Weil divisor $D = \sum a_i D_i$ such that all $a_i \geq 0$.
  We denote $D \geq 0$.

Def: • A principal divisor is a divisor D such that there exists a non-zero rational function f on X such that $D = \mathrm{div}(f) := \underbrace{\sum_{D_i \text{ prime div.}} \nu_{D_i}(f) D_i}_{\text{automatically a finite sum}}$

$\mathrm{div}(f) = \mathrm{div}_0(f) - \mathrm{div}_\infty(f)$ with $\underbrace{\mathrm{div}_0(f)}_{\text{zeros}} \geq 0$ and $\underbrace{\mathrm{div}_\infty(f)}_{\text{poles}} \geq 0$

Ex: • $X = \mathbb{A}^2_{\mathbb{C}} = \mathrm{Spec}(\mathbb{C}[x,y])$, $\mathrm{div}\left(\frac{x^2}{y^3}\right) = 2D_{<x>} - 3D_{<y>}$.
• $X = \mathbb{P}^2_{\mathbb{C}} = \mathrm{Proj}(\mathbb{C}[x,y,z])$, $\mathrm{div}\left(\underbrace{\frac{x^2 z}{y^3}}_{\in (\text{degree } 0)}\right) = 2D_{<x>} - 3D_{<y>} + D_{<z>}$. ← (hom. rational fct of degree 0)

Def: • $\mathrm{Div}_0(X)$ is the group of principal divisors on X and $Cl(X) = \mathrm{Div}(X)/\mathrm{Div}_0(X)$ is the (divisor) class group of X.
• $D \sim E$ (linearly equivalent) if $D - E \in \mathrm{Div}_0(X)$.
(The class group of X is the group of classes for linear equivalence.)

Ex. • $Cl(\mathbb{A}^n_{\mathbb{C}}) = 0$.
• $Cl(\mathbb{P}^n_{\mathbb{C}}) \xrightarrow[\deg]{\sim} \mathbb{Z}$ with $\deg(\sum_i a_i D_i) = \sum_i a_i \underbrace{\deg(D_i)}$ (if $D_i = \{f_i = 0\}$ then $\deg(D_i) = \deg(f_i)$)
(number of points of intersection (counted with multiplicity) of D_i with a line in general position)
(↑ computable via the Hilbert polyn.).

Nb: $(\sum_i a_i D_i)_{|U} = \sum_{\substack{i \\ D_i \cap U \neq \emptyset}} a_i \underbrace{D_i \cap U}_{\text{prime div.}}$

Def: • The sheaf associated to a Weil divisor D on X is $\mathcal{O}_X(D)$ with:
$$\begin{cases} \mathcal{O}_X(D)(U) = \{ f \in \mathbb{C}(X)^* \mid (\mathrm{div}(f) + D)_{|U} \geq 0 \} \cup \{0\} \\ r_{U \to V}(f) = f. \end{cases}$$
• A reflexive sheaf on X is a coherent sheaf $\mathcal{L}$ on X such that $\mathcal{L} \xrightarrow{\sim} (\mathcal{L}^\vee)^\vee$. ($\mathcal{L}^\vee = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ dual; iso. can. morphism)
• A coherent sheaf $\mathcal{L}$ on X is*

Thm: • $\mathcal{O}_X(D)$ is a reflexive sheaf of rank 1.
• If $\mathcal{L}$ is a refl. sheaf of rank 1 on X then $\mathcal{L} \simeq \mathcal{O}_X(D)$ for some Weil div. D on X
• $\mathcal{O}_X(D) \simeq \mathcal{O}_X(E) \iff D \sim E$.

* of rank 1 if the global sections of $\mathcal{L} \otimes_{\mathcal{O}_X} K_X$ form a one-dim. vector space over $\mathbb{C}(X)$. (K_X ↑ constant sheaf at $\mathbb{C}(X)$)

• $\mathcal{O}_X(D+E) \simeq ((\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(E))^\vee)^\vee$ and $\mathcal{O}_X(-D) \simeq \mathcal{O}_X(D)^\vee$.

