Peter Scholze awarded the Fields medal

Ulrich Görtz

Bonn, October 1, 2018

Most important research prize in mathematics

John Charles Fields

Since 1936,

59 medals awarded.

Age limit: 40 years

Most important research prize in mathematics

Der 30-jährige Peter Scholze darf sich über 10.000 Euro Preisgeld freuen.
(Foto: picture alliance/dpa)

Mittwoch, 01. August 2018

"Nobelpreis der Mathematik"

Peter Scholze gewinnt Fields-Medaille

Der Bonner Peter Scholze ist mit einem der höchsten Preise der Mathematik geehrt

Most important research prize in mathematics

SPIEGEL ONLINE

Fields-Medaille

Peter Scholze bekommt weltweit höchste Auszeichnung für Mathematiker

Peter Scholze bekommt als erster Deutscher seit 32 Jahren eine Fields-Medaille. Die Auszeichnung gilt als Nobelpreis für Mathematik.

Aus Rio de Janeiro berichtet Holger Dambeck

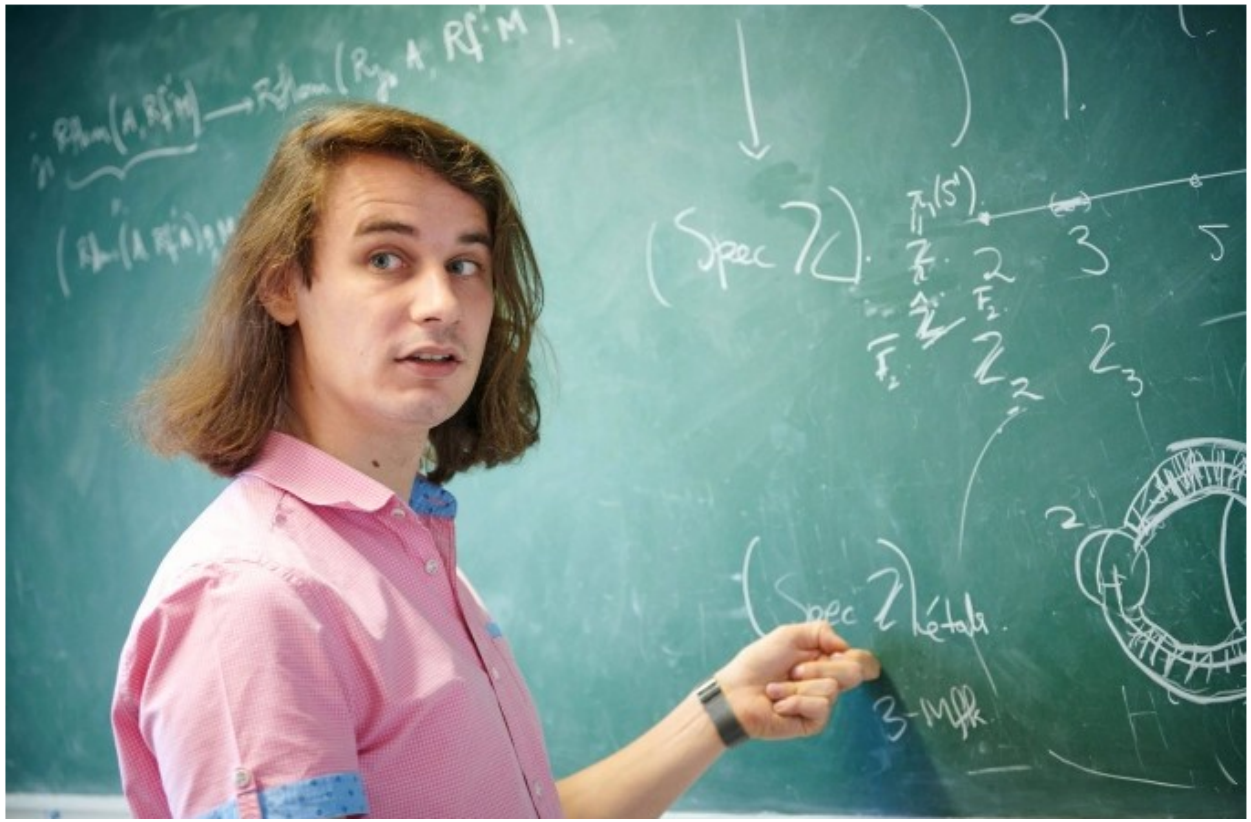
Most important research prize in mathematics

Frankfurter Allgemeine
Computer & Mathematik

MATHEMATIKER PETER SCHOLZE

Räume, die vor ihm niemand sah

VON ULF VON RAUCHHAUPT - AKTUALISIERT AM 06.08.2018 - 08:27



Goal of this talk

Some impression of the area, provide context for non-experts.

Urbano Monte's map of the earth, 1587

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David Rumsey Map Collection CC-BY-NC-SA 3.0

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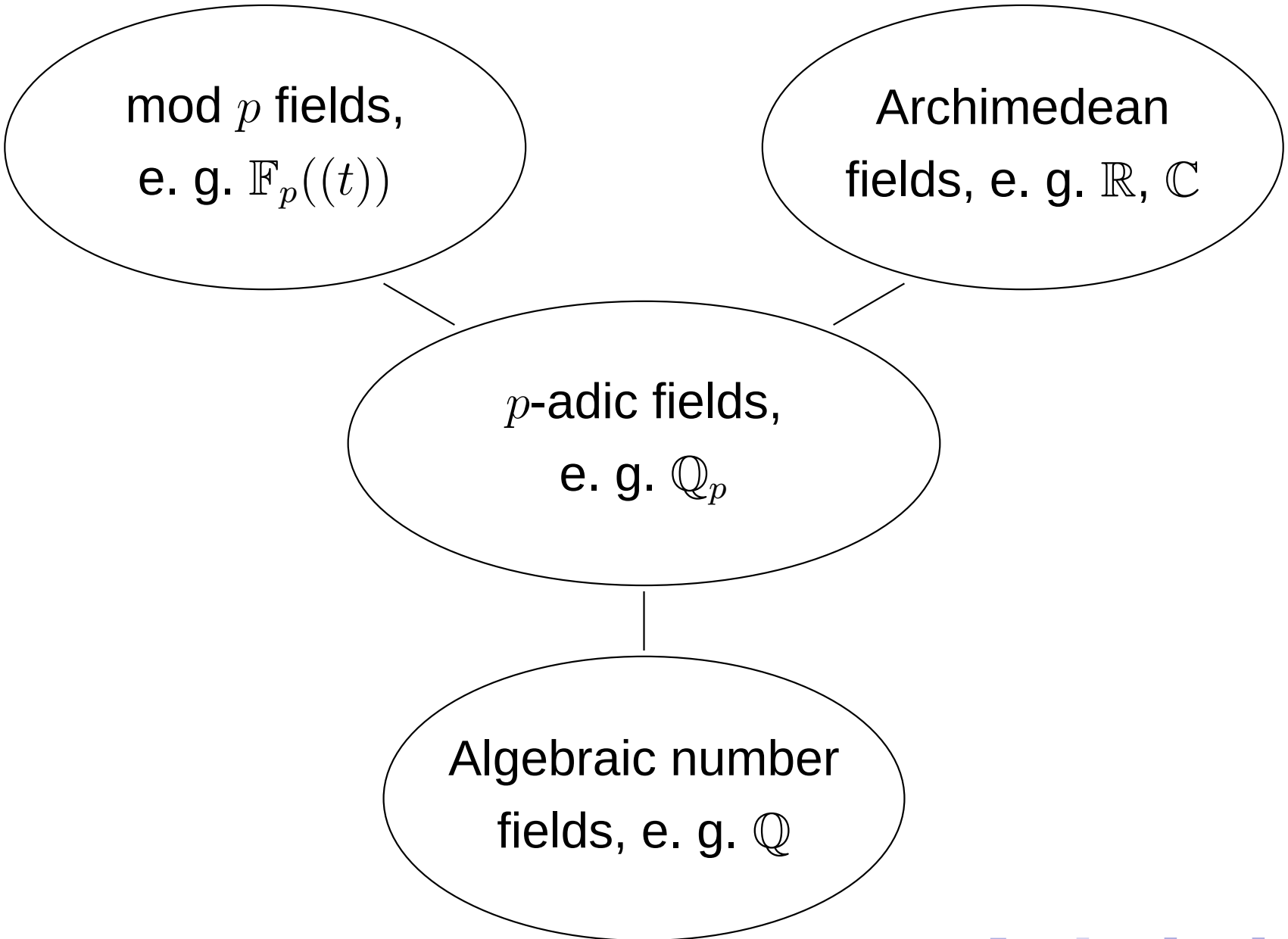

