Let us use the notation set up in the proof of Theorem 14.93 in [Görtz-Wedhorn] (all references below also refer to that book). We are only interested in the situation over $\bar{k}$, so everything is supposed to be over $\bar{k}$ and the subscript $\bar{k}$ is omitted.

The case $n = 1$ should actually be handled separately (the proof given does not work in that case as written, because E would be just one point, and a “hyperplane” in E would be empty), but this case is actually easier, because then the k-rational point x defines a divisor on $X = X'$ and the corresponding line bundle is $\mathscr{O}(1)$ (over $\bar{k}$).

Now let $n > 1$ and let $\pi\colon X' \to \mathbb{P}^n$ be the blow-up of the point $(1 : 0 : \cdots : 0)$. We identify X' with the closed subscheme

$$X' = V_+(T_iX_j - T_jX_i;\ 1 \leq i, j \leq n) \subset \mathbb{P}^{n-1} \times \mathbb{P}^n,$$

where $T_0, \ldots, T_n$ are homogeneous coordinates on the second factor $\mathbb{P}^n$, and $X_1, \ldots, X_n$ are homogeneous coordinates on the first factor $\mathbb{P}^{n-1}$ in this product.

Let $E \subset X'$ be the exceptional divisor and let $\mathscr{O}_{X'}(E)$ the associated line bundle. We want to show that the line bundle $\mathscr{M}_d := \pi^*\mathscr{O}_{\mathbb{P}^n}(d) \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{X'}(-dE)$ is globally generated for every $d \geq 1$, and want to understand the morphism $X' \to \mathbb{P}(H^0(X', \mathscr{M}_d))$ that it defines.

The key is to understand the case $d = 1$, so this is what we will do first.

The case $d = 1$. We start by defining some map and show afterwards that it is in fact the map given by $\mathscr{M}_1$.

So consider the restriction of the projection $\mathbb{P}^{n-1} \times \mathbb{P}^n \to \mathbb{P}^{n-1}$ to the first factor to X'. This is a morphism $r\colon X' \to \mathbb{P}^{n-1}$. To show that it corresponds to $\mathscr{M}_1$, we need to show the following claim:

Claim. We have

$$r^*\mathscr{O}_{\mathbb{P}^{n-1}}(1) \cong \mathscr{M}_1.$$

Proof of claim. We will express both sides as the line bundles attached to certain Cartier divisors. It is then enough to show that these Cartier divisors are linearly equivalent. (In fact, with the choices we will make they will even turn out to be equal.)

For $\mathscr{O}_{\mathbb{P}^n}(1)$, we can fix any i and then can view it as the line bundle of the Cartier divisor given by the tuple $(D_+(T_j), \frac{T_i}{T_j})_j$. (This corresponds to the Weil divisor $V_+(T_i)$.) Cf. Example 11.45 and Section (13.4). Below we will choose $i = 1$, so we get $(D_+(T_j), \frac{T_1}{T_j})_j$. Similarly, we can express $\mathscr{O}_{\mathbb{P}^{n-1}}(1)$ as the line bundle for the divisor $(D_+(X_i), \frac{X_1}{X_i})_i$.

To describe E in a similar way, we denote by $U_{ij} \subset X'$ the open subscheme where X_i and T_j are invertible. These open subschemes cover X'. The exceptional divisor E is given on U_{ij} by the equation $\frac{T_i}{T_j} = 0$ (because together with the equations $T_iX_j - T_jX_i$ and the invertibility of X_i this implies $T_1 = \cdots = T_n = 0$). (In particular, as a Weil divisor/closed subscheme, $E \cap U_{ij} = \emptyset$ for all $i > 0$.)

Now let us compare the two line bundles $r^*\mathscr{O}_{\mathbb{P}^{n-1}}(1)$ and $\mathscr{M}_1$ in terms of the Cartier divisors we have just described.

