Pro-étale cohomology for schemes

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October 29, 2024

The goal of this course is to give an introduction to (pro-)étale cohomology for schemes and explain how this theory defines a good notion of ℓ-adic cohomology. The course will be in two parts: a first one about étale cohomology (with an introduction to sheaf theory, including sheaf cohomology and a few facts about derived functors, study of the étale site of a scheme, some properties of étale sheaves). In the second part, I will (partially) explain the paper "The pro-étale topology for schemes" of Bhatt and Scholze (notion of locally weakly contractible topoi, replete topoi, weakly étale morphisms, comparison between étale and pro-étale and if time permits, constructible sheaves and 6-functors formalism in this setting).

Main references: The main reference for this course is the paper of Bhatt and Scholze [BS13]. For the étale cohomology I will mostly be using the books of Tamme [Tam94] and Milne [Mil80]. The chapters about étale cohomogy and pro-étale cohomology of the Stack Project [StackProject] can also be useful.

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1 Introduction: the Weil conjectures

One of the motivations for the introduction of étale cohomology comes from the so-called Weil conjectures. Those conjectures were stated by André Weil in the 40s and concern the number of points on varieties defined over finite fields. In this section, we briefly review the statement of these conjectures and how the construction of a "good" cohomology theory can help to solve them.

In this section X will be a smooth projective variety over a finite field $\mathbf{F}_q$, with $q = p^r$ for some prime p and $r \in \mathbf{N}_{\geq 1}$. We would like to count the $\mathbf{F}_{q^n}$-points of X, for $n \in \mathrm{N}$. This set is given by

$$X(\mathbf{F}_{q^n}) := \mathrm{Hom}_{\mathrm{Spec}(\mathbf{F}_q)}(\mathrm{Spec}(\mathbf{F}_{q^n}), X).$$

The above question can be reformulated using polynomials: for $f_1, \dots, f_m$ in $\mathbf{F}_q[t_0, \dots, t_d]$ homogeneous polynomials[1], we want to determine how many solutions the equations

$$f_1 = \dots = f_m = 0$$

have in $\mathbf{F}_{q^n}$, for each $n \in \mathrm{N}$. To solve this problem, let us introduce the zeta function associated to the variety X:

$$Z(X,t) := \exp\Big(\sum_{n=1}^{\infty} |X(\mathbf{F}_{q^n})| \frac{t^n}{n}\Big) \in \mathbf{Q}[|t|].$$

Note that if the function $Z(X,t)$ is known, then the numbers $|X(\mathbf{F}_{q^n})|$ can be recovered via the formula:

$$|X(\mathbf{F}_{q^n})| = \frac{1}{(n-1)!} \frac{d^n}{dt^n} \log(Z(X,t))\Big|_{t=0}.$$

So it is enough to compute the function $Z(X,t)$.

Before stating the conjectures, we give some examples where the zeta function is known.

Example 1.1 (The affine space)**.** Recall that the affine space $\mathbb{A}^d_{\mathbf{F}_q}$ of dimension d over $\mathbf{F}_q$ is the space $\mathrm{Spec}(k[x])$ endowed with its Zariski topology. So we have $|\mathbb{A}^d_{\mathbf{F}_q}(\mathbf{F}_{q^n})| = q^{nd}$ and this gives:

$$Z(\mathbb{A}^d_{\mathbf{F}_q}, t) = \frac{1}{1 - q^d t}.$$

Example 1.2 (The projective space)**.** An $\mathbf{F}_{q^n}$-point in $\mathbb{P}^d(\mathbf{F}_{q^n})$ can be described by its homogeneous coordinates $[x_0, x_1, \dots, x_d]$, with $x_i \in \mathbf{F}_{q^n}$ and at least one of the x_i's is non-zero. Two sets of coordinates give the same point if and only if one is the multiplication of the other by an element of $\mathbf{F}_{q^n}^{\times}$. This gives $|\mathbb{P}^d(\mathbf{F}_{q^n})| = \frac{q^{n(d+1)}-1}{q^n-1}$ and

$$Z(\mathbb{P}^d, t) = \frac{1}{(1-t)(1-qt)\dots(1-q^d t)}.$$

[1]They are the polynomials such that the variety X is given by $X := \mathrm{Proj}\big(\frac{\mathbf{F}_q[t_0,\dots,t_d]}{\langle f_1,\dots,f_m\rangle}\big)$.

Example 1.3. There are other cases where the zeta function for the varieties X is known. For an elliptic curve E, it can be shown that the zeta function can be computed via the formula

$$Z(E,t) = \frac{(1-\alpha t)(1-\beta t)}{(1-t)(1-qt)}$$

where α and β are conjugated in $\mathbf{C}$ and with absolute value $q^{\frac{1}{2}}$ (see for example [Sil09, Chapter 5]). More generally, if X is a curve of genus g then $Z(X,t)$ can be written

$$Z(X,t) = \frac{f(t)}{(1-t)(1-qt)}$$

with $f(t) \in \mathbf{Z}[t]$ of degree $2g$.

These computations lead to the following conjectures:

Conjecture 1.4 (Weil Conjectures)**.** Let X be a smooth connected projective variety of dimension d over $\mathbf{F}_q$. Then the zeta function $Z(X,t)$ satisfies the following property:

(i). **Rationality**: $Z(X,t)$ is a rational function in the variable t, with coefficients in $\mathbf{Q}$. More precisely,
$$Z(X,t) = \frac{P_1 ... P_{2d-1}}{P_0 ... P_{2d}},$$
with $P_i(t) \in \mathbf{Z}[t]$. Moreover we have $P_0(t) = 1-t$, $P_{2d}(t) = 1-q^d t$ and for $1 \leq i \leq 2d-1$, $P_i(t)$ is of the form $\prod_j (1-\alpha_{i,j} t)$.

(ii). **Functional equation**: there exists an integer $N \in \mathbf{N}$ such that $Z(X, q^{-d}t^{-1}) = \pm q^{\frac{Nd}{2}} t^N Z(X,t)$.

(iii). **Riemann hypothesis for finite fields**: the $\alpha_{i,j}$'s have absolute values $q^{\frac{-i}{2}}$.

(iv). **Relation to topology**: If X comes from a smooth projective variety over some $R \subset \mathbf{C}$, i.e. if X can be written $Y \otimes_R \mathbf{F}_q$ where R surjects onto $\mathbf{F}_q$ and Y is smooth and projective over $\mathbf{C}$, then
$$\deg P_i(t) = \dim_{\mathbf{Q}} H^i_{\text{sing}}(X(\mathbf{C}), \mathbf{Q}).$$

This conjecture was stated by Weil in 1949 and he proved it for curves and for abelian varieties. Dwork showed the rationality of the zeta function using methods from p-adic functional analysis. The introduction and study of ℓ-adic cohomology by Artin and Grothendieck, then allowed to prove the functional equation and later, in 1973, Deligne used it to prove the Riemann hypothesis for finite fields. This ℓ-adic cohomology will be the main object of study of this course.

Let us now explain how cohomology can be useful to prove these conjectures. Denote by $\mathrm{Var}_{\mathbf{F}_q}$ the category of algebraic varieties over $\mathbf{F}_q$. For the moment, let us assume that there exists a cohomology theory:

$$H^\bullet : \mathrm{Var}_{\mathbf{F}_q}^{\mathrm{op}} \to \{\text{graded } \mathbf{Q}\text{-vector spaces}\} \tag{1.0.0.1}$$

such that for a variety X smooth and projective of dimension d, the $\mathbf{Q}$-vector space $H^i(X)$ is finite dimensional for all i and $H^i(X) = 0$ for $i > 2d$. Write $X_{\overline{\mathbf{F}}_q}$ the base change of X to an algebraic closure $\overline{\mathbf{F}}_q$ of $\mathbf{F}_q$. The variety $X_{\overline{\mathbf{F}}_q}$ is equipped with a Frobenius morphism $\varphi : X_{\overline{\mathbf{F}}_q} \to X_{\overline{\mathbf{F}}_q}$. Assume that the cohomology $H^\bullet$ satisfies the following formula (called "Lefschetz trace formula"):

$$|X(\mathbf{F}_{q^n})| = \sum_{i=0}^{2d} (-1)^i \mathrm{tr}(H^i(\varphi^n)), \quad \text{for all } n \geq 1.$$

Note that $X(\mathbf{F}_{q^m})$ corresponds to the set of fixed points of the morphism $\varphi^m : X_{\overline{\mathbf{F}}_q} \to X_{\overline{\mathbf{F}}_q}$. Now we can plug in this formula into the definition of the zeta function. This yields

$$\begin{aligned} Z(X,t) &= \exp\Big(\sum_{n=1}^{\infty} |X(\mathbf{F}_{q^n})| \frac{t^n}{n} \Big) = \exp\Big(\sum_{n=1}^{\infty} \big(\sum_{i=0}^{2d} (-1)^i \mathrm{tr}(H^i(\varphi^n)) \frac{t^n}{n} \big) \Big) \\ &= \prod_{i=0}^{2d} \Big(\exp\big(\sum_{n=1}^{\infty} \mathrm{tr}(H^i(\varphi^n)) \frac{t^n}{n} \big) \Big)^{(-1)^i} = \prod_{i=0}^{2d} \det\big(\mathrm{Id} - t \cdot H^i(\varphi^n) \big)^{(-1)^{i+1}}. \end{aligned}$$

This would prove the rationality of $Z(X,t)$ and the proof of the Riemann Hypothesis for finite fields is reduced to the study of the eigenvalues of $H^i(\varphi)$. If moreover the cohomology theory satisfies Poincaré duality, i.e. there exists a trace isomorphism $H^{2d}(X) \xrightarrow{\sim} \mathbf{Q}$ that induces a natural perfect pairing of $\mathbf{Q}$-vector spaces:

$$H^i(X) \times H^{2d-i}(X) \to \mathbf{Q}$$

giving $H^i(X) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Q}}(H^{2d-i}(X), \mathbf{Q})$, then similar computations prove the functional equation for $Z(X,t)$. If in addition $H^\bullet$ can be compared with singular homology, we would get the point (iv) of the Weil conjectures. So we see that cohomology can be used as a tool to transform an algebraic geometry problem to a problem of linear algebra.

