Sheet 7: Weakly étale versus ind-étale

Recall that a morphism of rings $A \to B$ is called weakly étale if both $A \to B$ and the multiplication morphism $B \otimes_A B \to B$ are flat. The goal of this sheet is to prove the following theorem:

Theorem 1 *Let $f : A \to B$ be weakly étale. Then there exists a faithfully flat ind-étale morphism $g : B \to C$ such that $g \circ f : A \to C$ is ind-étale.*

Exercise 1.

(1) Prove that if $f : A \to B$ is ind-étale then f is weakly étale.

(2) Prove that if $f : A \to B$ and $g : B \to C$ are weakly étale then $g \circ f$ is weakly étale. Conversely, prove that if $g \circ f$ and f are weakly étale then g is weakly étale.

(3) Let $A \to A'$ be a faithfully flat map. Prove that $f : A \to B$ is weakly étale if and only if $f \otimes_A A' : A' \to B \otimes_A A'$ is weakly étale.

Exercise 2. Let $f : A \to B$ be a map of rings. Reasoning as in the proof of Theorem 6.2 from the lecture notes, prove that there exists a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ B & \longrightarrow & B' \end{array}$$

with $A \to A'$ and $B \to B'$ faithfully flat and ind-étale, A' and B' w-strictly local and f' w-local.

Exercise 3.

(1) Prove that any map $f : X \to Y$ of w-local spectral spaces admits a canonical factorization $X \to Z \to Y$ in $\mathcal{S}^{\mathrm{wl}}$ with $Z \to Y$ a pro-(Zariski localization) and $X \to Z$ inducing a homeomorphism $X^c \simeq Z^c$.

(2) Prove that any w-local map $f : A \to B$ of w-local rings admits a canonical factorization $A \xrightarrow{a} C \xrightarrow{b} B$ with C w-local, a a w-local ind-(Zariski localization) and b a w-local map inducing $\pi_0(\mathrm{Spec}(B)) \simeq \pi_0(\mathrm{Spec}(C))$.

Exercise 4. Assuming the following result (see Olivier's paper *Fermeture intégrale et changements de base absolument plats*):

Theorem 2 *Let A be a strictly henselian local ring, and let B be a weakly étale local A-algebra. Then $f : A \to B$ is an isomorphism.*

prove that if $f : A \to B$ is a w-local weakly étale map between w-local rings with A w-strictly local, then f is an ind-(Zariski localization).

Exercise 5. Use Exercise 2 and Exercise 4 to prove Theorem 1.