Further directions in the topic of Eigenvarieties

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1 How to visualize the existence of families of modular form through the Eigencurve

We recall that around ten talks ago we had fixed a prime p. We keep it fixed also for today.

First we give an idea of why the existence of the Coleman-Mazur eigencurve $\kappa : \mathscr{E} \to \mathcal{W}$ gives information about p-adic congruences of modular forms.

Proposition 1. *Let f be a normalized eigenform of level $\Gamma := \Gamma_0(p) \cap \Gamma_1(N)$ and weight k new away from p of slope $\nu < k-1$. Moreover, supposing that, if α is the eigenvalue for U_p, then $\alpha^2 \neq \chi_N(p)p^{k+1}$, where χ_N is the tame part of the character of f. Then for every $n >> 0$ there exists $k' \neq k$ and an eigenform f' of level Γ new away from p such that $f \equiv f' \mod p^n$ (meaning that $a_\ell(f) \equiv a_\ell(f') \mod p^n$ for every prime $\ell \nmid Np$). More precisely: there exists an open affinoid $B' \subset \mathscr{E}$ containing k and rigid analytic functions $a_n(T) \in \mathcal{O}(B') \cong \mathcal{O}_E\langle T\rangle$ (for E some finite extension of $\mathbb{Q}_p$) such that for every $k' \in B \cap \mathbb{Z}$ such that $k' > \nu + 1$, the evaluation at k' of the q-expansion*

$$F(q)(T) := \sum_{n \geq 1} a_n(T) q^n$$

is the q-expansion of a cuspidal classical eigenform new away from p, of weight k'. Moreover, the evaluation of $F(q)(s)$ at $s = k$ gives back f.

Proof. This is [Col97, Corollary B5.7.1]. We want to give a sketch of the proof knowing that the Coleman-Mazur eigenvariety exists. By the general construction we know that there is an affinoid $B = Sp(R)$ around k that is ν-adapted. The corresponding local piece of the eigenvariety is $Sp(\mathcal{T}_{B,\nu}) \to B$, where $\mathcal{T}_{B,\nu}$ is the eigenalgebra of $(M_k(\Gamma)^\dagger)^{\leq \nu} \subseteq (M_k(\Gamma)^\dagger)^{<k-1} = M_k(\Gamma)^{<k-1}$, where the last equality is a deep result known as Coleman classicality theorem. We also know that the map $Sp(T_{B,\nu}) \to B$ is unramified at every classical point (because the action of T_ℓ, S_ℓ for $\ell \nmid Np$ is semisimple on classical forms and the condition $\alpha^2 \neq \chi_N(p)p^{k+1}$ ensures that also U_p acts semisimply, by [Bel21, Exercise 2.6.21] and the fact that f has conductor N, so that $N/N_0 = p$ in that exercise notation). In particular, up to an étale extension, there exists a section.

The existence of a section means that one has a map $\mathcal{T}_{B,\nu} \otimes R' \to R' \leftarrow R$ (where the last is an étale morphism of rings) where $R' \cong \mathcal{O}_E\langle T\rangle_s$ (localization at some element s) for some finite extension $E/\mathbb{Q}_p$. In particular one has a map $T_\ell \to a_\ell(T) = \sum_i a_{\ell,i}T^i \in \mathcal{O}(B')$ and Coleman shows that B' can be taken so that $|a_\ell| \leq 1$ (i.e. $|a_{\ell,i}| \leq 1$ for every i), see the discussion after [Col97, Theorem B5.7] .

But now the congruence is clear: suppose that n is large enough so that an integer k' satisfying $k' \equiv k \mod (p-1)p^n$ belongs to B (such n exists because classical points are Zariski-dense in $\mathcal{W}$). As evaluating at the character $z \mapsto z^k$ corresponds to evaluate at $T = (1+p)^k - 1$ in terms of T, one has the following estimate for the absolute value of $a_\ell(k') - a_\ell(k)$:

$$|\sum a_{\ell,i}((1+p)^{k+c(p-1)p^n} - 1 - ((1+p)^k - 1))| \leq |(1+p)^{c(p-1)p^n} - 1| \leq p^{-n-1}.$$

From this (sketch) of proof it is not clear that f' can be taken to be primitive away from p. To obtain this one should construct the eigencurve using Banach modules of p-new overconvergent modular forms, and the result would follow. A local version of this is discussed in the proof of [Col97, Corollary B5.7.1]. $\square$

2 Why studying p-adic interpolation of modular forms

We now give some insight of how the existence of p-adic congruences of Fourier coefficients is used in practice.