We can view $r^*\mathscr{O}_{\mathbb{P}^{n-1}}(1)$ as the line bundle attached to the Cartier divisor $(U_{ij}, \frac{X_1}{X_i})_{i,j}$.

For $\mathscr{M}_1 = \pi^*\mathscr{O}_{\mathbb{P}^n}(1) \otimes \mathscr{O}(-E)$ we get $(U_{ij}, \frac{T_1}{T_j} \cdot \frac{T_j}{T_i})_{i,j}$, or in other words $(U_{ij}, \frac{T_1}{T_i})_{i,j}$.

But $\frac{X_1}{X_i} = \frac{T_1}{T_i}$ on X' (see the equations defining X'), so the two Cartier divisors are actually equal and in particular the corresponding line bundles are isomorphic.

We can also write this down in coordinates. The map r is given by

$$((x_1 : \cdots : x_n), (t_0 : \cdots : t_n)) \mapsto (x_1 : \cdots : x_n).$$

Now let us take the specific hyperplane $H = V_+(X_n) \subset \mathbb{P}^{n-1} = E$. Then $r^{-1}(r(H))$ consists of all points

$$((x_1 : \cdots : x_{n-1} : 0), (t_0 : \cdots : t_n)) \in X'.$$

All those points, lying on X', also satisfy $x_it_n = x_nt_i = 0$. If we had $t_n \neq 0$, we would obtain $x_1 = \cdots = x_n = 0$ which is not possible for a point in projective space. Therefore $t_n = 0$ for all those points, and one sees that $\pi(r^{-1}(r(H))) = V_+(T_n)$, a hyperplane in $\mathbb{P}^n$.

Remark. In more geometric terms, the map r has the following description. Clearly, the restriction $r_{|E}$ (where we now view E as a closed subscheme of X') is an isomorphism. Now let $x \in X' \setminus E$ be a closed point. Let g be the unique line in $\mathbb{P}^n$ which connects $\pi(x)$ with $(1 : 0 : \cdots : 0)$. The strict transform of g in X (i.e., the closure of $\pi^{-1}(g \setminus \{(1 : 0 : \cdots : 0)\})$) is a "line" (a closed subscheme of X' isomorphic to $\mathbb{P}^1$) which intersects E in a unique point y. Then r maps all points on this line to the point $r(y)$.

In particular, if $H \subset E$ is a hyperplane, then $r^{-1}(r(H))$ consists of all points on all those lines in X' as above which meet E in a point of H. If we view E as the projective space of lines in the tangent space of $\mathbb{P}^n$ at $(1 : 0 : \cdots : 0)$ (see the remark on top of p. 416), then we can describe $\pi(r^{-1}(r(H)))$ as the union of all lines in $\mathbb{P}^n$ through $(1 : 0 : \cdots : 0)$ which give rise to a tangent vector lying in H.

The case of general $d \geq 1$.

From the above discussion, we easily can pass to the general case. In fact, let $\iota\colon \mathbb{P}^{n-1} \to \mathbb{P}^N$ be the d-fold Veronese embedding, i.e., the morphism induced by the line bundle $\mathscr{O}_{\mathbb{P}^{n-1}}(d)$ (i.e., $\iota^*\mathscr{O}_{\mathbb{P}^N}(1) \cong \mathscr{O}_{\mathbb{P}^{n-1}}(d)$).

Then for $r\colon X' \to \mathbb{P}^{n-1}$ as above, we have $(\iota \circ r)^*\mathscr{O}_{\mathbb{P}^N}(1) \cong \mathscr{M}_1^{\otimes d} \cong \mathscr{M}_d$.

Now ι is a closed embedding, and therefore $(\iota \circ r)^{-1}((\iota \circ r)(H)) = r^{-1}(r(H))$, so that part of the discussion does not change when we replace r by $\iota \circ r$.

(But we need the result for arbitrary d because we do not know a priori, in the proof of Theorem 14.93, that we can find a line bundle $\mathscr{L}$ before passing to $\bar{k}$ that would give rise to $d = 1$.)