A cohomology theory satisfying this kind of "nice" properties (finiteness, vanishing in higher degrees, Poincaré duality, some kind of Lefschetz trace formula) is called a Weil cohomology theory. When working with varieties over $\mathbf{C}$, singular cohomology satisfies the axioms of Weil cohomology. More generally, in characteristic zero, the de Rham cohomology also defines a Weil cohomology theory. However, when working over finite fields, a Weil cohomology theory as written in (1.0.0.1) does not exist: this is due to the existence of supersingular elliptic curve (Serre). Indeed, assuming that for an elliptic curve E, such a cohomology exists this would give an anti-homomorphism

$$(\mathrm{End} E) \otimes \mathbf{Q} \to \mathrm{End}(H^1(E, \mathbf{Q})).$$

But if E is supersingular, this implies that the quaternion algebra $(\mathrm{End}\, E) \otimes \mathbf{Q}$ is non-split at p and ∞ and we obtain that $(\mathrm{End}\, E) \otimes \mathbf{R}$ is the Hamilton quaternions algebra $\mathbf{H}$. Extending the scalar to $\mathbf{R}$ in the formula above, we could get an anti-homomorphism $\mathbf{H} \to \mathcal{M}_2(\mathbf{R})$, but this does not exist. In fact, this argument shows that it is not possible to define a Weil cohomology with values in $\mathbf{R}$ and $\mathbf{Q}_p$, but it is still possible to work with $\mathbf{Q}_\ell$-coefficients, for $\ell \neq p$: this uses étale cohomology.

We will see that for a variety X over $\mathbf{F}_q$, the étale cohomology $H^\bullet_{\text{ét}}(X, \mathbf{Z}/\ell^n\mathbf{Z})$ of X with $\mathbf{Z}/\ell^n\mathbf{Z}$-coefficients define a cohomology theory with nice properties. This cohomology groups are $\mathbf{Z}/\ell^n$-modules and we obtain a $\mathbf{Q}_\ell$ vector space by taking the limit over n and inverting ℓ:

$$H^i_{\text{ét}}(X, \mathbf{Q}_\ell) := \varprojlim_n H^i_{\text{ét}}(X, \mathbf{Z}/\ell^n\mathbf{Z}) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell \quad \text{for } i \geq 0. \tag{1.0.0.2}$$

This definition of the ℓ-adic cohomology works relatively well and was used for many years. However, the fact that it is defined using inverse limit can cause problems and makes it difficult to handle. In [BS13], Bhatt and Scholze have introduced a new topology, the pro-étale topology, that gives a setting in which inverse limits behave well. We will see that the ℓ-adic pro-étale cohomology theory extends the étale one, giving a good definition of ℓ-adic cohomology in the cases where the definition (1.0.0.2) is defective and recovering it in the cases where it works well.

Remark 1.5 (Weil cohomology with p-adic coefficients). For E a supersingular elliptic curve, as mentioned before, $(\operatorname{End} E) \otimes \mathbf{Q}_p$ is not split leading to the impossibility to construct a Weil cohomology theory with coefficients in $\mathbf{Q}_p$. However, the algebra $(\operatorname{End} E) \otimes F$ where F is the fraction field of the Witt vector ring $W(\overline{\mathbf{F}}_q)$ is split. This suggests that it should be possible to define a Weil cohomology in the p-adic case, as long as we work with F-coefficients instead of $\mathbf{Q}_p$ ones, and it is indeed the case. The crystalline cohomology defines such a cohomology theory.

2 Sheaf theory

2.1 Grothendieck topologies and presheaves

Definition 2.1. Let $\mathscr{C}$ be an arbitrary category. A Grothendieck topology on $\mathscr{C}$ is the data, for any object U in $\mathscr{C}$, of a set $\mathrm{Cov}(U)$ of families $\{\varphi_i : U_i \to U\}_{i\in I}$ of morphisms in $\mathscr{C}$, called the coverings of U, satisfying the following axioms:

(i). **Isomorphism**: If $\varphi : U' \to U$ is an isomorphism in $\mathscr{C}$ then $\{\varphi : U' \to U\}$ is in $\mathrm{Cov}(U)$.

(ii). **Locality**: If $\{\varphi_i : U_i \to U\}_{i\in I}$ is in $\mathrm{Cov}(U)$ and for all i there are coverings $\{\psi_{i,j} : V_{i,j} \to U_i\}_{j\in J_i}$ in $\mathrm{Cov}(U_i)$ then $\{\varphi_i \circ \psi_{i,j} : U_{i,j} \to U\}_{(i,j)\in\prod_{i\in I}\{i\}\times J_i}$ is in $\mathrm{Cov}(U)$.

(iii). **Base change**: If $\{\varphi_i : U_i \to U\}_{i\in I}$ is in $\mathrm{Cov}(U)$ and $U' \to U$ is a morphism in $\mathscr{C}$ then,

a) for all $i \in I$, $U_i \times_U U'$ exists in $\mathscr{C}$,

b) the family $\{U_i \times_U U' \to U'\}_{i\in I}$ is in $\mathrm{Cov}(U')$.

A site is the data of a category $\mathscr{C}$ together with a Grothendieck topology on $\mathscr{C}$. The set of all coverings in $\mathscr{C}$ is denoted by $\mathrm{Cov}(\mathscr{C})$.

Example 2.2. (i). Let X be a topological space. The category of open subsets of X together with the usual coverings (i.e. the families $\{U_i \subset U\}_{i\in I}$ such that $U = \bigcup_{i\in I} U_i$) defines a Grothendieck topology. For two open subsets U_1 and U_2 inside an open subset V of X, the fibre product $U_1 \times_W U_2$ is the intersection $U_1 \cap U_2$.

(ii). Let X be a topological space. Consider $\mathrm{Top}|_X$ the category of topological spaces over X: the objects are pairs (Y, f) where Y is a topological space and $f : Y \to X$ is a continuous map, and the morphisms are the continuous maps $Y \to Z$ such that the following diagram commutes:

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ \downarrow & \swarrow & \\ X & & \end{array} .$$

For Y in $\mathrm{Top}|_X$, we say that a family of continuous maps $\{\varphi_i : Y_i \to Y\}_{i\in I}$ is a covering if $Y = \bigcup_{i\in I} \varphi_i(Y_i)$. This defines a Grothendieck topology on $\mathrm{Top}|_X$. The same holds if we require moreover the $\varphi_i : Y_i \to Y$ to be open immersions.

From now on, Ab will denote the category of abelian groups.

Definition 2.3. Let $\mathscr{C}$ be a category. A presheaf of sets (respectively, an abelian presheaf) on $\mathscr{C}$ is a functor $\mathscr{F} : \mathscr{C}^{\mathrm{op}} \to \mathrm{Set}$ (respectively, Ab). If U is an object in $\mathscr{C}$, we write $\Gamma(U, \mathscr{F}) := \mathscr{F}(U)$ and the elements in $\Gamma(U, \mathscr{F})$ are called sections of $\mathscr{F}$ on U. For $\varphi : V \to U$ a morphism in $\mathscr{C}$, and s a section in $\mathscr{F}(V)$ we write

$$\mathscr{F}(\varphi)(s) = s|_V$$

and the map $\mathscr{F}(\varphi)$ is called the restriction map.

The category of presheaves (where the morphisms are the natural transformations of functors) of sets on $\mathscr{C}$ is denoted by $\mathrm{PreShv}(\mathscr{C})$. The category of abelian presheaves is denoted by $\mathrm{PreShvAb}(\mathscr{C})$.

Example 2.4. Let $\mathscr{C}$ be a category and X an object in $\mathscr{C}$. The following functor defines a presheaf on $\mathscr{C}$:

$$h_X : \begin{cases} \mathscr{C}^{\mathrm{op}} & \to \mathrm{Set} \\ U & \mapsto h_X(U) := \mathrm{Hom}_{\mathscr{C}}(U, X). \end{cases}.$$

The Yoneda lemma states that for two objects X, Y in $\mathscr{C}$, there is a natural bijection:

$$\mathrm{Hom}_{\mathscr{C}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{PreShv}(\mathscr{C})}(h_X, h_Y).$$

2.2 Sheaves

2.2.1 Definition

Definition 2.5. Let X, Y, Z be sets, and let $\alpha : X \to Y$ and $\beta, \gamma : Y \to Z$ be maps. We say that the diagram

$$X \xrightarrow{\alpha} Y \underset{\gamma}{\overset{\beta}{\rightrightarrows}} Z$$

is exact if α is injective and the image of α is equal to the equalizer of (β, γ), that is

$$\mathrm{Im}(\alpha) = \{y \in Y \mid \beta(y) = \gamma(y)\}.$$

Note that if X, Y, Z are abelian groups and α, β, γ are linear, then the diagram above is exact if and only if the sequence

$$0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta-\gamma} Z \to 0$$

is exact.

Definition 2.6. Let $\mathscr{C}$ be a site, and let $\mathscr{F}$ be a presheaf of sets or abelian groups on $\mathscr{C}$. We say that $\mathscr{F}$ is a sheaf if for every covering $\{U_i \to U\}_{i \in I}$ in $\mathrm{Cov}(\mathscr{C})$, the diagram

$$\mathscr{F}(U) \longrightarrow \prod_{i \in I} \mathscr{F}(U_i) \rightrightarrows \prod_{(i,j) \in I^2} \mathscr{F}(U_i \times_U U_j) \tag{2.2.1.1}$$

is exact, where the two arrows on the right are given by $(s_i)_i \mapsto (s_i|_{U_i \times_U U_j})_{i,j}$ and $(s_i)_i \mapsto (s_j|_{U_i \times_U U_j})_{i,j}$ respectively.

If $\mathscr{C}$ is a site and X an object in $\mathscr{C}$, then we define the site $\mathscr{C}_X$ in the following way. The objects of $\mathscr{C}_X$ are morphisms $Y \to X$ with Y an object of $\mathscr{C}$. Morphisms between objects $Y \to X$ and $Y' \to X$ are morphisms $Y \to Y'$ in $\mathscr{C}$ that make the obvious diagram commute and a family of morphisms $\{Y_i \to Y\}_i$ of objects over Y is a covering in $\mathscr{C}_X$ if and only if it is a covering in $\mathscr{C}$.

For the empty covering (i.e. when $I = \varnothing$), this implies that $\mathscr{F}(\varnothing)$ is an empty product, which is a final object in the corresponding category (so, a singleton for Set and Ab). We denote $\mathrm{Shv}(\mathscr{C})$ (respectively $\mathrm{ShvAb}(\mathscr{C})$) the full subcategory of $\mathrm{PreShv}(\mathcal{C})$ (respectively $\mathrm{PreShvAb}(\mathscr{C})$) which objects are sheaves.