2.1 Galois representations

We use the following notation: E denotes a finite extension of $\mathbb{Q}_p$. We call $\mathcal{O}_E$ its ring of integers and we choose ϖ a uniformizer of $\mathcal{O}_E$ and set $\mathbb{F} := \mathcal{O}_E/\varpi$.

We follow [Bel11, §1] and [Bel21].

We start with defining p-adic representations:

Definition 1. Let K be a field. A p-adic representation of the Galois group $G_K := \mathrm{Gal}(\overline{K}/K)$ with coefficients in E is the giving of a finite dimensional E-vector space V with a continuous action of G_K.

From this we immediately get a continuous group homomorphism

$$\rho : G_K \to GL(V).$$

Indeed, we could have taken the giving of such a morphism as the definition of a Galois representation.

Usually K is taken to be a local or a global field.

Lemma 2. *Given a E-valued p-adic representation V, it is always possible to find a G_K-stable lattice T, i.e. a $\mathcal{O}_E$-submodule T of V such that $T \otimes_{\mathcal{O}_E} E = V$ and the map ρ_V factors through $\mathrm{Aut}_{\mathcal{O}_E}(T)$.*

Proof. For any lattice T, the subgroup $GL(T) \subset GL(V)$ is open. In particular $\rho^{-1}(GL(T))$ is open in G_K profinite and hence of finite index. Denote with $\{g_1, \dots, g_n\}$ a system of representatives for the quotient. Then $\sum_{i=1}^{n} \rho(g_i)T$ is a G_K-stable lattice. $\square$

Choosing a lattice amounts to choosing a basis for V (just take the $\mathcal{O}_E$ span of that basis). In general, once we have chosen a basis, we can view the representation as having values in $GL_n(E)$ for n the dimension of V. In this language, the above lemma is saying that, up to conjugation, we can assume that the representation actually has values in $GL_n(\mathcal{O}_E)$.

Definition 2. In general, if R is a topological ring, we can speak of an (n-dimensional) R-valued Galois representation of G_K, defined to be a continuous group homomorphism

$$\rho : G_K \to GL_n(R).$$

We say that ρ_1, ρ_2 are equivalent if one is obtained from the other by conjugation by an element $\gamma \in GL_n(R)$.

In particular it is clear that any choice of basis of an E-valued p-adic representation V gives equivalent representations, but this does not have to be true for $\mathcal{O}_E$, i.e. different choices of T might give non-equivalent representations. However, the following is true:

Proposition 3. *Suppose that ρ is 2-dimensional and that the representation $\overline{\rho}_{T,1} : G_K \to GL_2(T) \to GL_2(T/\varpi)$ is absolutely irreducible. Then any two choices of a lattice give equivalent representations.*

Proof. We consider the set of homothety classes of lattices. We say that two lattices Λ, Λ' in E^2 in different homothety classes are at distance n if n is the minimal positive integer such that $\varpi^n \Lambda \subseteq \Lambda' \subseteq \Lambda$ (Once the lattice Λ in the first class is fixed, there is a unique maximal Λ' contained in Λ in the second class). Since the definition of distance in independent of the elements of the classes, we give the correspondent definition for classes of lattices. Suppose that Λ, Λ' are G_K stable classes of lattices. Then their elements are G_K stable lattices because G_K cannot scale them by powers of ϖ (it would not act continuously). If two lattices are homotethic and G-stable, then their associated representations are equivalent. We prove by induction on $n \geq 1$ that if two lattices at distance n induce non-equivalent representations, then the residual representation cannot be (absolutely) irreducible. Suppose that the classes of Λ and Λ' are at distance 1, then the induced representation $\rho_1 : G_K \to GL(\Lambda/\varpi)$ is not irreducible, because it has the stable subspace $\Lambda'/\varpi\Lambda$. Suppose now that Λ, Λ' are at distance n and G stable, then apply the induction hypothesis to $\Lambda' + \varpi^{n-1}\Lambda$ which is clearly G-stable, at distance $n-1$ form Λ and at distance 1 form Λ'. $\square$

Then, once we have chosen a lattice T, we can also reduce it mod ϖ^n for every n, thus obtaining a collection of representations

$\rho_{T,n} : G_K \to GL(T/\varpi^n) \cong GL_m(\mathcal{O}_E/\varpi^n)$ for some m.

Definition 3. When K is a local field,we say that ρ is unramified if its restriction to the inertia group is trivial (equivalently: it factors through the Galois group $\mathrm{Gal}(K^{\mathrm{unr}}/K)$, where K^{unr} is the maximal unramified extension). If K is a number field, we say that ρ is unramified at a prime ℓ if its restriction to any decomposition group at ℓ is unramified (notice that this does not depend on the choice of a decomposition group, because they are all conjugate).

