SEMINAR MIRROR SYMMETRY AND THE BREUIL-MÉZARD CONJECTURE

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1. Talks

1.1. **Introduction [16.04].** (Vytas)

1.2. **Galois representations 1 [23.04].** Recall the notion of potentially semistable representation ρ, inertial (or Galois) type τ associated to it, and the p-adic Hodge type $\mathbf{v}$ (the Hodge–Tate weights) associated to it, following [8, §1.1]. (Saverio)

Explain the statement of the geometric version of the Breuil–Mézard conjecture for deformation rings, [2, Conjecture 4.2.1]. In particular, we need to see that one can attach a smooth irreducible representation of $\mathrm{GL}_n(\mathcal{O}_F)$ $\sigma(\tau)$ to the inertial type and an algebraic representation to the Hodge–Tate weight . In [4] only tame τ are considered. In this case the action of $\mathrm{GL}_n(\mathcal{O}_F)$ factors on $\sigma(\tau)$ factors through the quotient $\mathrm{GL}_n(k_F)$ and $\sigma(\tau)$ is described explicitely in [5, Proposition 9.2.1], see also [6, Proposition 6.14]. Give some examples for GL_2, either in [8] or [2].

1.3. **Galois representations 2 [30.04].** Kisin modules and lattices in crystalline Galois representations. Explain the content of [7], mainly Theorem 0.1; in particular, explain how the condition on the connection describes the essential image in Corollary 1.3.15. Mention that in Barsotti-Tate case the monodromy condition does not appear. (Sebastian?)

1.4. **Galois representations 3 [07.05].** Introduce (φ, Γ)-modules with coefficients [3, Definition 2.7.6], mention relation to Galois representations, Introduce Emerton-Gee stack for the group GL_n [3, Definition 3.2.1], state main properties [3, Theorem 1.2.1], in particular that irreducible components are indexed by Serre weights, for $\lambda \in X^*(T)^+$ and inertial type τ, introduce $\mathcal{X}^{\lambda,\tau} \hookrightarrow \mathcal{X}^{EG}$ [3, Theorem 4.8.12], if λ regular these are of relative dimension $n(n-1)/2$ otherwise smaller. Mention relation to Galois-deformation rings as versal rings from the first talk.

If time permits, discuss the case $n = 1$ ([3, Chapter 7] and [10]) (Pretend we all know what a formal algebraic stack is but get ready for questions :)).

1.5. **Local model theory 1 [21.05].** Usual affine Grassmanian for GL_n, Representability, see [11]. Families of affine Flag varieties, Pappas–Zhu local model for GL_n, Relation of Pappas–Zhu local model and moduli of Breuil-Kisin modules, Relation to Emerton–Gee stack ([9, Section 5]).

1.6. **Local model theory 2 [28.05].** Appendix B in v3 (the version before the last one on arXiv, which is done for GL_n, [4]). The aim is to explain the map in Definition B.3.2.

1.7. **Affine Springer fibres and monodromy-centralizer action [11.06].** Introduce affine Springer fibres following [12, Lecture II], representability, examples SL_2, affine Springer action. In reality we need actions on equivariant Borel–Moore homology [4, §5.6]. Stick to groups like SL_n or GL_n when it makes the exposition easier.

1.8. **BMR localisation [18.06].** Explain [4, Example 6.5.2] in detail, via [1, Theorem 5.3.1]. Explain what happens to baby Verma module, [1, Example 5.3.3 0)] and [1, 3.1.4]. The discussion of the centre in [4, §6.2] should appear. Explain relation to representation of Frobenius kernels [4, §6.4].

1.9. **Degenerations of local models [25.06].** Define deformed affine Springer fibers as in [4, Section 3.2]. Mention that for $\varepsilon = 0$ these spaces are related to affine Springer fibers and for $\varepsilon = 1$ to the local models appearing in Talk 6. We want to understand the map in [4, Definition 4.2.4] that moves cycles from the fiber $\varepsilon = 0$ to $\varepsilon = 1$ and for this we need to understand why [4, Lemma 4.2.3] is true.

1.10. **Mirror symmetry I [02.07].** Here it would make sense to work with reductive groups G since we pass from G to the Langlands dual $G^{\vee}$. Explain the content of [4, Section 7]. We need most of all [4, Proposition 7.0.1]. Maybe change content later?

1.11. **Mirror symmetry II or Recognition Principle [09.07].** Continue with the explanation of [4, Section 7] or say something about the key [4, Theorem 8.4.1]. Maybe change content later?

1.12. **Construction of the cycles [16.07].** State [4, Theorem 10.1.1]. For this, first recall the map constructed in Talk 6. Then (try to) explain the construction in [4, Definition 10.5.2]. If time remains, say something about the Breuil-Mézard relations.

1.13. **[23.07] Programm discussion for next term.**

Literatur

[1] Roman Bezrukavnikov, Ivan Mirković, and Dmitriy Rumynin. Localization of modules for a semisimple Lie algebra in prime characteristic (with an appendix by R. Bezrukavnikov and S. Riche, Computation for sl(3)). *Ann. Math. (2)*, 167(3):945–991, 2008. doi:10.4007/annals.2008.167.945.

[2] Matthew Emerton and Toby Gee. A geometric perspective on the Breuil-Mézard conjecture. *J. Inst. Math. Jussieu*, 13(1):183–223, 2014. doi:10.1017/S147474801300011X.

[3] Matthew Emerton and Toby Gee. *Moduli stacks of étale* (Φ, Γ)*-modules and the existence of crystalline lifts*, volume 215 of *Ann. Math. Stud.* Princeton, NJ: Princeton University Press, 2023. doi:10.1515/9780691241364.

[4] Tony Feng and Bao Le Hung. Mirror symmetry and the breuil–mézard conjecture. URL: https://arxiv.org/abs/2310.07006.

[5] Toby Gee, Florian Herzig, and David Savitt. General Serre weight conjectures. *J. Eur. Math. Soc. (JEMS)*, 20(12):2859–2949, 2018. doi:10.4171/JEMS/826.

[6] Florian Herzig. The weight in a Serre-type conjecture for tame n-dimensional Galois representations. *Duke Math. J.*, 149(1):37–116, 2009. doi:10.1215/00127094-2009-036.

[7] Mark Kisin. Crystalline representations and F-crystals. In *Algebraic geometry and number theory. In Honor of Vladimir Drinfeld's 50th birthday*, pages 457–496. Basel: Birkhäuser, 2006. doi:10.1007/978-0-8176-4532-8_7.

[8] Mark Kisin. The structure of potentially semi-stable deformation rings – proceedings of icm. II:294–311, 2010. URL: https://people.math.harvard.edu/%7Ekisin/dvifiles/licm.pdf.

[9] Daniel Le, Bao V. Le Hung, Brandon Levin, and Stefano Morra. Local models for Galois deformation rings and applications. *Invent. Math.*, 231(3):1277–1488, 2023. doi:10.1007/s00222-022-01163-4.

[10] Dat Pham. The Emerton-Gee stack of rank one (φ, Γ)-modules. *Doc. Math.*, 29(3):733–746, 2024. doi:10.4171/DM/947.

[11] Timo Richarz. Basics on affine grassmannians. URL: https://www.mathematik.tu-darmstadt.de/media/algebra/homepages/richarz/Notes_on_affine_Grassmannians.pdf.

[12] Zhiwei Yun. Lectures on Springer theories and orbital integrals. In *Geometry of moduli spaces and representation theory*, volume 24 of *IAS/Park City Math. Ser.*, pages 155–215. Amer. Math. Soc., Providence, RI, 2017.