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Problem sheet 4

Due date: Nov. 18, 2025.

Problem 12 Let $(I, \leqslant)$ be a partially ordered set. An *inductive system* of sets (with index set I) is a family X_i, $i \in I$, of sets together with maps $\varphi_{ji}\colon X_i \to X_j$ for all pairs $i, j \in I$ with $i \leqslant j$, such that

$$\varphi_{ii} = \mathrm{id}_{X_i}, \quad \varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}$$

for all $i \leqslant j \leqslant k \in I$.

A set C together with maps $\psi_i\colon X_i \to C$ such that $\psi_j \circ \varphi_{ji} = \psi_i$ for all $i \leqslant j$ is called a *colimit* (or *direct limit* or *inductive limit*) if it satisfies the following "universal property": For every set T together with maps $\xi_i\colon X_i \to T$ such that $\xi_j \circ \varphi_{ji} = \xi_i$ for all $i \leqslant j$, there exists a unique map $\chi\colon C \to T$ such that $\chi \circ \psi_i = \xi_i$ for all i. The colimit is also denoted by $\mathrm{colim}_{i\in I} X_i$ or by $\varinjlim_{i\in I} X_i$. (The maps φ_{ij} and ψ_i are usually omitted from the notation.)

(a) Suppose that I is *directed*, i.e., for all $i, j \in I$ there exists $k \in I$ with $i \leqslant k$ and $j \leqslant k$. Let (X_i, φ_{ji}) be an inductive system of sets, let U be the disjoint union

$$U = \coprod_{i\in I} X_i,$$

and consider the following relation on U: for $x, y \in U$, say $x \in X_i$, $y \in X_j$, we set $x \sim y$ if and only if there exists $k \geqslant i, j$ with $\varphi_{ki}(x) = \varphi_{kj}(y)$. Prove that $\sim$ is an equivalence relation and that the set $U/\sim$ of equivalence classes together with the natural maps $X_i \to U/\sim$ is a colimit of the system (X_i, φ_{ji}).

(b) Assume that I is directed, that all X_i are subsets of a set X, that $i \leqslant j$ if and only if $X_i \subseteq X_j$, and that the maps φ_{ji} are the inclusion maps. Prove that the union $C := \bigcup_i X_i$ with the inclusion maps $X_i \to C$ is a colimit of the X_i.

Problem 13 Let I be a partially ordered set.

(a) Define the notion of *colimit* (with index set I) in an arbitrary category. (If you do not know the notion of category, you can replace this problem by defining a suitable notion of colimit for (a) abelian groups, (b) rings, (c) topological spaces.)[1]

(b) Suppose that I is directed. Prove that all colimits with index set I in the category of abelian groups exist.

(c) Suppose that I is directed. Prove that the functor colim_i on the category of abelian groups is exact, i.e.: Let (A_i, φ_{ji}), (B_i, ψ_{ji}), (C_i, ξ_{ji}) be inductive systems of abelian groups indexed by I. Suppose that for each i, there are short exact sequences

$$0 \to A_i \to B_i \to C_i \to 0$$

which are compatible with the maps φ_{ji}, ψ_{ji}, ξ_{ji} in the obvious sense. Prove that these sequences induce a sequence

$$0 \to \mathrm{colim}_i A_i \to \mathrm{colim}_i B_i \to \mathrm{colim}_i C_i \to 0$$

which is again exact.

Problem 14

Let A be a ring. We call an element $e \in A$ *idempotent* if $e^2 = e$. Show that the following conditions are equivalent:

(i) $\operatorname{Spec} A$ is not connected.

(ii) There exists an idempotent element $e \in A$ different from 0 and 1.

(iii) There exists a ring isomorphism $A \cong A_1 \times A_2$ with non-zero rings A_1, A_2.

[1] It is not required that you prove the existence of colimits in these cases, but it may be enlightening to try to find a situation where colimits do not exist.

Problem 15 Let A be a ring. Consider the set [2]

$$\mathrm{Spec}'(A) = \{\varphi : A \to K | \varphi \text{ a ring homomorphism, } K \text{ a field}\} / \sim$$

where $\sim$ is the following equivalence relation: $\varphi_1 : A \to K_1, \varphi_2 : A \to K_2$ satisfy $\varphi_1 \sim \varphi_2$ if there exist a field K_0, and ring homomorphisms $A \to K_0, K_0 \to K_1, K_0 \to K_2$ such that the diagram

$$\begin{tikzcd} & K_1 \\ A \arrow[ur, "\varphi_1"] \arrow[r] \arrow[dr, "\varphi_2"'] & K_0 \arrow[u] \arrow[d] \\ & K_2 \end{tikzcd}$$

commutes.

(1) Show that $\sim$ is an equivalence relation.

(2) Show that there exists a bijection

$$\mathrm{Spec}'(A) \to \mathrm{Spec}(A).$$

[2]There are some set theoretic issues here which we will ignore.