Remark 4. Let K be a number field and S a finite set of places of K (usually taken to contain p). We sometimes write the product of the elements of S instead of S itself. A continuous p-adic representation $\rho : G_K \to GL_n(E)$ is unramified at prime $\ell \notin S$ (with the definition above) if and only if ρ factors through the Galois group $G_S := \mathrm{Gal}(K_S/K)$, where K_S is the maximal unramified extension of K unramified outside S.

Indeed for a place v of K_S above $\ell \notin S$, we have that $(K_S)_v \subseteq \mathbb{Q}_\ell^{\mathrm{unr}}$, i.e $\rho_\ell = \rho|_{D_\ell}$ (where D_ℓ is a decomposition group at ℓ) factors through $\mathrm{Gal}(\mathbb{Q}_\ell^{\mathrm{unr}}/\mathbb{Q}_\ell)$, i.e. it is trivial on I_ℓ.

Viceversa, if it is unramified outside S and does not factor through K_S, then there is an induced (w.l.o.g faithful) non trivial representation of $\mathrm{Gal}(L/K_S)$ for some non trivial Galois extension L of K_S ramified at some $\ell \notin S$. But then $\mathrm{Gal}(L/K_S)$ contains a non trivial quotient of I_ℓ and this is not possible, because of unramifiedness at ℓ and faithfulness.

Definition 4 (Semisimple representation)**.** A representation $\rho : G \to GL_n(R)$ (over a ring R) is semisimple if every G invariant subspace $W \subseteq R^n$ is a direct summand (as $R[G]$-module, where $R[G]$ is the group algebra).

The following theorem is taken from [Bel21, Theorem 2.6.22]:

Theorem 5. *Let $f \in M_k(\Gamma_1(N), \chi)$ be a normalized primitive eigenform, so in particular its Fourier coefficients are algebraic integers (χ is a Dirichlet character mod N and $f \in M_k(\Gamma_1(N), \chi)$ means* *that $S_\ell f = \chi(\ell)\ell^{k-2} f$, for $\ell \nmid Np$ a prime). Take $S = Np$. Then there exists a unique (up to equivalence) semisimple Galois representation*

$$\rho_f : G_{\mathbb{Q},S} \to GL_2(\overline{\mathbb{Q}}_p)$$

such that the characteristic polynomial of a Frobenius element $\mathrm{Fr}_\ell \in G_{\mathbb{Q},S}$ at $\ell \nmid Np$ is given by

$$X^2 - a_\ell(f)X + \chi(\ell)\ell^{k-1}.$$

Equivalently, there exits a unique (up to equivalence) semisimple representation

$$\rho_f : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_p)$$

such that it is unramified outside $S = Np$ and the characteristic polynomial of a Frobenius element $\mathrm{Fr}_\ell \in G_{\mathbb{Q}}$ is as above.

Proof. I will say a few words later about the construction of such representations. The unicity statement follows from the following two facts (and the fact that ρ is continuous):

- (Brauer-Nesbitt theorem) If $\rho_1, \rho_2 : G \to GL_n(E)$ are two **semisimple** representations of a group G valued in a field E, with characteristic polynomials of $\rho_1(g)$ and $\rho_2(g)$ equal for every $g \in G$, then $\rho_1 \cong \rho_2$. If moreover the characteristic of E is either 0 or $\geq n+1$, then the condition $Tr(\rho_1(g)) = Tr(\rho_2(g))$ is enough.
- (Chebotarev density theorem) The union of all the Frobenius conjugacy classes for primes v of a number field K such that $v \notin S$ are dense in $G_{K,S}$.

□

A slightly variation of the statement might be more enlightening, in view of the notations that we have followed in the seminar:

Theorem 6. *Let $f \in M_k(\Gamma_1(N))$ be a normalized primitive eigenform and let $\lambda_f : \mathbb{T}_0 \to \overline{\mathbb{Q}}_p$ be the associated system of eigenvalues ($\mathbb{T}_0$ being the good eigenalgebra of $M_k(\Gamma_1(N))$). There exists a unique (up to equivalence) semisimple representation*

$$\rho_f : G_{\mathbb{Q},S} \to GL_2(\overline{\mathbb{Q}}_p)$$

such that the characteristic polynomial of a Frobenius element $\mathrm{Fr}_\ell \in G_{\mathbb{Q},S}$ at $\ell \nmid Np$ is given by

$$X^2 - \lambda_f(T_\ell)X + \lambda_f(\ell S_\ell).$$

The following is one of the most interesting reasons for studying p-adic congruences of Hecke systems of eigenvalues:

Theorem 7. *Theorem Suppose that $f \in S_k(\Gamma_1(N), \mathcal{O}_E)$ and $f' \in S_{k'}(\Gamma_1(N), \mathcal{O}_E)$ are congruent mod ϖ^n (that is $a_\ell(f) \equiv a_\ell(f')$ for each $\ell \nmid Np$). Moreover, suppose that the reduction mod ϖ of their associated Galois representations $V_f, V_{f'}$ are absolutely irreducible (that's why I've reduced to cusp forms: Eisenstein series have reducible Galois representations associated to them). Choose lattices $T_f, T_{f'}$ in $V_f, V_{f'}$ respectively, then their associated $\bmod \varpi^n$ Galois representations are isomorphic:*

$$\rho_{T_f,n} \cong \rho_{T_{f'},n}.$$

Proof. The theorem follows from applying the unicity statement in point two of theorem 9 to the Henselian local ring $\mathcal{O}_E/\varpi^n$ and pseudocharacters $\mathrm{Tr}(\rho_{T_f,n})$ and $\mathrm{Tr}(\rho_{T_f,n})$. □

2.1.1 Construction of the modular representation

Let me start with the easy one. Let ψ be a Dirichlet character of conductor dividing N satisfying $\psi(-1) = (-1)^k$. We have the (normalized) Eisenstein series $E_{k,\psi} \in M_k(\Gamma_1(N))$ associated to ψ whose q-expansion is given by ([Bel21, (2.6.14)])

$$E_{k,\psi}(q) = c_0 + \sum_{n\geq 1}(\sum_{m|n}\psi(m)m^{k-1})q^n.$$

In particular for a prime ℓ we have

$$a_\ell(E_{k,\psi}) = 1 + \psi(\ell)\ell^{k-1}$$

It is clear what the associated Galois representation should be: just take the representation

$$\rho : G := G(\mathbb{Q}(\mu_{Np}^\infty)/\mathbb{Q}) \to GL_2(\overline{\mathbb{Q}}_p)$$

given by

$$g \to \begin{pmatrix} 1 & 0 \\ 0 & \psi'(g)\chi_{\mathrm{cyc}}(g) \end{pmatrix}$$

where $\psi(g)$ is computed by taking the quotient $\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$ of G. Thus, representations of Eisenstein series are just sums of characters (and in particular are not irreducible).

More interesting is to associate a Galois representation to cusp forms.

Let me first do the general construction (this works at least for $k \geq 2$, for $k = 1$ the procedure is different and in fact it uses congruences to deduce the result from higher weights) which is due to Deligne [Del]: for this fix $f \in S_k(\Gamma_1(N))$ whose coefficients are in a finite extension $L/\mathbb{Q}_p$.

One considers the universal elliptic curve $\pi : \mathscr{E} \to Y(\Gamma_1(N))$ and the relative cohomology lisse étale sheaf

$$\mathscr{T} := R^1\pi L(1).$$

The representation space V_f is then the maximal L-quotient of

$$H^1_{\text{ét}}(Y(\Gamma_1(N))_{\overline{\mathbb{Q}}}, \mathrm{Symm}^{k-2}(\mathscr{T})(1))$$

on which Hecke operators (appropriately defined) act as multiplication by their eigenvalues on f.

Why should you expect that this has anything to do with modular forms? ([Wie, §6.4]) This follows from the Shimura isomorphism, which I now explain. First of all notice that $\mathscr{T}$ is locally free of rank two (the stalk at any point is the Tate module of the fiber) and one can identify (for any ring R) $\mathrm{Symm}^{k-2}(R^2)$ with the space $V_{k-2}(R)$ of homogeneous polynomials of degree $k-2$ in two variables X, Y. Next, one has the properly called Shimura isomorphism

$$M_k(\Gamma_1(N), \mathbb{C}) \oplus \overline{S_k(\Gamma_1(N), \mathbb{C})} \cong H^1(\Gamma_1(N), V_{k-2}(\mathbb{C}))$$

(where H^1 denotes group cohomology and the action on polynomials is precomposition by the action of a matrix on the row vector (X, Y)) given by

$$(f, \overline{g}) \mapsto [\gamma \to \int_{z_0}^{\gamma z_0} f(z)(Xz+Y)^{k-2}dz + \int_{z_0}^{\gamma z_0} \overline{g}(z)(Xz+Y)^{k-2}d\overline{z}].$$

(Here $z_0 \in \mathbb{H}$ is fixed).

