Algebraic Geometry II
SS 2026

Prof. Dr. Ulrich Görtz
Dr. Andreas Pieper

Problem sheet 12

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Problem 34

Let X be a topological space. A sheaf $\mathscr{F}$ on X is called *flasque* (or flabby, in German: welk) if for all $V \subseteq U$ open in X, the restriction map $\Gamma(U, \mathscr{F}) \to \Gamma(V, \mathscr{F})$ is surjective.

(1) Show that a constant sheaf on an irreducible topological space is flasque.

(2) Let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of sheaves of abelian groups on X. Assume that $\mathscr{F}'$ is flasque. Show that the sequence

$$0 \to \Gamma(X, \mathscr{F}') \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}'') \to 0$$

is exact.

(3) Let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of sheaves of abelian groups on X. Assume that $\mathscr{F}'$ and $\mathscr{F}$ are flasque. Prove that $\mathscr{F}''$ is flasque.

Hint. For part (2), let $s'' \in \Gamma(X, \mathscr{F}'')$. Use Zorn's Lemma to show that there exists a maximal open subset U such that $s''_{|U}$ is in the image of $\Gamma(U, \mathscr{F}) \to \Gamma(U, \mathscr{F}'')$. Now assume there is a point $x \in X \setminus U$ and use that locally around x, s'' has a preimage in $\mathscr{F}$, and that $\mathscr{F}'$ is flasque to obtain a contradiction.

Problem 35 Let R be a ring. We have seen in Problem 31 that the category (R-Mod) of R-modules has enough injective objects. For any R-module M, the functor $\operatorname{Hom}_R(M, \cdot)$ is left exact and hence has a derived functor denoted $\operatorname{Ext}^i_R(M, \cdot)$. Now let M and N be R-modules. The goal of this exercise is to show that the first Ext-group $\operatorname{Ext}^1_R(M, N)$ classifies extensions of M by N, i.e., R-modules X together with a short exact sequence $0 \to N \to X \to M \to 0$. We say that two extensions X, X' are isomorphic if there is an isomorphism $X \xrightarrow{\sim} X'$ such that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & X' & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

commutes. Denote by $\operatorname{EXT}_R(M, N)$ the set of extensions of M by N modulo isomorphism.

(1) Let $N \hookrightarrow I$ be an embedding into an injective R-module and denote by $q\colon I \to Q$ the map onto the cokernel. Prove that there is an isomorphism $\xi\colon \operatorname{Ext}^1_R(M,N) \cong \frac{\operatorname{Hom}_R(M,Q)}{\operatorname{Hom}_R(M,I)}$.

(2) Show that the there is a well defined map $\frac{\operatorname{Hom}_R(M,Q)}{\operatorname{Hom}_R(M,I)} \to \operatorname{EXT}_R(M,N)$ which sends a homomorphism $\varphi\colon M \to Q$ to

$$X := \{(i,m) \in I \oplus M \mid q(i) = \varphi(m)\}\,,$$

viewed as an extension of M by N in the obvious way.

(3) Prove that ξ is a bijection by constructing an inverse map.
Hint: Given an extension X, construct a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & I & \longrightarrow & X' & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

Show that the bottom row splits and apply the snake lemma.

Problem 36 Let k be a field and consider $\mathbb{P}^1_k$ and an arbitrary point $P \in \mathbb{P}^1_k(k)$. Show that $\mathrm{H}^1(\mathbb{P}^1_k, \mathscr{O}_{\mathbb{P}^1_k}(-2)) \neq 0$ by looking at the short exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^1_k}(-2) \to \mathscr{O}_{\mathbb{P}^1_k}(-1) \to \kappa_P \to 0$$

constructed in Problem 33.