It can be showed that for a site $\mathscr{C}$, the categories $\mathrm{PreShvAb}(\mathscr{C})$ and $\mathrm{ShvAb}(\mathscr{C})$ are abelian[2] (see for example [Tam94, §3]).

Example 2.7 (Sheaves vs presheaves)**.** Let X be a topological space. Then, (1) the presheaf $U \mapsto \{ \text{ functions } U \to \mathbf{Z}\}$ is a sheaf.

(2) the presheaf $U \mapsto \{ \text{ constant functions } U \to \mathbf{Z}\}$ is not a sheaf (the glueing does not always exist).

(3) the presheaf $U \mapsto \begin{cases} 0 & \text{if } U \neq X \\ \mathbf{Z} & \text{if } U \neq X \end{cases}$ is not a sheaf (the glueing in not necessarily unique).

Example 2.8 (Sheaves on $G - \mathrm{Set}$)**.** This example is important and will come back later in the course. Let G be a group. We denote by $G{-}\mathrm{Set}$ the category whose objects are sets endowed with a left G-action and morphisms are equivariant maps. We endow $G - \mathrm{Set}$ with the Grothendieck topology in which the coverings are the families $\{\varphi_i : U_i \to U\}_{i\in I}$ such that $U = \bigcup_{i\in I} \varphi_i(U_i)$. Note that G is itself an object in $G - \mathrm{Set}$ (the action is given by left translations). Let us denote by $\mathcal{T}_G$ this site.

Lemma 2.9. *The functor*

$$\begin{cases} \mathrm{Shv}(\mathcal{T}_G) & \to G - \mathrm{Set} \\ \mathscr{F} & \mapsto \mathscr{F}(G) \end{cases} \tag{2.2.1.2}$$

defines an equivalence of categories.

Proof. We first check that it is well-defined, i.e. $\mathscr{F}(G)$ is in $G - \mathrm{Set}$. Using the isomorphism

$$\begin{cases} G & \xrightarrow{\sim} \mathrm{Aut}_G(G) \\ g & \mapsto (h \mapsto hg) \end{cases}$$

we see that any $g \in G$ gives rise to an element of $\mathrm{Aut}_{G-\mathrm{Set}}(G)$ and so to a map $\mathscr{F}(G) \to \mathscr{F}(G)$. Hence, we get a left action of G on $\mathscr{F}(G)$.

[2]This means that the hom-sets are abelian groups, we can define kernels and cokernels and they behave nicely.

To prove it is an equivalence of categories, we will show that the functor

$$\begin{cases} G - \mathrm{Set} & \to \mathrm{Shv}(\mathcal{T}_G) \\ Z & \mapsto h_Z : U \mapsto \mathrm{Hom}_G(U, Z) \end{cases}$$

defines a quasi-inverse for (2.2.1.2). Let Z be a G-set. The isomorphism $h_Z(G) \xrightarrow{\sim} Z$ is given by the map $\varphi \mapsto \varphi(1_G)$. Conversely, let $\mathscr{F}$ be in $\mathrm{Shv}(\mathcal{T}_G)$, we want to prove that we have an isomorphism of sheaves

$$\mathscr{F} \xrightarrow{\sim} \mathrm{Hom}_G(-, \mathscr{F}(G)).$$

Let Z be a G-set. The set $\{G \xrightarrow{\varphi_z} Z\}_{z \in Z}$ where $\varphi_z(g) = g \cdot z$ for all $z \in Z$ and $g \in G$, is a covering in $\mathcal{T}_G$. So, by definition, the following diagram is exact

$$\mathscr{F}(Z) \longrightarrow \prod_{z \in Z} \mathscr{F}(G) \rightrightarrows \prod_{(z_1, z_2) \in Z \times Z} \mathscr{F}(G \times_Z G)$$

Note that the term is the middle $\prod_{z \in Z} \mathscr{F}(G)$ is equal to $\mathrm{Hom}(Z, \mathscr{F}(G))$ (with no G-structure in the hom-set). To finish the proof it suffices to prove that the kernel of the right map in the diagram above is equal to the subset $\mathrm{Hom}_G(Z, \mathscr{F}(G))$ of $\mathrm{Hom}(Z, \mathscr{F}(G))$. But this follows from the definition of the maps in the diagram, noting that for z_1, z_2 in Z, the product of the two corresponding copies of G is equal to G if there exists $g \in G$ such that $z_2 = g \cdot z_1$ and empty otherwise. □

Replacing the category of sheaves of set by the category of abelian sheaves, we obtain:

Corollary 2.10. *The category of left G-modules is equivalent to the category of abelian sheaves on the canonical topology $\mathcal{T}_G$. The equivalence is given by the quasi-inverse functors $\mathscr{F} \mapsto \mathscr{F}(G)$ and $M \mapsto \mathrm{Hom}_G(-, M)$.*

2.2.2 Sheafification

Let $\mathscr{T}$ be a site. The goal of this section is to define a sheafifacation functor $(-)^\sharp : \mathrm{PreShvAb}(\mathscr{T}) \to \mathrm{ShvAb}(\mathscr{T})$, which is left-adjoint to the inclusion functor $i : \mathrm{ShvAb}(\mathscr{T}) \to \mathrm{PreShvAb}(\mathscr{T})$. As a first approximation of the sheafification of the presheaf $\mathscr{F}$, we introduce the following definition:

Definition 2.11. Let $\mathscr{F}$ be a presheaf on the site $\mathscr{T}$ and $\mathscr{U} = \{U_i \to U\}$ in $\mathrm{Cov}(\mathscr{T})$. We define the 0-th Čech cohomology group of $(\mathscr{U}, \mathscr{F})$ by

$$\check{H}^0(\mathscr{U}, \mathscr{F}) = \left\{ (s_i)_{i \in I} \in \prod_{i \in I} \mathscr{F}(U_i) \mid s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j} \right\}.$$

Note that there is a natural map $\mathscr{F}(U) \to \check{H}^0(\mathscr{U}, \mathscr{F})$. We would like to make the covering $\mathscr{U}$ in $\check{H}^0(\mathscr{U}, \mathscr{F})$ vary. To do this, we need the notion of refinement of a covering: for $\mathscr{U} = \{U_i \to U\}_{i \in I}$ in $\mathrm{Cov}(\mathscr{T})$, a covering $\mathscr{V} = \{V_j \to U\}_{j \in J}$ is a refinement of $\mathscr{U}$ if there exists a map $\alpha : J \to I$ and for all $j \in J$, a commutative diagram

$$\begin{array}{ccc} V_j & \xrightarrow{f_j} & U_{\alpha(j)} \\ \downarrow & \swarrow & \\ U & & \end{array} .$$

Note that for every refinement $f : \mathscr{V} \to \mathscr{U}$ in $\mathrm{Cov}(U)$, we get a canonical map

$$\check{H}^0(\mathscr{U}, \mathscr{F}) \to \check{H}^0(\mathscr{V}, \mathscr{F}),$$

given by $(s_i)_{i \in I} \mapsto ((s_{\alpha(j)})|_{V_j})_{j \in J}$. We can show that this map is independent of the choices of α and the f_j's.

Definition 2.12. Let $\mathscr{F}$ be an abelian presheaf on $\mathscr{T}$. For every $U \in \mathscr{T}$, we define

$$\mathscr{F}^+(U) = \varinjlim_{\mathscr{U} \in \mathrm{Cov}(U)} \check{H}^0(\mathscr{U}, \mathscr{F}).$$

Let $V \to U$ be a morphism in $\mathscr{T}$. If $\mathscr{U} = \{U_i \to U\}_{i \in I}$ is a covering of U, then $\mathscr{V} = \{U_i \times_U V \to V\}_{i \in I}$ is a covering of V and we get a morphism $\check{H}^0(\mathscr{U}, \mathscr{F}) \to \check{H}^0(\mathscr{V}, \mathscr{F})$. Taking the colimit, we get a morphism $\mathscr{F}^+(U) \to \mathscr{F}^+(V)$. This gives to $\mathscr{F}^+$ the structure of presheaf on $\mathscr{T}$.

Proposition 2.13. *Let $\mathscr{T}$ be a site and $\mathscr{F}$ an abelian presheaf on $\mathscr{T}$. Then $\mathscr{F}^\sharp := (\mathscr{F}^+)^+$ is a sheaf and the canonical map induces a functorial isomorphism*

$$\mathrm{Hom}_{\mathrm{PreShv}(\mathscr{C})}(\mathscr{F}, \mathscr{G}) = \mathrm{Hom}_{\mathrm{Shv}(\mathscr{C})}(\mathscr{F}^\sharp, \mathscr{G})$$

for any $\mathscr{G} \in \mathrm{Shv}(\mathscr{T})$.

The proof of the proposition uses the notion of separated presheaf: a presheaf $\mathscr{F}$ is separated if for every U in $\mathscr{C}$ and $\mathscr{U}$ in $\mathrm{Cov}(U)$, the canonical map

$$\mathscr{F}(U) \to \prod_{i \in I} \mathscr{F}(U_i)$$

is injective.

Sketch of proof. The proof is in three steps:

(1) There is a canonical map of presheaves $\mathscr{F} \to \mathscr{F}^+$.

(2) If $\mathscr{F}$ is a separated presheaf then $\mathscr{F}^+$ is a sheaf and the map $\mathscr{F} \to \mathscr{F}^+$ is injective.

(3) The presheaf $\mathscr{F}^+$ is separated.

Details can be read in [Tam94, §3]. □

Theorem 2.14. *Let $\mathscr{C}$ be a site. The category $\mathrm{ShvAb}(\mathscr{C})$ of abelian sheaves on $\mathscr{C}$ is an abelian category. The inclusion functor $i : \mathrm{ShvAb}(\mathscr{C}) \to \mathrm{PreShvAb}(\mathscr{C})$ is left exact and the sheafification functor $(-)^\sharp : \mathrm{PreShvAb}(\mathscr{C}) \to \mathrm{ShvAb}(\mathscr{C})$ is exact.*

Proposition 2.15 (Examples and properties). *(i). If $\mathscr{F}$ is a sheaf then $\mathscr{F} \simeq \mathscr{F}^\sharp$.*

(ii). If $f : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves then the presheaf

$$\mathrm{Ker}(f) := U \mapsto (f_U : \mathscr{F}(U) \to \mathscr{G}(U))$$

is a sheaf.