Finally one has a comparison isomorphism between étale cohomology (after choosing embedding $L \to \mathbb{C}$ and base changing to $\mathbb{C}$) with singular cohomology and singular cohomology of an affine curve with group cohomology of the fundamental group.

Actually for $k = 2$ and f with integer coefficients, the construction is much easier and was known earlier than the general one ([Mil96, Theorem 6.3]): over $\mathbb{C}$ a cusp form f corresponds to a unique holomorphic differential ω_f on $X_1(N)(\mathbb{C})$. Fix a point $P_0 \in X_1(N)(\mathbb{C})$. There is a well defined map $\alpha : X_1(N)(\mathbb{C}) \to \mathbb{C}/\Lambda_f$ given by

$$P \mapsto \int_{P_0}^{P} \omega_f$$

where Λ_f is the lattice in $\mathbb{C}$ given by the image of integration of ω_f along paths in $H_1(X_1(N)(\mathbb{C}), \mathbb{Z})$.

Then the complex elliptic curve $E_f := \mathbb{C}/\Lambda_f$ can be proven to have a model over $\mathbb{Q}$ and the Galois representation attached to f is just $T_p(E_f) \otimes \mathbb{Q}_p$ where $T_p(E_f) := \varprojlim_n E_f[p^n]$ is the Tate module of E_f.

2.2 Pseudocharacters

Definition 5. A 2-dimensional pseudocharacter of a group H valued in a commutative ring A is a pair of functions $\tau : H \to A$, $\delta : H \to A^\times$ satisfying

1. δ is a group homomorphism,
2. $\tau(xy) = \tau(yx)$ for every $x, y \in H$,
3. $\tau(1) = 2$,
4. $\tau(xy) + \delta(y)\tau(xy^{-1}) = \tau(x)\tau(y)$.

Remark 8. Suppose that one has a representation $\rho : H \to GL_2(A)$, then the pair $(\mathrm{Tr}(\rho), \det(\rho))$ is a 2 dimensional pseudocharacter.

Indeed, for matrices $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, y = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ we have

$$xy = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix}, \quad xy^{-1} = \det(y^{-1}) \begin{pmatrix} ad' - bc' & ba' - ab' \\ cd' - dc' & da' - cb' \end{pmatrix}$$

and the fourth relation becomes the identity

$$(aa' + bc' + cb' + dd') + (ad' - bc' - cb' + da') = (a + d)(a' + d').$$

In fact, if 2 is invertible in A, then it is enough to know $\mathrm{Tr}(\rho)$ because of the relation

$$2\det(x) = \mathrm{Tr}(x)^2 - \mathrm{Tr}(x^2)$$

for every 2-dimensional matrix x (the same in fact holds for point 4 of the above definition: evaluate at $y = x$).

One has the following reconstruction theorem ([Bel21, Theorem 2.6.24]):

Theorem 9.

- *Suppose that (τ, δ) is a 2-dimensional pseudocharacter of a group H valued in a field L. Then there exists a finite extension L' of L and a semisimple representation $\rho : H \to GL_2(L')$, unique up to isomorphism, such that $\mathrm{Tr}(\rho) = \tau, \det(\rho) = \delta$. If L is a local field and (τ, δ) are continuous, then ρ is also continuous.*
- *Suppose that (τ, δ) is valued in a henselian local ring A with maximal ideal $\mathfrak{m}$ and residue field k and denote with $\pi : A \to k$ the projection. Suppose that there exists an absolutely irreducible representation $\overline{\rho} : H \to GL_2(k)$ such that $(\pi \circ \tau, \pi \circ \delta) = (\mathrm{Tr}(\overline{\rho}), \det(\overline{\rho}))$, then there exists a representation $\rho : H \to GL_2(A)$ such that $(\mathrm{Tr}(\rho), \det(\rho)) = (\tau, \delta)$. Moreover, ρ is unique up to isomorphism and it is continuous if A is a complete DVR and (τ, δ) are continuous.*

Let E be a finite extension of $\mathbb{Q}_p$ as before and consider the space $M_k(\Gamma_1(N), \mathcal{O}_E)$ of modular forms of level $\Gamma_1(N)$ and weight k whose q-expansions are in $\mathcal{O}_E$. Let $\mathcal{H}_0$ denote the abstract good Hecke algebra and $\mathbb{T}_0$ the associated eigenalgebra over $\mathcal{O}_E$ acting faithfully over $M_k(\Gamma_1(N), \mathcal{O}_E)$ (of course $\mathbb{T}_0$ depends on N, k even if it does not appear from the notation).