Solving equations

Important problem in mathematics:

Understand set of solutions of an equation.

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Understand set of solutions of an equation.

- Do solutions exist?
- Are there only finitely many solutions? Can we count them? Can we write them down explicitly?
- If there are infinitely many solutions, does the set of solutions have a (geometric) structure?

Where are we looking for solutions?

Natural numbers $\mathbb{N} = \{0, 1, 2, 3, \dots\}$.

Integers *(add negative numbers)*

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

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Real numbers *(add all decimal numbers)*

$$-1,\ -0.5,\ 0,\ 0.333\dots,\ 1,\ 2,\ 3.14159265\dots \in \mathbb{R}$$

Can we detect cases without solutions?

This is often a very hard problem, for instance:

Theorem ("Fermat's Last Theorem", A. Wiles)

Let $n > 2$ be an integer. Then the equation

$$x^n + y^n = z^n$$

has no solutions with integers $x, y, z \geq 1$.

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For certain equations, however, it is easy to show that there are no solutions in the integers.

Understand set of solutions in real numbers

Trivially: If no solutions in $\mathbb{R}$, then no solutions in $\mathbb{Z}$.

Sometimes: Good understanding of solutions in real numbers

$\rightsquigarrow$ understand solutions in integers.

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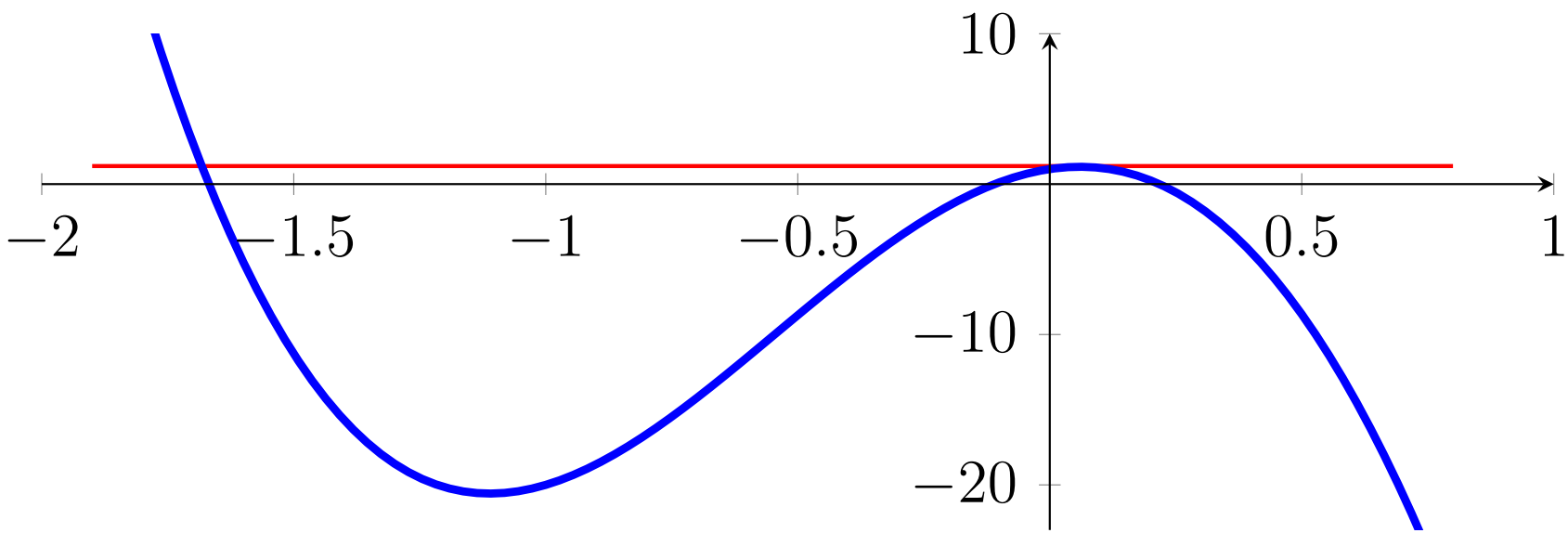
Over $\mathbb{R}$, can use analytic methods (Differential calculus, derivatives, …)



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left hand side $x^4 + 17$:
2, 3, 7 or 8,

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More powerful: Look at more final digits.

In other words: division with remainder by 10, 100, 1000, …, 10^i.

Looking at the final digit, refined version

Could also do division with remainder by other numbers $n = 2, 3, 4, \ldots$.

For instance consider the equation

$$x^4 - 17 = 7y^2$$

This is “solvable mod 10” (both sides can have final digit 3, for instance).

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But division with remainder by 7 gives remainder 1, 3, 4, or 5 on the left, and 0 on the right.

Division with remainder = n-adic final digit

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$$23$$

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7-adic expression:

$$23 = 21 + 2 = 3 \cdot 7^1 + 2 \cdot 7^0 = 32_7.$$

$23 \equiv 2 \mod 7$.

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Key point: passing to limit.

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A sequence of real numbers “coming arbitrarily close to each other” converges to a limit in $\mathbb{R}$.

- …not interesting in $\mathbb{Z}$,
- …does not work in $\mathbb{Q}$: We can approximate $\sqrt{2}$ by rational numbers, but it is not rational itself.

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Two real numbers are close to each other if the differences lie far to the right of the decimal point:

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A limit always exists because, naively speaking, we allow infinitely many digits to the right of the decimal point.

Setting up an analogy

“Solving” an equation so that the final 5 digits match is more difficult than having only the final digits match.

Having more digits match is a “better approximation” of the solution from this point of view.

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Example (Lind-Reichardt equation: $x^4 - 17 = 2y^2$)

- $x = 5$, $y = 8$: 608 versus 128
- $x = 85$, $y = 548$: 52 200 608 versus 600 608

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10-adic numbers $\mathbb{Z}_{10}$:
Allow infinitely many digits, extending to the left.

Computing with 10-adic numbers

$$\mathbb{Z}_{10} = \{\ldots a_2 a_1 a_0;\ a_i \in \{0, 1, \ldots, 9\}\}.$$

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Surprising things may happen:

$$\ldots 999 + 1 = 0, \quad \text{hence } \ldots 999 = -1.$$

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Properties

- all integers are 10-adic numbers,
- $\mathbb{Z}_{10}$ has operations $+$, $-$, $\cdot$.
- Even some fractions are 10-adic: $\ldots 6667 \cdot 3 = 1$.