(iii). If $f : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, we define the image of f, denoted by $\mathrm{Im}(f)$ as the sheafification of the presheaf:

$$U \mapsto \mathrm{Im}(f_U : \mathscr{F}(U) \to \mathscr{G}(U)).$$

(iv). Let $\mathscr{C}$ be a site and $f : X \to Y$ in $\mathscr{C}$. The direct image functor is defined as

$$f_* : \begin{cases} \mathrm{Shv}(\mathscr{C}_X) \to \mathrm{Shv}(\mathscr{C}_Y) \\ \mathscr{F} \mapsto f_*\mathscr{F} = (U \mapsto \mathscr{F}(U \times_Y X)) \end{cases}.$$

(As an exercise: check that $f_*\mathscr{F}$ is indeed a sheaf.)

(v). Let $\mathscr{C}$ be a site and $f : X \to Y$ in $\mathscr{C}$. The inverse image $f^{-1}\mathscr{F}$ of a sheaf $\mathscr{F}$ over Y is defined as the sheafification of $U \mapsto \mathrm{colim}_V \mathscr{F}(V)$ where the colimit is over the schemes $V \to Y$ such that there is a map $U \to X \times_Y V$. The inverse image functor is the functor

$$f^{-1} : \begin{cases} \mathrm{Shv}(\mathscr{C}_Y) \to \mathrm{Shv}(\mathscr{C}_X) \\ \mathscr{F} \mapsto f^{-1}\mathscr{F} \end{cases}.$$

3 Crash-course on derived categories

Let $\mathscr{C}$ and $\mathscr{D}$ be abelian category and $F : \mathscr{C} \to \mathscr{D}$ a left exact functor. This means that if we have an exact sequence of object in $\mathscr{C}$:

$$0 \to A \to B \to C \to 0$$

after applying F, we get an exact sequence

$$0 \to F(A) \to F(B) \to F(C).$$

We would like to extend this exact sequence further. To do that we will define the higher derived functors of F: they are functors $\mathrm{R}^i F$ for all $i \geq 0$, such that we have a long exact sequence

$$0 \to F(A) \to F(B) \to F(C) \to \mathrm{R}^1 F(A) \to \mathrm{R}^1 F(B) \to \mathrm{R}^1 F(C) \to \mathrm{R}^2 F(A) \to \cdots$$

When we are in the case $\mathscr{C} = \mathrm{ShvAb}(\mathscr{T})$ for some site $\mathscr{T}$ with a base X, $\mathscr{D} = \mathrm{Ab}$ and $F = \Gamma(X, -)$, these derived functors will define the cohomology of a sheaf $\mathscr{F}$:

$$H^i(X, \mathscr{F}) := \mathrm{R}^i \Gamma(X, \mathscr{F}),$$

in other words, for a short exact sequence of sheaves $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$, we obtain a long exact sequence

$$0 \to H^0(X, \mathscr{F}) = \mathscr{F}(X) \to H^0(X, \mathscr{G}) \to H^0(X, \mathscr{H}) \to H^1(X, \mathscr{F}) \to H^1(X, \mathscr{G}) \to H^1(X, \mathscr{H}) \to \cdots$$

Many details in this section will be skipped. More precise statements and proofs can be read in [Wei94] or [StackProject].

3.1 Definition of derived category

3.1.1 The homotopy category

In this section $\mathscr{A}$ will always be an abelian category. A chain complex $K^\bullet$ is a sequence

$$\cdots \xrightarrow{d} K^{i-1} \xrightarrow{d} K^i \xrightarrow{d} K^{i+1} \xrightarrow{d} \cdots$$

such that the composition $d \circ d$ is zero. A morphism of chain complexes $f : K^\bullet \to L^\bullet$ is a sequence of morphisms $\{f_i : K^i \to L^i\}_{i \in \mathbf{Z}}$ such that $d \circ f_i = f_{i+1} \circ d$. We will denote by $\mathrm{Ch}(\mathscr{A})$ the category of chain complexes in $\mathscr{A}$. It can be showed that since $\mathscr{A}$ is abelian, then $\mathrm{Ch}(\mathscr{A})$ is also abelian. We will also write $\mathrm{Ch}^+(\mathscr{A})$ (respectively $\mathrm{Ch}^-(\mathscr{A})$) for the full subcategory of bounded below (respectively bounded above) chain complexes, i.e. those $K^\bullet$ with $K^i = 0$ for $i << 0$ (respectively, $i >> 0$). The full subcategory of bounded (below and above) chain complexes will be denoted by $\mathrm{Ch}^b(\mathscr{A})$. In the category $\mathrm{Ch}(\mathscr{A})$, let us define the following operation:

- For $K^\bullet$ a chain complex and i an integer, the shift $K^\bullet[i]$ of K by i, is the chain complex such that the n-th term is K^{i+n}. Alternatively, $K^\bullet$ can be viewed as the tensor product $K^\bullet \otimes S^i$ where S^i is the chain complex whose terms are all zero except in degree $-i$ where it is $\mathbf{Z}$ and the tensor product in $\mathrm{Ch}(\mathscr{A})$ is defined by the formula:

$$(K^\bullet \otimes L^\bullet)^n = \bigoplus_{i+j=n} A^i \otimes B^j \quad \text{for all } n \in \mathbf{Z}.$$

- For a morphism of chain complexes $f : K^\bullet \to L^\bullet$, the cone of f is the chain complex $\mathrm{Cone}(f)$ such that the n-th term is $K^{n+1} \oplus L^n$ and the differentials are given by $\begin{pmatrix} d_K & 0 \\ f & d_L \end{pmatrix}$. Note that we obtain a short exact sequence of chain complexes

$$0 \to L^\bullet \to \mathrm{Cone}(f) \to K^\bullet[1] \to 0.$$

Alternatively, $\mathrm{Cone}(f)$ can be defined as the push-out

$$\begin{array}{ccc} L^\bullet \otimes S^0 & \longrightarrow & L^\bullet \otimes D^1 \\ \downarrow & & \downarrow \\ K^\bullet \otimes S^0 & \longrightarrow & \mathrm{Cone}(f) \end{array}$$

where S^0 is the chain complex whose terms are all zero except in degree 0 where it is $\mathbf{Z}$ and D^1 is the chain complex whose terms are all zero except in degree -1 and 0 where they are $\mathbf{Z}$ (and the differential between the two copies of $\mathbf{Z}$ is the identity).

- For $K^\bullet$ a chain complex and i an integer, the i-th cohomology of $K^\bullet$ is defined by the formula

$$H^i(K^\bullet) = \mathrm{Ker}(K^i \to K^{i+1})/\mathrm{Im}(K^{i-1} \to K^i) \quad \text{for all } i \in \mathbf{Z}.$$

The group $\mathrm{Ker}(K^i \to K^{i+1})$ is called the group of i-cocyles of $K^\bullet$ and we denote it by $Z^i(K^\bullet)$ while $\mathrm{Im}(K^{i-1} \to K^i)$ is the group of i-coboundaries of $K^\bullet$ and is denoted by $B^i(K^\bullet)$. If $0 \to A \to B \to C \to 0$ is a short exact sequence of chain complexes then taking cohomology yields a long exact sequence

$$\cdots \to H^{i-1}(C) \to H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to \cdots$$

Definition 3.1. Let $f : K^\bullet \to L^\bullet$ be a morphism of chain complexes. We say that f is a quasi-isomorphism if for all integer i, the morphisms $H^i(f) : H^i(K^\bullet) \to H^i(L^\bullet)$ are isomorphisms.

There is a notion of chain homotopy between two morphisms of chain complexes (see for example [Wei94, §1.4] for a definition). If f and g are morphisms of chain complexes $K^\bullet \to L^\bullet$ such that there exists a chain homotopy between f and g, we write $f \sim g$. This defines an equivalence relation on $\mathrm{Hom}_{\mathrm{Ch}(\mathscr{A})}(K^\bullet, L^\bullet)$. Note that if $f \sim g$ then $H^i(f) = H^i(g)$ for all $i \in \mathbf{Z}$. We say that two complexes $K^\bullet$ and $L^\bullet$ are homotopy equivalent if there exist $f : K^\bullet \to L^\bullet$ and

$g : L^\bullet \to K^\bullet$ such that $f \circ g \sim \mathrm{Id}$ and $g \circ f \sim \mathrm{Id}$. It can be proved that if $K^\bullet$ and $L^\bullet$ are homotopy equivalent then they are quasi-isomorphic (the converse is not true).

We define the homotopy category of $\mathscr{A}$ as the category $K(\mathscr{A})$ whose objects are chain complexes and sets of morphisms are the homotopy equivalence classes of maps of chain complexes, i.e. $\mathrm{Hom}_{K(\mathscr{A})}(K^\bullet, L^\bullet) = \mathrm{Hom}_{\mathrm{Ch}(\mathscr{A})}(K^\bullet, L^\bullet)/\sim$. Note that $K(\mathscr{A})$ satisfies the following universal property: for any functor $F : \mathrm{Ch}(\mathscr{A}) \to \mathscr{D}$ sending homotopy equivalence to isomorphism there exist a unique functor $\overline{F} : K(\mathscr{A}) \to \mathscr{D}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Ch}(\mathscr{A}) & \xrightarrow{F} & \mathscr{D} \\ \downarrow & \nearrow_{\overline{F}} & \\ K(\mathscr{A}) & & \end{array}.$$

Denote by $K(\mathscr{A})^+$, $K(\mathscr{A})^-$ and $K(\mathscr{A})^b$ the subcategories corresponding to $\mathrm{Ch}(\mathscr{A})^+$, $\mathrm{Ch}(\mathscr{A})^-$ and $\mathrm{Ch}(\mathscr{A})^b$.

Exact triangles in the homotopy category. Let $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ be a sequence of morphisms in $K(\mathscr{A})$. We say that the triangle (u, v, w) is exact if there exist f, g, h homotopy equivalences such that there is a commutative diagram:

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & A[-1] \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle f[-1]}\downarrow \\ A' & \xrightarrow{u'} & B' & \longrightarrow & \mathrm{Cone}(u') & \longrightarrow & A'[-1]. \end{array}$$

In particular, note that this implies that we have a long exact sequence:

$$\cdots \to H^{i-1}(C) \to H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to \cdots$$

Remark 3.2. The category $K(\mathscr{A})$ is called a triangulated category. More generally a triangulated category if an additive category $\mathscr{D}$ equipped with a functor $[1] : \mathscr{D} \to \mathscr{D}$ defining an auto-equivalence and a class of exact triangles $\mathscr{T}$ satisfying certain axioms. See for example [Wei94, §10.2] for the precise definition of triangulated category and exact triangles.