We can pack all the pseudocharacters attached to eigenforms in $M_k(\Gamma_1(N), \mathcal{O}_E)$ into a single one valued in $\mathbb{T}_0$ ([Bel21, Theorem 2.8.5]):

Theorem 10. *There exists a unique continuous 2-dimensional pseudocharacter*

$$\tau, \sigma : G_{\mathbb{Q},Np} \to \mathbb{T}_0 \otimes E$$

such that $\tau(\mathrm{Fr}_\ell) = T_\ell$ for every prime $\ell \neq Np$. Moreover, $\tau(c) = 0$ for c any complex conjugation and $\delta(\mathrm{Fr}_\ell) = \ell S_\ell$.

Proof. We give a sketch: it is enough to prove for a finite extension L/E (indeed, by continuity, the result would hold for E as well). Take L such that $\mathbb{T}_0 \otimes L \cong L^n$. Any factor corresponds to an eigenform and we know out to attach to them a Galois representation. Now take trace and determinant of their product. $\square$

In general we cannot do better than that. However, if $f \in M_k(\Gamma_1(N), \mathcal{O}_E)$ is an eigenform whose associated $\mod \varpi$ representation is absolutely irreducible, then we can apply point 2 of theorem 9 and get ([Bel21, Theorem 2.8.7]):

Theorem 11. *Assume that $\mathfrak{m}$ is a maximal ideal of $\mathbb{T}_0$ such that the composition of the pseudocharacter of theorem 10 with the reduction mod $\mathfrak{m}$ arises from an absolutely irreducible representation. Then there exists a unique continuous representation*

$$\rho_{\mathfrak{m}} : G_{\mathbb{Q},Np} \to GL_2(\mathbb{T}_{0,\mathfrak{m}})$$

such that $\mathrm{Tr}(\mathrm{Fr}_\ell) = T_\ell$ for every prime $\ell \neq Np$. Moreover, $\mathrm{Tr}(c) = 0$ for c any complex conjugation and $\delta(\mathrm{Fr}_\ell) = \ell S_\ell$.

2.3 Interpolating Galois representations along the eigencurve

With the eigenvariety at hand, we can actually construct a big p-adic family of Galois representations, and in particular we can attach them also to non-classical modular forms. For this denote with $\mathscr{E}$ the Coleman-Mazur eigenvariety of tame level N, denote with $w : \mathbb{Z}_p^\times \to \mathcal{O}(\mathscr{E})^\times$ the continuous character corresponding to the weight map $\kappa : \mathscr{E} \to \mathcal{W}$ and denote with ψ the algebra homomorphism $\phi : \mathcal{H}_0 \to \mathcal{O}(\mathscr{E})$ associated with the eigenvariety construction ($\mathcal{H}_0$ being the algebra generated by U_p and good Hecke algebra of level Np). Here are the results. The first one is [Bel21, Theorem 7.4.1]:

Theorem 12. *There exists a unique pseudocharacter*

$$(\tau, \delta) : G_{\mathbb{Q},Np} \to \mathcal{O}(\mathscr{E})$$

such that

$$\tau(\mathrm{Fr}_\ell) = \psi(T_\ell)$$

for every prime $\ell \nmid Np$*. Moreover,* $\tau(c) = 0$ *for every complex conjugation* c *and* δ *factors through the quotient* $\mathrm{Gal}(\mathbb{Q}(\zeta_{Np^\infty})/\mathbb{Q}) \cong \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ *and its restriction to* $\mathbb{Z}_p^\times$ *is the character*

$$\mathbb{Z}_p^\times \ni t \mapsto t \cdot w(t)$$

and its restriction to $(\mathbb{Z}/N\mathbb{Z})^\times$ *is the character*

$$(\mathbb{Z}/N\mathbb{Z})^\times \ni a \mapsto \langle a \rangle.$$

I remark that the construction of the eigencurve in [Bel21] is not the same as the one we have seen, and the relation is given in [Bel21, Theorem 7.2.3].

Definition 6. Recall ([Che05, §3.4]) that a subset of points Z of a rigid analytic space T is Zariski dense if for every closed analytic subvariety X of T (i.e. a subset given by the zero locus of a finite number of function on each element of an admissible covering) such that $Z \subset X$, then $X = T$.