Variant: p-adic numbers

Although we can compute in the set $\mathbb{Z}_{10}$ of 10-adic numbers, it has some less nice features:

$$\ldots 8212890625 \cdot \ldots 1787109376 = 0.$$

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That means: use p-adic expression, and allow it to extend infinitely to the left.

$$\mathbb{Z}_2 = \{\ldots a_2 a_1 a_0;\ a_i \in \{0, 1\}\}$$
$$\mathbb{Z}_7 = \{\ldots a_2 a_1 a_0;\ a_i \in \{0, 1, \ldots, 6\}\}$$

Geometry of the p-adic numbers

Absolute value on $\mathbb{Z}_p$

$$|x|_p = \frac{1}{p^n},$$

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Example

- $|48|_2 = |110000_2|_2 = 1/2^4 = 1/16,$
- $|23|_7 = |32_7|_7 = 1.$

Geometry of the p-adic numbers

Absolute value on $\mathbb{Z}_p$

$$|x|_p = \frac{1}{p^n},$$

where n is the number of zeros at the end of p-adic expression

We regard x close to y, if $|x - y|$ small.

Some unusual features:

- Every triangle is isosceles.
- Any two circles are disjoint or concentric.

The field of p-adic numbers

The field $\mathbb{Q}_p$: Enlarge $\mathbb{Z}_p$ by allowing finitely many digits after decimal point.

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Über eine neue Begründung der Theorie der algebraischen Zahlen.

Von **K. Hensel** in Berlin.

Die Analogie zwischen den Resultaten der Theorie der algebraischen Functionen einer Variabeln und der der algebraischen Zahlen hat mir schon seit mehreren Jahren den Gedanken nahe ge-

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Im allgemeinen schreiten alle diese Entwickelungen nach Potenzen von p mit ganzzahligen Exponenten fort, d. h. sie können folgendermafsen geschrieben werden:

$$(1) \qquad X = \frac{A_{-h}}{p^h} + \cdots + \frac{A_{-1}}{p} + A_0 + A_1 p + \cdots;$$

für diese Zahlen erhält man also genau dieselben Entwickelungen wie für eine algebraische Function in der Umgebung einer regulären Stelle.

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$$\mathbb{Z}_p = \left\{ \sum_{i=0}^{\infty} a_i p^i; \quad a_i \in \{0, 1, \ldots, p-1\} \right\},$$

$$\mathbb{Q}_p = \left\{ \sum_{i=i_0}^{\infty} a_i p^i; \quad i_0 \in \mathbb{Z}, \quad a_i \in \{0, 1, \ldots, p-1\} \right\}.$$

p-adic geometry

Tate (around 1962): Rigid analytic spaces

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$\rightsquigarrow$ reasonable notion of p-adic manifold/space.

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> *Peter Scholze has revolutionized the field of p-adic geometry.*
>
> *M. Rapoport, Laudatio for P. Scholze, ICM 2018*

p-adic and complex geometry are similar

Theorem (Scholze)

Let $C/\mathbb{Q}_p$ be complete and algebraically closed. Let X be a smooth proper rigid analytic space over C. For all $i \geq 0$, we have

$$\sum_{j=0}^{i} \dim_C H^{i-j}(X, \Omega_X^j) = \dim_C H^i_{dR}(X/C) = \dim_{\mathbb{Q}_p} H^i_{et}(X, \mathbb{Q}_p)$$

The local-global principle

Theorem (Hasse-Minkowski)

Let $n \geq 1$ and let $a_i \in \mathbb{Q}$, $1 \leq i \leq n$. Then the equation

$$a_1 x_1^2 + a_2 x_2^2 + \cdots + a_n x_n^2 = 1$$

has a solution $x_i \in \mathbb{Q}$, if and only if it has a solution in $\mathbb{R}$ and in every field $\mathbb{Q}_p$.

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Example (Hasse-Minkowski for $n = 1$)

$ax^2 = 1$ solvable in $\mathbb{Q} \Leftrightarrow a$ is a square $\neq 0 \Leftrightarrow a > 0$ and every prime p occurs with even exponent in factorization of a

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…can we write them down?

Linear: $2x - 6 = 0, \qquad x = \frac{6}{2} = 3.$

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Quadratic:

$$ax^2 + bx + c = 0,\ a \neq 0,$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \in \mathbb{R}.$$

Formulas for degrees 3, 4

(del Ferro, Tartaglia, Cardano, Ferrari $\approx$ 1500)

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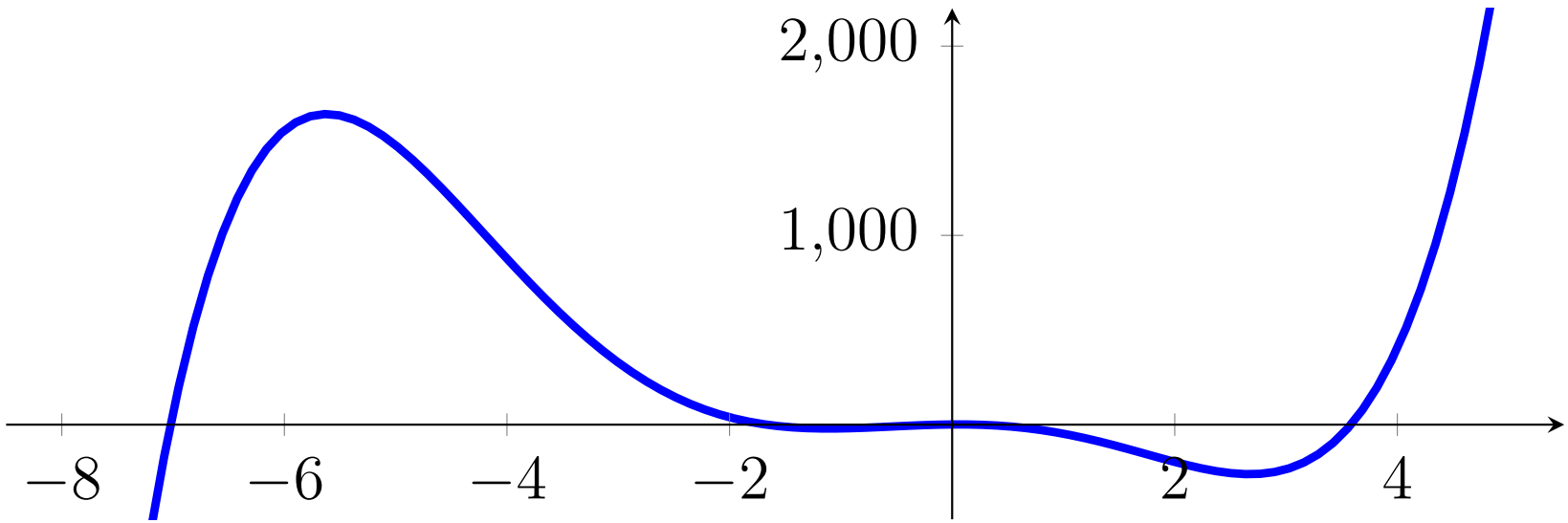
Even worse: For example, the zeros of the polynomial

$x^5 + 5x^4 - 20x^3 - 40x^2 + 5x + 1$

cannot be expressed in terms of $+$, $-$, $\cdot$, $/$ and $\sqrt[n]{-}$.