Rk: The class group of X is the group of classes (isomorphism) of reflexive sheaves of rank 1, the group law being the double dual of the tensor product.
(You can see that the inverse is the dual which is also $\mathcal{O}_X$ of minus the divisor.)

(⚠ The tensor product of reflexive sheaves is not necessarily reflexive.)

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(b) On toric varieties.

(1) (Last time, Herman introduced) Polyhedron $P \rightsquigarrow$ Toric variety X_P (4) (associated to them) (5). (X_P is a normal and irred. var. over $\mathbb{C}$) (6)

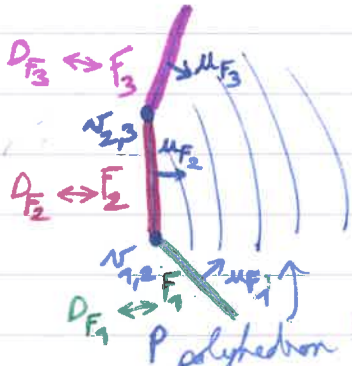
(i.e. (2) finite intersections of affine half-spaces of $M_{\mathbb{R}}$, and the (3) M is the character lattice of X_P and $m \in M \Leftrightarrow \underset{\text{nonzero}}{\chi^m}$ rational function on X_P.)

Recall the lemma: F face of $P \Leftrightarrow$ closed immersion $X_F \hookrightarrow X_P$.

In part., F facet of P (i.e. F codim 1 face of P) $\Leftrightarrow$ torus-invariant prime divisor $X_F \hookrightarrow X_P$ (which we denote) $D_F :=$

Ex:

$\mathbb{P}^2_{\mathbb{C}} = \mathrm{Proj}(\mathbb{C}[x,y,z])$



Recall that $P = \{m \in M_{\mathbb{R}}, \forall F \text{ facet of } P \ \langle m, u_F \rangle \geq -a_F\}$ $\quad$ // $\{m \in M_{\mathbb{R}}, \langle m, u_F \rangle = -a_F\}$

rational means that we can choose $u_F \in N$

Def: The minimal generator of F is the minimal $u_F \in N$ s.t. $F = \{m \in M_{\mathbb{R}}, \langle m, u_F \rangle = -a_F\}$ for some $a_F \in \mathbb{R}$.

Minimal means: $\forall k \in \mathbb{Z}$, $k \geq 2 \Rightarrow \frac{1}{k} u_F \notin N$.

From now on, the u_F are minimal (which also fixes the a_F).

we require $a_F \in \mathbb{Z}$ (We restrict ourselves to these polyhedra.)

Lemma: $\mathrm{div}(\chi^m) = \sum\limits_{F \text{ facet of } P} \langle m, u_F \rangle D_F$.

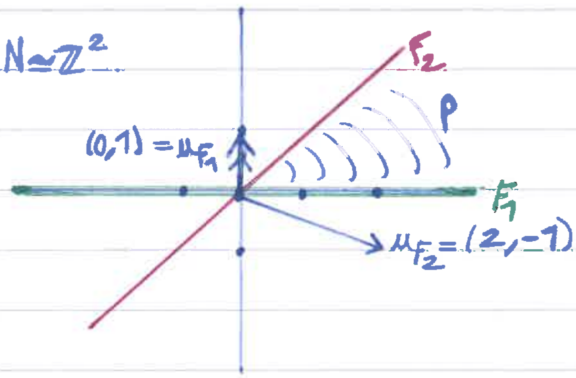
Thm: We have the exact sequence:

$$M \to \underbrace{\mathrm{Div}_T X_P}_{\substack{\text{torus-inv. div.} \\ \text{(i.e. div. invariant} \\ \text{under the torus action)}}} = \bigoplus_{F \text{ facet of } P} \mathbb{Z} D_F \to \mathrm{Cl}(X_P) \to 0$$

$m \mapsto \mathrm{div}(\chi^m)$ $\qquad$ $D \mapsto [D]$ class of D in $\mathrm{Cl}(X_P)$

(This allows us to compute $\mathrm{Cl}(X_P)$.)