Proof. We sketch a proof. We use the following facts: $\psi(T_\ell) \in \mathcal{O}(\mathscr{E})^0$ (the subring of power-bounded elements) for all $\ell \nmid pN$ and $\mathcal{O}(\mathscr{E})^0$ is compact (as a subspace of $\mathcal{O}(\mathscr{E})$ endowed with the coarsest locally convex topology such that each restriction map to open affinoids is compact). We consider the classical structure determined by taking $\mathbb{N} \subset \mathcal{W}(\mathbb{Q}_p)$, which is very Zariski dense in $\mathcal{W}$ by [Bel21, Lemma 6.7.3], and for $k \in \mathbb{N}$ we take as M_k^{cl} the space M_k of modular forms of weight k and level $\Gamma_1(N) \cap \Gamma_0(p)$. Since we have, for every $\nu \in \mathbb{R}$, that $(M_k^\dagger)^{\leq \nu} \hookrightarrow M_k^{cl}$ for every $k \geq \nu + 2$ (i.e. for every ν we are only excluding a finite number of k), then this satisfies the definition of a classical structure.

Denote with Z the set of classical points of $\mathscr{E}$ coming from the classical structure just defined. They are very Zariski-dense in $\mathscr{E}$ ([Bel21, proposition 3.8.6]). Moreover, for each $z \in Z$ corresponding to a modular form f of weight k, we have a pseudocharacter $\tau_z : G_{\mathbb{Q},Np} \to \overline{\mathbb{Q}}_p$ that satisfies $\tau_z(\mathrm{Fr}_\ell) = \psi(T_\ell)(z) =: ev_z(\psi(T_\ell))$ (and the other conditions in the statement), where ev_z is evaluation at z: the trace of the associated Galois representation (recall that $\psi(T_\ell)(z) = a_\ell(f)$).

We consider the map

$$ev_Z = \prod_{z \in Z} ev_z : \mathcal{O}(\mathscr{E})^0 \to \prod_{z \in Z} \overline{\mathbb{Q}}_p$$

this is injective by density and the image is a closed subspace. Moreover ev_Z is a homeomorphism on the image by compactness.

Since $\psi(T_\ell) \in \mathcal{O}(\mathscr{E})^0$, we have that the image of $\prod_{z\in Z} \tau_z$ is contained in $\prod_{z\in Z} ev_z$ and thus we can define $\tau := ev_Z^{-1} \circ \prod_{z\in Z} \tau_z$.

In particular we have a commutative diagram

$$\begin{tikzcd} \mathcal{O}(\mathscr{E})^0 \arrow[r, "\prod_z ev_z"] & \prod_{z\in Z} \overline{\mathbb{Q}}_p \\ & G_{\mathbb{Q},Np} \arrow[ul, "\tau"] \arrow[u, "\prod_{z\in Z} \tau_z"'] \end{tikzcd}$$

Since $ev_z(\tau(1)) = 2$ for every $z \in Z$ and $ev_z(\tau(xy) - \tau(yx)) = 0$ for every $z \in Z$, the same holds for every point of $\mathscr{E}$ and thus τ is the first part of a pseudocharacter. Since 2 is invertible in $\mathcal{O}(\mathscr{E})$, δ is completely determined from τ. $\square$

The second one is [Bel21, Corollary 7.4.2]:

Corollary 13. *Let $L/\mathbb{Q}_p$ be a finite extension and let $x \in \mathscr{E}(L)$. Then there exists a unique semisimple representation*

$$\rho_x : G_{\mathbb{Q},Np} \to GL_2(L)$$

with the property that

$$\mathrm{Tr}(\rho_x(\mathrm{Fr}_\ell)) = T_\ell(x)$$

for all primes $\ell \nmid Np$. Moreover, $\mathrm{Tr}(\rho_x(c)) = 0$ for every complex conjugation c and $\det(\rho_x)$ factors through the quotient $\mathrm{Gal}(\mathbb{Q}(\zeta_{Np^\infty})/\mathbb{Q}) \cong \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ and its restriction to $\mathbb{Z}_p^\times$ is the character

$$\mathbb{Z}_p^\times \ni t \mapsto t \cdot w(t)$$

and its restriction to $(\mathbb{Z}/N\mathbb{Z})^\times$ is the character

$$(\mathbb{Z}/N\mathbb{Z})^\times \ni a \mapsto \langle a \rangle.$$

Proof. By the reconstruction theorem there is a unique such representation with values in some finite extension L' of L and one has to show that this can be defined over L. $\square$

Moreover ([Bel21, lemma 7.4.9])

Proposition 14. *The Hodge-Tate weights of ρ_x are 0 and $-dw(x)-1$ where $dw(x)$ is the derivative of the character $w_x : \mathbb{Z}_p^\times \to \mathcal{O}(\mathscr{E})^\times \to L^\times$ at 1, if Hodge-Tate weights are normalized so that the weight of the cyclotomic character is -1.*