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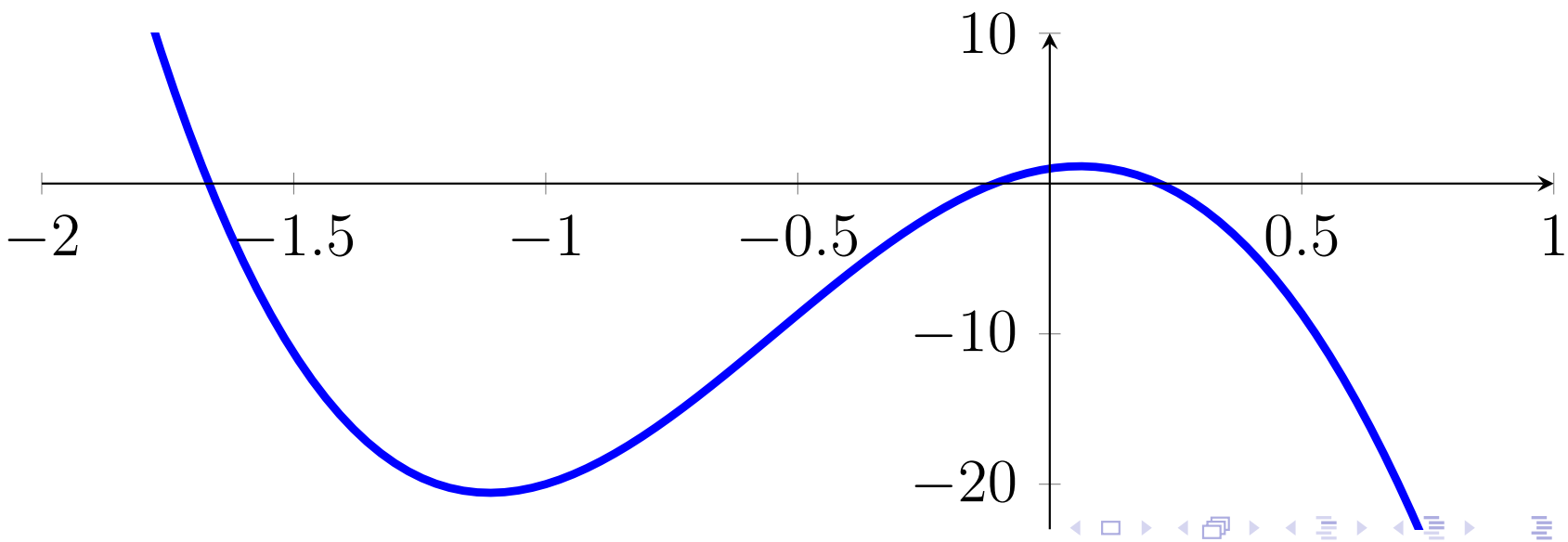

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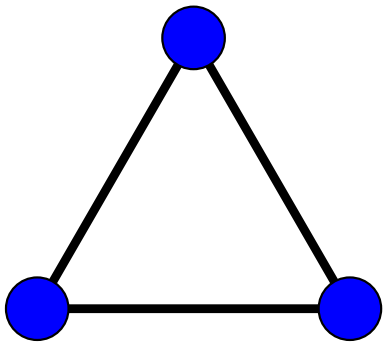
Distinguish geometric objects by their “symmetry group”



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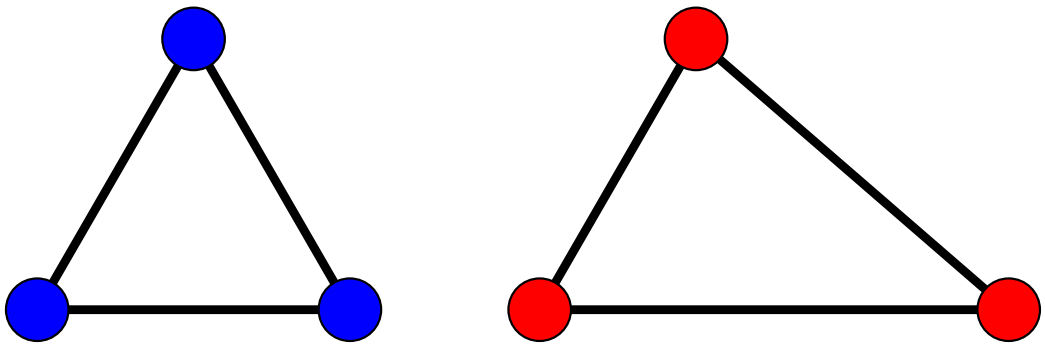

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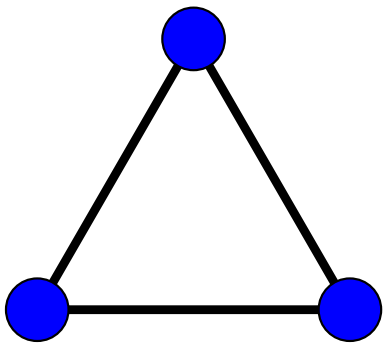

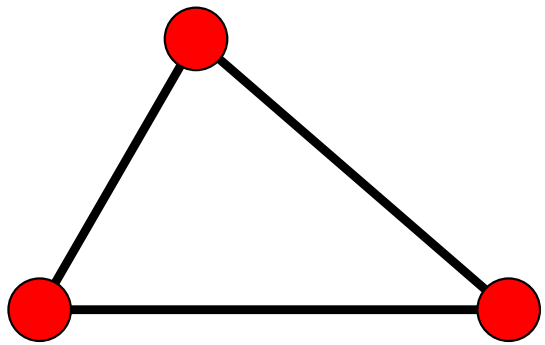

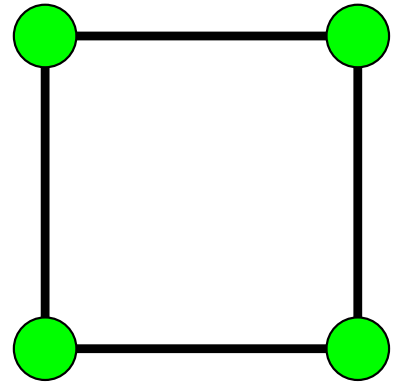

Galois groups

Distinguish kind of solution of polynomial by their symmetry group

Definition (informal)

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Definition

Let f be a polynomial with coefficients in a field K. Let L be the smallest field containing K and all zeros of f (in some algebraically closed extension field).
The Galois group of f is the group of field automorphisms $L \to L$ which fix all elements of K.

Solvability in terms of Galois groups

Theorem

If f is a polynomial over $\mathbb{Q}$ whose solutions can be expressed in terms of $+$, $-$, $\cdot$, $/$ and $\sqrt[n]{-}$ starting from rational numbers, then the Galois group of f is solvable.

Example

The Galois group of the polynomial

$$x^5 + 5x^4 - 20x^3 - 40x^2 + 5x + 1$$

is the symmetric group S_5 which is not solvable.

Can we understand Galois groups?

Definition (Absolute Galois group)

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Example ($K = \mathbb{Q}_p$)

$G_{\mathbb{Q}_p}$ is somewhat easier to understand, but still complicated.

The absolute Galois group of a finite field

Definition (Finite field with p elements)

Let p be a prime number. We let

$$\mathbb{F}_p := \{0, 1, \ldots, p-1\}$$

with addition and multiplication "modulo p".

In particular: $\underbrace{1 + \cdots + 1}_{p \text{ summands}} = 0$ in $\mathbb{F}_p$. ("Characteristic p")

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Remark

Let K be a field of characteristic p. Then

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Consequence

The absolute Galois group $G_{\mathbb{F}_p}$ is isomorphic to $\widehat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$. It is topologically generated by the Frobenius automorphism.