Ex: $M \simeq \mathbb{Z}^2$, $N \simeq \mathbb{Z}^2$



Gen. of $Cl(X_P)$: D_{F_1}, D_{F_2}

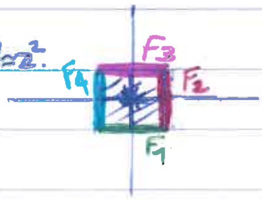
Rel. of $Cl(X_P)$: $\mathrm{div}(\chi^{(1,0)}) = 0$ and $\mathrm{div}(\chi^{(0,1)}) = 0$ and $\mathrm{div}(\chi^{(-1,0)}) = 0$ and $\mathrm{div}(\chi^{(0,-1)}) = 0$ ($\chi^{(m_1,m_2)} = \chi^{(m_1,0)}\chi^{(0,m_2)}$ etc.)

i.e. gen. by
$$\begin{cases} <(1,0),(0,1)> D_{F_1} + <(1,0),(2,-1)> D_{F_2} = 0 \text{ i.e. } 2D_{F_2} = 0 \\ <(0,1),(0,1)> D_{F_1} + <(0,1),(2,-1)> D_{F_2} = 0 \text{ i.e. } D_{F_1} - D_{F_2} = 0 \\ <(-1,0),(0,1)> D_{F_1} + <(-1,0),(2,-1)> D_{F_2} = 0 \\ <(0,-1),(0,1)> D_{F_1} + <(0,-1),(2,-1)> D_{F_2} = 0 \end{cases}$$

Thus, $Cl(X_P) \simeq \mathbb{Z}/2\mathbb{Z}$.

More gen., if $u_{F_1} = (0,1)$ and $u_{F_2} = (d,-1)$ with $d \in \mathbb{Z}$ at least 2 : D_{F_1}, D_{F_2} gen. with $dD_{F_2} = 0$ and $D_{F_1} - D_{F_2} = 0$, thus $Cl(X_P) \simeq \mathbb{Z}/d\mathbb{Z}$.

Ex: $M \simeq \mathbb{Z}^2$, $N \simeq \mathbb{Z}^2$.



$\mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$

Gen. of $Cl(\mathbb{P}^1 \times \mathbb{P}^1)$: $D_{F_1}, D_{F_2}, D_{F_3}, D_{F_4}$

Rel. of $Cl(\mathbb{P}^1 \times \mathbb{P}^1)$: $\mathrm{div}(\chi^{(1,0)}) = 0$ and $\mathrm{div}(\chi^{(0,1)}) = 0$

i.e.
$$\begin{cases} <(1,0),(0,1)> D_{F_1} + <(1,0),(-1,0)> D_{F_2} + <(1,0),(0,-1)> D_{F_3} \\ \quad + <(1,0),(1,0)> D_{F_4} = 0 \text{ i.e. } -D_{F_2} + D_{F_4} = 0 \\ <(0,1),(0,1)> D_{F_1} + 0 + <(0,1),(0,-1)> D_{F_3} + 0 = 0 \\ \text{i.e. } D_{F_1} - D_{F_3} = 0 \end{cases}$$

Thus, $Cl(\mathbb{P}^1 \times \mathbb{P}^1) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

2) Cartier divisors

a) In general.

Def: A Cartier divisor on X is a Weil divisor on X which is locally principal, i.e. $\exists (U_i)_{i \in I}$ opens in X, $X = \bigcup_{i \in I} U_i$, $D_{|U_i} = \mathrm{div}(f_i)$ for some rational function f_i on U_i, $\forall i \in I$.

$CDiv(X)$ is the group of Cartier divisors on X (for $+$).

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Rk: Principal divisors are Cartier divisors, i.e. $\mathrm{Div}_0(X) \subset \mathrm{CDiv}(X)$.

Def: The Picard group of X is $\mathrm{Pic}(X) = \mathrm{CDiv}(X)/\mathrm{Div}_0(X)$. $\mathrm{Pic}(X) \hookrightarrow \mathrm{Cl}(X)$.
(The Picard group is the group of classes of Cartier divisors for linear equivalence.)