3 p-adic Langlands

3.1 Characters again

Recall that we had also constructed the eigenvariety of tame level N for GL_1: $\mathcal{W}_N \to \mathcal{W}$. Each of the classical points of $\mathcal{W}_N$ corresponded to a character of the form $\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times \ni z \mapsto z^k\psi(z)\overline{\mathbb{Q}}_p^\times$ where ψ is a finite character of conductor dividing Np^r for some $r \in \mathbb{N}$. We can certainly associate Galois representations on them: just compose the character with the isomorphism

$$\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty N})/\mathbb{Q}) \cong \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times.$$

The Hodge-Tate weights of such characters are integers.

A similar argument to theorem 12 shows that we can attach (non Hodge-Tate) representations to any point of $\mathcal{W}_N$. In fact, we do not need the argument to do that: it is enough to use the isomorphism

$$\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty N})/\mathbb{Q}) \cong \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times.$$

3.2 Interpolating p-adic Langlands at p-adic weights

We just say few words on how the story goes on. Fix a connected reductive linear algebraic group over $\mathbb{Q}$. The two examples above (for $G = GL_1, GL_2$) are the first historical examples that pointed to the (conjectural existence) of a p-adic Langlands correspondence. This is some sort of correspondence between

$$\{ \text{ algebraic Hecke eigenforms for } G \,\} \leftrightarrow \left\{ \text{ geometric continuous representations } G_{\mathbb{Q}} \to {}^L G(\overline{\mathbb{Q}}_p) \right\}$$

where geometric means that they are unramified outside a finite number of primes and de Rham at p.

Here by Hecke eigenform one usually means a function on the double coset ([CE, §1.5], and I haven't considered their A_∞° for simplicity)

$$X_r := G(\mathbb{Q}) \setminus G(\mathbb{A})/K_\infty^\circ K^{\infty,p} G_r$$

where K_∞° is the connected component of the identity in a maximal compact of real points, $K^{\infty,p}$ is an open compact subgroup of tame level, G_r some compact open subgroup of $G(\mathbb{Q}_p)$.

For example for $G = GL_1$ and $K^{\infty,p} = U_N$ (in the notation of Linda's talk, for some N prime to p), we get

$$\mathbb{Q}^\times \setminus \mathbb{A}^\times/\mathbb{R}_+^\times U_N \cong \mathbb{Z}_p^\times/G_r \times (Z/N\mathbb{Z})^\times$$

and for $G = GL_2$ and $K_{\infty,p} = K_1(N) := \left\{ (x_v)_v \in \mathbb{A}^{\infty,p} \;\middle|\; x_v \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod v \right\}$, we get

$$GL_2(\mathbb{Q}) \setminus GL_2(\mathbb{A})/SO_2(\mathbb{R})K_1(N)G_r$$

which are the $\mathbb{C}$ point of a modular curve of some level depending on G_r. To see how to construct a Hecke eigenform in the above sense form $f \in S_k(\Gamma_1(N), \mathbb{C})$ see the end of [DT94, §1].

Elements in both sides of the above correspondence have some integral invariant associated to them: Hodge numbers on the left (see discussion in [CE, §1.8] and references there) and Hodge-Tate weights on the right.

We have seen above that interpolating Galois representations along the eigencurve(s) give Galois representations which are non de Rham at p. This poses the question of how to enlarge the left hand side to include objects corresponding to representations that are non de Rham at p. This conjectural enlarged set is what one would call a space of p-adic automorphic forms for G.

Definitions of such objects that work in practice are given in terms of so called *completed cohomology*. Emerton in his paper [Eme06] introduces the theory of completed cohomology and gives a different construction of the Coleman-Mazur eigencurve by using completed cohomology of the modular curve (no gluing process involved!). The advantage of this theory is that one can use representation theoretic methods to study it.

Somewhat unusually, I conclude the presentation with the general definition ([CE, §1])

Definition 7. Let G_0 be a profinite group (in our examples $G(\mathbb{Z}_p)$ with a countable basis given by normal open subgroups $\cdots \subset G_r \subset \ldots G_1 \subset G_0$. Consider a tower of topological spaces given by finite coverings $\cdots \to X_r \to \cdots \to X_1 \to X_0 \subset$ with G_0 action such that G_r acts trivially on X_r and realizes X_0 as a G_0/G_r-torsor.

The completed cohomology for this data is

$$\tilde{H}^* := \varprojlim_s \varinjlim_r H^*(X_r, \mathbb{Z}/p^s).$$

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