How far apart are characteristic 0 and p?

Compare

$$\mathbb{Q}_p = \left\{ \sum_{i=i_0}^{\infty} a_i p^i;\ i_0 \in \mathbb{Z},\ a_i \in \{0, 1, \ldots, p-1\} \right\}.$$

versus

$$\mathbb{F}_p((t)) = \left\{ \sum_{i=i_0}^{\infty} a_i t^i;\ i_0 \in \mathbb{Z},\ a_i \in \{0, 1, \ldots, p-1\} \right\}.$$

These descriptions look similar, but addition is very different!

Perfectoid fields and tilting

Definition (Scholze)

A *perfectoid field* is a field K, complete with respect to a non-discrete non-archimedean valuation, with residue characteristic $p > 0$ with ring of integers $\mathcal{O}_K = \{x \in K;\ |x| \leq 1\}$, such that the map

$$\mathcal{O}_K/p \to \mathcal{O}_K/p, \quad x \mapsto x^p,$$

is surjective.

Example

$\mathbb{Q}_p(p^{1/p^\infty})^\wedge$, $\mathbb{Q}_p(\mu_{p^\infty})^\wedge$, $\mathbb{F}_p((t))(t^{1/p^\infty})^\wedge$.

Tilting: Switch from char. 0 to positive characteristic

Every perfectoid field K has a tilt $K^\flat$.

$$K^\flat = \text{Frac}(\varprojlim_{x \mapsto x^p} \mathcal{O}_K/p).$$

The tilt $K^\flat$ has characteristic p: $1 + \cdots + 1 = 0$ in $K^\flat$.

Theorem (Fontaine, Wintenberger)

$$G_K \cong G_{K^\flat}.$$

Perfectoid spaces

Definition (Scholze)

Let K be a perfectoid field. A perfectoid space over K is an adic space which is locally isomorphic to an affinoid adic space, i.e., an adic space of the form $Spa(R, R^+)$ where R is a perfectoid K-algebra.

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Tilting for perfectoid spaces

Every perfectoid space X has a tilt $X^\flat$, and both have "the same étale covers":

Theorem (Scholze)

$$\pi_1(X) \cong \pi_1(X^\flat)$$

The Langlands program

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Theorem (Quadratic Reciprocity Law, Gauß)

Let $p \neq q$ *be prime numbers* > 2, $p \equiv 1 \mod 4$. *The equation*

$$x^2 \equiv q \mod p \quad \textit{has a solution}$$

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Example ($p = 5$, $q = 67$)

The equation $x^2 \equiv 67 \equiv 2 \mod 5$ has no solution.

Hence $x^2 \equiv 5 \mod 67$ has no solution.

Class field theory

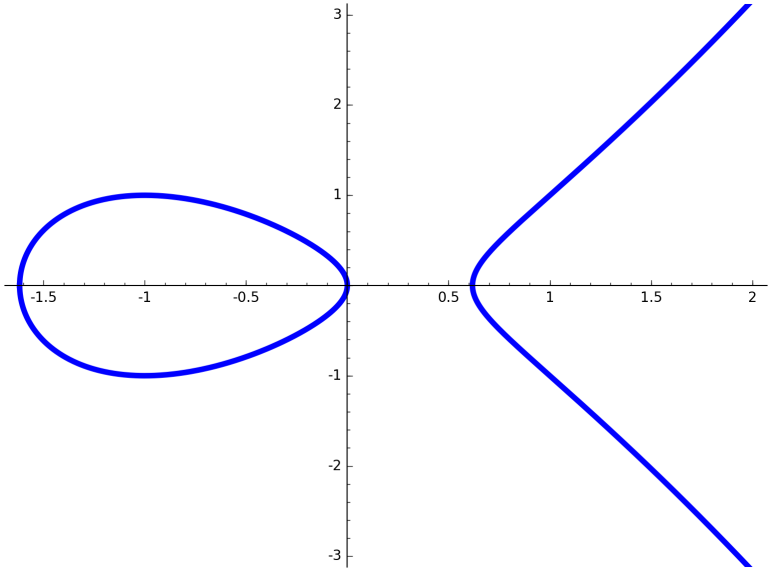
Describe the maximal abelian quotients $G_{\mathbb{Q}}^{\mathrm{ab}}$ and $G_{\mathbb{Q}_p}^{\mathrm{ab}}$.

> *La théorie du corps de classes a une réputation de difficulté qui est en partie justifiée. Mais il faut faire une distinction: il n'est peut-être pas en effet dans la science de théorie où tout à la fois les démonstrations soient aussi ardues, et les résultats d'une aussi parfaite simplicité et d'une aussi grande puissance.*
>
> *J. Herbrand, 1936*

Particular instance: Modularity

Elliptic curve

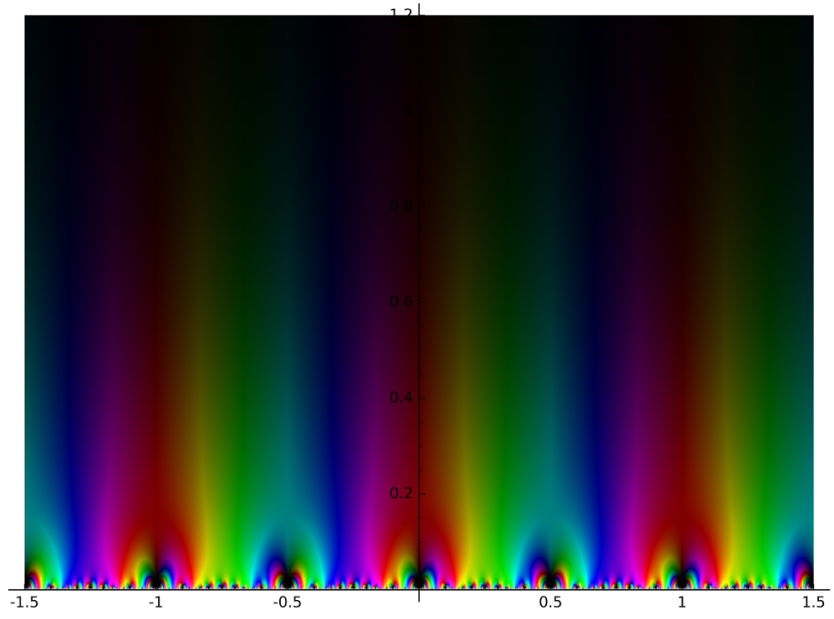
$$y^2 = x^3 + x^2 - x$$



Modular form

$$f : \mathbb{H} \to \mathbb{C}$$

holom., “highly symmetric”



Theorem (Wiles, …)

Every elliptic curve E over $\mathbb{Q}$ is modular.