Rk: If X is smooth then every Weil divisor is a Cartier div., hence $\mathrm{Pic}(X) \simeq \mathrm{Cl}(X)$.

Def: An $\mathcal{O}_X$-module $\mathcal{L}$ is invertible (called an invertible sheaf) if it has an inverse for $\underset{\mathcal{O}_X}{\otimes}$.

Rk: If $\mathcal{L}$ is invertible then $\mathcal{L}^{-1} = \mathcal{L}^\vee$. $\leftarrow$ dual

Def: $\mathcal{L}$ is locally free of rank 1 if: $\forall x \in X\ \exists x \in U$ open in X, $\mathcal{L}_{|U} \simeq \mathcal{O}_U$ as an $\mathcal{O}_U$-mod.

Prop: $\mathcal{L}$ is invertible if and only if $\mathcal{L}$ is locally free of rank 1.

Thm:
- If D is a Cartier divisor then $\mathcal{O}_X(D)$ is invertible.
- If $\mathcal{L}$ is an invertible sheaf on X then $\mathcal{L} \simeq \mathcal{O}_X(D)$ for some Cartier div. D on X.
- $\mathcal{O}_X(D) \simeq \mathcal{O}_X(E) \iff D \sim E$.
- $\mathcal{O}_X(D+E) \simeq \mathcal{O}_X(D) \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X(E)$ and $\mathcal{O}_X(-D) \simeq \mathcal{O}_X(D)^\vee$ for all Cartier div. D and E on X.

Rk: The Picard group of X is the group of (isomorphism) classes of invertible sheaves, the group law being the tensor product (or equivalently the double dual of the tensor product, since here $\mathcal{O}_X(D) \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X(E)$ is reflexive since it is iso. to $\mathcal{O}_X(D+E)$).

Def: A line bundle over X is a morphism $\pi: V \to X$ such that there exist:
- an open cover $(U_i)_{i\in I}$ of X (i.e. the U_i are opens in X s.t. $X = \bigcup_{i\in I} U_i$)
- $\forall i \in I$ an iso. $\varphi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \underset{\mathbb{C}}{\times} \mathrm{Spec}(\mathbb{C}[t])$ s.t. $p_{U_i} \circ \varphi_i = \pi^{|U_i}_{|\pi^{-1}(U_i)}$
- $\forall i, j \in I$ a linear map $g_{i,j} : \mathbb{C}[t] \to \mathbb{C}[t]$ (i.e. $g_{i,j}(t) = a_{i,j} t$ for some $a_{i,j} \in \mathbb{C}^*$ and $g_{i,j}$ is a morphism of $\mathbb{C}$-algebras) such that the following diagram commutes:

$$\begin{tikzcd}
 & (U_i \cap U_j) \underset{\mathbb{C}}{\times} \mathrm{Spec}(\mathbb{C}[t]) \\
\pi^{-1}(U_i \cap U_j) \arrow[ur, "\varphi_i|"] \arrow[dr, "\varphi_j|"'] & \circlearrowleft \\
 & (U_i \cap U_j) \underset{\mathbb{C}}{\times} \mathrm{Spec}(\mathbb{C}[t]) \arrow[uu, "\mathrm{Id}_{U_i\cap U_j} \underset{\mathbb{C}}{\times} \mathrm{Spec}(g_{i,j})"']
\end{tikzcd}$$

$(\varphi_i)|^{\ldots} \circ (\varphi_j^{-1})|^{\pi^{-1}(U_i\cap U_j)}_{(U_i\cap U_j)\underset{\mathbb{C}}{\times}\mathrm{Spec}(\mathbb{C}[t])}$