3.1.2 The derived category

Recall that we work with $\mathscr{A}$ an abelian category.

Definition 3.3. Let $\mathscr{C}$ be a category and S a class of morphisms in $\mathscr{C}$. The localisation of the category $\mathscr{C}$ with respect to S is the universal functor $Q : \mathscr{C} \to \mathscr{C}[S^{-1}]$ sending elements of S to isomorphisms: i.e. for any functor $F : \mathscr{C} \to \mathscr{D}$ sending elements of S to isomorphisms, there exists a unique functor $\overline{F} : \mathscr{C}[S^{-1}] \to \mathscr{D}$ making the following diagram commutative:

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{F} & \mathscr{D} \\ \downarrow & \nearrow_{\overline{F}} & \\ \mathscr{C}[S^{-1}] & & \end{array} .$$

In particular, the homotopy category $K(\mathscr{A})$ is the localisation of $\mathrm{Ch}(\mathscr{A})$ with respect to homotopy equivalences.

Example 3.4. The name "localisation" comes from the following example: let R is a ring and $S \subset R$ a multiplicatively closed subset and consider the category $\mathscr{C}_R$ whose only object is a point $*$ and the set of morphisms $\mathrm{Hom}(*, *)$ is equal to R (the composition being the multiplication in R). Then $\mathscr{C}_R[S^{-1}] = \mathscr{C}_{R[S^{-1}]}$.

Definition 3.5. The derived category $\mathscr{D}(\mathscr{A})$ is defined as the localisation of $K(\mathscr{A})$ with respect to the class of quasi-isomorphisms:

$$Q : K(\mathscr{A}) \to \mathscr{D}(\mathscr{A}) := K(\mathscr{A})[\mathrm{qis}^{-1}].$$

It can be proved that the derived category $D(\mathscr{A})$ is a triangulated category. More generally, we have the following proposition:

Proposition 3.6. *Let $(\mathscr{C}, [1], \mathscr{T})$ be a triangulated category. Then there exists a unique structure of a triangulated category on $S^{-1}\mathscr{C}$ such that $[1] \circ Q = Q \circ [1]$ and the localization functor $Q : \mathscr{C} \to S^{-1}\mathscr{C}$ sends exact triangles to exact triangles.*

For a proof of the above proposition, see for example [StackProject, 05R6].

We denote by $D^+(\mathscr{A})$, $D^-(\mathscr{A})$ and $D^b(\mathscr{A})$ the subcategories corresponding to $K^+(\mathscr{A})$, $K^-(\mathscr{A})$ and $K^b(\mathscr{A})$.

Remark 3.7. Let $\mathscr{C}$ be a abelian category and S a saturated multiplicative system [3]. For Y an object of $\mathscr{C}$, we define Y/S as the category whose objects are morphisms $s : Y \to Y'$ in S and a morphism between two objects $s : Y \to Y'$ and $t : Y \to Y''$ is a morphism $Y' \to Y''$ in $\mathscr{C}$ (not necessarily in S) making the obvious diagram commute. Then, the sets of morphisms in the category $S^{-1}\mathscr{C}$ can be described as follows:

$$\mathrm{Hom}_{S^{-1}\mathscr{C}}(X, Y) = \mathrm{colim}_{(s:Y \to Y') \in Y/S} \mathrm{Hom}_{\mathscr{C}}(X, Y').$$

Dually, for an object X of $\mathscr{C}$, the category S/X is defined as the category whose objects are morphisms $s : X' \to X$ in S and the Hom-sets are defined in a similar way as above. As before, we have:

$$\mathrm{Hom}_{S^{-1}\mathscr{C}}(X, Y) = \mathrm{colim}_{(s:X' \to X) \in (S/X)^{\mathrm{op}}} \mathrm{Hom}_{\mathscr{C}}(X', Y).$$

[3] See [StackProject, 04VB] for a definition of "saturated multiplicative system". In the following we will apply this to S the set of quasi-isomorphisms in the homotopy category.

3.2 Derived functors

3.2.1 Derived functors in general

Consider $F : \mathscr{D} \to \mathscr{D}'$ a functor between two triangulated category and S a saturated multiplicative system in $\mathscr{D}$. We will first define the notion of right and left derived functor $\mathrm{R}F$ and $\mathrm{L}F$ for such a functor F.

Definition 3.8. Let X be an object in $\mathscr{D}$.

(1) We say that the right derived functor $\mathrm{R}F$ is defined at X if the diagram

$$\begin{cases} (X/S) & \to \mathscr{D}' \\ (s : X \to X') & \mapsto F(X') \end{cases}$$

is essentially constant[4]. If $\mathrm{R}F$ is defined at X, we denote by $\mathrm{R}F(X)$ its value.

(2) Dually, we say that the left derived functor $\mathrm{L}F$ is defined at X if the diagram

$$\begin{cases} (S/X) & \to \mathscr{D}' \\ (s : X' \to X) & \mapsto F(X') \end{cases}$$

is essentially constant. If $\mathrm{L}F$ is defined at X, we denote by $\mathrm{L}F(X)$ its value.

It can be shown that if $s : X \to Y$ is in S, then $\mathrm{R}F$ (respectively $\mathrm{L}F$) is defined at X if and only if it is defined at Y and $\mathrm{R}F(X) \xrightarrow{\sim} \mathrm{R}F(Y)$. Also, $\mathrm{R}F$ is defined at $X \in \mathscr{D}$ if and only if it is defined at $X[1]$ and in that case, $\mathrm{R}F(X)[1] = \mathrm{R}F(X[1])$. Moreover, if (X, Y, Z) is an exact triangle in $\mathscr{D}$ and $\mathrm{R}F$ is defined at two of the three of X, Y, Z then it is defined at the third one and $(\mathrm{R}F(X), \mathrm{R}F(Y), \mathrm{R}F(Z))$ is an exact triangle. We get:

Proposition 3.9. *The full subcategory $\mathscr{E}$ of $\mathscr{D}$ consisting of objects where $\mathrm{R}F$ is defined is a triangulated category and $\mathrm{R}F$ defines a functor $\mathscr{E} \to \mathscr{D}$ sending exact triangles to exact triangles. Elements of S with source or target in $\mathscr{E}$ are morphisms of $\mathscr{E}$, $\mathrm{R}F$ sends elements of $S_{\mathscr{E}} := \mathrm{Arrows}(\mathscr{E}) \cap S$ to isomorphisms and it induces a functor of triangulated categories $\mathrm{R}F : S_{\mathscr{E}}^{-1}\mathscr{E} \to \mathscr{D}$ (sending exact triangles to exact triangles).*

We have a similar result replacing $\mathrm{R}F$ by $\mathrm{L}F$.

We will say that an object X in $\mathscr{D}$ computes $\mathrm{R}F$ (respectively $\mathrm{L}F$) if $\mathrm{R}F$ (respectively $\mathrm{L}F$) is defined at X and $F(X) \xrightarrow{\sim} \mathrm{R}F(X)$ (respectively $\mathrm{L}F(X) \xrightarrow{\sim} F(X)$).

Lemma 3.10. *If for all X in $\mathscr{D}$, there exists $s : X \to X'$ (respectively $s : X' \to X$) in S such that X' computes $\mathrm{R}F$ (respectively $\mathrm{L}F$) then $\mathrm{R}F$ (respectively $\mathrm{L}F$) is defined everywhere.*

[4]This means that in the associated ind-category $\mathrm{Ind}(\mathscr{D}')$, it is isomorphic to a filtered diagram consisting of a single object Y and the morphisms are all equal to identity.

3.2.2 Derived functors on the derived category

The above results can be made more explicit in the case where we consider the homotopy categories of abelian categories.

Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories and $F : \mathscr{A} \to \mathscr{B}$ an additive functor. We write F (respectively F^+, F^-) for the induced functor $F : K(\mathscr{A}) \to D(\mathscr{B})$ (respectively $K^+(\mathscr{A}) \to D^+(\mathscr{B})$, $K^-(\mathscr{A}) \to D^-(\mathscr{B})$).

Lemma 3.11. *(i).* $\mathrm{R}F$ *is defined at* $X \in K^+(\mathscr{A})$ *if and only if it* $\mathrm{R}F^+$ *is defined at* X *and in that case, they have the same values.*

(ii). $\mathrm{L}F$ *is defined at* $X \in K^-(\mathscr{A})$ *if and only if it* $\mathrm{L}F^-$ *is defined at* X *and in that case, they have the same values.*

(iii). For $X \in K^+(\mathscr{A})$*,* X *computes* $\mathrm{R}F$ *if and only if it computes* $\mathrm{R}F^+$.

(iv). For $X \in K^-(\mathscr{A})$*,* X *computes* $\mathrm{L}F$ *if and only if it computes* $\mathrm{L}F^-$.

We defined the right (respectively left) derived functor as the functor $\mathrm{R}F$ (respectively $\mathrm{L}F$) going from a full subcategory of $D(\mathscr{A})$ to $D(\mathscr{B})$. We say that an object A in $\mathscr{A}$ is right (respectively, left) acyclic for F if $A[0]$[5] computes $\mathrm{R}F$ (respectively $\mathrm{L}F$).

Definition 3.12. Assume $\mathrm{R}F$ is defined everywhere on $D(\mathscr{A})^+$. Let $i \in \mathbf{Z}$. The i-th derived functor of F is the functor

$$\mathrm{R}^iF = H^i \circ \mathrm{R}F : D(\mathscr{A})^+ \to \mathscr{B}.$$

The following lemma explains why we will mostly be interested in left exact functor when computing right derived functor.

Lemma 3.13. *With the assumptions from Definition 3.12, then* $\mathrm{R}^iF = 0$ *for* $i < 0$*,* R^0F *is left exact and the map* $F \to \mathrm{R}^0F$ *is an isomorphism if and only if* F *is left exact. Moreover, if* A *is an object in* $\mathscr{A}$ *then* A *is right acyclic if and only if* $F(A) \xrightarrow{\sim} \mathrm{R}^0F(A)$ *and* $\mathrm{R}^iF(A) = 0$ *for* $i > 0$.