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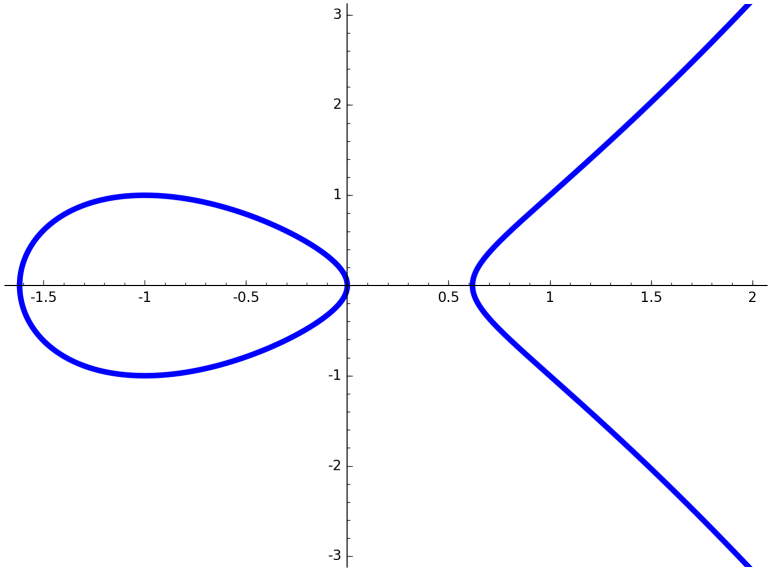

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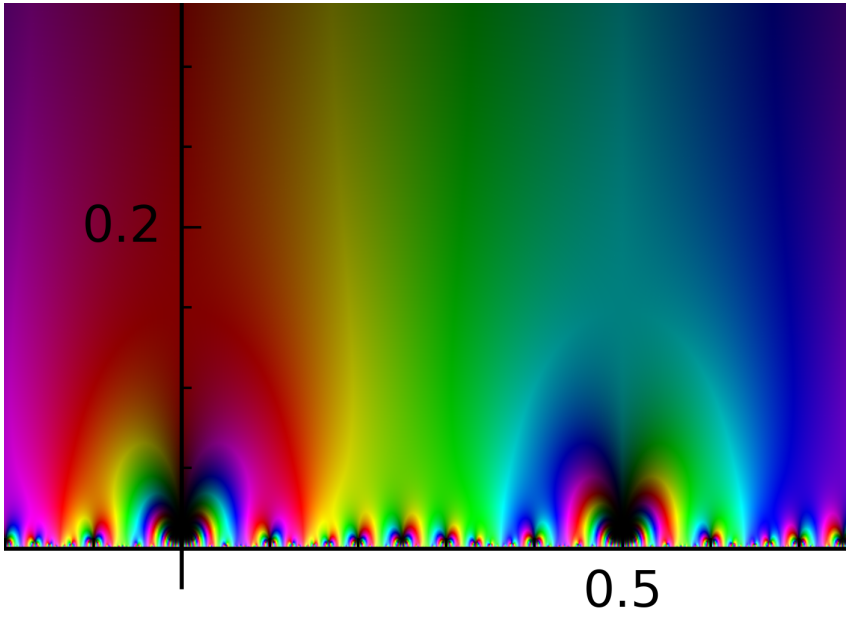

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$$\#\{(x,y);\ 0 \le x,y \le p-1,\ y^2 \equiv x^3+x^2-x \bmod p\}$$

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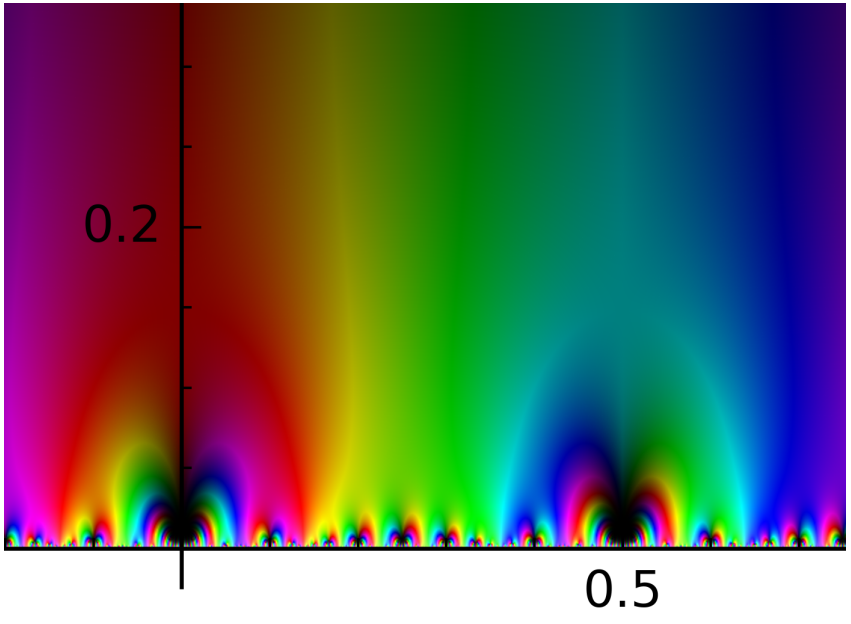

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$\rightsquigarrow$ L-function $L(E/\mathbb{Q},s)$

Modular form

Fourier expansion, $q = \exp(2\pi i z)$

$$q-2q^3-q^5+2q^7+q^9+2q^{13}+\cdots$$

$\rightsquigarrow$ L-function $L(f,s)$

Theorem (Wiles, ...)

Every elliptic curve E over $\mathbb{Q}$ is modular.

Corollary

The L-function $L(E/\mathbb{Q}, s)$ has a holomorphic continuation to $\mathbb{C}$.

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Theorem (Allen, Calegari, Caraiani, Gee, Helm, Le Hung, Newton, Scholze, Taylor, Thorne)

Let E be an elliptic curve over a CM field K. Then the L-function of E over K has a meromorphic continuation to $\mathbb{C}$.

Other direction: automorphic $\to$ Galois

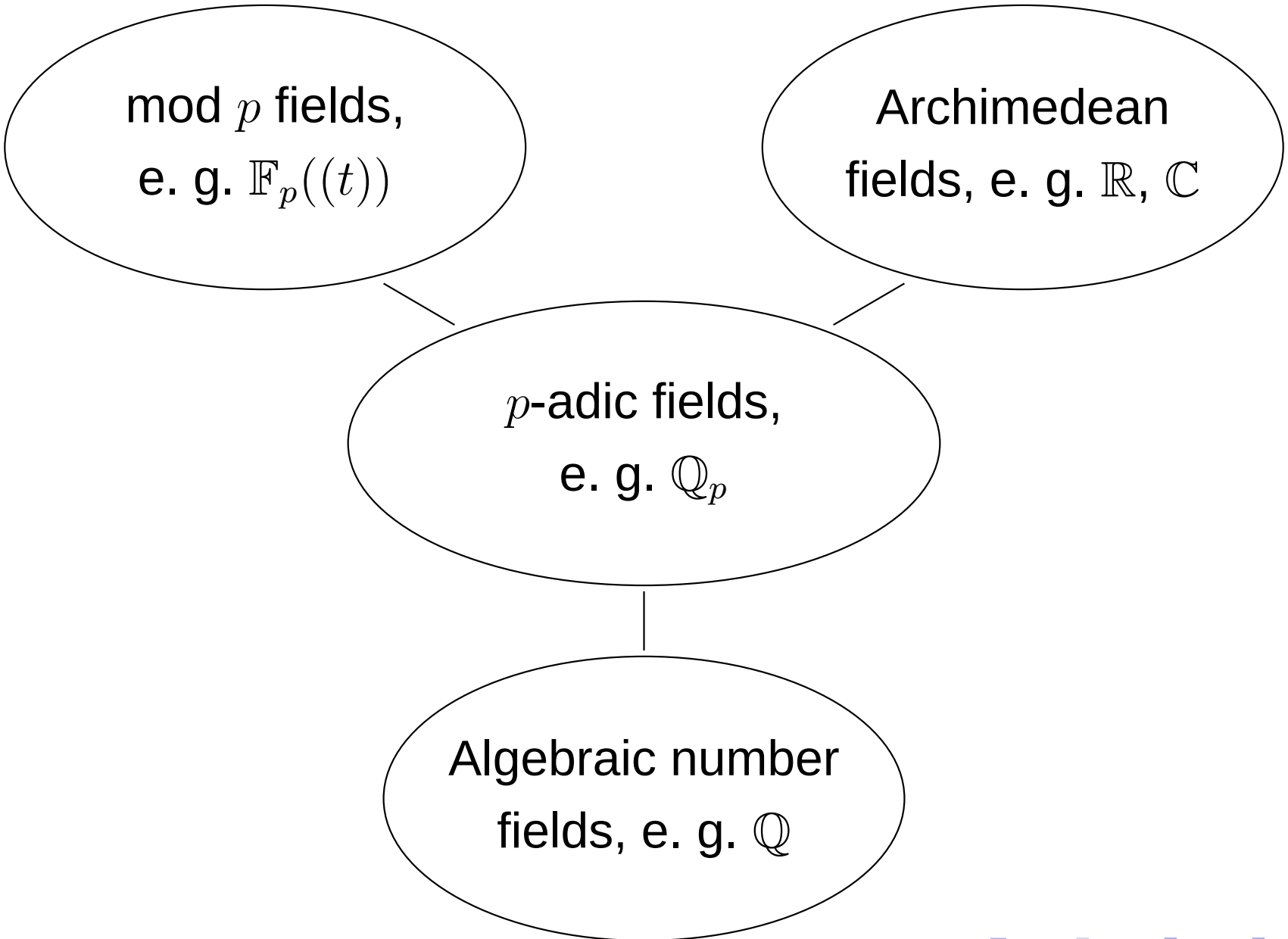
Theorem (Scholze)

