Algebraic Geometry I — Prof. Dr. Ulrich Görtz
WS 2025/26 — Dr. Andreas Pieper

Problem sheet 5

Due date: Nov. 25, 2025.

Problem 16

Let A be a ring, $f \in A$, M an A-module. Consider the inductive system of A-modules (with index set $\mathbb{Z}_{\geqslant 0}$)

$$M \to M \to M \to \dots$$

where all transition maps are given by multiplication by f. Show that there exists a natural isomorphism between the colimit $\mathrm{colim}_i M$ and the localization M_f of the A-module M with respect to the element f.

Problem 17

(1) Let $\psi\colon A \to B$ be an injective ring homomorphism between reduced rings. Show that every minimal prime ideal of A is in the image of ${}^a\psi : \operatorname{Spec} B \to \operatorname{Spec} A$. Give an example where ${}^a\psi$ is not surjective.

(2) Let $\varphi\colon A \to B$ be a ring homomorphism, and consider the associated map $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$. Assume that f is bijective and that f reflects specialization, i.e., for $x, x' \in \operatorname{Spec} B$ we have $f(x') \in \overline{\{f(x)\}}$ if *and only if* $x' \in \overline{\{x\}}$. Show that f is a homeomorphism.

Hint. In (2), we need to show that f is closed. Let $\mathfrak{b} \subset B$ be a radical ideal. Apply (1) to the ring homomorphism $A/\varphi^{-1}(\mathfrak{b}) \to B/\mathfrak{b}$ induced by φ and then use that f reflects specialization to show that $f(V(\mathfrak{b})) = V(\varphi^{-1}(\mathfrak{b}))$.

Problem 18 Let X be a topological space, and M an abelian group. We define the *constant presheaf* $\underline{M}^{\mathrm{pre}}$ to be the presheaf with

$$\underline{M}^{\mathrm{pre}}(U) = M$$

for any open $U \subset X$.

(1) Assume now that X is irreducible. Show that $\underline{M}^{\mathrm{pre}}$ is a sheaf.

(2) Assume still that X is irreducible. Let $\mathcal{F}$ be a sheaf of abelian groups on X. We say that $\mathcal{F}$ is *locally constant* if there exists an open covering $X = \bigcup_{i \in I} U_i$ and abelian groups M_i, $i \in I$ such that

$$\mathcal{F}_{|U_i} \cong \underline{M_i}^{\mathrm{pre}}_{|U_i}, \ \forall i \in I.$$

Show that any locally constant sheaf on an irreducible topological space is constant.