(**Rk:** $\pi' : V' \to X$ is iso. as a vector bundle to $\pi: V \to X$ iff $\forall i \in I\ \forall i' \in I'\ \exists$ linear map $g_{i,i'} : \mathbb{C}[t] \to \mathbb{C}[t]$ s.t. $(\varphi_i)|^{\ldots} \circ (\varphi_{i'}'^{-1})|^{\ldots}_{\ldots \times \mathrm{Spec}(\mathbb{C}[t])} = \mathrm{Id} \underset{\mathbb{C}}{\times} \mathrm{Spec}(g_{i,i'})$

Rk: The Picard group of X is iso. (as a group) to the group of (isomorphism) classes of line bundles, the group law being the fibre product. Pick your favourite iso. between:

- $[\mathcal{L}] \mapsto [\mathrm{Spec}(\mathrm{Sym}(\mathcal{L}))]$ (of inverse $[\pi] \mapsto [\mathrm{Sec}^\vee]$ with Sec the sheaf of sections of π) ← functorial (a m. of v.b. is sent to a m. of sheaves)

  functorial → (a m. of sheaves is sent to a m. of vector bundles)

- $[\mathcal{L}] \mapsto [\mathrm{Spec}(\mathrm{Sym}(\mathcal{L}^\vee))]$ (of inverse $[\pi] \mapsto [\mathrm{Sec}]$) ← more intuitive

Def: $\mathrm{Pic}^0(X)$ is the connected component of the neutral element of $\mathrm{Pic}(X)$.

Thm: The subgroup Pic^0 of the Picard group of a normal proper variety X over $\mathbb{C}$ is an abelian variety (called the Picard variety).

6) On toric varieties.

$P = \{m \in M_{\mathbb{R}}, \forall F \text{ facet of } P \ \langle m, u_F \rangle \geq -a_F\}$ (u_F: min. gen. of F)

lattice polyhedron (its vertices are in M)

$P \rightsquigarrow X_P$, F facet of $P \leftrightarrow D_F$ prime torus-inv. divisor on X_P (torus-invariant)

Thm: A torus-invariant Weil div $D = \sum_{F \text{ facet of } P} c_F D_F$ ($c_F \in \mathbb{Z}$) on X_P is a torus-inv. Cartier div. on X_P iff for each vertex v in P there exists $m_v \in M$ s.t. for each facet F of P containing v, $\langle m_v, u_F \rangle = -c_F$.

Rk: On an affine toric subvariety of X_P which is "around the vertex v", D is $\mathrm{div}(\chi^{-m_v})$.
(↑ which ⇔ $\mathrm{div}(\chi^{-v})$ around v.) (↑ this affine toric variety ⇔ maximal cone ⇔ vertex) (↑ recall Herman's talk) duality between the polyhedron and the fan

Cor: $D_P := \sum_{F \text{ facet of } P} a_F D_F$ is a torus-inv. Cartier div. on X_P (which makes X_P a projective toric var.)

Rk: If P is a polytope (i.e. a bounded polyhedron) then $\sum_{F \text{ facet of } P} a_F D_F \nsim 0$. (↑ i.e. its class in $\mathrm{Pic}(X_P)$ is $\neq 0$)

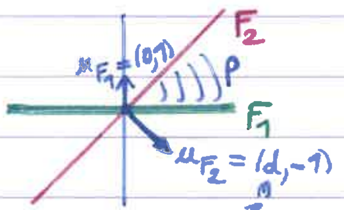
Thm: We have the exact sequence:

$$M \to \underbrace{\mathrm{CDiv}_T X_P}_{\text{torus-inv. Cartier div.}} \to \mathrm{Pic}(X_P) \to 0$$

$m \mapsto \mathrm{div}(\chi^m)$ $\quad D \mapsto [D]$ class of D in $\mathrm{Pic}(X_P)$

(This allows us to compute $\mathrm{Pic}(X_P)$.)

Ex: $M \simeq \mathbb{Z}^2, N \simeq \mathbb{Z}^2$.