Let F be totally real or CM, let $G = \mathrm{Res}_{F/\mathbb{Q}}(GL_n)$, and let X_K be the locally symmetric space attached to G and a compact open subgroup $K \subset G(\mathbb{A}_f)$. For every system of Hecke eigenvalues occurring in the cohomology $H^i(X_K, \overline{\mathbb{F}}_p)$, there exists a continuous Galois representation

$$\rho\colon \mathrm{Gal}(\overline{F}/F) \to GL_n(\overline{\mathbb{F}}_p)$$

such that Hecke eigenvalues and Frobenius eigenvalues match.

Goal of this talk



What is this good for?

Why are we doing this?

Fascinating to

- understand problems that have been studied for more than 2000 years,
- gain conceptual understanding of surprising patterns,
- teach the subject to others.

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(Polynomial) equations are everywhere:

- Elliptic curve cryptography
- Theoretical physics
- Computer science
- Biochemistry
- …

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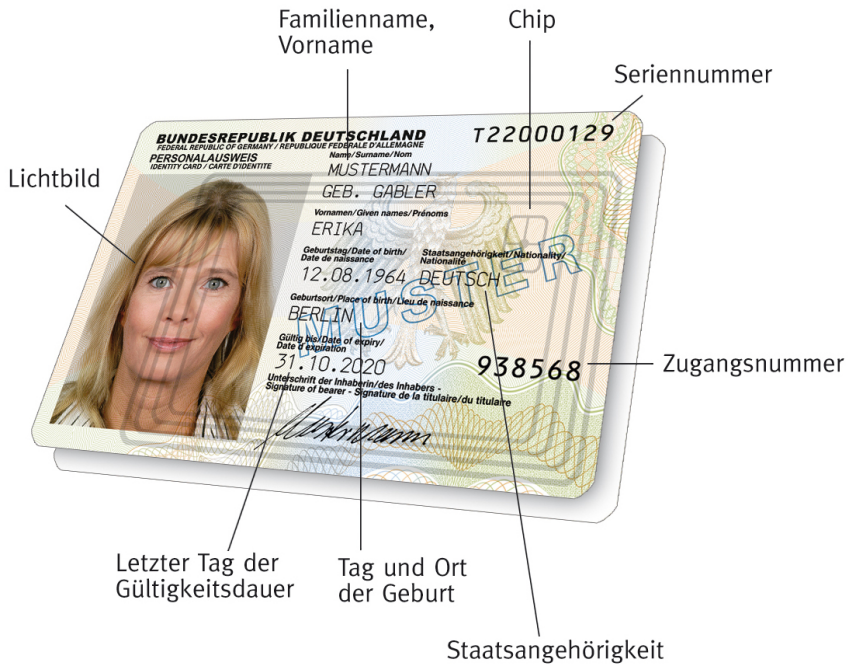

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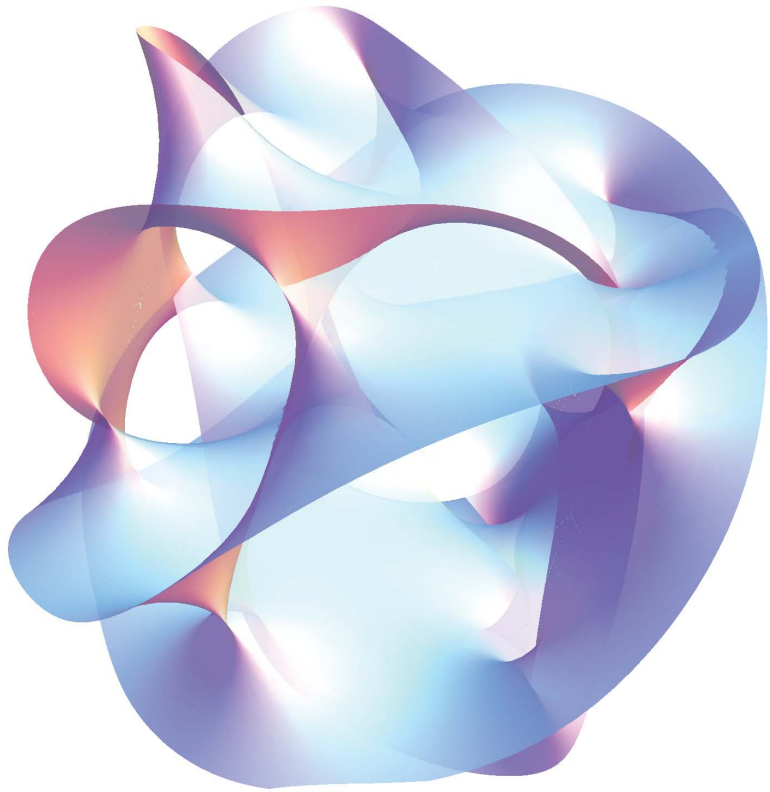