To compute right derived functors, our main tool will be the following result (and its corollary below):

Proposition 3.14 (Leray's acyclicity)**.** *Let* $F : \mathscr{A} \to \mathscr{B}$ *be an additive functor between abelian categories. Let* $K^\bullet$ *be a bounded below complex of right* F*-acyclic objects such that* $\mathrm{R}F$ *is defined at* $K^\bullet$*. Then, the canonical map* $F(K^\bullet) \to \mathrm{R}F(K^\bullet)$ *is an isomorphism in* $D^+(\mathscr{B})$*, i.e.,* $K^\bullet$ *computes* $\mathrm{R}F$.

[5] i.e. the complex whose all terms are zero except in degree 0 where it is A.

This result combined with Lemma 3.10 gives:

Corollary 3.15. *Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor of abelian categories.*

(i). If every object of A injects into an object acyclic for $\mathrm{R}F$, then $\mathrm{R}F$ is defined everywhere on $K^+(\mathscr{A})$ and we obtain a functor $\mathrm{R}F : D^+(\mathscr{A}) \to D^+(\mathscr{B})$ sending exact triangles to exact triangles. Moreover, any bounded below complex $K^\bullet$ whose terms are acyclic for $\mathrm{R}F$ computes $\mathrm{R}F$.

(ii). If every object of A is a quotient of an object acyclic for $\mathrm{L}F$, then $\mathrm{L}F$ is defined everywhere on $K^-(\mathscr{A})$ and we obtain a functor $\mathrm{L}F : D^-(\mathscr{A}) \to D^-(\mathscr{B})$ sending exact triangles to exact triangles. Moreover, any bounded below complex $K^\bullet$ whose terms are acyclic for $\mathrm{L}F$ computes $\mathrm{L}F$.

3.3 Sheaf cohomology

We will now apply the previous construction to the category of abelian sheaves. The acyclic objects will be given by complexes of injective sheaves. The i-th cohomology group of a sheaf is then defined as the i-th right derived functor of the global section functor.

3.3.1 Injective objects and resolutions

Recall that for an object I of an abelian category $\mathscr{A}$, the contravariant functor $A \mapsto \mathrm{Hom}_{\mathscr{A}}(A, I)$ is left exact. We say that an object I in $\mathscr{A}$ is injective if the functor $A \mapsto \mathrm{Hom}_{\mathscr{A}}(A, I)$ is exact. Equivalently, I is injective if for any object A with a subobject $A' \subset A$ and a morphism $A' \to I$, then this morphism can be extended to a morphism $A \to I$. We will see later a criterion for an abelian group to be injective (see Proposition 3.20).

Definition 3.16. Let $\mathscr{A}$ be an abelian category.

(i). If A is an object of $\mathscr{A}$, an injective resolution of A is a chain complex $I^\bullet$ together with a map $A \to I^0$ such that $I^n = 0$ for $n < 0$, the objects I^n are injective for all n and the cohomology of the complex is computed by

$$A \xrightarrow{\sim} \ker(d_I^0) \text{ and } H^i(I^\bullet) = 0 \text{ for } i > 0.$$

In other words, $A[0] \to I^\bullet$ is a quasi-isomorphism.

(ii). If $K^\bullet$ is in $D(\mathscr{A})$, an injective resolution of $K^\bullet$ is a chain complex $I^\bullet$ together with a map $\alpha : K^\bullet \to I^\bullet$ such that $I^n = 0$ for $n << 0$, the objects I^n are injective for all n and α is a quasi-isomorphism.

Definition 3.17. We say that $\mathscr{A}$ has enough injectives if for all A in $\mathscr{A}$ there exists a monomorphism $A \to I$ with I injective.

Proposition 3.18. *Assume that $\mathscr{A}$ has enough injectives. Then,*

(i). any object A in $\mathscr{A}$ admits an injective resolution and,

(ii). if $K^\bullet$ is a chain complex such that $H^n(K^\bullet) = 0$ for $n << 0$ then $K^\bullet$ admits an injective resolution.

Note that if $H^n(K^\bullet) = 0$ for $n << 0$ then there exists a quasi-isomorphism $K^\bullet \to L^\bullet$ with $L^\bullet$ bounded below: it suffices to take $L := \tau_{\geq n} K^\bullet$ where the truncation $\tau_{\geq n}$ is defined by $\tau_{\geq n} K = (\cdots \to 0 \to 0 \to \mathrm{coker}(d_{n-1}) \to K_{n+1} \to K_{n+2} \to \cdots)$.

Sketch of proof of Proposition 3.18. For the first point, let A be an object in $\mathscr{A}$ and take a monomorphism into an injective object $A \hookrightarrow I^0$. Consider the object I^0/A and choose I^1 injective such that I^0/A injects into I^1. Write d^0 for the map $I^0 \to I^1$. Let us now consider the object $A/\mathrm{im}(d^0)$. As before we can take I^2 injective such that $I^1/\mathrm{im}(d^0)$ injects into I^2 and denote by d^1 the natural map $I^1 \to I^2$. Iterating the construction, we obtain the complex $I^\bullet$ as wanted.

For the second point, we proceed by induction on the degree. Let a be an integer such that $K^i = 0$ for $i < a$. Consider the following induction hypothesis, for $n \geq a$:

(IH_n)
For $i \leq n$ there is a complex $\big(I_a \to I_{a+1} \to \cdots \to I_{n-1} \to I_n\big)$ with a map $\alpha : K^\bullet \to I^\bullet$ such that $H^i I^\bullet \simeq H^i K^\bullet$ for $i < n$ and $K^{n+1} \to K^n \oplus I^{n-1} \to I^n$ is exact.

Define C as the cokernel of the map $K^n \oplus I^{n-1} \to K^{n+1} \oplus I^n$ sending (x, y) to $(d(x), d(y) - \alpha(x))$. Choose I^{n+1} injective such that C injects into I^{n+1}. Then, using I^{n+1} we obtain (IH_{n+1}). □

Proposition 3.19. *Let $\mathscr{A}$ be an abelian category and I in $\mathscr{A}$ an injective object. Then I is right acyclic for any additive functor $F : \mathscr{A} \to \mathscr{B}$ (with $\mathscr{B}$ an abelian category).*

Sketch of proof. We more generally prove that a bounded below complex of injectives $I^\bullet$ computes the derived functor $\mathrm{R}F$. By definition, it suffices to prove that

$$\begin{cases} I^\bullet/\mathrm{Qis}^+(\mathscr{A}) & \to D^+(\mathscr{B}) \\ (I^\bullet \xrightarrow{\sim} K^\bullet) & \mapsto F(K^\bullet) \end{cases}$$

is essentially constant with value $F(I^\bullet)$. This comes from the fact that since the I^n are injective objects, each $\alpha : I^\bullet \xrightarrow{\sim} K^\bullet$ has a left inverse (see [StackProject, 013P]). □

3.3.2 Application to the category of abelian sheaves

Proposition 3.20. *An abelian group M is injective if and only if M is divisible, that is, for every integer $n \in \mathrm{N}_{\geq 1}$, the multiplication by n from M to M is surjective.*

Proof. Assume first that M is an injective abelian group. Let m an element of M and $n \geq 1$ an integer. Consider the morphism from $f : n\mathbf{Z} \to M$ sending n to m. Since M is injective f can be extended in a morphism $\tilde{f}$ from $\mathbf{Z}$ to M. Then, since $\tilde{f}$ is linear,

$$m = f(n) = \tilde{f}(n) = n \cdot \tilde{f}(1)$$

and m is divisible by n.

Now, let M be a divisible abelian group. Let N be an abelian group and N' be a subgroup of N. Let $f' : N' \to M$ be a linear map, we need to extend f' to a linear map $f : N \to M$. Consider the set of all morphisms $\tilde{f} : \tilde{N} \to M$ extending f', where $\tilde{N}$ is an intermediate subgroup between N' and N. This is partially ordered and every chain has an upper bound so it admits at least one maximal element $\tilde{f} : \tilde{N}_0 \to M$. We will show that $\tilde{N}_0 = N$. Assume the inclusion is strict and choose an element x in $N \setminus \tilde{N}_0$. Consider its projection $\bar{x}$ to $N/\tilde{N}_0$. If $\bar{x}$ has infinite order then the group generated by $\tilde{N}_0$ and x is isomorphic to $\tilde{N}_0 \oplus \mathbf{Z}$ so $\tilde{f}$ can be extended to $\langle \tilde{N}_0, x \rangle$, which is a contradiction. So $\bar{x}$ must have finite order n with $n \geq 2$. Since M is divisible, there exists m in M such that $n \cdot m = \tilde{f}(nx)$ and again, we can extend $\tilde{f}$ to $\langle \tilde{N}_0, x \rangle$. This gives a contradiction, so M is injective. □

Example 3.21. The abelian groups $\mathbf{Q}$ and $\mathbf{Q}/\mathbf{Z}$ are injective.

Theorem 3.22. *The abelian category* Ab *has enough injectives.*

Proof. Let N be an abelian group, we want to embed N into an injective abelian group M. Take $M := (\mathbf{Q}/\mathbf{Z})^{\mathrm{Hom}(N,\mathbf{Q}/\mathbf{Z})}$. Since $\mathbf{Q}/\mathbf{Z}$ is injective and arbitrary products of injective objects are injective, M is injective. Consider the map

$$\begin{cases} N & \to M \\ x & \mapsto (f(x))_{f \in \mathrm{Hom}(N,\mathbf{Q}/\mathbf{Z})}. \end{cases}$$

We will prove that this map is injective. Take $x \neq 0$ in N, it sufficed to find $f : N \to \mathbf{Q}/\mathbf{Z}$ such that $f(x) \neq 0$. Consider the subgroup $\mathbf{Z} \cdot x$ of N. If the order of x is finite, we can take $f(x) = \frac{1}{n}$. If the order of x is not finite, sending x to any non-zero element of $\mathbf{Q}/\mathbf{Z}$ gives such a map f. □

More generally, we have:

Theorem 3.23. *Let $\mathscr{A}$ be an abelian category.*

(i). If $\mathscr{A}$ has (arbitrary) direct sums, satisfies (Ab5)[6] *and has a generator then $\mathscr{A}$ has enough injectives.*

[6] i.e. filtered colimits are exact.