$Cl(X_P) \simeq \mathbb{Z}/d\mathbb{Z}$

Gen. of $\mathrm{Pic}(X_P)$: $c_{F_1} D_{F_1} + c_{F_2} D_{F_2}$ ($c_{F_1}, c_{F_2} \in \mathbb{Z}$) such that:

$\exists (m_1, m_2) \in \mathbb{Z}^2$ $\quad m_1 \times 0 + m_2 \times 1 = -c_{F_1}$ and $m_1 \times d + m_2 \times (-1) = -c_{F_2}$

$\Leftrightarrow d \mid c_{F_1} + c_{F_2}$ (ie $c_{F_1} + c_{F_2}$ is a multiple of d)

Rel. of $\mathrm{Pic}(X_P)$: $d D_{F_2} = 0$ and $D_{F_1} = D_{F_2}$

Thus, $\mathrm{Pic}(X_P) = 0$.

Hence, X_P is not smooth.

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Thm: X_P (toric var. ass. to P) is smooth iff $\mathrm{Pic}(X_P) \simeq Cl(X_P)$.

③ Global sections of the sheaf ass. to a torus-inv. Weil divisor on a toric variety.

Recall that if D is a Weil divisor on X then $\mathcal{O}_X(D)(X) = \{f \in \mathbb{C}(X)^* \mid \underbrace{div(f)+D \geq 0}_{\text{effective}}\} \cup \{0\}$.

Prop: If D is a torus-inv. Weil div. on X_P then $\mathcal{O}_X(D)(X) = \bigoplus_{\substack{m \in M, \\ div(\chi^m)+D \geq 0}} \mathbb{C}\cdot\chi^m$.

Def: The polyhedron P_D associated to the Weil divisor $D = \sum_{F \text{ facet of } P} c_F D_F$ is:
$P_D = \{m \in M_{\mathbb{R}} \mid \langle m, u_F\rangle \geq -c_F \text{ for each facet } F \text{ of } P\}$.

Rk: $P_{D_P} = P$. **Rk:** $\forall k \in \mathbb{N}$ $P_{kD} = kP_D := \{km, m \in P_D\}$.

Prop: If D is a torus-inv. Weil div. on X_P then $\underbrace{\mathcal{O}_X(D)(X)}_{H^0(X, \mathcal{O}_X(D))} = \bigoplus_{m \in M \cap P_D} \mathbb{C}\cdot\chi^m$.

next page (not ③ ... : new talk ② Projective toric varieties)

Ⅱ The Iitaka fibration. [Ch.6 in [CLS11], Ch.II, §7 in [Hartshorne], Ch.1 and Ch.2 §1 in [Lazarsfeld]] Birationality in AG I

① In general.

Def: Let $\mathcal{L}$ be an invertible sheaf (sometimes called line bundle) on X. Its section ring (or ring of sections) is the graded $\mathbb{C}$-alg. (algebra) $R(X,\mathcal{L}) := \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})$ (with $\mathcal{L}^{\otimes 0} = \mathcal{O}_X$, $\mathcal{L}^{\otimes 1} = \mathcal{L}$, $\mathcal{L}^{\otimes 2} = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}$, etc.).

Def: An invertible sheaf $\mathcal{L}$ on X is generated by a family $(s_i)_{i \in I}$ of global sections of $\mathcal{L}$ if each stalk $\mathcal{L}_x$ of $\mathcal{L}$ is generated (as an $\mathcal{O}_{X,x}$-module) by $((s_i)_x)_{i \in I}$.
($\mathcal{L}$ is finitely gen. by global sec. if it is gen. by a finite family of global sections.)

Ex: If $\varphi: X \to \mathbb{P}^n_{\mathbb{C}} = \mathrm{Proj}(\mathbb{C}[x_0,\dots,x_n])$ is a morphism then $\varphi^*(\mathcal{O}_{\mathbb{P}^n}(1))$ is gen. by $(\varphi^*(x_0), \dots, \varphi^*(x_n))$. (global sections: homogeneous pol. of degree 1)

