**Algebraic Geometry I** Prof. Dr. Ulrich Görtz
WS 2025/26 Dr. Andreas Pieper

## Problem sheet 2

Due date: Nov. 4, 2025.

**Problem 5** Let $k$ be a field and $n \in \mathbb{N}$, $n \geqslant 1$. Consider a polynomial

$$f = \sum_{i=0}^{n} c_i x^i \in k[x]$$

of degree $n$, i.e. $c_n \neq 0$. The goal of this exercise is to show that there exists a polynomial expression in the $c_i$ called the *discriminant* of $f$, denoted $\mathrm{disc}(f)$ such that: $\mathrm{disc}(f) = 0$ if and only if $f$ has a multiple root.

(1) Denote
$$k[x]_{\leqslant d} := \{h \in k[x] \colon \deg(h) \leqslant d\}.$$
For a polynomial $g \in k[x]_{\leqslant m}$ we define the *Sylvester matrix* $M(f, g)$ as the matrix of the linear map
$$\varphi\colon k[x]_{\leqslant m-1} \oplus k[x]_{\leqslant n-1} \to k[x]_{\leqslant m+n-1},\ (a, b) \mapsto af + bg.$$
(we equip these spaces with the obvious bases consisting of powers of $x$).

Define the *resultant* as $\mathrm{res}(f, g) = \det(M(f, g))$. Show that $\mathrm{res}(f, g) = 0$ if and only if $\gcd(f, g) \neq 1$.

*Hint:* When is $\varphi$ injective?

(2) Define[1] $\mathrm{disc}(f) = \mathrm{res}(f, f')$ (for $m = n - 1$) and observe that this is in fact a polynomial expression in the coefficients of $f$. Show that $\mathrm{disc}(f) = 0$ if and only if $f$ has a multiple root.

[1]Depending on the order of the bases chosen above, there is a sign ambiguity and for the usual definition of *the* resultant of two polynomials and *the* discriminant of a polynomial a specific ordering and hence a specific sign are chosen, and then $\mathrm{disc}(f) = (-1)^{n(n-1)/2}\mathrm{res}(f, f')$ but this is irrelevant for our purposes and we ignore this issue here.

**Problem 6**

Let $k$ be an algebraically closed field. Let $n \geqslant 1$. We identify the space $M := \mathrm{Mat}_{n\times n}(k)$ of $(n \times n)$-matrices with entries in $k$ with $k^{n^2}$ and equip it with the Zariski topology. By Problem 2 (2), it is irreducible.

(1) Prove that the subset of $M$ consisting of matrices $A$ such that $\mathrm{charpol}_A(A) = 0$ is closed in $M$ (without using the Theorem of Cayley-Hamilton).

(2) Use Problem 5 to prove that the subset of diagonalizable matrices with $n$ different eigenvalues in $k$ is open in $M$.

(3) Prove the Theorem of Cayley-Hamilton, i.e., prove that the subset defined in (1) equals all of $M$.

**Problem 7** The goal of this exercise is to prove the analogue of the abc-conjecture for the polynomial ring $\mathbb{C}[x]$:

**Theorem 0.1.** *Let $a, b, c \in \mathbb{C}[x]$ be such that $a + b = c$. We assume that $a$, $b$, $c$ are coprime, and not all constant. Then*

$$\max(\deg(a), \deg(b), \deg(c)) < |\{\alpha \in \mathbb{C}\colon (abc)(\alpha) = 0\}|.$$

(1) Define $\Delta(x) := \det \begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$. Show that $\Delta \neq 0$ and

$$\Delta(x) = \det \begin{pmatrix} a & c \\ a' & c' \end{pmatrix} = \det \begin{pmatrix} c & b \\ c' & b' \end{pmatrix}$$

(2) Show that

$$\deg(\Delta) \leqslant \min(\deg(a)+\deg(b)-1, \deg(a)+\deg(c)-1, \deg(c)+\deg(b)-1).$$

(3) Use the expression $\Delta = ab' - ba'$ to show that $a$ divides

$$\Delta(x) \prod_{\alpha\in\mathbb{C}\colon a(\alpha)=0} (x - \alpha)$$

and show an analogous statement for $b, c$.

(4) Deduce that $abc$ divides

$$\Delta(x) \prod_{\alpha \in \mathbb{C}:\ (abc)(\alpha)=0} (x - \alpha)$$

(5) Combine (3) and (5) to prove Theorem 0.1.