(ii). If $\mathscr{A}$ satisfies the conditions of the previous point and $\mathscr{B}$ is any category, then $\mathrm{Fun}(\mathscr{B}, \mathscr{A})$ *also satisfies the above conditions and, in particular,* $\mathrm{Fun}(\mathscr{B}, \mathscr{A})$ *has enough injectives.*

Corollary 3.24. *Let $\mathscr{T}$ be a category. The category* $\mathrm{PreShvAb}(\mathscr{T})$ *has enough injectives.*

In the above result, a generator of $\mathscr{A}$ is an object X of $\mathscr{A}$, such that for all Y in $\mathscr{A}$, there exists an epimorphism

$$\bigoplus_I X \to Y \to 0$$

with I arbitrary. Let $\mathscr{T}$ be a site. Let us give example of generators for $\mathrm{PreShvAb}(\mathscr{T})$ and $\mathrm{ShvAb}(\mathscr{T})$. For U an object in $\mathscr{T}$, we define the presheaf

$$\mathbf{Z}_U(V) = \bigoplus_{\mathrm{Hom}(V,U)} \mathbf{Z}.$$

In particular, for any abelian presheaf $\mathscr{F}$ there is a canonical isomorphism $\mathscr{F}(U) \simeq \mathrm{Hom}(\mathbf{Z}_U, \mathscr{F})$. Then the presheaf $\mathbf{Z} := \bigoplus_U \mathbf{Z}_U$ defines a generator of the category $\mathrm{PreShvAb}(\mathscr{T})$. Taking the sheafification $\mathbf{Z}^\sharp$, we get a generator for the category $\mathrm{ShvAb}(\mathscr{T})$.

We can then deduce from the first point of Theorem 3.23 the following result:

Theorem 3.25. *Let $\mathscr{T}$ be a site. The category* $\mathrm{ShvAb}(\mathscr{T})$ *has enough injectives.*

We can now define sheaf cohomology. Let $\mathscr{T}$ be a site. Note that the functor

$$\Gamma(U, -) : \mathrm{ShvAb}(\mathscr{T}) \to \mathrm{Ab}$$

is left exact as the composition of the left exact functor $\mathrm{ShvAb}(\mathscr{T}) \to \mathrm{PreShvAb}(\mathscr{T})$ and the exact functor $\Gamma(U, -) : \mathrm{PreShvAb}(\mathscr{T}) \to \mathrm{Ab}$. For $\mathscr{F}$ an abelian sheaf and i an integer, the i-the cohomology group of $\mathscr{F}$ is defined as the i-th derived functor of the global section functor:

$$H^i(U, \mathscr{F}) := \mathrm{R}^i\Gamma(U, \mathscr{F})$$

for any object U of $\mathscr{T}$.

4 Étale site

We would like to define a cohomology theory that is an algebraic geometry version of the singular cohomology for varieties over $\mathbf{C}$. A first guess would be to use the Zariski topology. However, this topology has not enough open sets: for example, for a complex variety, any two Zariski open sets meet. So, when computing the cohomology of a constant sheaf (i.e. the sheafification of a constant presheaf) we obtain that the restriction maps are surjective. This implies $H^i_{\text{zar}}(X, \mathscr{F}) = 0$ for $i > 0$ and $\mathscr{F}$ constant, and the Zariski topology does not detect cohomology in higher degrees. Hence, we need to find a finer topology. To do that, we will first define morphisms that are algebraic analogues to local homeomorphisms. There are two obstructions for a morphism of complex varieties to be a local homeomorphism: firstly, there cannot be branch points and secondly, the dimensions of the fibers cannot vary. In the algebraic geometry world, a morphism with no branch points will be called unramified and a morphism with fibers of locally constant dimension will be called flat. An étale morphism will be a morphism that is flat and unramified.

4.1 Étale morphisms

We will assume that all rings are noetherian and all schemes are locally noetherian. Before defining unramified morphisms, let us recall a few facts about flat morphisms.

Definition 4.1. We say that a morphism of rings $f : A \to B$ is flat if the functor $- \otimes_A B : \mathrm{Mod}_A \to \mathrm{Mod}_B$ is exact. A morphism of schemes $f : X \to Y$ is flat if for all $y \in Y$ the map $\mathscr{O}_{X,f(y)} \to \mathscr{O}_{Y,y}$ is flat.

Note that equivalently, a morphism $f : X \to Y$ is flat if and only if for any open affines U of X and V of Y such that $f(V) \subset U$, the morphism $\Gamma(U, \mathscr{O}_X) \to \Gamma(V, \mathscr{O}_Y)$ is flat. Open immersions are flat. The property of being flat is stable by base change and by composition.

Example 4.2. (i). If K is a field, every K-module is flat.

(ii). If A is a ring and $S \subset A$ is a multiplicatively closed subset, then the localization $A \to A[S^{-1}]$ is flat. An A-module M is flat if and only if for every prime ideal $\mathfrak{p} \subset A$ (respectively every maximal ideal $\mathfrak{m} \subset A$), the $A_\mathfrak{p}$-module $M_\mathfrak{p}$ (respectively the $A_\mathfrak{m}$-module $M_\mathfrak{m}$) is flat.

(iii). Let A be a ring. Then $A[X_1, \dots, X_d]$ is flat over A (in other words: the affine space $\mathbb{A}^d_A$ is flat over $\mathrm{Spec}(A)$).

(iv). Let Z be an hypersurface in $\mathbb{A}^d_A$, i.e. a scheme of the form $\mathrm{Spec}(A[X_1, \dots, X_d])/(P)$ with $P \neq 0$. Then Z is flat over A if and only if for all maximal ideal $\mathfrak{m}$ in A, $Z \otimes_A k(\mathfrak{m})$ is not equal to $\mathbb{A}^d_{k(\mathfrak{m})}$. In other words, an hypersurface in $\mathbb{A}^d_A$ is flat if and only if its closed fibers over A all have the same dimension.

(v). Standard examples of non-flat morphisms are given by blowups. Consider for example the blowup $\widetilde{\mathbb{A}}^2_k$ of $\mathbb{A}^2_k = \mathrm{Spec}(k[x,y])$ at the origin. The points of $\widetilde{\mathbb{A}}^2_k$ can be described by the pairs $((x,y),[X:Y]))$ in $\mathbb{A}^2_k \times \mathbb{P}^1_k$ such that $xY = yX$. The fiber of $\widetilde{\mathbb{A}}^2_k \to \mathbb{A}^2_k$ over a point $(x,y) \neq (0,0)$ is given by a single point in $\mathbb{P}^1_k$ while the fiber over the origin is the entire projective line. The blowup $\widetilde{\mathbb{A}}^2_k$ can be covered by the two open affines $\mathrm{Spec}(k[x,\frac{y}{x}])$ and $\mathrm{Spec}(k[\frac{x}{y},y])$. The morphisms $k[x,y] \to k[x,\frac{y}{x}]$ and $k[x,y] \to k[\frac{x}{y},y]$ are not flat.

Definition 4.3. We say that a morphism of rings $f : A \to B$ of finite-type is unramified at a prime $\mathfrak{q} \in \mathrm{Spec}(B)$ if the ideal $\mathfrak{p} := f^{-1}(\mathfrak{q})$ generates the maximal ideal in $B_{\mathfrak{q}}$ (i.e. $\mathfrak{q}B_{\mathfrak{q}} = f(\mathfrak{p})B_{\mathfrak{q}}$) and $k(\mathfrak{q})$ is a finite separable field extension of $k(\mathfrak{p})$. We say that f is unramified if f is unramified at every prime. A morphism of schemes $f : Y \to X$ that is locally of finite-type is unramified at $y \in Y$ if $\mathcal{O}_{Y,y}/\mathfrak{m}_x\mathcal{O}_{Y,y}$ is a finite separable extension of $k(x)$. It is unramified if it is unramified at all $y \in Y$.

In particular, a morphism $f : Y \to X$ is unramified if and only if for all $x \in X$, its fibers $Y_x \to \mathrm{Spec}(k(x))$ is unramified and it can be proved that this is true if and only if all geometric fibers of f are unramified (see [Mil80, Proposition 3.2]). Open immersions are unramified. Moreover, the property of being unramified is stable by base change and composition.

Example 4.4. (i). Let k be a field. We denote by $\overline{k}$ an algebraic closure of k. Recall that a finite k-algebra A is separable over k if and only if it is isomorphic to a finite product of separable field extensions of k and this is true if and only if $A \otimes_K \overline{k}$ is isomorphic to a finite product of copies of $\overline{k}$. Using that, it can be proved that $f : Y \to X$ is unramified if and only if for all $x \in X$, the fiber Y_x is isomorphic to a co-product $\amalg_i \mathrm{Spec}(k_i)$, where the k_i are finite separable field extensions of $k(x)$.

(ii). Let k be a field. The morphisms $k[x] \to k[x,y]/(x+y)(x-y)$ is unramified everywhere except at the origin. The same goes for $k[x] \to k[x], x \mapsto x^2$.

(iii). Take $k := \mathbf{F}_p(t)$. The morphism $k[x] \to k[x,y]/(y^p - xy - t)$ is unramified everywhere except at $(x, y^p - t)$ where it becomes the inseparable extension $\mathbf{F}_p(t) \to \mathbf{F}_p(t^{\frac{1}{p}})$.

The following alternative definition of unramified morphism can also sometimes be useful:

Proposition 4.5. *Let $f : Y \to X$ be locally of finite type. We have the following equivalences:*

(i). f is unramified.

(ii). The sheaf $\Omega^1_{Y/X}$ is zero.

(iii). The diagonal morphism $\Delta_{Y/X} : Y \to Y \times_X Y$ is an open immersion.

Sketch of the proof. We just recall the main ideas for the proof, more details can be found in [Mil80, Chapter I, Proposition 3.5] or [StackProject, 02G3]. Assume assertion (i). Using the compatibility of $\Omega^1_{Y/X}$ with base change and localisation, the proof of (ii) can be reduced to prove that $\Omega_{B/A} = 0$ for $A \to B$ a local morphism between local rings. Using Nakayama's lemma, we see that it suffices then to check that $\Omega_{L/K}$ is zero for L/K a finite separable extension of fields. To prove that the second point implies the third one, first note that the diagonal morphism is always locally closed. So, we can find some open U such that $\Delta_{X/Y} : Y \to U$ is closed and we denote by $\mathscr{I}$ the associated ideal. Using that $\mathscr{I}/\mathscr{I}^2 \simeq \Omega^1_{Y/X}$, we can then find an open V in U such that $\mathscr{I}|_V = 0$ and $Y \simeq V \to U \to Y \times_X Y$ gives the open immersion we want. Assume now that $\Delta_{Y/X} : Y \to Y \times_X Y$ is an open immersion. Passing to geometric fibers, we can assume $X = \mathrm{Spec}(k)$ with k an algebraically closed field. If $y \to Y$ is a closed point of Y, we can use the hypothesis to prove that the diagonal morphism associated to $\mathrm{Spec}(\mathscr{O}_{Y,y}) \to \mathrm{Spec}(k)$ is an open immersion. By a dimension argument, this yields $\mathrm{Spec}(\mathscr{O}_{Y,y}) \simeq \mathrm{Spec}(k)$ and it follows from the first point in Example 4.4 that f is unramified.