Prop: If $\mathcal{L}$ is gen. by $s_0, \dots, s_n \in H^0(X,\mathcal{L})$ then: $\exists!\ \varphi: X \to \mathbb{P}^n_{\mathbb{C}}$ $\begin{cases} \varphi^*(\mathcal{O}_{\mathbb{P}^n}(1)) \overset{\theta}{\simeq} \mathcal{L} \\ \forall i \in \{0,\dots,n\} \quad \varphi^*(x_i) \overset{\theta}{\leftrightarrow} s_i \end{cases}$
(Use $\mathbb{P}^n_{\mathbb{C}} = \bigcup_{i=0}^n \{x_i \neq 0\}$ ($\uparrow P,\ x_i(P) \neq 0$) and $X = \bigcup_{i=0}^n \{P \in X, (s_i)_P \notin m_P \mathcal{L}_P\}$ (m_P max. ideal of $\mathcal{O}_{X,P}$) to prove it.)

Def: If $R(X,\mathcal{L})$ is finitely gen. as a $\mathbb{C}$-algebra then the Iitaka fibration ass. to $\mathcal{L}$ is the rational map $\Phi: \begin{cases} X \dashrightarrow \mathrm{Proj}(R(X,\mathcal{L})) \\ x \mapsto \{\sigma, \sigma(x) = 0\} \end{cases}$. **Rk:** $\mathbb{P}^n_{\mathbb{C}} = \mathbb{P}(H^0(X,\mathcal{L}))$ above and the Iitaka fibration is $x \mapsto \{\sigma, \sigma(x) = 0\}$ ($\sigma \in H^0(X,\mathcal{L})$, $\sigma(x) \in \mathcal{L}_x / m_x \mathcal{L}_x$) ← linear hyperplane in $H^0(X,\mathcal{L})$.
(If you fix homogeneous coordinates, this corresponds to φ.)

Prop: Φ is dominant.

② Projective toric varieties.

$*$: ③ of last page (without the title).

$d = \dim(P) = \dim(M_{\mathbb{R}}) \geq 2$

or rather, as soon as kP is very ample with $\tilde{P} := kP$, $k \geq d-1$

vertices in M

Recall that when P is a full-dimensional lattice polytope, X_P is a projective (hence complete) toric variety. In this case, P_D is a polytope for each torus-invariant Weil divisor D on X_P (hence $M \cap P_D$ is finite since M is discrete).

The $(\chi^m)_{m \in M \cap P_D}$ give a morphism $\varphi : X_P \to \mathbb{P}_{\mathbb{C}}^{|M \cap P_D| - 1}$ as before.
Our favourite one is the one for $D = kD_P$ (hence the $(\chi^m)_{m \in M \cap kP}$).
↑ conventional one $\quad$ $k \geq d-1$ (or rather, as soon as kP is very ample)

III The ample cone. [Ch. 6 in [CLS11] and Ch. 1 in [Lazarsfeld]]

We assume that X is complete. ① Ample and semiample divisors.

Def: Let D be a Cartier divisor on X.

- D is very ample if there exist $n \geq 1$ and $\varphi : X \hookrightarrow \mathbb{P}_{\mathbb{C}}^n$ a closed immersion such that $\mathcal{O}_X(D) = \varphi^*(\mathcal{O}_{\mathbb{P}^n}(1))$.
- D is ample if $\mathcal{O}_X(D)^{\otimes n}$ is very ample for some $n \geq 1$.
- D is basepoint free if $\mathcal{O}_X(D)$ is generated by global sections.
- D is semiample if $\mathcal{O}_X(D)^{\otimes n}$ is basepoint free for some $n \geq 1$.

Rk: Very ample $\Rightarrow$ basepoint free and Ample $\Rightarrow$ semiample

Prop: If D is semiample then $R(X, \mathcal{O}_X(D)) = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(D)^{\otimes n})$ is finitely gen. as a $\mathbb{C}$-algebra.

P full-dim. lattice polytope → Rk: D_P is ample and basepoint free. It is called the polarisation of X_P.
(D_P very ample $\Leftrightarrow P$ very ample) $\quad P_{f^*(\varphi^*(\mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}}(1)))} = P_{(\varphi \circ f)^*}$

Recall that there is a polarised projective morphism of toric var. $f : X_P \to X_Q \xrightarrow{\varphi} \mathbb{P}^n_{\mathbb{C}}$ iff $\exists Q', k \geq 1$ $kP = Q + Q'$.