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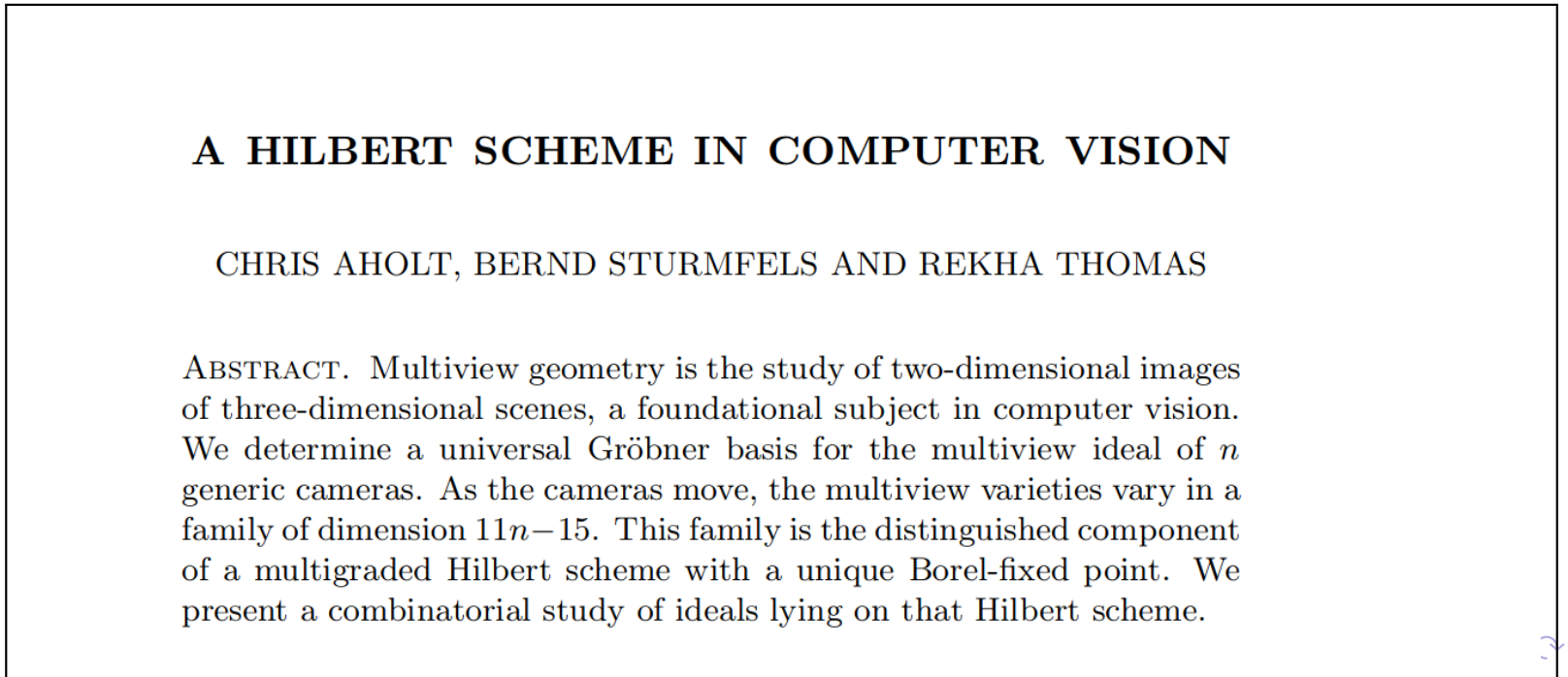

A HILBERT SCHEME IN COMPUTER VISION

CHRIS AHOLT, BERND STURMFELS AND REKHA THOMAS

ABSTRACT. Multiview geometry is the study of two-dimensional images of three-dimensional scenes, a foundational subject in computer vision. We determine a universal Gröbner basis for the multiview ideal of n generic cameras. As the cameras move, the multiview varieties vary in a family of dimension $11n-15$. This family is the distinguished component of a multigraded Hilbert scheme with a unique Borel-fixed point. We present a combinatorial study of ideals lying on that Hilbert scheme.

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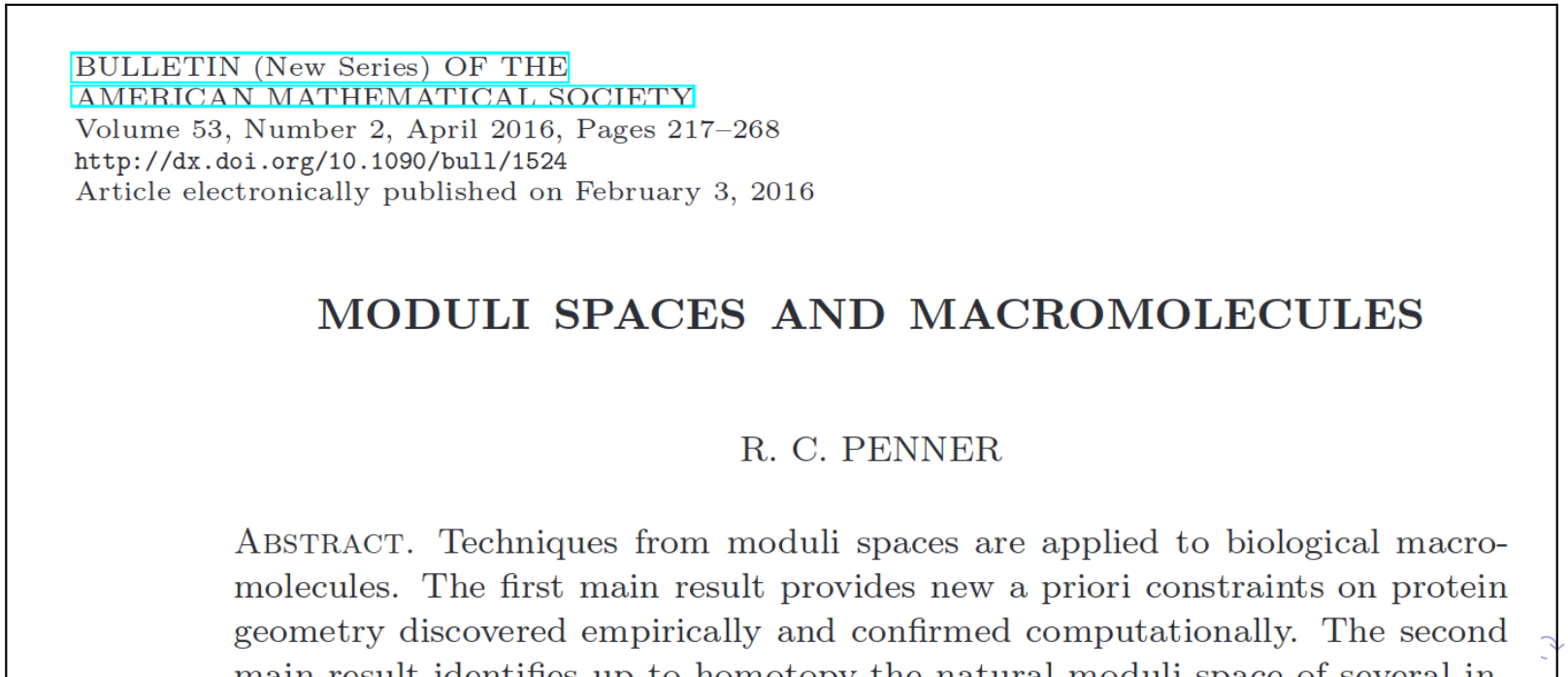

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Article electronically published on February 3, 2016

MODULI SPACES AND MACROMOLECULES

R. C. PENNER

ABSTRACT. Techniques from moduli spaces are applied to biological macromolecules. The first main result provides new a priori constraints on protein geometry discovered empirically and confirmed computationally. The second main result identifies up to homotopy the natural moduli space of several in-

Congratulations, Peter!

Seminar: Etale Kohomologie, WS 2007/08

"Programm": Wir haben die ersten drei Kapitel des Artikels von Deligne in SGA 4 1/2 gelesen

Vorträge

1	Treuflacher Abstieg	P. Scholze
2	Grothendieck-Topologien	S. Hähne
3	Etale Morphismen	T. Richarz
4	Die etale Fundamentalgruppe	R. Kucharczyk
5	Henselsche Ringe	A. Müller
6	Etale Garben	P. Hartwig
7	Halme, direktes Bild	A. Ivanov
8	Die Brauer-Gruppe	R. Kucharczyk
9	Der Satz von Tsen	F. Hellmann

Congratulations, Peter!