□

Note that, when working with affine schemes, the diagonal is always a closed immersion. But a closed immersion is open if and only if it is flat (see for example [StackProject, 0819]). So a morphism $A \to B$ of finite type is unramified if and only if $B \otimes_A B \to B$ is flat.

Definition 4.6. A morphism of schemes (or rings) is étale if it is flat and unramified.

Open immersions are étale. Moreover, the property of being étale is stable by base change and composition.

Example 4.7. (i). Let k be a field and $k \to A$ a finite k-algebra. Then A is étale if and only if $A \simeq L_1 \times \cdots \times L_n$ for some finite separable field extensions L_i/k.

(ii). **Jacobian criterion:** We say that a morphism of rings $A \to B$ is standard smooth if there exist integers $c \leq n$ and a presentation

$$B \simeq A[x_1, ..., x_n]/\langle f_1, \dots, f_c \rangle$$

such that $\det\left(\frac{\partial f_i}{\partial x_j}\right)_{1 \leq i,j \leq c}$ is invertible in A. An étale morphism $A \to B$ is standard smooth. More precisely, a morphism $A \to B$ is étale if and only if there exists a presentation as above with $c = n$.

(iii). Suppose $Y \to X$ is a morphism of smooth affine $\mathbf{C}$-varieties. Then $Y \to X$ is étale if and only if $Y(\mathbf{C}) \to X(\mathbf{C})$ is a local homeomorphism of topological spaces.

Remark 4.8 (Relation between étale and smooth morphisms). There exists an equivalent definition of étale morphism, using the notion of relative dimension: for a morphism of schemes $f : X \to Y$ locally of finite type, we say that f has relative dimension $d \geq 0$ if every non-empty fiber X_y for $y \in Y$ has pure dimension d. For example, for every integer $d \geq 0$, the morphisms $\mathbb{A}^d_S \to S$ and $\mathbb{P}^d_S \to S$ have relative dimension d. For every ring A and any integer $n \geq 1$, the morphism $f : \mathbb{A}^1_A \to \mathbb{A}^1_A$ given by $x \mapsto x^n$ has relative dimension 0. More generally, for any finite A-algebra B, the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ has relative dimension 0.

For $f : X \to Y$ is a morphism of affine schemes, we say that f is standard smooth if the induced ring map $\mathscr{O}(Y) \to \mathscr{O}(X)$ is standard smooth. If $f : X \to Y$ is a morphism of (arbitrary) schemes, we say that f is smooth at $x \in X$ if there exist affine open subsets $U \subset X$ and $V \subset Y$ with $x \in U$ and $f(U) \subset V$ such that the induced map $f|_U : U \to V$ is standard smooth. We say that f is smooth if it is smooth at every point of X. A morphism of schemes $f : X \to Y$ is smooth if and only if f is locally of finite presentation, flat and for every $y \in Y$, the geometric fibre $X_y = X \times_Y \mathrm{Spec}(k(y))$ is a non-singular variety (see [StackProject, 01VD, 01V7, 01V8]).

Using the Jacobian criterion, we obtain:

Proposition 4.9. *Let $f : X \to Y$ be a morphism of schemes. Then f is étale if and only if f is smooth of relative dimension* 0*.*

Moreover, it can be proved that smooth schemes are étale-locally like affine spaces: a morphism of schemes $f : X \to Y$ is smooth if and only if locally on the source and target, f can be written as follows:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & \mathbb{A}^d_Y \\ & {\scriptstyle f}\searrow & \downarrow \\ & & Y \end{array}$$

where $d \geq 0$ is an integer and φ is étale.

4.2 The étale topology

Let X be a scheme. We denote by $\mathrm{\acute{E}t}|_X$ the category of étale X-schemes[7]. Note that $\mathrm{\acute{E}t}|_X$ has finite fiber products and any morphisms between étale X-schemes is étale. We say that a family of morphisms $\{\varphi_i : U_i \to U\}_{i \in I}$ in $\mathrm{\acute{E}t}|_X$ is a covering if $U = \bigcup_{i \in I} \varphi_i(U_i)$. This defines a Grothendieck topology on $\mathrm{\acute{E}t}|_X$ and we write $X_{\text{ét}}$ the site defined that way.

Remark 4.10. The site $X_{\text{ét}}$ is the small étale site. We can also define the big étale site $\mathrm{Sch}|_{X,\text{ét}}$: it is the category of all X-schemes endowed with the Grothendieck topology in which the coverings are the families of étale morphisms $\{\varphi : U_i \to U\}_{i \in I}$ such that $U = \bigcup_{i \in I} \varphi_i(U_i)$. Since a morphism between étale X-schemes is étale, there is a canonical morphism from $X_{\text{ét}}$ to $\mathrm{Sch}|_{X,\text{ét}}$. If $\mathscr{F}$ is an abelian sheaf on $\mathrm{Sch}|_{X,\text{ét}}$, then $\mathscr{F}|_{X_{\text{ét}}}$ is a sheaf on $X_{\text{ét}}$ and $H^i(X_{\text{ét}}, \mathscr{F}|_{X_{\text{ét}}}) = H^i(X, \mathscr{F})$ for all $i \geq 0$.

[7] In particular, X is a final object in $\mathrm{\acute{E}t}|_X$.

4.2.1 The fpqc topology

The fpqc topology is coarser than the étale topology (see Lemma 4.14 below) but finer than the Zariski topology. In particular, we have than if a presheaf $\mathscr{F}$ is a sheaf for the fpqc topology, it will also be a sheaf for the étale topology. The fpqc topology has already been studied during the problem sessions (see Exercise Sheets 1 and 2). For clarity, we quickly summarize here what are the main results we have proved.

Definition 4.11. Let X be a scheme.

(i). Let U be a scheme over X. A Zariski covering of U is a family of morphisms $\{\varphi_i : U_i \to U\}_{i\in I}$ of schemes such that each φ_i is an open immersion and such that $U = \bigcup_{i\in I} \varphi_i(U_i)$. This defines a Grothendieck topology on $\mathrm{Sch}|_X$.

(ii). Let U be a scheme over X. An fpqc covering of U is a family $\{\varphi_i : U_i \to U\}_{i\in I}$ such that each φ_i is a flat morphism, $U = \bigcup_{i\in I} \varphi_i(U_i)$ and for each affine open $T \subset U$ there exists a finite set K, a map $\iota : K \to I$ and affine opens $T_{\iota(k)} \subset U_{\iota(k)}$ such that $T = \bigcup_{k\in K} \varphi_{\iota(k)}(T_{\iota(k)})$. This defines a Grothendieck topology on $\mathrm{Sch}|_X$.

Note that any Zariski covering is an fpqc-covering. If A is a ring, a A-module M is called faithfully flat if a sequence of A-modules $N_1 \to N_2 \to N_3$ is exact if and only if the sequence $M \otimes_A N_1 \to M \otimes_A N_2 \to M \otimes_A N_3$ is exact. We say that a morphism of schemes $f : X \to Y$ is faithfully flat if it is flat and surjective. A morphism of affine scheme $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$ is an fpqc covering if and only if $A \to B$ is faithfully flat.

Lemma 4.12. *Let X be a scheme. For a presheaf $\mathscr{F}$ of sets (or abelian groups) on the fpqc site the following are equivalent:*

(i). $\mathscr{F}$ is an fpqc sheaf.

(ii). The gluing property is satisfied for fpqc coverings of the following types:

a) $\{U_i \to U\}_{i\in I}$ a surjective family of open immersions,

b) $\{V \to U\}$ a single surjective morphism of affine schemes.

The above lemma can be used to prove that any representable presheaf is a sheaf in the fpqc topology. We say that the fpqc topology is subcanonical. More precisely, we have:

Proposition 4.13. *(i). Let R' be a faithfully flat R-algebra, and let $R'' = R' \otimes_R R'$. Consider the two maps $R' \to R''$ given by $x \mapsto x \otimes 1$ and $x \mapsto 1 \otimes x$. The following diagram is exact:*

$$R \longrightarrow R' \rightrightarrows R".$$

(ii). Let $f : S' \to S$ a faithfully flat and quasi-compact morphism of schemes and X and Y schemes over S. Denote by X', Y' (respectively X'', Y'') their base changes to S'

(respectively $S'' = S' \times_S S'$). Then the following diagram is exact:

$$\mathrm{Hom}_S(X, Y) \to \mathrm{Hom}_{S'}(X', Y') \rightrightarrows \mathrm{Hom}_{S''}(X'', Y'').$$

4.2.2 Étale sheaves

We will now give some example of étale sheaves. To be able to use the results from the preceding section, let us first prove the following lemma:

Lemma 4.14. *Any étale covering is an fpqc-covering.*

Proof. Let $\{U_i \xrightarrow{\varphi_i} U\}_{i \in I}$ be an étale covering. An étale morphism is flat and by construction, an étale covering is a family of jointly surjective morphisms, so we only have to check the quasi-compactness. Let $V \subset U$ be an affine open, and write $\varphi_i^{-1}(V) = \bigcup_{j \in J_i} V_{i,j}$ for some affine opens $V_{i,j} \subset U_i$. Since φ_i is open (étale morphisms are flat and locally of finite presentation so they are open), we obtain that $V = \bigcup_{i \in I} \bigcup_{j \in J_i} V_{i,j}$ is an open covering of V. But V is quasi-compact, so this covering admits a finite refinement. This concludes the proof. □

The étale topology being finer than the fpqc one, we obtain that any fpqc sheaf is an étale sheaf. In particular, we obtain:

Proposition 4.15. *The étale topology is subcanonical. More precisely, for X a scheme and Z an (arbitrary) X-scheme, the functor $U \mapsto \mathrm{Hom}_X(U, Z)$ is a sheaf of sets on $X_{\text{ét}}$.*

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