Prop: D on X_P is semiample $\Rightarrow \exists Q', k \geq 1$ $kP = P_D + Q' \Leftrightarrow \exists$ pol. mor. of toric var. $f : X_P \to X_{P_D}$.

Thm: On a smooth complete toric variety, a divisor is ample iff it is very ample.

Def: Let P be a full-dim. lattice polytope, e be an edge of P (i.e. a 1-dim. face of P) v_1 and v_2 be the vertices at the border of e and F be a facet of P (i.e. a face of codim. 1 in P) which contains v_2 but not v_1. A Cartier divisor $D = \sum_{F \text{ facet of } P} c_F D_F = (\operatorname{div}(\chi^{-m_\sigma}))$ "around σ" satisfies the edge inequality (called wall inequality in fan-language) $I(e,F)$ if

$$\langle m_{\sigma_{v_1}}, u_F \rangle > -c_F .$$

↑ comes from the Cartier data of D $\quad$ ↑ min. gen. of F

Thm: A Cartier divisor D on X_P is ample iff it satisfies all the edge inequalities $I(e,F)$ (e edge of P, F facet of P containing exactly one of the vertices at the ∂ of e).
= border

Research seminar on VGIT: Talk 2

② The ample cone.

We further assume that X is regular.

Def: The Néron-Severi group of X is $NS(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$. (It can be defined as the classes for algebraic equivalence.)

Thm: $NS(X)$ is a finitely gen. abelian group.

Def: The Picard number of X is the rank of $NS(X)$.

Not: $N^1_{\mathbb{R}}(X) := NS(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

Prop: If D is an ample Cartier div. on X and D' is in the class of D in $NS(X)$ then D' is ample.

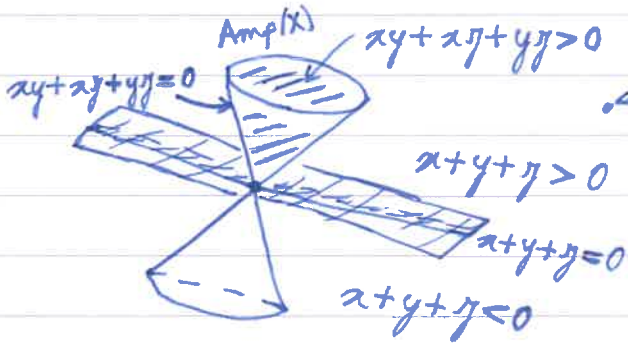
Def: The ample cone of X, denoted $\mathrm{Amp}(X)$, is the cone in $N^1_{\mathbb{R}}(X)$ (i.e. the subset of $N^1_{\mathbb{R}}(X)$ stable under multiplication by positive (>0) scalars) generated by classes of ample Cartier div. on X.

⚠ It is not necessarily a polyhedral cone.

Ex: Let E be a sufficiently generic (it has "general moduli") elliptic curve, $X = E \times_{\mathbb{C}} E$, $P \in E$.
Let f_1 (resp. f_2, δ) be the class in $N^1_{\mathbb{R}}(X)$ of $\{P\} \times_{\mathbb{C}} E$ (resp. $E \times_{\mathbb{C}} \{P\}$, Δ the diagonal).

(f_1, f_2, δ) is a basis of $N^1_{\mathbb{R}}(X)$, which we now identify with $\mathbb{R}^3$.

The ample cone of X is $\{(x,y,z) \in \mathbb{R}^3,\ xy + xz + yz > 0 \text{ and } x+y+z>0\}$.



Circular cone (ice-cream cone): not polyhedral!

Rk: If E is any elliptic curve, then $\mathrm{Amp}(X)$ is the intersection of the previous cone with a linear subspace of $N^1_{\mathbb{R}}(X)$ (which may be bigger, i.e. (f_1, f_2, δ) is still a free family but is not necessarily a basis). $\mathrm{Amp}(X)$ is still not polyhedral

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